Manipulative Advertising by a Monopolist*

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PRELIMINARY DRAFT

Abstract

Firms spend significant amounts of money on advertising. Some of these ads take the form of truthful information disclosure about product attributes, while some others are designed to mislead consumers. This paper studies the latter type of advertising using a signal jamming framework under asymmetric information. Consumers are uncertain about the quality of an experience good supplied by a monopolist. Deceptive ads by the monopolist increase the mean of a noisy but otherwise unbiased signal about the quality level. Ad levels are not perfectly observable and depend on the type of monopolist. With some random noise in signals, consumers face a signal filtering problem. Even though they perfectly anticipate the advertising strategies of each type and process their private signal accordingly, we show that in equilibrium deceptive ads influence consumer demand. Intensity of manipulative advertising does not necessarily increase with quality, and its effect on consumer surplus may be positive or negative depending on the price level.

JEL: D8, K4, L1, L4.
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1 Introduction

Deceptive advertising is ubiquitous. People are constantly exposed to messages by politicians and firms that are misleading by design. When the interests of senders and receivers are not fully aligned, these messages may harm the receivers to the extent they are persuasive. Even though voters and consumers are typically aware of such conflicts of interest and do not take senders’ messages at face value, oftentimes, they end up voting for the wrong candidate or making a purchase that is ex-post inferior. One common and intuitive way to explain such failures is to attribute the persuasive power of ads to behavioral biases or bounded rationality on the part of receivers (e.g. Mullainathan et al. 2008). Indeed, some evidence suggests that consumers are more easily persuaded by messages that appeal to intuition rather than reason (Bertrand et al. 2010). Another explanation would be that our prior beliefs are biased toward inferior choices to begin with, and ads simply reinforce what we already want to believe in.

This paper proposes an alternative explanation in which consumers are fully rational, immune from behavioral biases, and hold beliefs that are a priori unbiased. We study false advertising about the quality of an experience good. Persuasive (i.e. deceptive) power of ads stems from an informational problem buyers face. Since consumers cannot verify the quality of the experience good before the purchase, they draw on various sources of information such as word-of-mouth advice or online reviews to make a decision. Yet consumers differ with respect to the sources of information they are exposed to. The social network one belongs to, the type of media one follows and many other factors can affect one’s information sampling process in ways that are both hard to predict (due to an idiosyncratic component) and that vary across different people. In other words, the opinions or reviews consumers hear or read about a product can be thought as containing random biases of different strength and direction that cannot be perfectly anticipated even by consumers themselves. The goal of this paper is to explore the implications of such an information collection process. The main question we tackle is whether under some circumstances firms can exploit this communication environment through misleading ads.\footnote{Empirical evidence suggests that in some sectors we indeed observe manipulative practices that are reminiscent of the mechanism we propose. For example, Mayzlin, Dover and Chevalier (2014) show that the evidence from online hotel reviews are consistent with the employment of fake promotional reviewers by hotels. Another example is pharmaceutical research, where pharmaceutical companies have incentives to manipulate the research process to get more favorable research outcomes (see e.g.}
argue that firms can indeed manipulate noisy information channels so as to generate a systematic inference problem for buyers.

We propose a game-theoretic model with asymmetric information. A monopolist tries to sell an experience good of either high or low quality to a continuum of consumers with unit demand at an exogenously given price. Quality is directly observable to the monopolist but not to consumers. Instead, consumers receive a noisy signal about quality. Manipulative advertising is modeled as an unobserved action by the monopolist that shifts the mean of this otherwise unbiased signal. Under this communication environment, although consumers perfectly anticipate the advertising strategies of each type of monopolist and update their beliefs accordingly, in equilibrium, the monopolist will be able to shape average beliefs via manipulation and consequently influence demand for its product. The systematic bias in consumer beliefs stems from the fact that private signals contain a random noise and advertisement levels are unobserved and state-dependent (differ by the type of monopolist). As a result, consumers not only remain uncertain about the state, but they also assign a different likelihood to each state than they would in the absence of manipulation. We demonstrate that the low [high] quality monopolist will enjoy higher demand under manipulative advertising whenever prices are lower [higher] than the quality level consumers expect under their common prior beliefs. As a result, the welfare consequences of manipulation also depends on the price level. In particular, the effect on ex-ante consumer surplus tends to be negative at low prices and positive at high prices. Firm profits on the other hand depend not only on sales but also costs of ads. We show that sellers' ability to successfully manipulate consumers inevitably leads to some wasteful spending on advertisement. The two types of monopolist engage in an implicit arms race where false ads by one type trigger more effort by the other to mislead consumers. We argue that the problem of designing an optimal advertising policy under such an environment is not trivial since it requires identification of market specific details.

While most of the theoretical literature on advertising focuses on truthful advertising, there are some recent models of deceptive advertising to which our paper relates. The existing work modeled misleading advertisement in various ways. Most of the time, deceptive messages are taken as given, instead of being derived as an equilibrium choice of firms, and consumers are assumed to take these messages at face value rather than rationally discounting them (e.g. Hattori and Higashida 2012). Some

Finucane and Boult 2004 and Sismondo 2008).
papers allow for false or unsubstantiated claims and study how various regulations on product advertisement influence the degree of information acquisition and disclosure by firms. Yet they take the strength of claims about product quality as exogenously given (e.g. Corts 2013). Also, in most of these papers, false claims are supported in equilibrium only when firms are uncertain of their own product quality, i.e. there is no intentional misinformation by firms (e.g. Corts 2014). Some recent papers feature intentionally false claims (Rhodes and Wilson 2015, Piccolo, Tedeshi and Ursino 2015, 2016). Some equilibria in these models feature deceptive advertising by low quality firm, which does affect consumers’ posterior beliefs about product quality. While these papers let the low-quality firm pretend to be a high-quality firm, they do not allow the high-quality firm to make false claims (i.e. exaggerate its product quality). This is a restrictive assumption since high-quality firm cannot use ads to separate itself from the low-quality firm. In contrast, the manipulation framework we have in mind allows for false advertisement by all firms. Our project is not the only study to model false advertisement as an unobservable bias sellers introduce to otherwise unbiased but noisy signals. Drugov and Troya-Martinez (2015) adopt a similar signal jamming approach. In their paper consumer tastes are heterogeneous, and the quality of the match between the firm’s product and an individual consumer’s taste is observable neither by the consumer nor the firm. The paper features positive bias in seller’s advice (or advertisement) in equilibrium. However, since the seller cannot condition this bias on the quality of a particular match, false advice does not affect total sales. In this project, we have a different motivation; namely to support an equilibrium in which false advertising has a real effect on demand. In contrast to Drugov and Troya-Martinez (2015), our project will feature asymmetric information between firms and consumers, such that firms will observe the quality of their product while consumers will be uncertain about it.

Information manipulation by hidden unobservable actions have been studied in various contexts. Earlier work such as by Matthews and Mirman (1983), and Fudenberg and Tirole (1986) concentrated on manipulation of unobservable pricing decision by an incumbent firm to deter a potential entrant to the market. More recently, Edmond (2013), Caselli, Cunningham, Morelli and Moreno Barreda (2014), and Aköz and Arbatli (2016) studied information manipulation in political context. We contribute to this literature by analyzing the impact of the additional variation (in our case this is price) in the benefit function of the manipulating player. Price has three types effects
on manipulation. Higher prices increase the marginal benefit of doing manipulation, which increase the incentive to increase manipulation by both types of monopoly. On the other hand, higher prices reduce the consumer surplus, therefore make harder for the consumers to be convinced to buy the product. Because of this secondary effect, we find that for extremely high prices manipulation by both types of monopoly are very close to zero. The third effect of price is to determine the relative advantage in the implicit competition between the types of monopoly. We find that lower (higher) prices enable the low(high)-quality type to manipulate a larger set of consumers.

We propose a special case of costly Bayesian persuasion. However, in our model the sender learns its type before committing to a manipulation plan and the cost of persuasion increases as the bias in the information that the receivers have. This is in contrast with the canonical model of Bayesian persuasion by Kamenica and Gentzkow (2011), and with follow up study in (2014), where providing a more precise information is costly, instead of manipulation. In our model, manipulation by each type increases the incentive to do more manipulation by the other type. Therefore, we find that both types of monopoly engages in manipulation even if consumers perfectly adjusts their posterior beliefs, canceling any effect of manipulation on revenues. This type of behavior is absent in the models by Kamenica and Gentzkow (2011) and (2014), since the commitment power that a monopoly has enables it to shut down manipulation by a particular type if manipulation is not effective.

The rest of the paper is organized as follows. Section 2 lays out the model and characterizes the equilibrium. Section 3 presents the main analytical and numeric results regarding the effect of manipulation on total demand and consumer surplus. Section 4 provides some intuition about why deceptive ads are effective in influencing average behavior, and section 5 concludes the paper.

2 Model

Consider a monopolist releasing a new product whose quality is unknown to consumers $i \in [0, 1]$. The monopolist can be one of two types $j \in \{L, H\}$ based on the quality of the product it supplies. In particular, each type produces a product of quality $v_j$ such that $0 \leq v_L < v_H$. We index the monopolist’s type by the product quality it supplies. The marginal costs of production do not depend on product quality and are normalized to zero without loss of generality.
At a certain cost, the monopolist of type \( j \) can send an advertising signal

\[
x_i = v_j + a_j + \varepsilon_i, \tag{1}
\]

where \( a_j \equiv a(v_j) \geq 0 \) is the degree of manipulative advertisement chosen by monopolist \( j \) and \( \varepsilon_i \sim F \) is a random noise in the advertising signal with a known cumulative distribution function \( F \). Prior to observing their private advertising signals, consumers hold a common prior belief \( G : \{v_L, v_H\} \to [0, 1] \) about the product quality (hence the type) of the monopolist they face. Following Assumption 1, states the restrictions we put on the noise distribution and parametrizes the prior distribution.

**Assumption 1** Density function \( f \) for the random noise in advertising signals is continuous, log-concave, symmetric around zero and unimodal with unbounded support and finite moments. \( G \) is a discrete distribution function such that \( Pr(v_L) = g \in (0, 1) \) and \( Pr(v_H) = 1 - g \).

Upon observing her private signal each consumer decides whether to buy one unit of the monopolist’s good. Ex-post utility of each consumer who purchased the product offered by monopolist \( j \) is given by

\[
u = v_j - p_j
\]

where \( p_j \) is the unit price announced by monopolist \( j \) and \( v_j \) is the product quality. If the product is not purchased, then \( u = 0 \). We denote the binary purchase decision of the consumer by a function \( s(x_i) \in \{0, 1\} \) so that \( i \) buys the good, i.e., \( s(x_i) = 1 \) if and only if \( E[v_j|x_i] \geq p_j \).

We assume that the market has a regulated price \( \bar{p} \in (v_L, v_H) \) and the monopoly commits to setting the price at \( \bar{p} \) prior to learning its product quality type. The regulation on the price, which could be exogenously regulated by a public authority or be the outcome of the decision of the monopoly before supplying to this market, strips the price off its informative function and reduces it to a parameter of transfer of surplus from consumers to the monopoly.\(^2\)

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\(^2\)If the monopoly could choose the price strategically, the price would not necessarily have any informative value. Indeed, it is possible to support every exogenous price as an outcome of a pooling equilibrium if the value of the low quality \( v_L \) equals 0. However, endogeneizing price has non-
Since the noisy signals are the only information sources for consumers, they base their decision solely on the information they receive. Therefore, the sales of monopolist of type \( j \) is given by

\[
S(v_j, a_j) = \int_0^1 s(x_i)di = \int_0^1 s(v_j + a_j + \varepsilon_i)di,
\]

where \( s(x_i) \) is indicator function for the purchasing decision made by the consumer.

The profit to the monopolist of type \( j \) is given by

\[
\pi_j = p_jS(v_j, a_j) - C(a_j),
\]

where \( C(.) \) is the cost of advertising and it is equal to

\[
C(a_j) = \begin{cases} 
0 & \text{if } a_j = 0 \\
\bar{c} + c(a_j) & \text{if } a_j > 0,
\end{cases}
\]

where \( \bar{c} > 0 \) is the fixed cost of manipulative advertising and \( c(.) \) is the variable cost of manipulative advertising. We assume for the benchmark model that the fixed cost of advertising to be 0. We discuss the implications of corner solutions caused by non-zero fixed cost of advertising in the “Extensions” section.

Following Assumptions 2 and 3 state the restrictions we put on the advertising costs. Assumption 2 guarantees rules out the trivial cases, where the monopolist does not prefer to do any advertising. Assumption 3 imposes strict concavity to the profit function of the monopoly and therefore a unique advertisement level at each quality level.

**Assumption 2** The cost function \( C(.) \) satisfies \( C'(0) = 0 \), and \( C''(a), C'(a) > 0 \) for all \( a > 0 \).

**Assumption 3** \( \min_{a \geq 0} C''(a) > v_H \max_x f'(x) \), where \( f \) is the pdf of the distribution function \( F \).

trivial implications for the characterization of the set of equilibria. Since we focus on manipulative advertising in this paper, we abstract away from any informative value of pricing by assuming that it is exogenously fixed.
We employ symmetric pure-strategy Perfect Bayesian Equilibrium (PBE) as the solution concept for our analysis. Intuitively, PBE requires sequential rationality and Bayesian update for posterior beliefs whenever possible. More formally a strategy profile is the advertisement choice of each type of monopoly \((a_L, a_H)\) and the purchasing decision of the consumer after observing the noisy signal \(x s(x)\). Then a strategy profile 

\[
\langle a^*_L, a^*_H, s^*(\cdot) \rangle
\]

accompanied with the posterior belief of a consumer \(\mu(x)\) who observed the signal \(x\) is a symmetric pure-strategy PBE if and only if

\[
\begin{align*}
a^*_L &\in \arg\max_{a_L \in \mathbb{R}^+} \bar{p}S(a_L, a^*_H, v_L) - C(a_L) \\
a^*_H &\in \arg\max_{a_H \in \mathbb{R}^2} \bar{p}S(a^*_L, a_H, v_H) - C(a_H) \\
S(a^*_L, a^*_H, v_j) &= \int_{-\infty}^{\infty} s(x)f(x - v_j - a^*_j)dx \\
s(x) &= \begin{cases} 
1 & \text{if } \sum_j (v_j - \bar{p})\mu(x)(v_j) \geq 0 \\
0 & \text{otherwise}
\end{cases} \\
\mu(x)(v_j) &= \frac{Pr(x|v = v_j)G(v_j)}{Pr(x|v = v_L)g + Pr(x|v = v_H)(1 - g)},
\end{align*}
\]

where \(Pr(x, p|v = v_j)\) is the probability that the signal \(x\) and the price \(p\) to be realized on the equilibrium path given that the type of the monopoly is \(v_j\).

### 2.1 Equilibrium Analysis

Assumption 1 puts some regularity conditions on the posterior beliefs that consumers could have. In particular, consumers’ posterior expectation of quality given the signal they receive is strictly increasing with the signal. Therefore, the purchasing decision of consumers admits a simple monotonic threshold structure. To show that we will first suppose such a monotonic strategy by consumers and calculate the advertising decision of the monopoly, and then we will confirm that consumers’ decisions confirm our supposition.

Suppose that the consumers follow a monotonic threshold strategy \(\bar{x}\) such that
s(x_i) = 1 if and only if x_i ≥ ¯x. Then the monopolist chooses the level of manipulative advertisement a_j that solves

\[
\max_{a_j \geq 0} \bar{p}[1 - F(\bar{x} - v_j - a_j)] - C(a_j) \quad \text{for all } j \in J
\]  

(7)

The first order condition to this problem is given by

\[
\bar{p}f(\bar{x} - v_j - a_j) = C'(a_j) \quad \text{for all } j \in J
\]  

(8)

Assumption 1 combined with Assumption 2 ensures that for any j an interior solution a_j > 0 to this problem exists. Moreover, if we further assume Assumption 3, we can ensure that the solution is also unique.

Given the levels of manipulative advertising, consumers form their equilibrium beliefs using Bayesian update as follows. When a consumer receives an advertising signal x, her posterior expectation of the product quality will be

\[
\mathbb{E}(v|x) = \frac{\sum_j v_j f(x - v_j - a_j)G(v_j)}{\sum_j f(x - v_j - a_j)G(v_j)}
\]  

(9)

Therefore, at price \(\bar{p}\), a consumer will be indifferent between buying and not buying the good if and only if she receives a signal \(\bar{x}\) which satisfies

\[
\mathbb{E}(v|\bar{x}) = \frac{\sum_j v_j f(\bar{x} - v_j - a_j)G(v_j)}{\sum_j f(\bar{x} - v_j - a_j)G(v_j)} = \bar{p} \iff \sum_j (v_j - \bar{p})f(\bar{x} - v_j - a_j)G(v_j) = 0.
\]  

(10)

Four equations given by (8) and (10) determine the equilibrium with manipulative advertising.

If there were no manipulative advertising, then consumers would form their posterior beliefs as

\[
\mathbb{E}(v|x) = \frac{\sum_j v_j f(x - v_j)G(v_j)}{\sum_j f(x - v_j)G(v_j)},
\]

where x is any signal that a consumer might receive. Then, given a price \(\bar{p}\), the
condition that determines the signal $x$ under which the consumer is indifferent between buying and not buying is given by

$$
\mathbb{E}(v|x) = \frac{\sum_j v_j f(x - v_j) G(v_j)}{\sum_j f(x - v_j)} = \bar{p} \iff \sum_j (v_j - \bar{p}) f(x - v_j) G(v_j) = 0.
$$

(11)

The consumers’ problem of estimating the quality level is not trivial. A Bayesian consumer knows that she is manipulated by the Monopoly, thus she has to adjust her posterior belief accordingly. By Assumption 1 if there were no manipulation in the information that the consumer receives, a higher signal would directly translate into a higher likelihood of the H-type Monopoly. However, when there is manipulation, a higher signal might mean a higher quality or higher manipulative advertisement by the Monopoly. For the posterior expectations to be monotonic in signals, the quality difference should dominate the difference in manipulation, which requires a sufficient increase in cost of advertisement compared to the response of consumers to higher signals. This way, the same noise value $\varepsilon$ would lead to a higher signal $x$ when the underlying quality is high. That is $v_H + a_H + \varepsilon > v_L + a_L + \varepsilon$. Following Lemma 1 shows that Assumption 3 or the weaker Assumptionas:monovav below are sufficient for that.

**Assumption 4** Suppose that cost function and the quality levels satisfy the following inequality

$$
C'(v_H - v_L) \geq v_H f(0).
$$

**Lemma 1** Suppose that Assumptions 4 and 3 hold. Assumption 3 or Assumption 4 implies that $a_H + v_H > a_L + v_L$.

We present the proofs in the Appendix A.

Note that Assumption 4 is a weaker condition than Assumption 3 since simple integration shows that Assumption 3 implies Assumption 4. Therefore, for the following Lemma, which states that the posterior expectation is monotonic in signal, to hold we do not require a strictly concave profit function but only a sufficient quality difference between two types of Monopoly.
Lemma 2 Suppose that Assumptions 1 and 2 and Assumption 4 hold. Then, $E[v|x]$ is strictly increasing in $x$.

The most immediate consequence of Lemma 2 is the existence and uniqueness of a threshold $\bar{x}$ for every manipulation pair $(a_L, a_H)$ by the Monopoly.

Corollary 1 Suppose that Assumptions 1, 2 and 4 hold. For any price $\bar{p} \in (v_L, v_H)$, there is a unique threshold $\bar{x}$ for advertising signals such that only those consumers with $x > \bar{x}$ will purchase the product. It is defined by the condition $E[v|\bar{x}] = \bar{p}$.

Combining the first order conditions (8) and Corollary 1 establishes the existence of an equilibrium with positive manipulative advertisement levels conditioned on a price, $\bar{p}$. Following Theorem provides conditions for existence and uniqueness of such equilibria.

Theorem 1 Suppose that Assumptions 1, 2 and 4 hold. For each price level $\bar{p} \in (v_L, v_H)$, there exists a Perfect Bayesian Equilibrium, characterized by manipulative advertisement levels $a_L, a_H > 0$, and a signaling threshold $\bar{x}$, such that any consumer $i$ who receives signal $x_i$ purchases the product if and only if $x_i \geq \bar{x}$.

Moreover, if Assumption 3 holds as well, for price level $\bar{p} \in (v_L, v_H)$, there exists a unique equilibrium.

One of the important implications of Theorem 1 is that there is always some advertising. The Monopoly chooses to spend some of its resources on information manipulation whether it produces a low quality product or a high quality one. This is one of the advantages of the information manipulation approach that we take in this model. This is in contrast with informative advertisement models and most of the variants of false advertisement models, where usually only one of the quality types engage in advertising (see Rhodes and Wilson, 2015 for example).

3 The Implicit Competition Between Types and Effective Manipulation

Theorem 1 implies that both types of the Monopoly do manipulative advertising. That is, the Monopoly spends some of its resources for manipulative advertising
whether it is of high or low quality. Actually, it is not optimal from an ex-ante perspective for the Monopoly to do advertising irrespective of its quality level. To see this, suppose that a regulator publicly announces to reduce the manipulative advertising by the Monopoly uniformly until the advertisement level by at least one type of Monopoly vanishes. That is, if the expected advertisement levels of the Monopoly are such that \( a_L > a_H > 0 \), the agency reduces the advertisement level by \( a_H \), so that \( L \)-type’s advertisement level is \( a_L - a_H \) and \( H \)-type’s advertisement level is 0.

Such an adjustment in the advertisement levels leads consumers to uniformly shift their posterior estimation of the quality since each signal contains less manipulation. Therefore, the threshold that the consumers employ also reduces to adjust for the uniform reduction in the manipulation levels. In particular, the posterior expectation would be

\[
\frac{\sum_j v_j f(\bar{x} - \min\{a_j\} - v_j - (a_j - \min\{a_j\}))G(v_j)}{\sum_j f(\bar{x} - \min\{a_j\} - v_j - (a_j - \min\{a_j\}))G(v_j)} = \frac{\sum_j v_j f(\bar{x} - v_j - a_j)G(v_j)}{\sum_j f(\bar{x} - v_j - a_j)G(v_j)} = \bar{p}.
\]

That is, the Monopoly achieves the same amount of sales by spending uniformly less on advertising, which makes the Monopoly better off. Then, why does the Monopoly engages in such an excessive level of advertising?

One of the factors that lead the Monopoly to increase its advertisement level even further is the implicit competition between the two types of Monopoly. When the Monopoly observes its quality level, it takes the advertisement level by the other type of Monopoly as given. If the consumers expect that the other type of Monopoly would engage in more aggressive manipulation, then the Monopoly would increase its manipulation level as well to balance the consumers’ lower level of estimated quality level. In other words, each type of the Monopoly behaves as if it is in an “arms race” with the other type of Monopoly. The more aggressive a type is in its advertisement decision, the more advantageous it is since consumers cannot make a type specific adjustment in their expectations but an average one.

This implicit arms race between the two types of Monopoly gives the price a second function apart from its direct effect on revenues: determining the competitive advantage of each type against the other type of Monopoly. We show below that the
low (high) quality type is more advantageous for low (high) prices. Intuitively, lower prices are associated with the lower risk of negative consumer surplus $\bar{p} - v_L$. In particular, when the price is lower than the prior expected quality level, the monopoly expects that majority of consumers will receive a favorable enough signal and decide to purchase the product, even if the quality is low and both types engages in the same level of manipulation. In such cases, the low quality type has more power to shift the expectations of consumers by manipulation. To see this consider the following modification of equation (10)

\[(v_H - \bar{p})(1 - g)f(\bar{x} - a_H - v_H) = (\bar{p} - v_L)gf(\bar{x} - a_L - v_L).\]  

(12)

When $(\bar{p} - v_L)g < (v_H - \bar{p})(1 - g)$ or equivalently

$$\bar{p} < (1 - g)v_H + gv_L,$$

the threshold signal $\bar{x}$ that makes consumers indifferent between purchasing and not purchasing the product is such that

$$f(\bar{x} - a_H - v_H) < f(\bar{x} - a_L - v_L),$$

which immediately implies by first order conditions (8) that $a_L > a_H$. Indeed, consumer indifference condition (12) gives us a complete characterization of the comparison of the manipulation levels in terms of price levels.

**Proposition 1** The manipulation levels $a_H = a_L$ if and only if $\bar{p} = (1 - g)v_H + gv_L$. $a_H > a_L$ if and only if $\bar{p} > (1 - g)v_H + gv_L$.

Low quality type uses the competitive advantage the lower prices give to engage in more aggressive manipulation than the high quality type. This in principle should increase the sales of the low quality type. However, low prices are also associated with higher demand. To isolate the impact of manipulative advertising on sales, we will compare the equilibrium sales when there is manipulative advertising and when the manipulative advertising is restricted to be 0 for both type at a fixed price. Specifically, we will tell that type $j$ does effective manipulation at price $\bar{p}$ if

$$1 - F(\bar{x} - a_j - v_j) > 1 - F(\bar{x} - v_j) \Leftrightarrow a_j > \bar{x} - \bar{x}. $$

(13)
Recall that the threshold $x$ is the signal that makes a consumer indifferent between purchasing and not purchasing the product, when there is no manipulation. It is uniquely defined by the equation (11). Therefore, $x - \bar{x}$ is the average adjustment by consumers to manipulation by the Monopoly. If Monopoly of type $j$ is doing more aggressive manipulation than the average adjustment by consumers, consumers fail to fully account for the manipulation by type $j$.

The map of effective manipulation over prices is closely related to the comparison between $a_L$ and $a_H$ that is laid out by Proposition 1. We show below that if type $j$ does more aggressive manipulation than type $k$, then type $j$ does effective manipulation. However, as intuitive as it may sound, effective manipulation by one type does not preclude the effective manipulation by the other type. Following Proposition 2 provides a map of effective manipulation.

**Proposition 2** Suppose that Assumptions 1, 2 and 4 hold. For cases in which $\bar{p} < (1-g)v_H + gv_L$ and $\bar{x} > v_L + (a_L + a_H)/2$ the only type that effectively manipulates is the $L$-type. For cases in which $\bar{p} < (1-g)v_H + gv_L$ and $\bar{x} < v_L + (a_L + a_H)/2$ both types can effectively manipulate.

For the cases in which $\bar{p} > (1-g)v_H + gv_L$ and $\bar{x} \le v_H + a_H$, the only type that effectively manipulates is the $H$-type. For the remaining cases, in which $\bar{p} > (1-g)v_H + gv_L$ and $\bar{x} > v_H + (a_L + a_H)/2$ both types can effectively manipulate.

Effective manipulation by type $j$ is the net effect of manipulative advertising on sales of the type $j$ Monopoly. If the only type that can effectively manipulate is the $L$-type, existence of manipulative advertising makes consumers worse-off, since the manipulative advertising increases the share of consumers who end up with a negative surplus if the Monopoly is of low quality type. If the only type that can effectively manipulate is the $H$-type, then consumers are better-off since the share of consumers who receive a positive surplus increases if the Monopoly is of $H$-type. To be more precise, we define the aggregate ex-ante surplus as follows:

$$
(1-g)(v_H - \bar{p})(1 - F(\bar{x} - a_H - v_H)) - g(\bar{p} - v_L)(1 - F(\bar{x} - a_L - v_L)),
$$

when there is manipulation. When there is no manipulation, ex-ante consumer surplus is defined as

$$
(14)
$$
We define the net effect of the manipulative advertising on consumer surplus as the difference between these two types of consumer surplus at a fixed price $\bar{p}$.

Following Corollary 2, which is a direct result of Proposition 2, shows that the relatively lower prices are associated with negative effect of manipulative advertising and higher prices are associated with a positive effect of it.

**Corollary 2** The net effect of manipulative advertising on consumer surplus is negative when $\bar{p} < (1 - g)v_H + gv_L$ and $\bar{x} > v_L + (a_L + a_H)/2$ and positive and $\bar{p} > (1 - g)v_H + gv_L$ and $\bar{x} \leq v_H + a_H$.

The cases that Corollary 2 leaves out are the ones where both types of Monopoly can do effective manipulation. In such cases, the net effect of manipulative advertising depends on the relative amount of manipulation by each type but also on the relative responsiveness of consumers to private signals. We provide a numerical example to show exactly how consumer surplus changes with manipulative advertising. We use normal distribution as the noise distribution, a quadratic function as the cost of manipulation and uniform distribution for the prior beliefs. Figure 1 illustrates that the net effect of manipulation is minimum at a price lower than the ex-ante expected value of the quality and maximum at a price higher than the ex-ante expected value of the quality.

Corollary 2 establishes that consumers may sometimes be better off if they simply received a noisy signal about product quality that is not biased through manipulative advertising. However, many times it may not be feasible (or it may be too costly) to acquire unbiased information about the quality of a product because informative messages are typically bundled together with biased statements, making it impossible to perfectly separate one from the other. In other words, if you choose to watch a TV commercial or browse the website of a company to get information about a newly released product, exposure to bias is typically a price you agree to pay. One natural question that pops up then is if you would rather ignore informative but biased advertising by switching the channel when a TV commercial appears or not visiting a web site that you know has an incentive to oversell a certain product. In a world where we are on a daily basis exposed to tons of implicit or explicit advertising content without our will, it is certainly hard, if not impossible, to insulate oneself
from manipulative messages. Yet even if you somehow could ignore all these ads and choose not to receive any signal, it is still not straightforward if you would want to do it. Translated into our framework, the question is if a consumer would ever prefer to merely rely upon her prior beliefs when making a purchasing decision instead of acting upon her posterior beliefs after a manipulative advertising signal. The following proposition addresses this question.

**Proposition 3** Suppose that Assumptions 1, 2 and 4 hold. Then, a consumer would never ignore the only available advertising signal $x_i = v_j + a_j + \varepsilon_i$ (where $j \in H, L$ depending on the state) and base her purchasing decision on her prior beliefs.

Proposition 3 establishes that no matter how high manipulative advertising by each potential type of monopolist is, consumers are better off by taking the advertising signals into account when making their decisions. At some level this result is to be expected, because despite the bias, advertising signals are informative about product quality. But at the same time, depending on the price level, manipulative advertising can worsen or improve ex-ante consumer surplus relative to non-manipulative advertising. In this sense, Proposition 3 is not so straightforward. Without any signal, a consumer always purchases the product when $\bar{p} \leq gv_L + (1 - g)v_H =: \tilde{p}$, and
never purchases it otherwise. Interestingly, \( \tilde{p} \) coincides with the price threshold below which L-type does greater manipulation than the H-type (\( a_L > a_H \)). Below this price, consumers overestimate quality when L-type is in charge and underestimate it when H-type is in charge. In contrast, when \( \tilde{p} > \tilde{p} \), manipulative advertising moves posterior beliefs in the welfare improving direction, but in the absence of any signal this is also when consumers (acting merely on their priors) will not purchase the product.

4 The nature of consumers’ inference problem

We show that depending on the price level and the state of the world (i.e. the type of the monopolist) manipulative advertising can improve or deteriorate consumers’ posterior beliefs about product quality. Essential for this result is the presence of idiosyncratic noise in signals and the fact that two monopoly types employ different levels of advertisement in equilibrium.

When the monopolist with low product quality (L-type) advertises more than the monopolist with a high product quality (H-type), the median consumer will expect product quality to be higher [lower] when L-type [H-type] is in charge compared to what she would expect in the absence of any manipulation. This result follows from the fact that distance between signal means in each state will typically be different from the corresponding distance under the benchmark case where signals do not contain a systematic bias. To build some intuition, consider the following example where, depending on the state of the world, a monopolist can either be L-type with product quality \( v_L \) or H-type with product quality \( v_H > v_L \). The random noise in signals has zero mean and has a unimodal and symmetric distribution. Monopolists can move the signal mean upward by an amount \( a \) through manipulative advertising at some cost \( C(a) \). Imagine that the state is realized and consumers face an L-type. Consider the median consumer who would receive an advertising signal of \( v_L + a_L \) when manipulation is allowed, and \( v_L \) when it is not allowed. Due to the noisy nature of the signals she cannot tell for sure if her signal comes from an L-type or an H-type monopolist. She instead uses her prior beliefs about the types and her knowledge of the noise distribution to assign posterior weights on these two events. If \( a_L > a_H \), mean signal levels for the two types will be closer to each other compared to the benchmark situation where distance between mean signals is \( \varepsilon = v_H - v_L > (v_H + a_H) - (v_L + a_L) = \varepsilon - (a_L - a_H) \). As a result, under manipulative advertising, the absolute magnitude of
the random noise that would move a signal from $v_H + a_H$ to $v_L + a_L$ is smaller than the corresponding noise that moves a signal from $v_H$ to $v_L$. Since the probability density of the noise distribution is symmetric around zero, the median consumer assigns a greater likelihood to the event that her signal comes an H-type when signals are manipulated such that $a_L > a_H$. When true type is $H$, by the same reasoning, she would assign greater likelihood to the event that her signal comes from an $L$-type. This inference problem would lead to a qualitatively opposite result if $a_L < a_H$.

5 Conclusion

We have studied the implications of deceptive advertising by a monopolist under asymmetric information and a noisy communication environment. We have modeled advertising as a signal jamming technology and shown that the ability to increase demand via advertising is not reserved for only one type of monopolist. Depending on the price level, both types can be effective manipulators. Under relatively low prices, the low-quality monopolist has more incentive to advertise than the high-quality monopolist and the signal jamming technology delivers higher sales for its product. The opposite is true when prices are relatively high. Beyond its informational consequences, which, depending on the context, might be harmful or beneficial for consumers, deceptive ads lead to an arms race between the two types of monopolists and socially wasteful spending. In ongoing work, we analyze the implications of different types of regulations on deceptive advertising.

A Proofs

Proof of Lemma 1 Suppose first that Assumption 3 holds. Then, each first order condition (8) has a unique solution, since Assumption 3 guarantees the strict concavity of the profit function globally.

Suppose for a moment that $a_L + v_L \geq a_H + v_H$, which would imply that $\bar{x} - a_L - v_L \leq \bar{x} - a_H - v_H$ and also $a_L > a_H$. Then having a unique solution to equation (8) implies that if $L$-type uses the lower manipulation level $a_H$, its marginal revenue should be greater than its marginal cost. That is,

$$\bar{p}f(\bar{x} - a_H - v_L) > C(a_H) = \bar{p}f(\bar{x} - a_H - v_H),$$
which implies

\[|\bar{x} - a_H - v_L| < |\bar{x} - a_H - v_H|,\]

since the noise pdf \( f(\cdot) \) is unimodal. Then, \( v_L < v_H \) implies that \( \bar{x} > a_H + v_H \). On the other hand, \( a_L > a_H \) implies by first order conditions (8) that

\[f(\bar{x} - a_L - v_L) > f(\bar{x} - a_H - v_H) \iff |\bar{x} - a_L - v_L| < |\bar{x} - a_H - v_H|,\]

which implies \( \bar{x} < a_H + v_H \). Hence, a contradiction.

Now, suppose that Assumption 4 holds and \( a_L + v_L \geq a_H + v_H \iff a_L > a_H + v_H - v_L \). But at such a manipulation level by the \( L \)-type, the marginal cost would always be greater than the marginal revenue by Assumption 4.

**Proof of Lemma 2** For notational simplicity, denote \( f_j \equiv f(x - v_j - a_j) \) and \( f'_j \equiv f'(x - v_j - a_j) \). Then,

\[
\frac{\partial E[v|x]}{\partial x} = \sum_{j \in J} v_j f'_j G(v_j) \left( \sum_{j \in J} f_j G(v_j) \right) - \sum_{j \in J} f'_j G(v_j) \left( \sum_{j \in J} v_j f_j G(v_j) \right) > 0 \iff \\
\sum_{j \in J} v_j f'_j G(v_j) \left( \sum_{j \in J} f_j G(v_j) \right) - \sum_{j \in J} f'_j G(v_j) \left( \sum_{j \in J} v_j f_j G(v_j) \right) > 0 \iff \\
g(1 - g) f'_H f_L > g(1 - g) f_H f'_L \iff \frac{f'((\bar{x} - a_H - v_H)}{f((\bar{x} - a_H - v_H)} > \frac{f'((\bar{x} - a_L)}{f((\bar{x} - a_L)}, (16)
\]

which holds, since \( f \) is log-concave by Assumption 1 and \( a_H + v_H > a_L + v_L \) by Lemma 1.

**Proof of Theorem 1** By Assumption 4 and Inverse Function Theorem, for each manipulative advertisement couple \( a_L, a_H \) there exists a unique \( \bar{x} \), and \( \bar{x}(a_L, a_H) \) is a continuously differentiable function. Therefore we can reduce the number of equations that define the equilibrium into the following two equations that are very similar to the first order conditions (8):
\[ \bar{p} f(\bar{x}(a_L, a_H) - v_j - a_j) = C'(a_j) \quad \text{for all } j \in J, \quad (17) \]

which has a positive solution by Assumptions 1 and Intermediate Value Theorem. Moreover, the solution is unique if Assumption 3 holds as well.

**Lemma 3** Suppose that Assumptions 1, 2 and 3 hold. When \( \bar{p} \) converges to \( v_L \), the signaling thresholds \( \bar{x} \) and \( \bar{x} \) diverge to \( -\infty \), and when \( \bar{p} \) converges to \( v_H \), the signaling thresholds \( \bar{x} \) and \( \bar{x} \) diverge to \( \infty \).

**Proof of Lemma 3** When \( \bar{p} < (v_H + v_L)/2 \), \( a_L > a_H \) and by first-order conditions (8)

\[
f(\bar{x} - a_L - v_L) > f(\bar{x} - a_H - v_H) \iff |\bar{x} - a_L - v_L| < |\bar{x} - a_H - v_H|,
\]

since \( f(\cdot) \) is unimodal. Then by Lemma 1 \( \bar{x} < a_H + v_H \).

On the other hand, when \( \bar{p} > (v_H + v_L)/2 \), \( a_L < a_H \) and by first-order conditions (8)

\[
f(\bar{x} - a_L - v_L) < f(\bar{x} - a_H - v_H) \iff |\bar{x} - a_L - v_L| > |\bar{x} - a_H - v_H|,
\]

then by Lemma 1 \( \bar{x} > a_L + v_L \).

The consumer indifference condition (12) can rewritten as follows

\[ \frac{v_H - \bar{p}}{\bar{p} - v_L} = \frac{f(\bar{x} - a_L - v_L)}{f(\bar{x} - a_H - v_H)}. \]

When \( \bar{p} \to v_L \), LHS of the equation above converges to \( \infty \) and therefore \( f(\bar{x} - a_H - v_H) \to 0 \), which implies \( \bar{x} \to \{-\infty, \infty\} \). But since \( \bar{x} \leq a_H + v_H < \infty \), \( \bar{x} \to -\infty \).

When \( \bar{p} \to v_H \), LHS of the equation above converges to 0 and therefore \( f(\bar{x} - a_L - v_L) \to 0 \), which implies \( \bar{x} \to \{-\infty, \infty\} \). But since \( \bar{x} \geq a_L + v_L > -\infty \), \( \bar{x} \to \infty \).

**Lemma 4** Suppose that Assumptions 1, 2 and 3 hold. The signaling threshold \( \bar{x} \) strictly increases with price \( \bar{p} \). That is, \( \partial \bar{x}/\partial \bar{p} > 0 \).

**Proof of Lemma 4** To simplify the notation let \( f(\bar{x} - a_L - v_L) = f_L \), \( f(\bar{x} - a_H - v_H) = f_H \), \( f'(\bar{x} - a_L - v_L) = f'_L \), \( f'(\bar{x} - a_H - v_H) = f'_H \).

Implicit differentiation of first-order conditions (8) enables us to calculate \( \partial a_L/\partial \bar{p} \) and \( \partial a_H/\partial \bar{p} \) as follows.
\begin{align*}
f_L + \bar{p} f'_L \frac{\partial \bar{x}}{\partial \bar{p}} = (\bar{p} f'_L + C''(a_L)) \frac{\partial a_L}{\partial \bar{p}} \Leftrightarrow \\
\frac{\partial a_L}{\partial \bar{p}} = \frac{f_L}{\bar{p} f'_L + C''(a_L)} + \frac{\bar{p} f'_L}{\bar{p} f'_L + C''(a_L)} \frac{\partial \bar{x}}{\partial \bar{p}} \quad \text{and,}
\end{align*}

\begin{align*}
f_H + \bar{p} f'_H \frac{\partial \bar{x}}{\partial \bar{p}} = (\bar{p} f'_H + C''(a_H)) \frac{\partial a_H}{\partial \bar{p}} \Leftrightarrow \\
\frac{\partial a_H}{\partial \bar{p}} = \frac{f_H}{\bar{p} f'_H + C''(a_H)} + \frac{\bar{p} f'_H}{\bar{p} f'_H + C''(a_H)} \frac{\partial \bar{x}}{\partial \bar{p}} \end{align*}

Implicit differentiation of the consumer indifference condition (12) yields \( \partial \bar{x}/\partial \bar{p} \) as follows.

\begin{align*}
-f_H (1 - g) + (1 - g)(v_H - \bar{p}) f'_H \left( \frac{\partial \bar{x}}{\partial \bar{p}} - \frac{\partial a_H}{\partial \bar{p}} \right) = g f_L + g (\bar{p} - v_L) f'_L \left( \frac{\partial \bar{x}}{\partial \bar{p}} - \frac{\partial a_L}{\partial \bar{p}} \right) \Rightarrow \\
\frac{\partial \bar{x}}{\partial \bar{p}} = \frac{g f_L + (1 - g) f_H}{(1 - g)(v_H - \bar{p}) f'_H - g (\bar{p} - v_L) f'_L} - \frac{g (\bar{p} - v_L) f'_L}{(1 - g)(v_H - \bar{p}) f'_H - g (\bar{p} - v_L) f'_L} \frac{\partial a_L}{\partial \bar{p}}.
\end{align*}

Now, suppose that \( \partial \bar{x}/\partial \bar{p} = 0 \). Then combining the calculations above, \( \partial \bar{x}/\partial \bar{p} = 0 \) implies that

\begin{align*}
\frac{g f_L + (1 - g) f_H}{(1 - g)(v_H - \bar{p}) f'_H - g (\bar{p} - v_L) f'_L} f_H
+ \frac{(1 - g)(v_H - \bar{p}) f'_H - g (\bar{p} - v_L) f'_L}{(1 - g)(v_H - \bar{p}) f'_H - g (\bar{p} - v_L) f'_L} p f'_L + C''(a_H)
- \frac{g (\bar{p} - v_L) f'_L}{(1 - g)(v_H - \bar{p}) f'_H - g (\bar{p} - v_L) f'_L} f_L = 0,
\end{align*}

which implies after reorganizing

\begin{align*}
0 &= \frac{g f_L}{(1 - g)(v_H - \bar{p}) f'_H - g (\bar{p} - v_L) f'_L} \left( 1 - \frac{\bar{p} - v_L}{\bar{p}} \frac{\bar{p} f'_L}{\bar{p} f'_L + C''(a_L)} \right) \\
+ \frac{(1 - g) f_H}{(1 - g)(v_H - \bar{p}) f'_H - g (\bar{p} - v_L) f'_L} \left( 1 + \frac{v_H - \bar{p}}{\bar{p}} \frac{\bar{p} f'_L}{\bar{p} f'_L + C''(a_L)} \right) \neq 0,
\end{align*}
which is a contradiction. To see this note that $C''(a_L) > 0$ implies that

$$0 < \bar{p} - v_L \frac{\bar{p} f'_L}{\bar{p} f'_L + C''(a_L)} < 1,$$

and

$$v_H - \bar{p} \frac{\bar{p} f'_H}{\bar{p} f'_H + C''(a_H)} < -1 \Rightarrow C''(a_H) < v_H f'_H,$$

which contradicts with Assumption 3.

Now, we established that $\partial \bar{x}/\partial \bar{p}$ cannot be 0. But this implies by Lemma 3 that $\partial \bar{x}/\partial \bar{p}$ is globally positive.

**Proof of Proposition 2** We will start with the effective manipulation by $L$-type. $L$-type does effective manipulation if and only if $a_L > \bar{x} - \bar{x}$, which is equivalent to $\bar{x} > \bar{x} - a_L$. By Corollary 1 this is equivalent to $\mathbb{E}(v|\bar{x} - a_L) < \bar{p}$ if manipulation is restricted to be 0. That is, when there is no manipulation, a consumer who receives a signal that equals to $\bar{x} - a_L$, should expect that the quality should be lower than the price $\bar{p}$. When we calculate the posterior expectation of such a consumer

$$(1 - g)v_H f(\bar{x} - a_L - v_H) + g v_L f(\bar{x} - a_L - v_L) < \bar{p},$$

$$(1 - g)(v_H - \bar{p}) f(\bar{x} - a_L - v_H) - g(\bar{p} - v_L) f(\bar{x} - a_L - v_L) < 0$$

$= (1 - g)(v_H - \bar{p}) f(\bar{x} - a_H - v_H) - g(\bar{p} - v_L) f(\bar{x} - a_L - v_L) \Rightarrow f(\bar{x} - a_L - v_H) < f(\bar{x} - a_H - v_H) \Leftrightarrow |\bar{x} - a_L - v_H| > |\bar{x} - a_H - v_H|.$

In sum,

$$a_L > \bar{x} - \bar{x} \Leftrightarrow |\bar{x} - a_L - v_H| > |\bar{x} - a_H - v_H|. \quad (18)$$

We will show below that this equivalence condition for $L$-type’s effective manipulation holds when $\bar{p} < (1 - g)v_H + g v_L$ or when $\bar{p} > (1 - g)v_H + g v_L$ and $\bar{x} > (a_L + a_H)/2 + v_H$.

Now, when $\bar{p} < (1 - g)v_H + g v_L$, $a_L > a_H$ by Proposition 1. Then by first-order
conditions \( [8] \)

\[
f(\bar{x} - a_L - v_L) > f(\bar{x} - a_H - v_H) \iff |\bar{x} - a_L - v_L| < |\bar{x} - a_H - v_H|,
\]
since \( f(\cdot) \) is unimodal. Then by Lemma \([1]\) the condition above is implies equivalent to

\[
\bar{x} < a_H + v_H,
\]

\[
\bar{x} - a_H - v_H < \bar{x} - a_L - v_L < a_H + v_H - \bar{x} \Rightarrow
\]

\[
\bar{x} < \frac{a_L + a_H}{2} + \frac{v_L + v_H}{2} < \frac{a_L + a_H}{2} + v_H
\]

On the other hand, when \( a_L > a_H \), condition \([18]\) is equivalent to

\[
\bar{x} < a_L + v_H
\]

\[
\bar{x} - a_L - v_H < \bar{x} - a_H - v_H < a_L + v_H - \bar{x} \iff
\]

\[
\bar{x} < \frac{a_L + a_H}{2} + v_H,
\]

which holds when \( \bar{p} < (1 - g)v_H + gv_L \).

When \( \bar{p} > (1 - g)v_H + gv_L \), \( a_H > a_L \) by Proposition \([1]\) Therefore, condition \([18]\) becomes equivalent to

\[
\bar{x} > a_L + v_H
\]

\[
a_L + v_H - \bar{x} < \bar{x} - a_H - v_H < \bar{x} - a_L - v_H \iff
\]

\[
\bar{x} > \frac{a_L + a_H}{2} + v_H,
\]

because of Lemma \([1]\) This completes the argument for effective manipulation by \( L \)-type.

By a similar argument as above, \( H \)-type does effective manipulation if and only if
\[ a_H > \bar{x} - \bar{a} \iff \bar{x} > \bar{x} - a_H \]

\[ (v_H - \bar{p})(1 - g)f(\bar{x} - a_H - v_H) - (\bar{p} - v_L)gf(\bar{x} - a_H - v_L) < 0 \]

\[ = (v_H - \bar{p})(1 - g)f(\bar{x} - a_H - v_H) - (\bar{p} - v_L)gf(\bar{x} - a_L - v_L) \iff \]

\[ f(\bar{x} - a_L - v_L) < f(\bar{x} - a_H - v_L) \iff |\bar{x} - a_L - v_L| > |\bar{x} - a_H - v_L|. \]

In sum,

\[ a_H > \bar{x} - \bar{a} \iff |\bar{x} - a_L - v_L| > |\bar{x} - a_H - v_L|. \tag{19} \]

When \( \bar{p} < (1 - g)v_H + gv_L \), \( a_H < a_L \), therefore condition \((19)\) is equivalent to

\[ \bar{x} < a_L + v_L, \]

\[ \bar{x} - a_L - v_L < \bar{x} - a_H - v_L < a_L + v_L - \bar{x} \iff \]

\[ \bar{x} < v_L + \frac{a_L + a_H}{2}. \]

When \( \bar{p} > (1 - g)v_H + gv_L \), \( a_H > a_L \), therefore condition \((19)\) is equivalent to

\[ \bar{x} > a_L + v_L \]

\[ a_L + v_L - \bar{x} < \bar{x} - a_H - v_L < \bar{x} - a_L - v_L \iff \]

\[ \bar{x} > v_L + \frac{a_L + a_H}{2}, \]

which holds as long as \( a_H > a_L \).

**Proof of Proposition 3** For notational simplicity let \( f_j := f(\bar{x} - v_j - a_j) \) and \( F_j := F(\bar{x} - v_j - a_j) \) for \( j \in \{H, L\} \). In the absence of advertising signal consumer purchases the product if and only if \( \bar{p} \leq gv_L + (1 - g)v_H \), i.e. whenever expected quality under prior beliefs exceeds the price. Therefore, ex-ante (expected) consumer utility (surplus) without any advertising signal is equal to
\[ E_{NS}(CS) = \begin{cases} g v_L + (1 - g)v_H, & \text{if } \bar{p} \leq g v_L + (1 - g)v_H \\ 0, & \text{if } \text{otherwise.} \end{cases} \]

On the other hand, when the consumer acts upon the biased advertising signal, expected consumer surplus will be

\[ E_S(CS) = g(1 - F_L)(v_L - \bar{p}) + (1 - g)(1 - F_H)(v_H - \bar{p}) \]

We need to show that \( E_S(CS) \geq E_{NS}(CS) \) always holds. When consumer decides based on advertising signals, the signal threshold for purchasing decision is given by equation 11 which in turn implies that

\[ \frac{(1 - g)(v_H - \bar{p})}{g(\bar{p} - v_L)} = \frac{f_L}{f_H}. \]  

(20)

Consider the first case, i.e, \( \bar{p} \leq g v_L + (1 - g)v_H \). Then,

\[ E_S(CS) \geq E_{NS}(CS) \iff -g(v_L - \bar{p})F_L - (1 - g)(v_H - \bar{p})F_H \iff \frac{F_L}{F_H} \geq \frac{(1 - g)(v_H - \bar{p})}{g(\bar{p} - v_L)} = \frac{f_L}{f_H} \iff \]

\[ \frac{f_L}{F_L} \leq \frac{f_H}{F_H} \]  

(21)

(22)

where the equality in 21 follows from equation 20. By Lemma 1, \( a_L + v_L < a_H + v_H \). Therefore, \( \bar{x} - v_L - a_L > \bar{x} - v_H - a_H \). Also, since \( f \) is log-concave by Assumption 1, \( F \) must also be log-concave. Combining these two observations we obtain the inequality in 22 as desired.
Now consider the remaining case that $\bar{p} > g v_L + (1 - g) v_H$. Then,

$$E_S(CS) \geq E_{NS}(CS) \iff \left( g(v_L - \bar{p})(1 - F_L) + (1 - g)(v_H - \bar{p})(1 - F_H) \geq 0 \right) \iff$$

$$\frac{1 - F_L}{1 - F_H} \leq \frac{(1 - g)(v_H - \bar{p})}{g(\bar{p} - v_L)} = \frac{f_L}{f_H} \iff \frac{f_L}{1 - F_L} \geq \frac{f_H}{1 - F_H} \tag{23}$$

where, as before, the equality in 23 follows from equation 20. Since $f$ is log-concave, $1 - F$ is also log-concave. This in turn implies that $f(x)/(1 - F(x))$ is increasing in $x$. Since $\bar{x} - v_L - a_L > \bar{x} - v_H - a_H$ always holds, we obtain the inequality in 24 as desired.
References


