Relative Evaluation Scheme for Teams and Multi-tasks

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Abstract

One of the solutions to the moral hazard in team production is to divide the firm into more than two teams and make them compete. We extend this relative evaluation scheme for teams so that we can pin down the optimal incentive power explicitly. We find that the firm should provide a higher-powered incentive when the firm’s size grows and the degree of complementarity of each team’s output increases, and this explains why large companies tend to have a more skewed remuneration menu. We further extend the model to the cases where workers have to deal with two kinds of multi-tasks: the first is to help their coworkers and the second is they face a general and team-specific jobs. We find that, in the former case, given a higher-powered incentive, a worker tends to put more effort into help, and in the latter case, he/she tends to put more effort into the team-specific job and less into the general job. This result implies that the firm sometimes would be better off by providing a lower-powered incentive.

Keywords—Team Production, Relative Performance Evaluation, Multi-tasks

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1 Introduction

An owner of a firm has two options to obtain intermediate goods: to buy them from a market or to make them by him/herself. When he/she decides to produce them, while he/she can save transaction costs (Coase 1937), he/she has to monitor and disciple workers; otherwise, they would shirk. In particular, when employees work as a team and each employee’s contribution cannot be measured, they work less than the level that maximizes total surplus of production (Holmström 1982). To resolve this problem he/she has to design an incentive scheme, and he/she has to answer the following two questions: what kind of system works and when he/she should provide a higher-powered incentive.

One of the solutions to the first problem is to divide the team into two groups and to make them compete (Marino and Zábojník 2004). This scheme gives bonuses to employees who belong to a group which outperforms the other, and when this is adopted, the firm can

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achieve the efficient allocation of effort that maximizes total surplus. The intuition is, under this system, employees work harder to make their team a winner.

This incentive scheme is realistic and persuasive, and the remaining task is to answer the second problem: when the firm should provide a higher-powered incentive. This question has been an important agenda both in academia and practice, however, there are a few frameworks that can provide the optimal incentive power analytically. An exception for a single agent and single principal case is the linear contract model by Holmström and Milgrom (1987) who showed that we could approximate the optimal incentive by a linear function of an agent’s output with an appropriate slope and intercept.

In this paper, we explore the relative evaluation scheme for teams in two ways. First, we propose a novel model that enable us to pin down the optimal incentive power explicitly and to conduct its comparative static analysis. To the best of our knowledge, this is the first model that enable us to do so. We find that the optimal incentive power is a function of team size and the elasticity of the team’s production function, and the firm should provide a higher-powered incentive when team size is larger, and the degree of complementarity of each group’s output is higher. Second, we extend the model to two cases where agents engage in multiple tasks, in particular, when workers can help their coworkers and when there are a general and group-specific jobs. We find that if help is a productive activity, employees tend to help their colleagues more when the firm uses a higher-powered incentive. We also discover that given a higher powered incentive, employees tend to put more effort into the group-specific job and less effort into the general job, and this could reduce the team’s total output when the latter type of job is more important. As far as we know, these extensions are also new to the literature.

The reason why we are interested in deriving the optimal incentive power and performing its comparative static analysis is to explain why larger companies tend to have a more skewed remuneration menu. In the major Japanese manufacturers, in which they tend to adopt the multi-division form, it is common that each employee’s wage depends on his/her division’s performance about others. For instance, he/she get bonuses when his/her units achieves its quota while others does not. In the large companies, employees in an outperforming division tend to be rewarded more, regarding both immediate lump-sum payments and future promotion opportunities. Our theory can explain this tendency because we can show that it is optimal for large manufacturers to use a higher-powered incentive.

We extend our model to the multi-tasks is to understand the scheme’s impact on the firm’s culture and limitation. As for help, it is known that on a shop floor in the major Japanese manufacturers, workers are allowed and willing to help their coworkers, and this could explain the high productivity of firms. Our theory can explain that this cooperative culture is consistent with the fact that, in such companies, workers’ payment sometimes depends on the relative performance of their factory. As for a general and group-specific jobs, our theory implies that there are certain cases in which the firm shouldn’t use a higher-powered incentive, which is similar to the multi-task problem as a single agent and single principal case (Holmström and Milgrom (1991). The intuition is when the firm provides a higher-powered
incentive; employees tend to put more effort into the task that affects their evaluation more efficiently, which is the group-specific job in our model. Therefore, if this type of work doesn’t improve the firm’s performance, the company is better off by using a lower powered incentive.

Our model is an extension of the classical model of team production proposed by Holmström (1982). We consider a firm that consists of risk neutral employees who has limited liability, which is divided into more than two teams. The total output of the company is a function of these teams’ output which is a function of the effort of employees who belong to the team. While each employee’s effort is not observable, each team’s output is observable and verifiable. The firm distributes its production to each employee, according to the following distribution scheme which satisfies the budget balance condition as a company. Each team receives its share of the total output and divides it equally to employees in the team. The share is a function of each team’s output, in which the team which performs better than the other receives more.

We specify the share function by a class of generalized Tullock function so that we can obtain the optimal incentive power analytically, and we find that it is increasing in the firm’s size and the degree of complementarity of each team’s output. The intuition is, without any incentive device, when team’s size increases, the free-riding problem becomes severer, and when the level of complementarity rises, the underperformance of each employee reduces the firm’s output further. Therefore, the firm has to use a higher-powered incentive to resolve these problems. We check the robustness of this result and find that production functions of the firm and the team can be general, and we can incorporate heterogeneity of workers regarding their cost and the team’s size.

In the first extension, we enable employees to help their colleagues. In our model, an employee can reduce his/her colleagues’ marginal cost of effort by paying a cost. We find that when help is productive, which means that help can reduce others’ cost efficiently, employees tends to put more effort to help others. In the second extension, we consider a case in which there are two kinds of tasks, a general and team-specific tasks. In this model, a team’s output is a function of efforts that employees put into the team-specific job, and the firm’s output is a function of each team’s output and efforts that employees put into the general job. We assume that the marginal cost of each task are increasing in the effort that the agent puts into the other task. We find that when the firm uses a higher-powered incentive, employees put more effort into the team-specific task and less to the general task.

The remainder of the article is organized as follows. In Section 2, we introduce our model and discuss the property of the sharing rule. In section 3, we present our results and perform comparative statics analysis. In Section 4, we extend our model to show that our primary result holds even when the number of groups is more than two, and the size of the group is different across groups. Section 5 investigates the interaction between the intergroup competition and multi-tasks. Finally, Section 6 concludes the article.
1.1 Related literature

Literature on the efficiency of the team production has been extensively developed. The free-rider problem in team production is pointed out by Alchian and Demsetz (1972), and there have been numerous contributions, starting from Holmström’s (1982) seminal work and followed by, among others, Mookherjee (1984), Rasmusen (1987), Legros and Matsushima (1991), Kandel and Lazear (1992), Legros and Matthews (1993), Miller (1997), and Strausz (1999).

Our work is most related to recent works that showed relative performance evaluation can solve the inefficiency problem. Greshkov, Li, and Schweinzwer (2009) showed that the free-rider problem can be solved in a wide range of production functions as long as a partial and noisy ranking of agents’ efforts is available. Marino and (2004) solved a puzzle that profit sharing is widely observed though it should suffer from the free-rider problem. They showed that the problem can be solved when a firm employs internal competition which makes subordinate organizations compete with each other. We share our focus with Marino and Zábojník (2004), the importance of intergroup competition as an incentive device, and we extend their results to more general production processes and improve their model by representing the incentive power by a parameter.

2 Model

Consider a team consisting of two groups, 1 and 2, each of which contains \(N \geq 2\) homogeneous risk-neutral individuals. Each agent’s effort is unobservable, and each group’s output is observable.

Individual \(i\) chooses effort level \(e_i \in A = [\delta, +\infty)\) where \(\delta\) is positive and arbitrarily close to zero. An agent’s cost function is \(c(e_i)\) where \(c\) is strictly convex, strictly increasing, twice continuously differentiable, and \(c(0) = 0\).

A group yields output from agents’ effort within groups. Let group \(J\)’s output be denoted by \(x_J\) and \(x_J = f(e_J)\) where \(f\) is concave, strictly increasing, twice continuously differentiable, and \(f(0) = 0\), and \(e_J\) is a profile of agents’ effort in group \(J\). We assume that \(f(e) = f(e')\) for any permutation of \(e\), denoted by \(e'\), i.e, each agents’ effort contributes to the groups’ output equally.

The team’s output \(y\) is a function of groups’ output, i.e. \(y(x) = y(x_1, x_2)\) and \(y\) is concave, strictly increasing, twice continuously differentiable, and \(y(0) = 0\). We assume that every group’s output affects the team’s output equivalently.
2.1 Efficiency and inefficiency benchmark

The first best effort level is defined as the solution to the following surplus maximization problem:

$$\max_{e} \ y(x_1, x_2) - \sum_{i=1}^{2N} c(e_i).$$

(1)

From the first order condition, the first best effort level of an agent which we denote $e^*$ satisfies

$$\frac{\partial y}{\partial x_j} \frac{\partial f(e_i^*, e_{-i}^*)}{\partial e_i} = c'(e_i^*)$$

(2)

for each $i$.

If agents split the total output equally, then the first best effort level cannot be attained at an equilibrium. In this case, an agent’s problem becomes

$$\max_{e_i} \ y(x_1, x_2) - \frac{c(e_i)}{2N},$$

(3)

and his/her effort level at a symmetric equilibrium $e$ is determined by

$$\frac{\partial y}{\partial x_j} \frac{\partial f(e)}{\partial e_i} \frac{1}{2N} = c'(e).$$

(4)

It is clear that agents exert less efforts as the team size increases, and the first best cannot be achieved as long as the group size is more than two.

2.2 Sharing function

We use a generalized Tullock function as a sharing function because this approach has two major advantages. First, we can guarantee that agent’s limited liability constraint and the team’s budget balance constraint are always satisfied. Second, we can represent the inventive power by a parameter, which enables us to perform comparative static analysis of the power on the changes in the size of the team and production technologies.

We let group $J$’s share of the total output be denoted by $s_J$, and $s_1$ is defined as

$$s_1 = \frac{x_1}{x_1 + x_2} = \frac{1}{1 + (x_2/x_1)^r},$$

(5)

where $r$ is a parameter chosen by a designer of this incentive scheme. We name $r$ the incentive power because the share of output becomes more sensitive to the relative performance of the group when this parameter becomes larger. Note that group $J$’s share of output exceeds its output $x_J$ when it becomes a winner, and it falls behind $x_J$ when it becomes a loser.

An agent’s wage is the group’s share of output divided by the size of the group, $s_J y/N$, and this is always positive.
3 Results

We focus on an agent in group 1 because each group and agent are symmetric. The utility maximization problem of agent $i$ in group 1 is

$$\max_{e_i} \frac{x_1^i}{x_1^i + x_2^i} \frac{y}{N} - c(e_i),$$

and its first order condition is

$$\frac{x_1^i}{x_1^i + x_2^i} \frac{\partial y}{\partial x_1} \frac{\partial f(e_i, e_{-i})}{\partial e_i} \frac{1}{N} + \frac{r x_1^{i-1} x_2^i}{(x_1^i + x_2^i)^2} \frac{\partial f(e_i, e_{-i})}{\partial e_i} \frac{y}{N} - c'(e_i) = 0.$$  

(7)

Under the sharing rule, a marginal change in effort affects the agents’ wage through two channels. First, it affects the total output, and second, it alters the share of output of his/her group. The first term in the equation represents the first effect, and the second term represents the second one.

We focus on the symmetric equilibrium as is standard in the literature. At a symmetric Nash equilibrium, in which every agent chooses the same level of effort $e$, the effort level is determined by

$$\frac{1}{2} \frac{\partial y}{\partial x_J} \frac{\partial f(e)}{\partial e_i} \frac{1}{N} + \frac{r}{4x_J} \frac{\partial f(e)}{\partial e_i} \frac{y}{N} = c'(e),$$

(8)

and we obtain the following results from this equation.

Lemma 1 Agents’ effort is increasing in the strength of the incentive power.

This lemma implies that the designer of the payment scheme can make agents put more effort by providing a higher-powered incentive, and provide the following results.

Proposition 1 There exists the optimal incentive power $r^*$ that achieves the first best. Moreover, $r^*$ is determined by

$$r^* = (2N - 1) \frac{2 \frac{\partial y(x^*)}{\partial x_J} x_J^*}{y^*}.$$  

(9)

We read the term of $2N - 1$ as the size of the team. The rest is the elasticity of total output on a group’s output multiplied by the number of groups, and we understand this as the degree of complementarity of groups’ output. It turns out that the optimal incentive power is increasing in the size of the team and the degree of complementarity.

Our interpretation is that the higher-powered incentive must be provided when the free-rider problem is more severe. It is intuitive that the size of the team represents the seriousness of the problem because agents exert less effort in equilibrium as the organization becomes larger. It is also natural that the free-rider problem becomes more imperative when the degree of complementarity of groups is large because under-performance of one group reduces the output of the team more significantly.

It is worth mentioning that we can allow for the heterogeneity of the agent’s cost function as long as two groups are symmetric. We can let agent $i$’s cost function be $c_i(e_i)$ while each
group is made up of the same combination of agents. The first best level becomes

$$\frac{\partial y}{\partial x_J} \frac{\partial f(e^*)}{\partial e_i} = c'_i(e^*_i)$$ (10)

for each $i$. The same $r^*$ remains optimal because the role of the incentive power $r$ is to make the agent’s marginal income equal to his/her marginal contribution to the output, and this is unaffected by the heterogeneity of the agent’s cost function.

**Corollary 1** As long as each team is symmetric, the optimal incentive power $r^*$ achieves the first best even when agent’s cost functions are heterogeneous.

The formal proof is in the appendix. The above corollary implies that as long as the designer knows that groups are proportional, he/she does not have to know about the shape of cost functions of each agent. However, we admit that the symmetricity assumption of agent’s effort in group’s output is crucial.

### 4 Extension

In this section, we extend our model to more general cases. We show that our primary result holds even when there are multiple groups, group sizes are different. We also demonstrate that the specification of the sharing function is not necessary.

#### 4.1 Multiple groups

Suppose that there are $M$ groups each of which contains $N$ homogeneous risk-neutral agents. The individual’s cost function and the group’s production function remains the same, and the team’s output is $y(x) = y(x_1, \ldots, x_M)$. The first best effort level $e^*$ is similarly defined:

$$\frac{\partial y}{\partial x_J} \frac{\partial f(e^*_i, e^*_i)}{\partial e_i} = c'_i(e^*_i).$$ (11)

The sharing function is modified as

$$s_J = \frac{x_J}{\sum_{K=1}^{M} x_K}. $$ (12)

The utility maximization problem of agent $i$ in group $J$ is

$$\max_{e_i} \frac{x_J}{\sum_{K=1}^{M} x_K} \frac{y}{N} - c(e_i),$$ (13)

and the first order condition is

$$\frac{x_J}{\sum_{K=1}^{M} x_K} \frac{\partial y}{\partial x_J} \frac{\partial f(e_i, e_{-i})}{\partial e_i} \frac{1}{N} + \frac{r x_J^{r-1} (\sum_{K \neq J} x_K^r)}{(\sum_{K=1}^{M} x_J^r)^2} \frac{\partial f(e_i, e_{-i})}{\partial e_i} \frac{y}{N} - c'(e_i) = 0.$$ (14)

We obtain a similar result as before, however, we need additional restrictions on the shape of production function to guarantee the existence of the optimal incentive power.
Proposition 2  If the optimal incentive power \( r^* \) exists, it is determined by
\[
r^* = \frac{M N - 1}{M - 1} M \frac{\partial y(x^*)}{\partial x_J} \frac{x_J^*}{y^*}.\tag{15}
\]
The interpretation of the optimal incentive power is the same as the benchmark model. One major difference is in the denominator in the first “team size” term. The reason why the size of the team is divided by the number of groups is because the winner’s reward is magnified by the number of rival groups, hence, the incentive could be lower-powered.

4.2 Group size heterogeneity

There are two groups and group \( J \) consists of \( N_J \). The production function of group \( J \) is \( f^J \), and that of the team is \( y(x_1, x_2) \). The first best effort level is the solution to the following problem:
\[
\max _{e} y(x_1, x_2) - \sum _i c(e_i) \tag{16}
\]
and the first best effort level for agents in group 1, \( e_1^* \) is determined by
\[
\frac{\partial y(x^*)}{\partial x_1} \frac{\partial f^1(e_1^*)}{\partial e_1} = c'(e_1^*), \tag{17}
\]
and that for agents in group 2 is
\[
\frac{\partial y(x^*)}{\partial x_2} \frac{\partial f^2(e_2^*)}{\partial e_2} = c'(e_2^*). \tag{18}
\]
The sharing rule is generalized as follows:
\[
s_J = \frac{N_J(x_J/x_J^*)^{r_J}}{N_1(x_1/x_1^*)^{r_1} + N_2(x_2/x_2^*)^{r_2}}, \tag{19}
\]
and this is interpreted as a reward scheme based on relative attainment level of quota. The agent’s problem is similar to the previous ones.

Proposition 3  If optimal incentive powers \((r_1^*, r_2^*)\) exist, they are given by
\[
r_1^* = (N_1 + N_2 - 1) \frac{N_1 + N_2}{N_2} \frac{\partial y(x^*)}{\partial x_1} \frac{x_1^*}{y^*}, \tag{20}
\]
and
\[
r_2^* = (N_1 + N_2 - 1) \frac{N_1 + N_2}{N_1} \frac{\partial y(x^*)}{\partial x_2} \frac{x_2^*}{y^*}. \tag{21}
\]
As in the benchmark, the incentive power is determined by the size of the team, and in addition, is adjusted according to the ratio of the group size. This is because relatively stronger incentives are needed for a larger group.

4.3 General sharing function

Our primary results holds without the specification of the sharing rule that only uses the output ratio. We can substitute the generalized Tullock function with a more general function, such as
\[
s_J = \frac{g(x_J)}{g(x_1) + g(x_2)},
\]
where \( g \) is differentiable, strictly increasing and \( g(0) = 0 \). The first order condition of an agent in group 1 becomes

\[
\frac{g'(x_1)}{g(x_1) + g(x_2)} \frac{\partial y}{\partial x_1} \frac{\partial f(e_i, e_{-i})}{\partial e_i} \frac{1}{N} + \frac{g'(x_1)g(x_2)}{(g(x_1) + g(x_2))^2} \frac{\partial f(e_i, e_{-i})}{\partial e_i} \frac{y}{N} = e'(e_i),
\]

and the first best level of effort is attainable if \( g(x) \) satisfies

\[
\frac{dg(x^*)}{dx} \frac{x^*}{g(x^*)} = (2N - 1)2 \frac{\partial y(x^*)}{\partial x} \frac{x^*}{y^*}.
\]

5 Multi-tasks

In this section, we analyze the impact of relative evaluation scheme on agents’ decision on their allocation across multiple tasks. In addition to the task to increase the group’s output, we consider two kinds of tasks. The first case is in which agents can help others on the same team and the second one is in which agents can also put their efforts into improving the total output of the entire team.

5.1 Help

In this section, we study the incentive of agents to help their coworkers under the relative evaluation scheme. In this model, an agent can reduce others’ marginal costs of efforts that they put to increase the group’s output by paying additional costs. This model captures situations in which workers are allowed to help their coworkers directly, such as helping them to solve troubles in the assembly line, or employees can share their skills and knowledge with each other to make the production process more efficiently.

In the following analysis, we are going to show that the relative evaluation scheme fosters an agent’s incentive to help. As we have shown in the basic model, an agent’s incentive to make his/her team perform better becomes stronger as a higher powered incentive is employed. In this situation, to improve his/her team’s performance, not only can the agent work harder but also can help his/her coworkers so that they work harder. Therefore a higher powered incentive makes the agent’s incentive to help stronger.

5.1.1 Model

Suppose that the total output is a linear function of each group’s output, in particular, let \( y = x_1 + x_2 \). Let \( a^k_i \) be the amount of efforts that agent \( i \) puts to help \( k \), \( a^i \) be the vector of help that \( i \) gives, and \( a^i \) be the help that \( i \) receives. We assume that agents can only help other members who belong to the same group. As in the basic model, \( e_i \) denotes the amount of efforts that agent \( i \) puts to increase his/her group’s output. The cost of \( i \), \( c_i \), is a function of \( e, a_i, \) and \( a^i \). We assume that \( c_i \) is increasing in \( e_i \) and \( a^k_i \) for all \( k \), and \( \partial c_i^2/\partial e_i \partial a_i^k \geq 0 \) which captures the fact that helping others makes an effort to increase production more costly. Moreover, \( \partial c_i^2/\partial e_i \partial a_i^k \leq 0 \), or the marginal cost of an effort is decreasing in \( a_i^k \), for all \( k \) which means that help from others are valuable.
To simplify our argument, we assume that the cost of agent \( i \) in group \( J \) be

\[
c_i(e_i, a_i, a^i) = \frac{c_1}{2} e_i^2 + \frac{c_2}{2} \sum_{k \in J, k \neq i} a_k^2 + c_3 e_i \sum_{k \in J, k \neq i} a_k - c_4 e_i \sum_{k \in J, k \neq i} a_k,
\]

where \( c_1, c_2, c_3, c_4 > 0 \). Parameters \( c_1 \) and \( c_2 \) denotes the slope of marginal costs of the effort and help, \( c_3 \) indicates the sensitivity of marginal costs of the effort to help, and \( c_4 \) measures the effectiveness of aid from others. When \( c_3 \) is larger, helping coworkers makes the marginal costs of the effort much higher, and when \( c_4 \) is greater, help from others are more useful. Note that we assume that \( \partial c_2^i / \partial a_k \partial a_l = 0 \) for any \( k \) and \( l \), which means that help from others has no direct impact on the cost of helping others. This assumption is not too restrictive when the primary reason of agents to help coworkers is to make them more productive, which makes his/her group produce more.

Agent \( i \) in group 1’s problem is to choose the level of effort and help to maximize

\[
\frac{1}{2N} \frac{x_1 x_2}{x_1 + x_2} - c_i(e_i, a_i, a^i),
\]

and its first order condition with respect to \( e_i \) is

\[
\frac{r x_1^{-1} x_2}{(x_1^2 + x_2^2)^2} \frac{x_1 + x_2}{N} + \frac{x_1}{x_1 + x_2 N} = c_1 e_i + c_3 \sum_{k \in J, k \neq i} a_k - c_4 \sum_{k \in J, k \neq i} a_k,
\]

and that for \( a_k^i \) is

\[
\left[ \frac{r x_1^{-1} x_2}{(x_1^2 + x_2^2)^2} \frac{x_1 + x_2}{N} + \frac{x_1}{x_1 + x_2 N} \right] \frac{\partial e_k}{\partial a_k^i} = c_2 a_k^i + c_3 e_i.
\]

From equation (25), we find

\[
\frac{\partial e_k}{\partial a_k^i} = \frac{c_4}{c_1},
\]

which means that with one unit of help, an agent puts additional \( c_4 / c_1 \) unit of effort. The interpretation of the ratio \( c_4 / c_1 \) is the real impact of help in reducing the marginal cost of an effort.

At a symmetric equilibrium where \( e_i = e \) for all \( i \) and \( a_k^i = a_k = a \) for all \( i \) and \( k \), the first order condition with respect to \( e \) becomes

\[
\frac{1}{2N} + \frac{r}{2N} = c_1 e - (c_4 - c_3)(N - 1)a.
\]

The right-hand side is the marginal benefit of effort, which consists of two components. The first term is the one without any incentive device, which is the marginal increase in output divided by the size of the whole team, and the second term is the one created by the relative evaluation scheme. The left-hand side is the marginal cost which is the marginal cost of effort adjusted by net help. The help term consists of the amount of help \( a \), the number of the agent’s colleagues \( N - 1 \) and a multiplier \( c_4 - c_3 \). The multiplier represents the net effectiveness of help in equilibrium, and when this is positive, help is productive.
Similarly, the first order condition with respect to \( a \) becomes

\[
\frac{r + 1}{2N} c_4 c_1 = c_2 a + c_3 e. \tag{29}
\]

The right-hand side represents the marginal benefit of help, which is how much the agent’s wage increases when others put a unit of effort multiplied by how much they do so with a unit of help. The left-hand side is the marginal cost of help, which is affected by the amount of effort. Given the incentive power \( r \), the agent chooses \( a \) and \( e \) so that the marginal benefit and cost are equal.

The equilibrium level of the effort and help is the solution to the system of equations (28) and (29). Although we provide full details of derivation in the appendix, it is useful to discuss it briefly to understand the intuition. From (29), the level of help is given by

\[
a = \frac{(r + 1)c_4}{2Nc_1c_2} - \frac{c_3}{c_2} e. \tag{30}
\]

The first term is the marginal benefit of help divided by the marginal cost of it and we call this the core level of help. The second term represents the amount of help that is dampened by effort. We can obtain a similar expression for effort:

\[
e = \frac{r + 1}{2Nc_1} - \frac{(c_4 - c_3)(N - 1)c_3}{c_1}. \tag{31}
\]

The first term is the marginal benefit of effort divided by the marginal cost of it and we call this the core level of effort. The second term represents the marginal change in the degree of effort caused by that of help.

When we plug (30) into (28), we obtain

\[
\frac{r + 1}{2N} c_4 c_1 = c_2 a + \frac{(N - 1)(c_4 - c_3)c_3}{c_2} e - \frac{(N - 1)(c_4 - c_3)(r + 1)c_4}{2Nc_1c_2}. \tag{32}
\]

The first term of the left-hand side is the direct marginal cost of effort, the second term is the amount of net help suppressed by effort, and the third term is the basic level of net help that the agent receives. Therefore, we can interpret that the left-hand side is the total marginal cost that incorporates both direct and indirect impact of effort. When we plug (32) into (29), we find an expression for \( a \):

\[
\frac{r + 1}{2N} c_4 c_1 = c_2 a + \frac{(N - 1)(c_4 - c_3)c_3}{c_1} a. \tag{33}
\]

The right-hand side is the marginal change in wage when the amount of help changes by one unit. When the level of effort rises, the wage increases by \((r + 1)/2N\) and the level of effort changes by \((c_4 - c_3)/c_1\) when that of help increases by one unit. The first term on the left-hand side is the direct marginal cost of help and the second term is the indirect marginal cost, which is the change in the marginal cost caused by the change in the level of effort. After rearranging terms, we obtain the following equations.
Lemma 2  In the symmetric Nash equilibrium, the level of effort and help is given by

\[ e = \frac{r + 1}{2Nc_1} \left( 1 + \frac{(N - 1)(c_4 - c_3)c_4}{c_1c_2} \right), \]  

and

\[ a = \frac{r + 1}{2Nc_2} \left( 1 + \frac{c_4 - c_3}{c_1} \right). \]

5.1.2 The incentive power and the level of help

From equations (34) and (35), we can obtain a straightforward sufficient condition in which agents puts more effort in both productions and help when the intergroup competition becomes more fierce.

Proposition 4 If help is productive in equilibrium, or \( c_4 - c_3 \) is positive, both \( e \) and \( a \) are increasing in the incentive power \( r \).

We provide the intuition of the impact of the incentive power on the level of effort and help by presenting an adjustment process of agents to the change in the power. Suppose that the organization is initially at equilibrium and the incentive power becomes greater. The initial reaction of the agent is to raise the core level of effort and help, defined in (30) and (28), because the shift in the incentive power increases the marginal benefit of effort and help. This initial adjustment triggers further adjustments. The rise in the level of help alters the marginal cost of an effort whose direction depends on the productivity of help, and this changes the marginal cost of help. The organization reaches a new equilibrium when this process stops. When the initial rise in the core level is significant enough, the amount of effort and help is greater at the new equilibrium.

Note that, even if help is unproductive in equilibrium, or \( c_4 - c_3 < 0 \), the amount of help is increasing in the higher power incentives if

\[ c_2 < \frac{(N - 1)(c_3 - c_4)c_3}{c_1}. \]

When help is unproductive, a decrease in the level of help increases the level of effort, which reduces the marginal cost of help. Therefore, the interpretation of the above condition is that the direct impact on the marginal cost is smaller than the indirect reduction of the cost.

5.1.3 The optimal level of effort and help

The first best level of effort and help are the ones that maximize a total surplus defined by

\[ y = \sum_i c_i(e_i, a_i, a^i). \]
The first order condition with respect to \( e_i \) is
\[
1 = c_1 e_i + c_3 \sum_{k \in J, k \neq i} a_k^i - c_4 \sum_{k \in J, k \neq i} a_k^i, \tag{38}
\]
where the right-hand side is the marginal productivity of effort and the left-hand side is its marginal cost. The first order condition with respect to \( a_k^i \) is
\[
c_4 e_k = c_2 a_k^i + c_3 e_i, \tag{39}
\]
where the left-hand side is the marginal equilibrium benefit of help and the right hand side is its marginal cost. While an individual chooses the level of help to make his/her colleagues work harder, the planner does so to alleviate pain.

Since agents are symmetric, the optimal level of effort and help are solutions to the following system of equations:
\[
1 = c_1 e - (N - 1)(c_4 - c_3)a \tag{40}
\]
and
\[
(c_4 - c_3)e = c_2 a. \tag{41}
\]
Notice that when help is productive, the amount of help is increasing in that of effort because the greater the level of effort, the greater the benefit of reducing its marginal cost. In contrast, when help is not productive, the amount of help is decreasing in that of effort.

Notice that when help is productive, the amount of help is increasing in that of effort because the benefit of help is to reduce the marginal cost of effort, which is strictly increasing. In contrast, when help is not productive, the amount of help is decreasing in that of effort.

When we plug (41) into the second term of the right-hand side of (40), we obtain
\[
(N - 1)(c_4 - c_3)a = \frac{(N - 1)(c_4 - c_3)^2}{c_2} e, \tag{42}
\]
which means that the optimal level of effort is such that its marginal productivity is equal to its cost, which is adjusted by help.

The level of effort and help that maximizes the total surplus are
\[
e^* = \frac{1}{c_1 - \frac{(N - 1)(c_4 - c_3)^2}{c_2}}, \tag{43}
\]
and
\[
a^* = \frac{(c_4 - c_3)}{c_2 - \frac{(N - 1)(c_4 - c_3)^2}{c_1}}. \tag{44}
\]
Given that \( e^* \) is positive, \( a^* \) is positive if and only if help is productive.
In general, it is not possible to achieve the first best level of effort and help simultaneously. Suppose that the designer’s target is to make the effort level equal to its first best level, then the optimal incentive power is given by

\[ r^* = 2N \frac{1 + \frac{(N - 1)(c_4 - c_3)}{c_1c_2}}{1 - \frac{(N - 1)(c_4 - c_3)^2}{c_1c_2}(1 + \frac{(N - 1)(c_4 - c_3)c_4}{c_1c_2})} - 1. \]  

(45)

When this level of incentive power is chosen, help is given by

\[ \hat{a} = \frac{r^* + 1}{2Nc_2} \frac{c_4 - c_3}{c_1 c_3} \frac{c_1}{c_1 c_2} \left(1 + \frac{(N - 1)(c_4 - c_3)c_3}{c_1 c_2}\right), \]  

(46)

and this has the following relationship with \( a^* \).

**Proposition 5** When \( e^* \) is positive, \( \hat{a} \) is greater than \( a^* \) if and only if \( c_1 < \frac{(N - 1)(c_3 - c_4)c_4}{c_2} \).

**Corollary 2** When \( e^* \) is positive, \( \hat{a} \) is smaller than \( a^* \) if help is productive.

The interpretation is when help is productive; the incentive power that achieves the first best level of effort is not high enough to achieve that of help.

### 5.2 Inter-group task

Suppose that the total production of the firm also depends on an agent’s another type of efforts which do not affect each group’s output. In other words, while a worker decides the amount of effort he/she puts to increase his/her group’s production, he/she also has to choose the level of contribution to the firm’s output which does not affect his/her team’s relative performance.

An example of this situation is research and development activities in multidivisional firms where each division has its research and development unit. A researcher has to decide the amount of time and resources he/she spends on each topic ranging from an applied one to develop a new product to a basic one which has little immediate commercial value. The result of the former kinds of research would improve his/her division’s performance immediately, but that of latter would not do so, though it might create a new business opportunity for the firm in the long run. Therefore, if the designer of organization provides a higher powered incentive based on the performance of divisions, he/she would spend more time on applied research and less time on basic research. If the value of the latter is significant, the designer had better provide a lower powered incentive. An alternative and realistic solution is to assign the different type of tasks to separate agents, such as establishing a basic research center.

Suppose that an agent faces two kinds of tasks, the one is for his/her team and the other is for the entire firm. We denote the level of effort that agent \( i \) puts for the former type of task by \( e_i \), and that of the latter by \( a_i \). We assume that team \( J \)'s output \( x_J \) is given by \( \sum_{i \in J} e_i \), and the total output \( y \) is given by \( y = x_1 + x_2 + \theta \sum_{i=1}^{2N} a_i \) where \( \theta \geq 0 \) is the relative
importance of each team’s output. Let agent $i$’s cost function be
\begin{equation}
  c_i(e_i, a_i) = \frac{c_1}{2}e_i^2 + \frac{c_2}{2}a_i^2 + c_3e_ia_i,
\end{equation}
where $c_3 > 0$ means that the marginal cost of each task is increasing in the amount of effort that he/she puts to the other one. We assume that $c_3$ is small enough relative to $c_1$ and $c_2$, in particular, $c_3 < \sqrt{c_1c_2}$.

Agent $i$ in group 1’s problem is to maximize
\begin{equation}
  x_{r^1} x_{r-1} N + \sum_{i=1}^{2N} a_i - \frac{c_1}{2}e_i^2 + \frac{c_2}{2}a_i^2 + c_3e_ia_i.
\end{equation}

The first order condition with respect to $e_i$ is
\begin{equation}
  (x_1^r - x_2^r) x_{r^1} + x_{r-1} + \theta \sum_{i=1}^{2N} a_i - \frac{c_1}{2}e_i^2 + \frac{c_2}{2}a_i^2 + c_3e_ia_i = 0,
\end{equation}
where the first term in the right-hand side represents the incentive provided by the relative evaluation scheme and the second term represents the motivation to work without any incentive device. The first order condition with respect to $a_i$ is
\begin{equation}
  \frac{x_{r^1}^r}{x_{r^1} + x_{r-1}^r} \theta = c_2a_i + c_3e_i,
\end{equation}
and this equation displays that the relative evaluation scheme provides no additional incentive for this kind of task.

At a symmetric equilibrium where $e_i = e$ and $a_i = a$ for all $i$, $e$ and $a$ are solutions to the following system of equations:
\begin{equation}
  \frac{r + 1}{2N} + \frac{\theta ra}{2Ne} = c_1e + c_3a,
\end{equation}
and
\begin{equation}
  \frac{\theta}{2N} = c_2a + c_3e.
\end{equation}

From (51), given $e$, $a$ is given by
\begin{equation}
  a = \frac{\theta}{2Nc_2} - \frac{c_3}{c_2}e,
\end{equation}
which implies that the agent puts less effort into this task as he/she does so into the other task and the team size grows. Notice that when $c_3 = 0$, the amount of $a$ is determined independent from that of $e$.

When we plug (52) into (50), we find
\begin{equation}
  \frac{r + 1}{2N} \left(1 - \frac{\theta c_3}{c_2}\right) + \frac{\theta^2 r}{4N^2c_2e} = c_1e - \frac{c_3^2}{c_2}e.
\end{equation}
The right-hand side is the marginal benefit of effort and the left-hand side is its marginal cost.
If we assume that $e$ cannot be negative, it is given by

$$e = \frac{r + 1}{2N} \left( 1 - \frac{\theta c_3}{c_2} \right) + \frac{1}{N} \sqrt{\frac{(r + 1)^2}{4} \left( 1 - \frac{\theta c_3}{c_2} \right)^2 + \frac{\theta^2 r}{c_2} \left( 1 - \frac{c_3^2}{c_2} \right)} + \frac{2}{c_1 - \frac{c_3^2}{c_2}} \right),$$

(54)

and $a$ is obtained when we plug this into (52).

5.2.1 The incentive power and output

In this subsection, we present that providing no incentive can be optimal in some cases. The intuition is when the designer provides a higher-powered incentive; agents put more effort into task $e$ and less effort into $a$, and this can reduce the total output when the latter type of effort is more valuable. Instead of providing a qualitative result, we present two numerical examples.

![Figure 1: The equilibrium level of effort and output when $\theta = 4$](image)

Figure 1 shows the relationship between level effort and output and the degree of the incentive power where $c_1 = c_2 = 1$, $c_3 = 1/2$, $N = 5$ and $\theta = 4$. The incentive power $r$ is on the horizontal axis and effort $e$ and $a$ and output $y$ are on the vertical axis. When the $r$ becomes larger, an agent increases $e$ and decreases $a$, and $y$ declines because the general job is more important than the team specific one.

Figure 2 shows the relationship when $c_1 = c_2 = 1$, $c_3 = 1/2$, $N = 5$ and $\theta = 1$. In this case, output increases because the general job is not crucial.
6 Conclusion

One of the central questions in organizational economics is how the owner should provide incentives to agents while saving the cost of information acquisition. In this paper, we show that when a firm can be decomposed into two separate groups whose output is observable, the owner can achieve the efficient outcome by making group compete with each other. We demonstrate that the incentive power should be strengthened when the team size is large and the degree of complementarity among each group’s output is substantial. We also examine the interplay between intergroup competition and peer monitoring, and find that intergroup competition fosters mutual monitoring of agents.

7 Appendix

Proof of lemma 1

At a symmetric equilibrium, the first order condition is

\[
\frac{1}{2} \frac{\partial y}{\partial x_j} \frac{\partial f(e_i, e_{-i})}{\partial e_i} \frac{1}{N} + \frac{r x_j^{-1} x_j^2}{(2x_j)^2} \frac{\partial f(e_i, e_{-i})}{\partial e_i} \frac{y}{N} - c'(e) = 0.
\]

and this becomes

\[
\frac{1}{2} \frac{\partial y}{\partial x_j} \frac{\partial f(e_i, e_{-i})}{\partial e_i} \frac{1}{N} + \frac{r}{4x_j} \frac{\partial f(e_i, e_{-i})}{\partial e_i} \frac{y}{N} = c'(e).
\]

The lemma follows from the above equation because the right hand side is increasing in \( r \).
Proof of proposition 1

To achieve the first best, the incentive power \( r^* \) must satisfy

\[
\frac{1}{2} \frac{\partial y}{\partial x_j} \left( \frac{\partial f(e^*_i, e^-_i)}{\partial e_i} \right) + \frac{r^*}{4x_j} \frac{\partial f(e^*_i, e^-_i)}{\partial e_i} \frac{y}{N} = \frac{\partial y}{\partial x_j} f_i(e^*_i) \]  

(57)

Therefore, \( r \) satisfies

\[
\frac{r}{4x_j} \frac{\partial f(e^*_i, e^-_i)}{\partial e_i} \frac{y}{N} = \frac{2N - 1}{2N} \frac{\partial y}{\partial x_j} \frac{\partial f(e^*_i, e^-_i)}{\partial e_i},
\]  

(58)

and we find

\[
r^* = (2N - 1) \frac{\partial y}{\partial x_j} \frac{x^*_j}{y^*}.
\]  

(59)

The rest is to check the second order condition. The derivative of the first order condition with respect to \( e_i \) is

\[r x_i^{-1} x_j \frac{\partial y}{\partial x_j} \left( \frac{\partial f(e_i, e_{-i})}{\partial e_i} \right) \frac{y}{N} = \frac{2}{x_j} x_i \frac{\partial y^2}{\partial x_j} \left( \frac{\partial f(e_i, e_{-i})}{\partial e_i} \right) \frac{1}{N} + \frac{\partial f(e_i, e_{-i})}{\partial e_i} \frac{1}{N} + \frac{\partial f(e_i, e_{-i})}{\partial e_i} \frac{y}{N}.
\]

(60)

We assume that \( f \) and \( y \) are concave, and \( c \) is strictly concave, thus terms which contains second order derivatives are non positive. Hence, it suffices to check the following term is non positive:

\[
2 \frac{r x_i^{-1} x_j}{x_i + x_j^2} \frac{\partial y}{\partial x_j} \left( \frac{\partial f(e_i, e_{-i})}{\partial e_i} \right) \frac{1}{N} + \frac{r(r - 1)x_i^{-2} x_j (x_i + x_j^2) - 2r^2 x_i^{2r-2} x_j (x_i + x_j^2) \left( \frac{\partial f(e_i, e_{-i})}{\partial e_i} \right) y}{(x_i + x_j^2)^3} \frac{1}{N} + \frac{r x_i^{-1} x_j}{x_i + x_j^2} \frac{\partial f(e_i, e_{-i})}{\partial e_i} \frac{y}{N}.
\]

(61)

At a symmetric equilibrium, the above equation becomes:

\[
2 \frac{r x_i^{2r-1}}{4x_i^{2r}} \frac{\partial y}{\partial x_j} \left( \frac{\partial f(e_i, e_{-i})}{\partial e_i} \right) \frac{1}{N} + \frac{r(r - 1)x_i^{3r-2} - 2r^2 x_i^{3r-2} \left( \frac{\partial f(e_i, e_{-i})}{\partial e_i} \right)}{(x_i^3)^3} \frac{y}{N}.
\]

(62)

After a manipulation, we find that the sign is determined by

\[
2 \frac{\partial y}{\partial x_j} - \frac{y}{x_i}. \]

(63)

Since \( y \) is a concave function, for any \( \mathbf{x} \) and \( \mathbf{x}' \),

\[
y(\mathbf{x}) - y(\mathbf{x}') \leq \nabla y(\mathbf{x}') \cdot (\mathbf{x} - \mathbf{x}').
\]
where $\nabla$ is the gradient vector. If we put 0 and $x$, we get
\[
-y(x) \leq \nabla y(x) \cdot (-x),
\] (64)
and when $x_1 = x_2$,
\[
y \geq \frac{\partial y}{\partial x_1} x_1 + \frac{\partial y}{\partial x_2} x_2 = 2 \frac{\partial y}{\partial x_1} x_1,
\] (65)
and this implies that
\[
2 \frac{\partial y}{\partial x_1} \frac{x_1}{y} \leq 1.
\] (66)
Therefore, the second order condition is satisfied.

**Proof of corollary 1**

The utility maximization problem of agent $i$ in team 1 is
\[
\max_{e_i} \frac{x_i^1}{x_i^1 + x_i^2 N} - c_i(e_i).
\] (67)
The first order condition of the agent’s problem is
\[
\frac{x_i^1}{x_i^1 + x_i^2 N} \frac{\partial y}{\partial x_1} \frac{\partial f(e_i, e_{-i})}{\partial e_i} \frac{1}{N} + \frac{r x_i^1 x_i^2}{(x_i^1 + x_i^2)^2} \frac{\partial f(e_i, e_{-i})}{\partial e_i} \frac{y}{N} - c_i'(e_i) = 0.
\] (68)
We show that a profile of strategies where each agent chooses his/her first best effort level is indeed a Nash equilibrium when $r^*$ is chosen as in the proposition 1. If everyone chooses his/her first best effort level, the first order condition of agent $i$ becomes
\[
\frac{1}{2} \frac{\partial y}{\partial x_j} \frac{\partial f(e_i^*, e_{-i}^*)}{\partial e_i} \frac{1}{N} + \frac{r}{4x_j} \frac{\partial f(e_i^*, e_{-i}^*)}{\partial e_i} \frac{y}{N} = c'_i(e_i^*),
\] (69)
and if $r$ is given by
\[
r^* = (2N - 1)2 \frac{\partial y(x^*) x_j^*}{\partial x_j y^*},
\] (70)
the first order condition satisfies
\[
\frac{1}{2} \frac{\partial y}{\partial x_j} \frac{\partial f(e_i^*, e_{-i}^*)}{\partial e_i} \frac{1}{N} + \frac{r^*}{4x_j} \frac{\partial f(e_i^*, e_{-i}^*)}{\partial e_i} \frac{y}{N} = \frac{\partial y}{\partial x_j} \frac{\partial f(e_i^*, e_{-i}^*)}{\partial e_i}.
\] (71)

**Proof of proposition 2**

To achieve the first best effort level, the incentive power $r$ must satisfy
\[
\frac{1}{M} \frac{\partial y(x^*)}{\partial x_j} \frac{\partial f(e_i^*, e_{-i}^*)}{\partial e_i} \frac{1}{N} + \frac{r^* x_j^{2r-1} (M - 1)}{M^2 x_j^{2r}} \frac{\partial f(e_i^*, e_{-i}^*)}{\partial e_i} \frac{y(x^*)}{N} = \frac{\partial y(x^*)}{\partial x_j} \frac{\partial f(e_i^*, e_{-i}^*)}{\partial e_i}.
\] (72)
This becomes,
\[
\frac{1}{M} \frac{\partial y(x^*)}{\partial x_j} \frac{1}{N} + \frac{r^* x_j^{2r-1} (M - 1)}{M^2} \frac{y(x^*)}{N} = \frac{\partial y(x^*)}{\partial x_j}.
\] (73)
Therefore, $r$ is determined by
\[
\frac{r^* x_j^{2r-1} (M - 1) y(x^*)}{M^2} = \frac{MN - 1}{M} \frac{\partial y(x^*)}{\partial x_j},
\] (74)
or
\[
r^* = \frac{MN - 1}{M - 1} \times M \frac{\partial y(x^*) x_j^*}{\partial x_j y^*}.
\] (75)
Proof of proposition 3

Agent \( i \) in team \( J \) solves the following maximization problem:

\[
\max_{c_i} s_j \frac{y}{N_j} - c(e_i). \tag{76}
\]

The first order condition of agent \( i \) in team 1’s problem becomes

\[
\frac{N_1(x_1/x_1^i)^{r_1}}{N_1(x_1/x_1^i)^{r_1} + N_2(x_2/x_2^i)^{r_2}} \frac{\partial y(x_1, x_2)}{\partial x_1} \frac{\partial f^1(e_i, e_{-i})}{\partial e_i} \frac{1}{N_1} + \frac{N_1 r_1 x_1^{r_1-1} x_1^i - r N_2(x_2/x_2^i)^{r_2}}{(N_1(x_1/x_1^i)^{r_1} + N_2(x_2/x_2^i)^{r_2})^2} \frac{\partial f^1(e_i, e_{-i})}{\partial e_i} \frac{y(x_1, x_2)}{N_1} - c'(e_i) = 0. \tag{77}
\]

The first order condition for an agent in team 2 is derived analogously, i.e.,

\[
\frac{N_2(x_2/x_2^*)^{r_2}}{N_1(x_1/x_1^i)^{r_1} + N_2(x_2/x_2^i)^{r_2}} \frac{\partial y(x_1, x_2)}{\partial x_2} \frac{\partial f^2(e_i, e_{-i})}{\partial e_i} \frac{1}{N_2} + \frac{N_2 r_2 x_2^{r_2-1} x_2^i - N_1(x_1/x_1^i)^{r_1}}{(N_1(x_1/x_1^i)^{r_1} + N_2(x_2/x_2^i)^{r_2})^2} \frac{\partial f^2(e_i, e_{-i})}{\partial e_i} \frac{y(x_1, x_2)}{N_2} - c'(e_i) = 0. \tag{78}
\]

To achieve the first best as an equilibrium, \( r_1^* \) must satisfy

\[
\frac{1}{N_1 + N_2} \frac{\partial y(x_1, x_2)}{\partial x_1} + \frac{r_1 N_2}{(N_1 + N_2)^2} \frac{y(x_1, x_2)}{x_1} = \frac{\partial y(x_1, x_1)}{\partial x_1}, \tag{79}
\]

and \( r_2^* \) must satisfy

\[
\frac{1}{N_1 + N_2} \frac{\partial y(x_1, x_2)}{\partial x_1} + \frac{r_2 N_1}{(N_1 + N_2)^2} \frac{y(x_1, x_2)}{x_2} = \frac{\partial y(x_1, x_1)}{\partial x_1}. \tag{80}
\]

Proof of lemma 2

From (29), we have

\[
a = \frac{1}{2} \left( r + \frac{1}{c_1} \frac{c_4}{c_3} - c_3 e \right), \tag{81}
\]

and when we plug this into (28), we find

\[
\frac{r + 1}{2N} = c_1 e - (c_4 - c_3)(N - 1) \frac{1}{2} \left( r + \frac{1}{c_1} \frac{c_4}{c_3} - c_3 e \right)
\]

\[
= \left( c_1 + \frac{(N - 1)(c_4 - c_3)c_3}{c_2} \right) e - \frac{(N - 1)(r + 1)(c_4 - c_3)c_4}{2N c_1 c_2}. \tag{82}
\]

Therefore,

\[
\frac{r + 1}{2N} + \frac{(N - 1)(r + 1)(c_4 - c_3)c_4}{2N c_1 c_2} = \left( c_1 + \frac{(N - 1)(c_4 - c_3)c_3}{c_2} \right) e, \tag{83}
\]
which becomes

\[
e = \frac{(r+1)c_2}{2N} + \frac{(N-1)(r+1)(c_4-c_3)c_4}{2Nc_1}c_1c_2 + (N-1)(c_4-c_3)c_3
\]

\[
e = \frac{r+1}{2N} \left( c_2 + \frac{(N-1)(c_4-c_3)c_4}{c_1} \right)
\]

\[
e = \frac{r+1}{2Nc_1} \left( \frac{c_1c_2 + (N-1)(c_4-c_3)c_4}{c_1c_2 + (N-1)(c_4-c_3)c_3} \right)
\]

\[
e = \frac{r+1}{2Nc_1} \left( 1 + \frac{(N-1)(c_4-c_3)c_4}{c_1c_2} \right).
\]

By plugging this into (28), we find

\[
a = \frac{1}{c_2} \left\{ \frac{r+1}{2N} \left( c_1c_2 + (N-1)(c_4-c_3)c_4 \right) \right\} - \frac{c_2}{c_3}\frac{r+1}{2N} \left( \frac{c_1c_2 + (N-1)(c_4-c_3)c_4}{c_1c_2 + (N-1)(c_4-c_3)c_3} \right). \]

\[
a = \frac{r+1}{2Nc_1c_2} \left\{ c_4 - c_3 \left( \frac{c_1c_2 + (N-1)(c_4-c_3)c_4}{c_1c_2 + (N-1)(c_4-c_3)c_3} \right) \right\}, \]

The term in the parentheses becomes

\[
c_4 - c_3 \left( \frac{c_1c_2 + (N-1)(c_4-c_3)c_4}{c_1c_2 + (N-1)(c_4-c_3)c_3} \right)
\]

\[
= \frac{c_1c_2c_4 + (N-1)(c_4-c_3)c_3c_4 - c_1c_2c_3 - (N-1)(c_4-c_3)c_3}{c_1c_2 + (N-1)(c_4-c_3)c_3}
\]

\[
= \frac{c_1c_2(c_4-c_3)}{c_1c_2 + (N-1)(c_4-c_3)c_3}
\]

Therefore \(a\) is determined by

\[
a = \frac{r+1}{2Nc_1c_2} \frac{c_1c_2(c_4-c_3)}{c_4 - c_3}
\]

\[
a = \frac{r+1}{2Nc_2} \frac{c_1}{1 + \frac{(N-1)(c_4-c_3)c_4}{c_1c_2}}.
\]

**Derivation of the optimal incentive power**

The optimal incentive power is such that

\[
\frac{1}{c_1} - \frac{(N-1)(c_4-c_3)^2}{c_1c_2} = \frac{r^* + 1}{2Nc_1} \left( \frac{1}{1 + \frac{(N-1)(c_4-c_3)c_4}{c_1c_2}} \right),
\]

therefore, we find that

\[
\frac{r^* + 1}{2N} = \frac{\left( \frac{1}{1 + \frac{(N-1)(c_4-c_3)c_4}{c_1c_2}} \right)}{\left( \frac{1}{1 - \frac{(N-1)(c_4-c_3)^2}{c_1c_2}} \right)}.
\]

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which implies
\[ r^* = 2N \frac{\left(1 + \frac{(N-1)(c_4 - c_3)c_3}{c_1c_2}\right)}{\left(1 - \frac{(N-1)(c_4 - c_3)^2}{c_1c_2}\right) \left(1 + \frac{(N-1)(c_4 - c_3)c_4}{c_1c_2}\right)} - 1. \] (90)

Proof of proposition 5

In equilibrium,
\[ \hat{a} = r^* + 1 \frac{1}{2N} c_2 \frac{c_4 - c_3}{c_1} \left(1 - \frac{(N-1)(c_4 - c_3)^2}{c_1c_2}\right) \left(1 + \frac{(N-1)(c_4 - c_3)c_4}{c_1c_2}\right), \] (91)
and when we plug (45) into this, we find
\[ \hat{a} = \frac{1}{c_2} \frac{c_4 - c_3}{c_1} \left(1 - \frac{(N-1)(c_4 - c_3)^2}{c_1c_2}\right) \left(1 + \frac{(N-1)(c_4 - c_3)c_4}{c_1c_2}\right). \] (92)
Recall that the optimal level of \( a \) is
\[ a^* = \frac{1}{c_2} \frac{(c_4 - c_3)}{c_1} \left(1 - \frac{(N-1)(c_4 - c_3)^2}{c_1c_2}\right). \] (93)
The gap between \( a^* \) and \( \hat{a} \) is
\[ a^* - \hat{a} = \frac{1}{c_2} \frac{(c_4 - c_3)}{c_1} \left(1 - \frac{(N-1)(c_4 - c_3)^2}{c_1c_2}\right) \left(1 + \frac{(N-1)(c_4 - c_3)c_4}{c_1c_2}\right). \] (94)
If \( c_4 - c_3 > 0 \), then \( 1 + \frac{(N-1)(c_4 - c_3)c_4}{c_1c_2} > 1 \), which implies that \( a^* > \hat{a} \). If \( c_4 - c_3 < 0 \) and \( 1 + \frac{(N-1)(c_4 - c_3)c_4}{c_1c_2} < 0 \), then \( a^* < \hat{a} \).

References

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