Network Effects, Aftermarkets and the Coase Conjecture: a Dynamic Markovian Approach

Didier Laussel*
Ngo Van Long†
Joana Resende‡

December 8, 2014

* Aix-Marseille University (Aix-Marseille School of Economics), CNRS & EHESS. Email: didier.laussel@univ-amu.fr.
† Department of Economics, McGill University, Montreal H3A 2T7, Canada. Email: ngo.long@mcgill.ca.
‡ Ce.fup, Economics Department, University of Porto. Email: jresende@fep.up.pt.
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Abstract: This paper investigates the expansion of the network of a monopolist firm that produces a durable good and is also involved in the corresponding aftermarket. We characterize the Markov Perfect Equilibrium of the continuous time dynamic game played by the monopolist and the forward-looking consumers, under the assumption that consumers benefit from the subsequent expansion of the network. The paper contributes to the theoretical discussion on the validity of the Coase conjecture, analyzing whether Coase’s prediction that the monopolist serves the market in a "twinkling of an eye" remains valid in our setup. We conclude that the equilibrium network development may actually be gradual, contradicting Coase’s conjecture. We find that a necessary condition for such a result is the existence of aftermarket network effects that accrue (at least partly) to the monopolist firm.

JEL-Classification: L12, L14

Keywords: durable good; network externalities; aftermarkets, Coase Conjecture
1 Introduction

In a seminal work, Coase (1972) argued that, in continuous time, under rational expectations, a monopolist that produces and sells a durable good will lose all her monopoly power, given her inability to commit to future prices and outputs. In equilibrium, the price is equal to the constant marginal cost, and the monopolist serves all her customers in a "twinkling of an eye". Subsequent literature (Stokey (1981), Bulow (1982), Gul et al. (1986)) has confirmed this conjecture, showing that all trade takes place instantaneously at a price equal to marginal cost (in the "no gap case") or a price equal to the valuation of the lowest-valuation consumer (in the "gap case").

Coase (1972) considered neither the possibility of network effects1, nor the fact that the value of durable goods may be enhanced by the subsequent consumption of complementary goods and services (CGS). Yet, there are important markets in which the durable good producers are increasingly involved both in a primary market (in which the production and sale of the durable good take place) and an aftermarket (where CGS are provided by the firm, possibly in the presence of some rival CGS producers). Examples of such markets include tablets/smartphones and applications, hardware and software, wireless services and phone calls, and so on (see e.g. Shapiro, 1995).

Another significant feature of these markets is the existence of network effects (both in the primary market and the aftermarket). Consider for instance the case of tablets (e.g. iPad) and video calls apps (e.g Facetime). In this example, the iPad can be seen as the durable good, whereas Facetime would be a CGS. Several types of network effects may arise in this context. First, there may exist primary market network effects (PMNE). These occur, for example, if consumers buy an iPad because, among other reasons, the device itself is considered fashionable on the eyes of other consumers, yielding a conspicuous consumption

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1These effects arise when the benefits derived from of a good are increasing in the total number of consumers buying that good. See the seminal papers by Rohlfis (1974), Katz and Shapiro (1985), Grilo et al. (2001), or, more recently, Amir and Lazzati (2011) and Griva and Vettas (2011).
effect\textsuperscript{2} akin to positive network externalities (as studied by Grilo \textit{et al.}, 2001). Second, there may also exist aftermarket network effects (AMNE): for example, we expect the utility of a given Facetime user to be increasing with the number of other Facetime users with whom she can communicate.\textsuperscript{3}

The present paper provides a theoretical investigation on the speed of expansion of the network of a monopolist firm that produces a durable good and is also involved in the corresponding aftermarket, under the assumption that buyers benefit from the subsequent expansion of the network. Does Coase’s prediction that the monopolist must serve the market in a "\textit{twinkling of an eye}" remains valid? The paper yields a theoretical contribution to this debate.\textsuperscript{4} In the context of our model, the network development may be gradual, contradicting Coase’s prediction. The existence of AMNE that (at least partly) accrue to the monopolist firm is a necessary condition for this outcome.

We analyze a continuous time dynamic game played by the monopolist and forward-looking consumers with heterogeneous valuations. Each consumer, correctly forecasting future prices and network expansions, finds out whether, given her type, it is advantageous to buy the durable good and, if so, when to buy it. We assume a continuum of infinitely lived consumers, ranked in order of their stand-alone valuation of the durable good. Each

\textsuperscript{2}Regarding this type of effect, Dritsa and Zacharias (2013) point that "Although there is a wide number of firms that produce such devices of similar quality, there are many consumers who prefer to buy an iPhone or a Blackberry as they are convinced that the specific gadgets will confer status to them. The Apple paradigm is also applied in the market for tablet PCs where iPad, amongst all other brand names, is acknowledged as the product that confers status to its purchasers."

\textsuperscript{3}Another illustrative example could be the case of the operating systems (OS) market. OS can be reasonably treated as a durable good, with the software programmes being the corresponding CGS (see Economides, 2000). As pointed out by Cabral (2011) there are primary market network effects since "If I use Windows OS then, when I travel, it is more likely I will find a computer that I can use (both in terms of knowing how to use it and in terms of being able to run files and programs I carry with me)." There are also aftermarket network effects since the utility of a given software often depends on the potential number of individuals with whom the consumers may exchange files. For example, Page and Lopatka (2000) argue that "the value to an individual of a particular word processing program, say WordPerfect, likely will depend in part on the number of others who select WordPerfect and with whom the individual expects to exchange files. This effect is diminished to the extent that conversion between programs is possible, but, so long as conversion is imperfect or costly, the effect persists."

\textsuperscript{4}The Coase Conjecture has spurred a lot of theoretical research, leading to many interesting contributions. However, as argued by Waldman (2003) the practical applicability of many models dealing with the Coase conjecture may be limited since in the real-world markets, firms often try (and succeed) to find strategies (e.g. contracts, product design,....) to overcome the commitment problem raised by Coase.
consumer demands at most one unit of the durable good, whose value depends on its intrinsic characteristics, the PMNE, and the value of the subsequent CGS purchases (which is influenced by AMNE). We make an assumption in favor of the Coase conjecture by considering non-stationary network effects that arise when the (forward-looking) buyer who purchases the good at time $t$ benefits from the network expansion after time $t$. In the iPad/ Facetime example, we may have non-stationary network effects: consumers may benefit from the future expansion of opportunities to communicate with other adopters using the Facetime app.

The monopolist’s problem consists in choosing a time path for the durable good’s output and its price so as to maximize the present value of the discounted stream of total profits that she earns both in the primary market and in the CGS market. We assume that the monopolist cannot make any commitment about future prices and output. Following Laussel and Long (2012) and Hilli et al. (2013), we focus on the Markov Perfect Equilibrium (MPE) of the game.

Our analysis of the speed of expansion of the network shows that, under positive AMNE, Coase’s prediction that the market is served in a "twinkling of an eye" remains valid if and only if the monopolist derives sufficiently large benefits from AMNE. In this situation there is a "large gap" between the lowest consumer valuation and the monopolist’s “effective marginal cost” of supplying her.\footnote{The effective marginal cost is the marginal production cost minus the per period marginal profitability (in the aftermarket) of supplying one additional unit of durable good.} In that case, it is in the interest of the firm to reap all these benefits at once, serving all consumers immediately, and the price of the durable good is higher than its production cost. In contrast, if the AMNE accruing to the monopolist are positive but not too strong, the equilibrium network development may be gradual, contradicting Coase’s conjecture. Then, in equilibrium, the monopolist prefers to slow down the adoption rate of the durable good since the expected price is a decreasing function of the network size. This result arises because, in this case, the expected price of the durable good equals the monopolist’s effective marginal cost of supplying an additional...
consumer, which is a decreasing function of the network size when the monopolist derives profits from the existence of AMNE.

It is worth noting that the gradual network expansion can take place both in the "no gap" case and in the "gap" case. However, in the latter, such network outcome can only occur when AMNE accruing to the monopolist firm are neither too weak nor too strong, implying a "small gap" between the lowest consumer valuation and the monopolist’s effective marginal cost of supplying her. To be more precise, in this "small gap case", the network expands gradually until a critical mass of users is reached. Afterwards, all the remaining consumers are served in a "twinkling of an eye" with a given positive probability, or remain unserved with the complementary probability.

Our model is connected to two strands of literature: the literature on durable good monopolies, with the Coasian Conjecture as a main theme, and the literature on dynamic monopoly pricing in network industries, which also touches on the Coasian Conjecture.

Starting with the seminal work of Coase (1972), a vast literature has studied the optimal monopoly pricing of durable goods. See, for example, Stokey (1981), Bond and Samuelson (1984), Kahn (1986), Gul et al. (1986), Ausubel and Deneckere (1989), Karp (1996a,b), Driskill (1997) among others.

Kuhn and Padilla (1996) have considered a monopolist selling both a durable and a non-durable good to a representative consumer with linear-quadratic preferences over the two goods (that may be complements or substitutes). There are no network effects and the monopolist cannot commit to future prices of the goods. They showed that the firm does not sell the stock of durables in one go. At the formal level, the non-durable good market in their model plays a similar role to the aftermarket in our paper, and the violation of the Coase Conjecture rests on a formally identical feature: the convexity of instantaneous equilibrium profits in the non-durable good market with respect to the stock of the durable good. There

\[ \text{6}\text{More recently, there was a boost on the literature studying dynamic pricing in oligopoly network industries- See, for example, Doganoglu (2003), Laussel et al. (2004), Mitchell and Skrzypacz (2006), Driskill (2007), Markovich (2008), Markovich and Moenius (2009), Chen et al. (2009), Cabral (2011), Laussel and Resende (2014), among others.} \]
are however substantial differences between the two papers beyond the formal similarities. In our paper, there is a continuum of consumers’ types instead of a representative consumer; buying one unit of durable is a necessary condition for consuming CGS in the aftermarket; and the stock of durables enters the instantaneous equilibrium aftermarket payoff functions via network effects rather than complementarity/substitutability in consumption.

In the literature on monopoly pricing in network industries, it has often be argued that the Coase Conjecture may fail when consumers do not benefit from further network developments after they have bought the durable good. Under such stationary network effects, the value of the durable good to later consumers increases as the network expands through time, and several authors have shown that access-pricing strategies may be time-consistent in network industries with these features. In some circumstances (see Cabral et al., 1999), a low introductory price is necessary to reach a critical mass of users and launch the market. However, almost all of these papers (Xie and Sirbu, 1995, Fudenberg and Tirole, 2000, Gabszewicz and Garcia, 2007, 2008) avoid important issues of Coasian dynamics by assuming that consumers may buy the good only when they are young, i.e., consumers do not optimize over the date of purchase. An exception is Bensaid and Lesne (1996) who allow consumers to choose the date of purchase, in a discrete time model. Considering only the “Gap case”, they show that (i) the price of the good may increase through time, and (ii) prices and profits are bounded below.

Mason (2000) is the closest paper to ours. He analyses the behavior of a monopolist selling a durable good to a continuum of consumers. The firm only participates in the primary market. Considering stationary PMNE, he concludes that such effects may lead to gradual network expansion at equilibrium. Unlike Bensaid and Lesne (1996), Mason (2000) finds that the price of the durable good does not increase through time.\(^7\)

Differently from Mason (2000), we consider non-stationary instead of stationary PMNE.\(^7\)

\(^7\)The difference in results between Mason (2000) and Bensaid and Lesne (1996) are due both to the fact that they consider different cases (No Gap versus Gap) and that the discrete time formulation implies that there is a non-degenerate time interval of commitment.
Moreover, as we allow the durable good monopolist to participate in the aftermarket as well, we introduce an additional source of (non-stationary) network effects, the AMNE. To our knowledge, ours is the first work to shed some light on how non-stationary primary and aftermarket network effects affect the dynamics of the problem faced by a monopolist producing a durable good. We show that the existence of non-stationary PMNE alone does not invalidate the Coase conjecture.

The rest of the paper is organized as follows. Section 2 describes the main ingredients of the model. Section 3 provides a detailed description of the primary market and the aftermarket. Section 4 characterizes the Markov perfect equilibrium for different magnitudes of the AMNE and Section 5 concludes.

2 The Basic Model

We consider a monopolist producing a perfectly durable good at a constant marginal cost $c$. The firm also participates in an aftermarket where complementary goods and services (CGS) are provided. In line with Mason (2000), we assume that (a) the durable good is not subject to depreciation; (b) there is no capacity constraint; (c) the monopolist must sell, rather than rent the output; and (d) the time horizon is infinite. Consumers are infinitely lived and each consumer demands at most one unit of the durable good.

Heterogeneous consumers, indexed by $\theta$, are uniformly distributed in the interval $[0,1]$. Higher $\theta$ types value the durable good more highly. The size of the monopolist’s customer base at instant $s$ is denoted by $D(s)$, which is correctly anticipated by consumers. The PMNE yielded at time $s$ are assumed to be proportional to the customers base and are denoted by $\omega D(s)$, where $\omega \geq 0$ represents the intensity of the PMNE. The term $\theta + \omega D(s)$ represents the instantaneous value of the durable good benefits (without any CGS) accruing to the consumer type $\theta$, at $s$.

The value of the durable good also depends on the benefits yielded by future consumption of CGS. The expected CGS consumer surplus obtained at instant $s$ is denoted by $Z^e(s)$.
Since all agents have rational expectations, firms and consumers’ beliefs about the future are confirmed in equilibrium, implying that the expected CGS consumer surplus, $Z^e(s)$, is equal to the actual one, denoted by $Z(s)$. This means that consumers are perfectly able to anticipate the evolution of the monopolist’s network (anticipating future AMNE).

The price of the durable good at instant $s$ is denoted by $p(s)$ and consumers correctly anticipate its evolution.

Denoting by $V(\theta, t)$ the discounted expected lifetime utility obtained by a consumer type $\theta \in [0, 1]$ who chooses to buy the durable good at instant $t$, we have

$$V(\theta, t) = \int_{t}^{\infty} [\theta + \omega D(s)] e^{-r(s-t)} ds + \int_{t}^{\infty} Z^e(s) e^{-r(s-t)} ds - p(t),$$

(1)

where $r$ stands for the discount rate and $s > t$.

3 The Aftermarket and the Primary market

In this section, we specify in more detail our assumptions about the primary market and the aftermarket. We start by studying the aftermarket because consumers’ anticipated valuations about future CGS consumption play an important role on their decisions concerning the purchase (or not) of the durable good. By deriving consumers’ optimal decision in the aftermarket, we will be able to compute $Z^e(s)$, conditional on consumers having purchased the durable good at a given time $t$. This will allow us to determine how the optimal time of purchase of the durable good varies across consumer types.

3.1 The aftermarket

When computing $V(\theta, t)$ consumers need to anticipate $Z^e(s)$, $s > t$. Because consumers are forward-looking and have rational expectations, $Z^e(s) = Z(s)$. The value of $Z(s)$ depends on the intrinsic characteristics of CGS, the AMNE and the price of CGS. We assume that the intrinsic characteristics of the CGS are exogenous and time-invariant. As far as AMNE are concerned, consumers need to correctly anticipate the future evolution of the
$D(s)$ (which is the basis for future AMNE) as well as the future price of CGS. Like in Kuhn and Padilla (1996), we assume that (i) CGS are non-durable and instantly consumed, and (ii) the CGS provider(s) is (are) unable to commit to future prices.

The no-commitment assumption, which implies static profit maximization in the aftermarket, consistently extends to the non-durables the usual assumption underlying the Coase Conjecture literature, according to which the durable good monopolist is unable to commit to future prices of the durable good. This rules out non-credible strategies in which the firm would promise at time $t$ to lower the price in the aftermarket at some time $t' > t$, in order to make the primary good more attractive to forward-looking consumers.

The non-durability of CGS is a simplifying assumption. It rules out the problem of consumers’ dynamic optimization over the time of CGS purchases. While this assumption is suitable for some aftermarkets (e.g. phone calls or the repairing services), there is some level of CGS durability to be reckoned with in some other aftermarkets (e.g applications or software). However, even in those cases, there generally remains a substantial difference between the lifetime lengths of CGS and of the corresponding durable goods. Hence, it is probably not too bad an approximation to assume that CGS are non-durable, and thus optimization over the purchase date applies only to the durable goods.

Let $Z(D(s))$ denote the equilibrium CGS consumer surplus at instant $s$ and suppose:

$$Z(D(s)) = \gamma_1 + \phi_1 D(s), \gamma_1 \geq 0,$$

with $\phi_1 \geq 0$, meaning that the equilibrium CGS surplus (weakly) increases with $D(s)$ (due to AMNE).

Analogously, let $\pi^A(D(s))$ denote the equilibrium instantaneous profit on CGS supplied by the monopolist producer of the durable good in the aftermarket at instant $s$. Then $\frac{\pi^A(D(s))}{D(s)}$ represents its equilibrium instantaneous aftermarket profit per customer. We suppose that:

$$\frac{\pi^A(D(t))}{D(t)} = \gamma_2 + \phi_2 D(t), \gamma_2 \geq 0,$$

where $\phi_2 \geq 0$, meaning that the monopolist’s equilibrium aftermarket profit per customer
increases with its consumer base.\textsuperscript{8}

Equation (3) corresponds to a reduced form modelling of the aftermarket. It is the maximum value function obtained from the profit maximization problem in the aftermarket. In Appendix A, we provide further details on some illustrative examples for which equations (2) and (3) hold. \textbf{Example 1 deals with oligopolistic competition in the CGS market where firms are Cournot rivals.} This example is suitable to study situations in which the producer of a certain OS (e.g. Windows OS) is also involved in the provision of a certain type of applications or software\textsuperscript{9} (e.g. a word processing software like Microsoft Word) but it faces the competition of independent software suppliers providing very close substitutes (e.g. Word Process, Word Perfect, Google Docs, ...). Example 2 relaxes the assumption of perfect substitutability of CGS. It is suitable to study the provision of horizontally differentiated software or applications. In these examples, the equilibrium instantaneous profit per consumer (corresponding to equation (3)) is linearly increasing in $D$ since, when selling CGS in the aftermarket, the durable good producer (as well as the other independent suppliers) takes advantage of a larger consumer base to set a higher price. Although an increase in $D$ results in higher prices for CGS, \textbf{we also obtain that the equilibrium consumer surplus in CGS increases linearly} with $D(s)$ since, despite the higher CGS price, consumers retain a part of the additional surplus generated by a larger network.

\subsection*{3.2 The primary market}

When computing $V (\theta, t)$, forward-looking consumers anticipate that $Z^e(s) = Z(D(s))$. This allows us to compute, for each consumer, the value of purchasing the durable good at time $t$. Plugging equation (2) into (1), we obtain the expected life-time utility, discounted back to time zero, of a type-$\theta$-consumer if she buys the durable good at time $t$:

\textsuperscript{8}In fact, the crucial assumption that leads to the failure of the Coase Conjecture is that equilibrium instantaneous profits per consumer in the aftermarket are increasing in the consumer base of the durable good. Assuming an affine function is only for the sake of tractability.

\textsuperscript{9}In the context of example 1, the quality and variety of applications or software is taken as given since we concentrate on direct AMNE. See Farrell and Klemperer (2007) for further details on the different nature of direct and indirect network effects.
\[ e^{-rt} V(\theta, t) = e^{-rt} \left[ \frac{\theta + \gamma_1}{r} + (\omega + \phi_1) \int_t^\infty D(s)e^{-r(s-t)}ds - p(t) \right]. \] (4)

Each type-\( \theta \) consumer has to decide whether she will ever buy the equipment and, if so, what is the optimal time to buy it. These two decisions are, in principle, distinct. Nonetheless, they are often confused in the literature. A consumer of type \( \theta \) will buy the durable good if and only if there exists some date \( t' \) such that \( V(\theta, t') \geq 0 \). In this case, the consumer needs to choose the optimal date of purchase \( t(\theta) \) in order to maximize (4), given her expectations about the equipment price path. Since consumers have rational expectations, the equilibrium evolution of the equipment price is determined by a simple arbitrage condition following from the first-order condition of this problem. The second-order condition implies that higher types of consumers should not buy the equipment later than lower types. The following Lemma provides further information on the non-arbitrage condition.

**Lemma 1** For a consumer of type \( \theta \), the optimal purchase date \( t(\theta) \) satisfies

\[
\frac{dp(t(\theta))}{dt} = rp(t(\theta)) - [\theta + \gamma_1 + (\omega + \phi_1) D(t(\theta))] \tag{5}
\]

where \( t(\theta) \) is non-increasing in \( \theta \), i.e. \( \frac{dt(\theta)}{d\theta} \leq 0 \).

Denote by \( \theta(t) \) the lowest consumer type who has already bought the equipment at time \( t \). From Lemma 1, \( \theta(t) \) is non-increasing in \( t \). The assumption of uniform distribution of types over \([0, 1]\) implies \( \theta(t) = 1 - D(t) \). Accordingly, the non-arbitrage condition stated in Lemma 1 leads consumers to formulate their expectations about the evolution of the price of the durable good according to the following rule:

\[
\frac{dp(t)}{dt} = rp(t) - b(D(t)), \tag{6}
\]

where we have defined

\[ b(D(t)) \equiv 1 - D(t) + \gamma_1 + (\omega + \phi_1) D(t), \] (7)
We can interpret \( b(D(t)) \) as representing the *instantaneous full benefit* at time \( t \) that the equipment confers to the *marginal* customer \( \theta(t) \).

Since the customer \( \theta(t) \) buys the equipment at instant \( t \), it must be the case that an infinitesimal postponement of the purchase involves a marginal cost that is just balanced by a marginal monetary benefit. The former corresponds to the consumer’s forgone enjoyment of the instantaneous full benefit of holding the equipment at moment \( t \), i.e. \( b(D(t)) \). It is the sum of three terms, as shown in equation (7): (i) the intrinsic "stand-alone benefit" of the equipment to the marginal consumer, \( \theta(t) = 1 - D(t) \); (ii) the PMNE, \( \omega D(t) \); and (iii) the instantaneous net benefit obtained from the consumption of CGS, \( \gamma_1 + \phi_1 D(t) \), which includes the AMNE. The marginal benefit of postponing infinitesimally the purchase of the equipment consists of the interest income \( r p(t) \) and the cost savings due to the reduction in the equipment price \( -\frac{d p(t)}{d t} \).

Notice that delaying the purchase would also imply that one would start enjoying the benefit flows with larger network effects: \( D(t + dt) \) versus \( D(t) \). However, this consideration does not appear in the arbitrage condition. This is because, in comparing the net benefits of purchasing the durable good at two points in time, \( t \) and \( t + \Delta t \), any terms that involve \( [D(t + \Delta t) - D(t)] \times \Delta t \) are ‘second order small’ and thus can be neglected as \( \Delta t \) becomes arbitrarily small.

Condition (6) shows that, in equilibrium, the marginal benefit and the marginal cost of postponing infinitesimally the purchase must cancel out for the marginal consumer \( \theta(t) \); for otherwise, she could improve her lifetime utility by changing the timing of her purchase.

Integrating (6) yields the equilibrium price path

\[
p(t) = \int_{t}^{\infty} b(D(s)) e^{-r(s-t)} ds,
\]

which is perfectly anticipated by forward-looking consumers. We derive the corresponding arbitrage equation for the case of stationary network effects in Remark B.1 in Appendix B.

Let us now show that under non-stationary network effects, in equilibrium, the consumer
who purchases the durable good at time \( t \) expects a positive life-time surplus that depends only on the equilibrium time path of the durable good producer’s network size. From (4),

\[
V(\theta(t), t) = \frac{\theta(t) + \gamma_1}{r} + (\omega + \phi_1) \int_t^\infty D(s) e^{-r(s-t)} ds - p(t),
\]

Substituting in the above equation for \( p(t) \) using its equilibrium value from (8), we obtain

\[
V(\theta(t), t) = \int_t^\infty [D(s) - D(t)] e^{-r(s-t)} ds
\]

where we have made use of the fact that \( \theta(t) = 1 - D(t) \).

**Remark 1:** The expected life-time surplus of a consumer who optimally purchases at time \( t \), \( V(\theta(t), t) \), is strictly positive when there is \( s > t \), such that \( D(s) > D(t) \).

This follows from the fact that in equilibrium the customer base is non-decreasing, \( D(s) \geq D(t) \) for all \( s \geq t \). Consumers who find it optimal to purchase the equipment at \( t \), rather than at some some later time \( t' > t \), derive from that purchase a positive expected lifetime utility. If the network size is strictly increasing over time, this utility is strictly positive. This means that some consumer types with valuations that are marginally below \( \theta(t) \) do not buy the equipment at \( t \) but they would strictly prefer buying the equipment to never buying it. They simply delay their purchase in order to benefit from future lower prices and greater network effects.

The monopolist’s instantaneous profits in the primary market are equal to

\[
\pi^{PM}(t) = q(t) [p(t) - c],
\]

where \( q(t) \) is the equipment quantity sold by the monopolist at instant \( t \). Since the durable good monopolist benefits from CGS sales, we can define the effective marginal cost of supplying one additional unit of the durable good as follows:

\[
m(D(t)) = rc - \gamma_2 - 2\phi_2 D(t), \quad (9)
\]
where $m(D(t))$ is equal to the amortized manufacturing cost per-period\(^{10}\) minus the marginal instantaneous profit (with respect to the customer base) derived from CGS sales, which is equal to $\gamma_2 + 2\phi_2 D(t)$.

In the following section, we derive the Markov Perfect Equilibrium (MPE) of this game. In a MPE, the players’ strategies depend only on the state variable, which is the size of the monopolist’s installed base of consumers.

The nature of the equilibrium turns out to depend crucially on the relative position of the curve representing the effective marginal cost of supplying one additional unit of the durable good, $m(D)$, and the curve representing the instantaneous full benefit of the equipment, $b(D)$, to the marginal customer. In what follows, in order to sharpen our analysis, we assume that $b(0) > m(0)$, i.e. the instantaneous full benefit of the equipment to the highest-valuation consumer, if she were the only consumer, exceeds the effective marginal cost of supplying the durable good to this consumer. This assumption is sufficient to ensure that it is worthwhile for the monopolist to produce a positive output. We state this as Assumption A1. In Remark B.2 in Appendix B we provide further details on what happens when the assumption is violated.

**Assumption A1:** $b(0) > m(0)$, i.e., $1 + \gamma_1 > rc - \gamma_2$.

For $\phi_2 > 0$, the per period marginal profitability (in the aftermarket) of supplying one additional unit of the durable good is increasing with the size of the monopolist’s network. Therefore, the effective marginal cost of supplying one additional unit of the durable good decreases with $D$, for $\phi_2 > 0$. If $\phi_2$ were zero, then $m(D)$ would be a horizontal line. As can be seen from equation (7), the graph of $b(D)$ is a straight line, and its slope can be negative or positive, depending on whether $\omega + \phi_1$ is smaller or greater than 1. In fact, when the network of the monopolist firm expands, two effects arise. First, when $D$ increases, consumers benefit from greater network effects. (Recall that $\omega + \phi_1$ measures the marginal full network benefit for consumers, considering both the PMNE and the AMNE.) However,

\(^{10}\)Producing one additional unit of durable good costs $c$ to the firm at moment $t$, when that unit is manufactured. This is equivalent to incurring a cost $rc$ per unit of time from $t$ until infinity.
this also affects the position of the marginal customer itself. In particular, lower types of consumers become interested in buying the good, which reduces the instantaneous benefit that the equipment confers to the marginal customer.

Notice that \( b(1) = \omega + (\gamma_1 + \phi_1) \) while \( m(1) = rc - \gamma_2 - 2\phi_2 \). **Recollecting that** \( b(0) > m(0) \), we distinguish three mutually exclusive and exhaustive cases: the No Gap case, the Small Gap case and the Large gap case, depending on the intensity of AMNE. The figure below illustrates where each regime holds on the space \((\omega, \phi_2)\). The figure is drawn for \( r = 0.07, c = 10, \gamma_1 = 0.05, \gamma_2 = 0.1, \phi_1 = 0.05 \). The parameters were chosen in order to allow us to capture all the cases described above in a single diagram, to save space. The figure shows that, provided \( \phi_2 > 0 \), the No Gap case (NG) arises for weak AMNE; the Small gap case (SG) arises for intermediate AMNE, whereas the Large Gap Case arises for strong AMNE.

**INSERT FIGURE 1** *(we could put old figure 2 here in order to cut one figure)*

The figure also shows that, provided \( \phi_2 > 0 \), i.e. there are strictly positive AMNE and the durable good producer is able to extract part of the surplus generated by them, the role of PMNE and AMNE in the configuration of the possible MPE regimes is similar. In fact, as long as \( \phi_2 > 0 \), the type of regime depends on the intensity of overall network effects (i.e. it depends on the sum of PMNE and AMNE).

**Case 1 (The No Gap Case)** The No Gap Case is defined to be the case where there exists a customer base \( D \in (0, 1] \) such that, at \( D \), the marginal customer’s instantaneous full benefit, \( b(D) \), is exactly equal to the monopolist’s effective marginal cost, \( m(D) \). We will show that, in this case, the durable good monopoly, starting with any initial customer base \( D_0 < D \), will restrict the size of its customer base to \( D \). Since \( b(D) \) and \( m(D) \) are linear and since \( b(0) > m(0) \) (by Assumption A1), the No Gap Case arises if and only if \( b(1) \leq m(1) \), which is equivalent to the condition

\[
\phi_1 + 2\phi_2 \leq rc - (\gamma_1 + \gamma_2) - \omega. \tag{10}
\]
Since \( \omega \) and \((\phi_1, \phi_2)\) are, respectively, the primary and aftermarket network parameters, the No Gap Case arises when the aftermarket network effects are \textit{weak}, in the sense of inequality (10).\(^{11}\)

When condition (10) is not satisfied, we have \( b(1) > m(1) \), i.e., there is a positive gap between the benefit to the lowest valuation consumer type, and the firm’s effective marginal cost \textbf{if the whole market is covered}. In the literature on durable good monopoly without network effects, this situation is known as the Gap Case, with the well-known results that the monopolist will immediately cover the whole market and make a strictly positive profit (Bulow, 1982). Interestingly, in our model with network effects, these results do not carry over without substantial qualifications. In fact, instead of a Gap Case, we have two different Gap Cases, with completely different equilibrium characteristics: The Large Gap Case and The Small Gap Case.

Given Assumption A1, a Gap Case implies that at all possible installed customer bases, \( b(D) > m(D) \). Therefore it may be tempting to conjecture that the monopolist will want to serve the whole potential market, i.e., to try achieve \( D = 1 \), either immediately, or asymptotically. However, upon reflection, a key consideration is whether the total cost of supplying the whole market instantaneously is lower than \textit{total revenue from durable good sales, at the non-discrimination price such that the price that least-valuation consumers are willing to pay for the equipment}.

The total cost of supplying the whole market instantaneously is the capitalized value of the area under the per-period marginal cost schedule, the area \( A \) defined by

\[
A \equiv \int_0^1 m(D)dD.
\]

\(^{11}\)Condition (10) can be re-written as \( \omega + \phi_1 + 2\phi_2 \leq rc - (\gamma_1 + \gamma_2) \), showing that the condition for the no gap case really depends on the overall network effects (i.e. the sum of PMNE and AMNE). Yet, the roles of PMNE and AMNE are substantially different because the existence of AMNE is a necessary condition for the Coase conjecture to fail, whereas the same is not true for PMNE. In subsection 4.2, we indeed show that non-stationary PMNE alone do not invalidate the Coase Conjecture. It is also worth noting that the roles played by \( \phi_1 \) and \( \phi_2 \) are not completely symmetric. In fact, to have a violation of the Coase conjecture, we need \( \phi_2 > 0 \), which requires the existence of strictly positive AMNE and, in addition, part of the surplus they create must be absorbed by the producer of the durable good.
Since $m(D)$ is linear, the area $A$ is exactly equal to $m(1/2)$, and its capitalized value is $\frac{1}{r}m(1/2)$. This is to be compared with the price that the lowest-valuation consumer is willing to pay, $\frac{1}{r}b(1)$. If $b(1) \geq m(1/2)$, we say that we are in the Large Gap Case. If $b(1) < m(1/2)$, we say that we are in the Small Gap Case.

**Case 2 (The Large Gap Case)**

The inequality $b(1) \geq m(1/2)$ is equivalent to the following condition

$$\phi_1 + \phi_2 \geq rc - (\gamma_1 + \gamma_2) - \omega. \quad (11)$$

When this condition holds, we say that the AMNE are strong.

Clearly, in the special case where $\phi_2 = 0$, whenever the Gap Case arises, i.e., $\phi_1 + 2\phi_2 > rc - (\gamma_1 + \gamma_2) - \omega$, we necessarily are in Large Gap Case.

**Case 3 (The Small Gap Case)**

By definition, a Small Gap arises if and only if $m(1/2) > b(1) > m(1)$. The first strict inequality part of this condition is equivalent to

$$rc - (\gamma_1 + \gamma_2) - \omega > \phi_1 + \phi_2 \quad (12)$$

while the second strict inequality, $b(1) > m(1)$ is simply the negation of the No Gap Condition (10). The Small Gap Case arises if and only if the network effects are “intermediate” in the sense that $\phi_1 + 2\phi_2 > rc - (\gamma_1 + \gamma_2) - \omega > \phi_1 + \phi_2$. Obviously, for a Small Gap Case to exist, one requires that $\phi_2 > 0$.

The properties of the Markov perfect equilibrium differ fundamentally across the three cases. This will be demonstrated in the next section.

**Remark 2.** Our analysis can be easily extended to the case of negative network effects (i.e. congestion effects). If all network effects were negative, there would exist only two possible cases: The No-Gap case and the Large Gap case. With congestion effects in the aftermarket, the Coase conjecture remains valid.\(^\text{12}\)

\(^{12}\)The proof of Remark 2 is available from the authors, upon request.
Figure 1 above illustrates the results in Remark 2. In the Large Gap case, the Coase conjecture of immediate sale to all customers holds both in the case of positive and negative network effects. The Small Gap case only exists for $\phi_2 > 0$. Finally, the No Gap case arises both under positive or negative network effects, but its properties are considerably different. As we will show in Section 4, when $\phi_2 > 0$, sales take place gradually (regardless of the positive or negative network effects arising in the primary market). On the contrary, when $\phi_2 < 0$, the Coase conjecture remains valid.

4 Markov Perfect Equilibrium

In a MPE, the monopolist uses a Markovian output strategy and the consumers’ expectations about the evolution of prices can be represented by a Markovian price function.

4.1 Some preliminary considerations

Consider first the Markovian price function representing consumers’ expectations about the evolution of the equipment price. Assume that all consumers have a common Markovian price expectation function $\zeta(D)$, where $\zeta$ is a function of the state variable $D$. When consumers have the ability to perfectly anticipate future market outcomes, condition (8) implies that the price expectations function must be rational, i.e.

$$\zeta(D(t)) = p(t) = \int_t^\infty b(D^*(s))e^{-r(s-t)}ds,$$

or, written in full,

$$\zeta(D(t)) = \int_t^\infty [1 - D^*(s) + \gamma_1 + (\omega + \phi_1) D^*(s)] e^{-r(s-t)}ds,$$

where $\{D^*(.)\}_t^\infty$ is the time path of the state variable $D$ induced by the strategic behavior of the monopolist from time $t$ on, when the state variable initially (at time $t$) takes the value $D(t)$.

Consider now the monopolist’s Markovian output strategy. Such a strategy specifies how the monopolist intends to sell the durable good. For example, in some intervals of the state
space, the monopolist may choose to sell the durable good gradually whereas in some other intervals of the state space, she may prefer to sell a lumpy amount. The output strategy is said to be Markovian if it is a rule $G$ which tells the firm the amount of the durable good to sell at time $t$, based only on the knowledge of its current customer base, $D(t)$. A strategy $G$ is a best reply to the consumers’ expectations rule $\zeta$ if it yields a time path of sales $q^*(t)$ that maximizes total discounted profits of the integrated firm for all starting (date, state) pairs $(t, D(t))$:

$$
\Pi(t) = \int_t^{\infty} \left[ \pi^A(s) + \pi^PM(s) \right] e^{-r(s-t)} ds.
$$

(15)

In other words, we look for a solution to the problem

$$
\max_{q(s)} \Pi(t)
$$

s.t.

$$
\frac{dD(s)}{ds} = q(s), \ D(t) \text{ given},
$$

that generates a price path that is consistent with (14).

Formally, a Markovian strategy $G$ is the specification of (i) a collection of disjoint intervals of the state space, $I_1, I_2, ..., I_m$, in which the monopolist plans to sell lumpy amounts of the durable good, where $I_i \equiv [a_i, b_i] \subset [0, 1]$, (ii) a lumpy sale function $L_i(.)$ corresponding to each interval $I_i$, such that $L_i(.) \geq 0$ specifies an upward jump in the state variable, so as to increase the customer base from $D$ to $D + L_i(D)$, with $0 \leq D + L_i(D) \leq 1$ where $D \in I_i$, and (iii) a gradual sales function $g(.)$ defined for all $D \notin I_i$, such that

$$
q(t) = \frac{dD(t)}{dt} = g(D(t)) \text{ for } D \notin I_i.
$$

**Definition 1** A Markov-perfect equilibrium is a pair $(G, \zeta)$ such that, (i) given the price function $\zeta$, the strategy $G$ maximizes the integrated firm’s payoff, starting at any (date,state) pair $(t, D(t))$ and (ii) given $G$ and $(t, D(t))$, the price function $\zeta$ satisfies the rational expectation property (14).
Having defined our equilibrium concept, we now turn to a characterization of the Markov Perfect Equilibrium.

Before stating our main results, it is useful to consider two benchmark scenarios. In the first benchmark scenario, there are no network effects ($\omega = 0$ and $\phi_1 = \phi_2 = 0$). In this case, only the No Gap and the Large Gap cases may arise, depending on the sign of the expression $rc - (\gamma_1 + \gamma_2)$. Lemma C.1 in Appendix C shows that traditional Coasian dynamics hold at equilibrium. The monopolist’s equilibrium strategy is to supply the whole market immediately, both in the (large) Gap and the No Gap cases. This makes the point that adding an aftermarket (without network effects) to an otherwise traditional durable goods model does not change the standard literature results.

We now turn to the second benchmark scenario, to show that traditional Coasian dynamics remain valid under PMNE only (i.e. $\omega > 0$ but $\phi_1 = \phi_2 = 0$). Again, only the No Gap and the Large Gap cases may arise. Results in Lemma C.1 remain valid, except for the size of the monopolist’s steady state network in the No Gap case, which is now equal to $D = \frac{1-rc+\gamma_1+\gamma_2}{1-\omega}$. This shows that the Coase Conjecture is consistent with the existence of non-stationary PMNE. This result is in sharp contrast to Mason (2000). Analyzing the No Gap case, Mason (2000) finds gradual network expansion under stationary PMNE. In addition, as long as the network size is expected to increase, Mason (2000) gets lower instantaneous expected prices than the ones we obtain under non-stationary network effects (see Remark B.1 in Appendix B for a formal comparison between the two arbitrage conditions).

In order to understand the intuition behind the difference between the results in Mason (2000) and in our work, note that in the former when consumers make their decision whether to purchase at time $t$ or to delay their purchase until time $t+\Delta t$ they are trading off, on the one hand, the cost of having to forgo the enjoyment of the good (including the associated network effects) during $\Delta t$, and, on the other hand, the gains from both the lower price at date $t+\Delta t$, and a larger constant perpetual flow.
of network benefits (where the constancy arises from the fact that, with stationary network effects, consumers do not benefit from further expansions of the network after the date of purchase). In our model with non-stationary network effects, that is no longer the case. Consumers enjoy the same stream of benefits from the future expansion of the network beyond date \( t + \Delta t \), whether they purchase at time \( t \) or at time \( t + \Delta t \).\(^{13}\) This difference explain why, in our model, with only non-stationary PMNE, and absent AMNE that accrues partly to the monopoly, the equilibrium price expectation function is independent of the consumer base \( D \), while in Mason’s model, where PMNE are stationary, the price expectation function is dependent on \( D \), leading the monopolist to practice intertemporal price discrimination.

As shown afterwards, the existence of aftermarket network effects that accrue (at least partly) to the firm will change this picture. In that case the monopolist’s effective marginal cost becomes decreasing in the network size, leading to consumer expectations of falling prices, which in turn motivates the monopolist from selling too quickly, in a twinkling of an eye (the following subsections provide a rigorous analysis of this point).

It is also worth noting that the standard Coasian results also apply if we suppose that \( \omega > 0 \) and \( \phi_1 > 0 \) but \( \phi_2 = 0 \), a purely hypothetical situation in which all the benefit from AMNE would accrue to consumers. This could correspond to a situation in which the aftermarket is perfectly competitive (e.g. because the number \( N \) of CGS suppliers tends to plus infinity in Example 1 or the degree of substitutability \( \theta \) tends toward 1, i.e. perfect substitutes, in Example 2), yielding zero aftermarket profits for the durable good producer. The source of the violation of the Coase Conjecture in our model is the fact that AMNE, when partly appropriated by the durable good producer, introduce a convexity of instantaneous equilibrium profits in the non-durable good market with respect to the stock of the durable good.

\(^{13}\)As we have mentioned in sub-section 3.2, the term \( \Delta D \times \Delta t \) which corresponds to the difference in network benefits that accrues over the small time interval \( \Delta t \) is insignificant for an infinitessimal \( \Delta t \).
In what follows, we assume $\phi_2 > 0$ and we will show we can get a failure of the Coase conjecture where we did not get one before. This finding is independent of the existence (or not) of PMNE. We also find that the the Markov Perfect Equilibrium may have different properties depending on the parameters of the model (and in particular on the intensity of AMNE). In the Large Gap Case, the standard Coasian result applies, with immediate coverage of the entire market, and positive profit for the monopolist. In the No Gap Case, but with AMNE reflected in $\phi_2 > 0$, we obtain the result that the monopolist sells the durable good gradually, and her profit equals zero. A new feature of MPE arises in the Small Gap Case (which requires $\phi_2 > 0$). Each of these cases is analyzed separately in the subsections below.

4.2 MPE with strong AMNE (the Large Gap case)

In this sub-section, we tackle the simplest case. As shown before, the Large Gap Case arises when $b(1) \geq m(1/2)$, or equivalently, $\phi_1 + \phi_2 \geq rc - \omega - \gamma_1 - \gamma_2$, implying strong AMNE. Recall that the Large Gap case requires that (i) the marginal benefit to the lowest valuation consumer, $b(1)$, exceeds the effective marginal cost, $m(1)$; and (ii) the total cost of serving the whole spectrum of consumers immediately in one go, $\frac{1}{r} m(1/2) = \frac{1}{r} \int_0^1 m(D)dD$ is lower than the revenue from selling durable goods to all customers in one go, $b(1)/r$.

It is easy to verify that the Markov-perfect equilibrium in the Large Gap Case has the following properties (See Appendix D for the formal proof). First, consumers expect that the price of the durable good is constant, equal to the present value of the stream of net

\[ b(1) = \frac{1}{r} \int_0^1 m(D)dD \]

\[ m(1/2) \]

\[ \frac{1}{r} \]

14In Kuhn and Padilla (1996) the violation of the Coase conjecture was also due to the convexity of instantaneous profits in the non-durable good market with respect to the stock of the durable good. They do not consider any kind of network effects but still obtained gradual sales. This might appear to be in contradiction with the results obtained for our benchmark scenarios without network effects. However, as Kuhn and Padilla (1996) made clear, the optimal policy would be a lumpy one if the two products are independent (that is neither substitutes nor complements). When aftermarket network effects are set to nil in the current paper, this is exactly the case as well. We gratefully acknowledge an anonymous referee for pointing out this result.
benefits to the lowest valuation consumer:

$$\zeta(D) = \frac{1}{r} (\gamma_1 + \phi_1 + \omega) = \frac{b(1)}{r} = p^*.$$  \hspace{1cm} (16)

Second, the monopolist’s equilibrium strategy is to supply the whole market immediately, i.e., $D(t) = 1$ for $t > 0$. In other words, starting from any $D \in [0, 1)$, the firm uses the lumpy sale strategy $L(D) = 1 - D$. As a result, the firm’s value in the Large Gap case is

$$J(D) = (1 - D)(p^* - c) + \frac{1}{r} [\gamma_2 + \phi_2] \geq 0$$  \hspace{1cm} (17)

where the term inside the square brackets is the capitalized value of the stream of profits in the aftermarket when all customers are served. Note that $J(0) = \frac{1}{r}(b(1) - m(1/2)) \geq 0$. Since, in the Large Gap case, $rc \leq \omega + \phi_1 + \phi_2 + \gamma_1 + \gamma_2$ it follows that the possibility that $b(1) \equiv \phi_1 + \gamma_1 + \omega < rc$ exists. This means the equilibrium price $p^*$ may be smaller than the production cost $c$. In such a case, we have $J'(D) = -(p^* - c) > 0$, and hence we still have $J(D) > 0$ for all $D > 0$, because $J(0) \geq 0$. In other words, any loss in the primary market is more than offset by the positive aftermarket profits.

Note also that the firm’s initial value, $J(D_0)$, is strictly positive,\(^\text{15}\) as can be seen from equation (A.8). This resembles the standard "gap case" in the literature on durable goods monopoly without network effects.

4.2.1 MPE with weak AMNE (the No Gap case)

In this sub-section, we consider the case where AMNE are sufficiently weak, so that there exists $D \in (0, 1]$ for which $b(D) = m(D)$. Condition (10) characterizing the No Gap Case is equivalent to $m(1) \geq b(1)$, guaranteeing that the effective marginal cost of delivering the equipment to the lowest-valuation consumer type $\theta = 0$ is larger than the marginal benefit yielded by the equipment, thereby excluding the possibility of “full market coverage.” The

\(^{15}\text{Except in the rasor edge case where the following conditions hold simultaneously, } b(1) = m(1/2) \text{ and } D_0 = 0.\)
bold line in Figure 2 depicts the equilibrium Markovian price function (multiplied by $r$) under weak network effects (it is drawn for $r = 0.05$, $\omega = 0.1$, $\gamma_1 = 0.05$, $\phi_1 = 0.1$, $\gamma_2 = 0.1$, $\phi_2 = 0.05$ and $c = 18$). The parameters were chosen in order to verify the no gap case condition (10).

The analytical characterization of the MPE under weak AMNE is provided in Proposition 1 below.

**Proposition 1 (The No Gap case, with AMNE)** Assume that the aftermarket network effects are positive, $\phi_2 > 0$, but weak enough so that $\phi_1 + 2\phi_2 \leq rc - (\gamma_1 + \gamma_2) - \omega$, or, equivalently, $m(1) \geq b(1)$, implying the existence of a unique $D \in (0, 1]$, such that $m(D) = b(D)$. Then the steady state network size is $D = \frac{1 - rc + (\gamma_1 + \gamma_2)}{1 - (\phi_1 + 2\phi_2) - \omega}$, and

(i) The Markovian equilibrium price function, for all $D \in [0, D]$, is

$$\zeta(D) = c - \frac{1}{r} (\gamma_2 + 2\phi_2 D) = \frac{m(D)}{r}, \quad (18)$$

(ii) The equilibrium value of the firm is

$$J(D) = \frac{\pi^A(D)}{r} = \frac{\gamma_2 D + \phi_2 D^2}{r} \quad (19)$$

(iii) The monopolist’s equilibrium strategy is to sell gradually the durable good: the rate of sale at time $t$ is

$$q(t) = \frac{r}{2\phi_2} [b(D(t)) - m(D(t))]. \quad (20)$$

This implies that $q(t)$ asymptotically approaches zero, as the network size asymptotically approaches $D$. Starting at $D(0) = 0$, the size of the monopolist’s network is increasing over time, satisfying the equation

$$D(t) = (1 - e^{-\psi t}) D. \quad (21)$$
The speed of convergence is \( \psi = \frac{(1-(\phi_1+2\phi_2)-\omega)}{2\phi_2} > 0 \), and \( D(t) \to D \in (0,1] \) asymptotically as \( t \to \infty \).

**Proof:** See the Appendix B.

From Proposition 1, we obtain that the equilibrium equipment price equals the monopolist’s effective marginal cost of supplying an additional consumer, and therefore in the Markov-perfect equilibrium, the price at any time is below the marginal production cost, \( c \). Moreover, the equilibrium equipment price is decreasing through time, until \( D \) is reached. The rationale behind this result is the following: At equilibrium, we have \( p(t) = \frac{m(D(t))}{r} \) and \( D(t) \) is increasing through time until \( D \) is reached. As \( D(t) \) expands, in the aftermarket, the marginal profitability of supplying an additional customer increases (due to the AMNE accruing to the monopolist, for \( \phi_2 > 0 \)). This means that the monopolist’s effective marginal cost of supplying an additional consumer becomes lower (see equation (9)) when \( D \) increases, leading to a reduction in the equilibrium price of the durable good. Hence the monopolist is practicing intertemporal price discrimination, since the price paid by early consumers is higher than the one paid by later consumers.

Proposition 1 also shows that, for \( \phi_2 > 0 \), the equilibrium strategy implies a gradual network expansion, meaning that, in this case, Coase’s prediction that the monopolist sells in one go does not hold. Instead, the monopolist’s strategy consists in selling the equipment gradually until the (generically) interior steady state \( D \) is reached.

As shown in Proposition 1, the steady state \( D \) is increasing with \( \phi_2 \). Intuitively, when the AMNE parameter \( \phi_2 \) is larger, the durable good producer gets a higher aftermarket profitability per customer. Therefore the monopolist is interested in selling her durable good to a larger set of consumers.

The speed of convergence to the steady state, \( \psi \), is decreasing in \( \phi_2 \). This means that, in equilibrium, the monopolist prefers to slow down the adoption of the durable good (i.e. to opt for a slower rate of price decrease) when the AMNE are stronger (provided they remain
sufficiently weak to be consistent with the No Gap case). In fact, here we have \( \zeta(D) = \frac{m(D)}{r} \).

When \( \phi_2 \) increases, the effective marginal cost schedule \( m(D) \) becomes steeper, implying that, a given increase in \( D \) decreases the effective marginal cost by more, the greater the value of \( \phi_2 \). Accordingly, for higher values of \( \phi_2 \), the monopolist (facing a steeper price schedule) is induced to increase \( D \) more slowly. When AMNE become stronger (in the sense that \( \phi_2 \) increases), the price reacts more sharply to any given increase in \( D \) and therefore, the monopolist is induced to sell at a slower path. In contrast, when the price curve becomes horizontal (\( \phi_2 = 0 \)) or even increasing (\( \phi_2 < 0 \)) the monopolist would be induced to sell in one go.

Proposition 1 also implies that the value of the firm evaluated at \( D \) is just equal to the discounted stream of net returns that would be obtained in the aftermarket if \( D \) were kept constant for ever. The firm initially incurs losses in the primary market but is is able to recoup them through aftermarket profits as the size of its network progressively expands. From (19), we have \( J(0) = 0 \), i.e., starting at \( D = 0 \). This means that despite the gradual network expansion, Coase’s prediction that the monopolist’s market power disappears remains valid in the No-Gap case.

The result that \( J(0) = 0 \) implies that the monopolist expects to gain nothing by selling gradually as compared with choosing a zero output forever. Nonetheless, starting at \( D = 0 \), refraining from production and sale is not a Markov perfect equilibrium. The rationale behind this result is the following: if the monopolist were to choose a zero output, from (8) it would follow that the expected equipment price would be constant for ever at \( \frac{b(0)}{r} \), which would of course induce her to sell, given Assumption A1. As a result, the monopolist prefers to sell the durable good and therefore, the firm’s value, which is initially zero at \( D(0) = 0 \), increases through time as the network of the durable good increases.

For the sake of completeness, Corollary 1 considers the hypothetical case where, perhaps because of some errors in the past, a fraction \( D^\# > D \) of consumers has bought the durable good.
**Corollary 1** In a hypothetical case under which the monopolist faces a network size $D^\#$ greater that the desired steady-state $\bar{D}$, if the monopolist cannot buy back what she has sold she will stay put at $D^\#$, and the value of the firm, starting from $D^\#$, is equal to:

$$J(D^\#) = \frac{\gamma_2 D^\# + \phi_2(D^\#)^2}{r} \text{ for } D^\# \in (\bar{D}, 1].$$

**Proof:** See the Appendix $D$.

The Corollary shows that as long as the monopolist is unable to buy back what she has sold, the (continuing) equilibrium strategy starting from this point $D^\#$(which is off the equilibrium path) consists in staying put at $D^\#$, and the value of the firm, starting from $D^\#$, is simply the capitalized value of the stream of the aftermarket profit flow. In this case, we must have that those consumers with low valuations such that $\theta < 1 - D^\#$ can only be in equilibrium if they expect to gain nothing in buying the durable good, which means that

$$\zeta(D) = \frac{m(D)}{r}, \text{ for all } D \in [0, D^\#] \text{ and } \zeta(D) = \frac{b(D)}{r} \text{ for all } D \in [D^\#, 1].$$

**4.2.2 MPE with intermediate AMNE (Small Gap case)**

In the Small Gap case (i) the marginal benefit to the lowest valuation consumer, $b(1)$, exceeds the effective marginal cost, $m(1)$; and (ii) the total cost of serving the whole spectrum of consumers immediately in one go is greater than the sale revenue from selling durable goods to all customers in one go. Therefore, this case arises when $b(1) < m(1/2)$, or equivalently, $\omega + \phi_1 + 2\phi_2 > rc - (\gamma_1 + \gamma_2) > \omega + \phi_1 + \phi_2$, which requires $\phi_2 > 0$.

The optimal strategy of the monopolist is quite interesting in this case. In the Proposition below, we show that there exists a critical value $\tilde{D} \in (0, 1)$ such that: (i) if the monopolist starts with some $D$ above it, the optimal strategy is to cover the whole market immediately; (ii) if she starts with some $D$ below it, the optimal strategy is to expand her customer base gradually, until $\tilde{D}$ is reached; (iii) at $\tilde{D}$, she plays a mixed strategy. With probability $\lambda$, she makes a lumpy sale $1 - \tilde{D}$ so that the whole market is covered immediately, while with
probability $1 - \lambda$, she stops selling the durable good. The two last actions yield the same payoff to the firm.

Let us first establish the existence of a unique critical value $\tilde{D} \in (0, 1)$ which satisfies the following condition

$$\frac{\gamma_2 \tilde{D} + \phi_2 (\tilde{D})^2}{r} = (1 - \tilde{D}) \left( \frac{b(1)}{r} - c \right) + \frac{\gamma_2 + \phi_2}{r} \quad (22)$$

The left-hand side is the capitalized value of the perpetual constant flow of profits in the aftermarket, $\gamma_2 \tilde{D} + \phi_2 (\tilde{D})^2$, if, starting at $\tilde{D}$, the monopolist refrains from selling. The right-hand side is the alternative payoff if, given that its existing customer base is $\tilde{D}$, the firm decides to cover the whole market instantaneously, by selling $(1 - \tilde{D})$ in one go, at the price $p = b(1)/r$. Under this alternative, the firm earns a higher perpetual constant flow of profits in the aftermarket, $\frac{\gamma_2 + \phi_2}{r} > \frac{\gamma_2 \tilde{D} + \phi_2 (\tilde{D})^2}{r}$, but at the cost of selling the durable goods at a loss, because $\frac{b(1)}{r} - c$ is negative.

It can be shown that there exists a unique $\tilde{D} \in (0, 1)$ that satisfies eq. (22), with

$$\tilde{D} = \frac{1}{\phi_2} \left( r c - (\gamma_1 + \gamma_2 + \phi_1 + \phi_2) - \omega \right), \quad (23)$$

which is positive since $m(1/2) > b(1) > m(1)$. Note that $\tilde{D} < 1$ because $rc - (\gamma_1 + \gamma_2 + \phi_1 + \phi_2) - \omega < \phi_2$ in this case. An increase in the intensity of AMNE, $\phi_1$ or $\phi_2$, will make $\tilde{D}$ smaller. When $\phi_1$ increases, $b(1)$ also increases because the instantaneous full benefit for the lowest valuation consumer is increasing with $\phi_1$. When $\phi_2$ increases, the full marginal cost of serving an additional consumer goes down, since $m(D)$ is decreasing with $\phi_2$. Accordingly, if we increase $\phi_1$ and/or $\phi_2$ sufficiently, $b(1)$ will approach $m(1/2)$, and $\tilde{D}$ will approach zero, meaning that the monopolist sells everything in a twinkling of an eye (as in the Large Gap case). In contrast, if we decrease $\phi_1$ or $\phi_2$ sufficiently, $b(1)$ goes down and $m(1)$ goes up, implying that and $\tilde{D}$ will approach 1 and, in the limit, the monopolist behaves as in No Gap case.

Note that $\tilde{D}$ is not a steady state. In fact, starting from any $D_0 \in \left[ 0, \tilde{D} \right)$, $D(t)$ will reach $\tilde{D}$ at some finite time $T$, and as soon as $\tilde{D}$ is reached, the firm plays the mixed strategy
described above. Any consumer who has bought a unit of the durable good prior to \( T \) can expect to re-sell it at time \( T \), at the price \( b(1)/r \) with probability \( \lambda \) or, with probability, \( 1 - \lambda \), at price \( b(\tilde{D})/r \), the price that the marginal consumer \( \tilde{\theta} = 1 - \tilde{D} \) would be willing to pay (if the monopolist stops producing at time \( T \)).

The expected price of the durable good at time \( T \) is then equal to

\[
E_p(T) = \lambda \frac{b(1)}{r} + (1 - \lambda) \frac{b(\tilde{D})}{r}.
\]

The probability \( \lambda \) which characterizes the mixed strategy is chosen so as to eliminate any arbitrage opportunities on the consumers’ side. Hence, the monopolist’s choice of \( \lambda \) must be such that no speculator can gain by buying just before \( T \) and re-selling at \( T \):

\[
\lim_{t\uparrow T} p(t) = \lambda \frac{b(1)}{r} + (1 - \lambda) \frac{b(\tilde{D})}{r} \tag{24}
\]

Since \( \zeta(D) = m(D)/r \) for \( D \in [0, \tilde{D}] \), condition (24) is equivalent to

\[
\frac{m(\tilde{D})}{r} = \lambda \frac{b(1)}{r} + (1 - \lambda) \frac{b(\tilde{D})}{r} \tag{25}
\]

To make precise the above intuitive discussion, we state the following proposition:

**Proposition 2** When \( \phi_2 > 0 \) and \( m(\tfrac{1}{r}) > b(1) > m(1) \) (Small Gap),

(i) The consumers’ equilibrium expected price function is

\[
\begin{align*}
\zeta(D) &= c - \frac{1}{r}(\gamma_2 + 2\phi_2 D) = \frac{m(D)}{r} \quad \text{if } D \in [0, \tilde{D}) \\
E:\zeta(D) &= c - \frac{1}{r}(\gamma_2 + 2\phi_2 \tilde{D}) = \frac{m(\tilde{D})}{r} \quad \text{if } D = \tilde{D} \\
\zeta(D) &= \frac{b(1)}{r} = \frac{1}{r}(\gamma_1 + \phi_1 + \omega) < \frac{m(D)}{r} \quad \text{if } D \in (\tilde{D}, 1]
\end{align*}
\]

Thus the expected price function is piece-wise continuous, and has a jump discontinuity immediately to the right of \( \tilde{D} \).

(ii) The monopolist’s equilibrium strategy is \( L(D) = 1 - D \), if \( D \in (\tilde{D}, 1] \); or gradually selling the durable good at a rate given by equation (A.13), if \( D \in [0, \tilde{D}) \). At \( D = \tilde{D} \), the monopolist uses a mixed strategy: (a) selling \( 1 - \tilde{D} \), with probability \( \lambda \), or stop selling, with probability \( 1 - \lambda \); where \( \lambda b(1) + (1 - \lambda)b(\tilde{D}) = m(\tilde{D}) \):

\[
\lambda \frac{1}{r}(\gamma_1 + \phi_1 + \omega) + (1 - \lambda) \frac{1}{r} \left( \gamma_1 + \phi_1 \tilde{D} + \omega \tilde{D} + 1 - \tilde{D} \right) = c - \frac{1}{r}(\gamma_2 + 2\phi_2 \tilde{D}). \tag{27}
\]
(iii) the firm’s value is

\[
\begin{align*}
J(D) &= \frac{\gamma_2 D + \phi_2 D^2}{r} \quad \text{for all } D \in [0, \bar{D}] \\
J(D) &= \frac{(1-D)(\gamma_1 + \phi_1 + \omega - rc) + \gamma_2 + \phi_2}{r} \quad \text{for all } D \in [\bar{D}, 1]
\end{align*}
\]

(28)

**Proof:** See the Appendix.

It is worth noting that the value function \(J(D)\) in the Small Gap case has two segments. Over the interval \([0, \bar{D}]\), \(J(D)\) is strictly convex, and it is equal to \(\pi^A(D)/r\) just as in equation (19) of the No Gap case. In that case, despite the fact we are in a (small) Gap case, the value of the firm is zero. Over the interval \((\bar{D}, 1]\), \(J(D)\) is linear, and it is equal to \((1-D)(\frac{b(1)}{r} - c) + \frac{\gamma_2 + \phi_2}{r}\), just as in equation (17) of the Large Gap case.

Figure 3 below depicts the Markovian equilibrium price function (in bold), in the Small Gap case. The figure is drawn for \(r = 0.05, \omega = 0.1, \gamma_1 = 0.05, \phi_1 = 0.1, \gamma_2 = 0.1, \phi_2 = 0.05\) and \(c = 10\). The parameters coincide with the ones used in the figure two, except for \(c\), which was now set to a lower value in order to be consistent with condition (12).

**PLEASE PLACE FIGURE 3 HERE**

## 5 Conclusion

This paper analyses the dynamic problem faced by a monopolist firm that produces a durable good (in the primary market) and also participates in the corresponding aftermarket, where CGS are provided. The consumption of the durable good is subject to both primary and aftermarket network effects, yielding strategic complementarities that are new to the literature on dynamic pricing of a durable good with an aftermarket.

We characterize the evolution of the monopolist’s equilibrium network and the equilibrium price trajectories, under *non-stationary* network effects. The paper provides a theoretical contribution by studying whether Coase’s prediction that all the sales take place in the twinkling of an eye remains valid under primary market and aftermarket network effects.
When AMNE take place and the monopolist is able to (at least partly) benefit from them, equilibrium outcomes may diverge from Coase’s conjecture. In that case, Coase’s prediction that all the sales take place in a twinkling of an eye only holds when there is a large gap between the lowest consumer valuation and the monopolist’s marginal cost of supplying her.

The first theoretically interesting result occurs in the No Gap case, under which we get an optimal gradual sales policy, implying intertemporal price discrimination and contradicting Coase’s conjecture that all the sales take place immediately.

The second theoretically interesting result occurs when there is a positive (though small) gap between the lowest consumer valuation and the monopolist’s marginal cost of supplying her. In that case, our results depend on whether the size of the monopolist’s network is above or below a critical threshold. While in the first case, the evolution of sales is similar to the Large Gap case, in the second case the dynamics correspond to the ones observed in the No Gap case, yielding a gradual optimal sales policy. When the monopolist’s consumer base is equal to the critical network size, the monopolist stops producing with a certain probability, and, with the complementary probability, it covers the whole market immediately.

Our model can be enriched by extension in several dimensions. In our future research, we intend to study the monopolist’s dynamic behavior under the threat of entry of an imperfectly substitute durable good. Other possible extensions would be worthwhile as well, namely the analysis of the durable good depreciation, and the study of of non-perishable CGS.

**Acknowledgement:** The authors are very grateful to the editor, Professor Ulrich Doraszelski, and three referees for their helpful comments. Joana Resende acknowledges financial support from Cef.up and FCT through research grant PTDC/EGE-ECO/115625/2009.
APPENDIX

APPENDIX A - AFTERMARKET NETWORK EFFECTS: EXAMPLES

Example 1: Direct AMNE and competition à la Cournot

Consider the OS/software markets. In Example 1, we take as given the variety/quality of the available software and we concentrate on direct network effects arising for a certain type of software.

Concerning the structure of the aftermarket, in this example we suppose that the monopolist producer of the durable good is also a CGS producer who competes à la Cournot with other independent suppliers of CGS.

For now, we assume that in the aftermarket the monopolist producer of the durable good and \( N - 1 \) independent firms provide homogenous CGS at a constant marginal cost, which, w.l.o.g., is normalized to zero.

Consumers who already own an equipment derive utility \( Z(t) \) from the consumption of \( k(t) \) units of CGS at time \( t \), with:

\[
Z(t) = \gamma [k(t) - \frac{1}{2}k^2(t)] + \phi D(t) [k(t) - \frac{1}{2}k^2(t)] - \rho(t)k(t),
\]

The term \( \gamma [k(t) - \frac{1}{2}k^2(t)] \), \( \gamma > 0 \) corresponds to the "stand alone value of CGS". The term \( \phi D(t) [k(t) - \frac{1}{2}k^2(t)] \) corresponds to the (direct) network benefit associated with the consumption of CGS. \( \rho(t) \) is the unit price of CGS. Assuming that consumers who already own a durable good maximize their instantaneous utility, we obtain the total demand for CGS at instant \( t \), given by \( D(t) \left[ 1 - \frac{\rho(t)}{\gamma + \phi D(t)} \right] \). Considering the outcome of the Cournot game played by the CGS suppliers, we obtain the instantaneous, equilibrium price of the CGS, equal to \( \frac{\gamma + \phi D(t)}{N+1} \). Accordingly, the model yields (i) the equilibrium instantaneous CGS utility specification (2), with \( \gamma_1 = \frac{\gamma}{2} \left( \frac{N}{N+1} \right)^2 \) and \( \phi_1 = \frac{\phi}{2} \left( \frac{N}{N+1} \right)^2 \); and (ii) the monopolist’s equilibrium profit per customer in (3), with \( \gamma_2 = \frac{\gamma}{(N+1)^2} \) and \( \phi_2 = \frac{\phi}{(N+1)^2} \).

\(^{16}\)In the literature about network effects, the "stand alone value" refers to the value of a good that is independent of its network size (Katz and Shapiro, 1985).
Example 2: Direct AMNE and price competition with differentiated CGS

In Example 1, it was assumed that CGS providers offer a homogeneous good. However, there are many situations in which the goods and services sold in the aftermarket are horizontally differentiated (for example, the software and/or applications available to a certain OS are often horizontally differentiated). The model described above can be easily adapted to deal with this possibility. In what follows, we show that the properties of the equilibrium utility and profit specifications (in equations (2) and (3), respectively) remain valid in the context of direct AMNE and price competition with differentiated products.

We assume the existence of \( N > 3 \) varieties of CGS and only one of these varieties is supplied by the durable good producer. Following Ottaviano, Tabuchi and Thisse (2002), we assume that consumers, facing CGS prices \( p_i(t) \), get the net utility level \( Z(t) \) from the consumption of \( k_i(t) \) units of each CGS \( i \) at time \( t \),

\[
Z(t) = \left[ \gamma + \phi D(t) \right] \left( \sum_{i=1}^{N} k_i(t) - \frac{1 - \theta}{2} \sum_{i=1}^{N} k_i^2(t) - \theta \left( \sum_{i=1}^{N} k_i(t) \right)^2 \right) - \sum_{i=1}^{N} p_i(t) k_i(t),
\]

where \( \gamma \) now measures the stand-alone value for any CGS consumption bundle \( k(t) = (k_1(t), k_2(t), ..., k_N(t)) \), \( \phi \) measures the intensity of the direct AMNE, \( \theta \) measures the degree of complementarity/substitutability among the CGS. We suppose \( \theta \in (\frac{1}{N}, 1) \), the two extreme values corresponding to the cases of perfect complements and substitutes, respectively.

Considering the Bertrand equilibrium, in which firms set \( p_i(t) \) non-cooperatively, we obtain the equilibrium *instantaneous CGS utility* specification (2), where \( \gamma_1 = \gamma \vartheta_1 \) and \( \phi_1 = \phi \vartheta_1 \), with \( \vartheta_1 = \frac{N(1+(N-2)\theta)(1+(4N-1)\theta)}{2(2+(N-3)\theta)(1+(N-1)\theta)} \). Analogously, each Bertrand competitor’s equilibrium profit per customer is given by (3), where \( \gamma_2 = \gamma \vartheta_2 \) and \( \phi_2 = \phi \vartheta_2 \), with \( \vartheta_2 = \frac{(1-\theta)(1+\theta(N-2))}{2(2+(N-3)\theta)(1+(N-1)\theta)} \).

**Appendix B - Some technical remarks**

**Remark B.1.** In the case of stationary network effects as in Mason (2000), equation
(4) becomes
\[ e^{-rt}V(\theta, t) = e^{-rt} \left[ \frac{\theta + \gamma_1}{r} + (\omega + \phi_1) \frac{D(t)}{r} - p(t) \right]. \] (A.1)

Using the same basic argument as in the proof of Lemma 1 (in Appendix D), we would obtain the arbitrage equation
\[ \frac{dp(t(\theta))}{dt} = rp(t(\theta)) - [\theta + \gamma_1 + (\omega + \phi_1) D(t(\theta))] + (\omega + \phi_1) \frac{1}{r} \frac{dD(t(\theta))}{dt}, \] (A.2)
which differs from (5) by the last term on the right-hand side. This would lead consumers to form their expectations about the evolution of the price of the durable good according to the following rule:
\[ \frac{dp(t)}{dt} = rp(t) - b(D(t)) + (\omega + \phi_1) \frac{1}{r} \frac{dD(t)}{dt}, \] (A.3)
instead of (6). The condition (A.2) exactly corresponds, mutatis mutandis, to Mason’s equation 4, page 1985. Integrating, we obtain:
\[ p(t) = \int_t^\infty (1 - D(s) + \gamma_1 + (\omega + \phi_1) D(t)) e^{-r(s-t)} ds \] (A.4)
or equivalently,
\[ p(t) = \int_t^\infty (b(D(s)) e^{-r(s-t)} ds + (\omega + \phi_1) \int_t^\infty [D(t) - D(s)] e^{-r(s-t)} ds. \] (A.5)

The divergence between the two price equations (A.5) and (8) results from the term
\[ (\omega + \phi_1) \int_t^\infty [D(t) - D(s)] e^{-r(s-t)} ds, \]
which is negative as long as the network size is expected to increase. This accounts for the fact that, under stationary effects, the customer does not benefit from the future expansion of the network.

Remark B.2. If Assumption A1 were violated, so that \( 1 + \gamma_1 \leq rc - \gamma_2 \), the strategy of never producing the durable good, coupled with an expected equipment price constant forever at \( \frac{b(0)}{r} \) would be an equilibrium for some set of parameter values, provided that \( b(0) \leq m(\frac{1}{2}) \). The proof is along the same lines as the proof of the Proposition regarding the Large Gap.
MPE. When, in addition to $b(0) < m\left(\frac{1}{2}\right)$, $m(1) < b(1)$, this equilibrium would co-exist, for some subset of parameter values, with an equilibrium in which the market is covered in finite time. This corresponds to situations when the production of durable goods may become profitable only thanks to the network effects. When both $b(0) > m\left(\frac{1}{2}\right)$ and $b(1) < m\left(\frac{1}{2}\right)$, finally, there would also be room for situations where no MPE would exist.

Appendix C - Benchmarks

Lemma C1

Assume the complete absence of any type of network effects, so that $\omega = \phi_1 = \phi_2 = 0$. Then

(I) When $(\gamma_1 + \gamma_2) \leq rc$, the No Gap case arises. The Markov Perfect Equilibrium has the following Coasian properties:

(i) The equilibrium price function is a constant: $\zeta(D) = c - \frac{\gamma_2}{r} = \frac{1}{r}b(D) = \frac{1}{r}m(D)$,

(ii) Starting at any $D < D$, the monopolist’s equilibrium strategy is the lumpy sale strategy $L(D) = D - D\frac{1}{rc} + (\gamma_1 + \gamma_2)$;

(iii) The firm’s value is $J(D) = \frac{\gamma_2 D}{r}$ for all $D \in [0, D]$ with, in particular, $J(0) = 0$.

(II) When $(\gamma_1 + \gamma_2) > rc$, the Large Gap case arises. The price is constant and it is equal to the present value of the stream of net benefits to the lowest valuation consumer. The monopolist’s equilibrium strategy is to supply the whole market immediately. The value of the firm is strictly positive.

Proof:

For the No Gap case, apply the proof of Proposition 1 in Appendix D, and take the limit $\phi_2 \to 0$. For the Large Gap case, the proof is formally a special case of the proof in the section MPE with strong network effects, by setting the relevant network effect parameters to zero.

Appendix D - Proofs
Proof of Lemma 1

(i) Differentiating (4) with respect to \( t \) yields the first order condition

\[
e^{-rt} \left[ rp(t) - \theta - \gamma_1 - (\omega + \phi_t) D(t) - \frac{dp(t)}{dt} \right] = 0. \tag{A.6}
\]

Multiplying both sides by \( e^{rt} > 0 \) and rearranging yields equation (5).

(ii) From (A.6), we obtain the implicit function

\[
\psi(t, \theta) \equiv rp(t) - \theta - \gamma_1 - (\omega + \phi_t) D(t) - \frac{dp(t)}{dt} = 0
\]

This gives us

\[
\frac{dt}{d\theta} = -\frac{\partial \psi / \partial \theta}{\partial \psi / \partial t}
\]

where \( \partial \psi / \partial t < 0 \) by the second order condition, and \( \partial \psi / \partial \theta = -1 \). Thus \( \frac{dt(\theta)}{d\theta} \leq 0 \).

It can be easily shown that the SOC hold at equilibrium for all the cases studied in the paper. The proof is available from the authors upon request.

Proof of the MPE with strong AMNE (the Large Gap case)

(i) Given the lumpy sale strategy \( L(D) = 1 - D \), rational expectations require that consumers hold the following price function \( \zeta(D) = \frac{b(1)}{r} \). This function is constant.

(ii) Given the consumers’ price expectation function \( \zeta(D) = \frac{b(1)}{r} \), the firm’s problem is, given any \( D_0 \in [0, 1] \), where \( D_0 \) stands for \( D(0) \), to choose the time path of sale to maximize the integrated firm’s value:

\[
J(D_0) = \max \int_0^\infty e^{-rt} \left[ \gamma_2 D(t) + \phi_2 D^2(t) + \frac{dD(t)}{dt} \frac{1}{r} (b(1) - rc) \right] dt
\]

Integration by parts, noting that \( D(t) \) is bounded, yields

\[
J(D_0) = \max \int_0^\infty e^{-rt} \left[ \gamma_2 D(t) + \phi_2 D^2(t) + (D(t) - D_0) (b(1) - rc) \right] dt. \tag{A.7}
\]

For \( \phi_2 > 0 \), the expression inside [\( \ldots \)] is strictly convex in \( D(t) \). Therefore the maximum of this expression with respect to \( D(t) \) occurs either at \( D = D_0 \) or at \( D = 1 \). It can be easily
seen that the maximum occurs at $D = 1$ for all possible $D_0 \in [0, 1]$ if and only if we are in the Large Gap case, so that $\phi_1 + \phi_2 \geq rc - \gamma_1 - \gamma_2 - \omega$.

Then the value of the firm, i.e., the maximized value of integral (A.7), which is obtained by setting $D(t) = 1$ for all $t > 0$, is

$$J(D_0) = \frac{H(1, D_0)}{r} \geq \frac{H(D_0, D_0)}{r} = \gamma_2 D_0 + \phi_2 D_0^2,$$  \hspace{1cm} (A.8)

where:

$$H(D, D_0) \equiv \gamma_2 D + \phi_2 D^2 + (D - D_0) (b(1) - rc)$$

Note that $J(0) = \frac{1}{r} (b(1) - m(1/2)) \geq 0$. Not surprisingly, the value function $J(D)$ is linear in $D$ and it is higher than the capitalized value of the instantaneous aftermarket profit, $(\gamma_2 D + \phi_2 D^2)$, for all $D \in [0, 1]$, with equality only at $D = 1$. ■

**Proof of Proposition 1** The Hamilton-Jacobi-Bellman equation for the monopolist is

$$rJ(D) = \max_q \{\gamma_2 D + \phi_2 D^2 + [\zeta(D) - c] q + J'(D) q\},$$  \hspace{1cm} (A.9)

where the time index has been omitted. Since this equation is linear in $q$, the optimal $q$ is finite if and only if

$$\zeta(D) - c + J'(D) = 0, \text{ for all } D \in (0, 1). \hspace{1cm} (A.10)$$

Substituting this into the HJB equation (A.9), we obtain

$$J(D) = \frac{\gamma_2 D + \phi_2 D^2}{r}, \text{ for all } D \in (0, 1). \hspace{1cm} (A.11)$$

In light of (A.10), the Markovian equilibrium price function is:

$$\zeta(D) = c - J'(D) = c - \frac{\gamma_2 + 2\phi_2 D}{r} = \frac{m(D)}{r}. \hspace{1cm} (A.12)$$

Thus $p(t) = \zeta(D(t)) = \frac{m(D(t))}{r}$. 

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Differentiating with respect to time we obtain

$$\dot{p}(t) = \frac{1}{r}m'(D)\dot{D}(t) = -\frac{2\phi_2}{r}q(t)$$

Taking into account equations $\dot{p} = rp - b(D)$, the equilibrium output rate is

$$q(t) = \frac{r}{2\phi_2} [b(D(t)) - m(D(t))]$$

$$= \frac{r}{2\phi_2} [(1 - rc + \gamma_1 + \gamma_2) - D(t)(1 - \phi_1 - 2\phi_2 - \omega)]$$

(A.13)

Note that $q$ becomes zero when $D$ reaches the value $\overline{D}$ defined by $b(D) = m(D)$:

$$\overline{D} = \frac{1 - rc + \gamma_1 + \gamma_2}{1 - \phi_1 - 2\phi_2 - \omega} \leq 1$$

(A.14)

Assumption A1 implies that the numerator is positive. The denominator is greater than the numerator because we are dealing with the No Gap Case, yielding $0 \leq \overline{D} \leq 1$. With $q(t) = \frac{r}{2\phi_2} [b(D(t)) - m(D(t))]$, we see that, given $\phi_2 > 0$, the output rate $q$ is positive if and only if $D < \overline{D}$. The stability of the steady state is ensured, and consequently sales are gradual. Replacing the optimal value of $q(t)$ in $D(t) = \int_0^t q(s) ds$, given $D(0) = 0$, we obtain $D(t) = (1 - e^{-\psi t})\overline{D}$. Finally, the convergence speed is $\psi = \frac{r}{2\phi_2} (1 - \phi_1 - 2\phi_2 - \omega)$ which is positive due to Assumption A1 and the hypothesis that the AMNE are weak. To finish the proof, note that all the necessary conditions for an equilibrium are satisfied under condition (A.12). It remains to verify that the monopolist, starting with any $D < \overline{D}$, is never interested in selling a discrete amount $\tilde{D} - D$, such that $D < \tilde{D} \leq \overline{D}$, prior to embarking on gradual sales according to (A.13). To demonstrate this, note that the value of selling a positive discrete amount $\tilde{D} - D$ at time $t = 0$ is given by $J(\tilde{D}) - J'(\tilde{D})(\tilde{D} - D)$ which is strictly lower than $J(D)$ since, from (A.11),

$$J(\tilde{D}) - J'(\tilde{D})(\tilde{D} - D) - J(D) = -\frac{\phi}{4r}(\tilde{D} - D)^2 < 0.$$ 

This shows that selling gradually is always better than selling discrete amounts.$\blacksquare$

**Proof of Corollary 1**
Given the conditions in Corollary 1, consumers with low valuations such that \( \theta < 1 - D^# \) are only in equilibrium if they do not expect to gain nothing in buying the durable good. This suggests the following price expectation function

\[
\begin{cases}
  \zeta(D) = \frac{m(D)}{r} & \text{for all } D \in [0, D^#] \\
  \zeta(D) = \frac{b(D)}{r} & \text{for all } D \in [D^#, 1]
\end{cases}
\]

To prove Corollary 1, we must verify that given the above price expectation function, the firm finds it optimal to choose the corner solution \( q(D) = 0 \) whenever \( D \in [D^#, 1] \). With the value function \( J(D) = (\gamma_2 D + \phi_2 D^2)/r \), it is indeed true that the corner solution \( q(D) = 0 \) whenever \( D \in [D^#, 1] \) does indeed satisfy the HJB equation:

\[
\max_{q \geq 0} \left\{ \gamma_2 D + \phi_2 D^2 + [\zeta(D) - c] q + J'(D)q \right\} = r J(D)
\]

where the FOC is satisfied:

\[
\frac{b(D)}{r} - c + \frac{\gamma_2 + 2\phi_2 D}{r} \leq 0 \text{ for } D \in [D^#, 1].
\]

**Proof of Proposition 2**

(i) Given that the firm’s lumpy sale strategy \( L(D) = 1 - D \) for all \( D \in (\tilde{D}, 1] \), rational expectations on the part of consumers imply that \( \zeta(D) = \frac{1}{r} (\gamma_1 + \phi_1 + \omega) = \frac{b(1)}{r} \) for all \( D \in (\tilde{D}, 1] \). And given the consumers’ function \( \zeta(D) = \frac{1}{r} (\gamma_1 + \phi_1 + \omega) = \frac{b(1)}{r} \) for all \( D \in (\tilde{D}, 1] \), the firm’s maximization problem, for any given \( D_0 > \tilde{D} \), is the same as in equation (A.7), hence its optimal strategy is \( L(D) = 1 - D \) for all \( D \in (\tilde{D}, 1] \). The value function is therefore

\[
J(D) = \frac{(1 - D)(\gamma_1 + \phi_1 + \omega - rc) + \gamma_2 + \phi_2}{r} \text{ for } D \in (\tilde{D}, 1]
\]

Turning to the interval \([0, \tilde{D})\), given to the output strategy \( g(D) \) defined by equation (A.13), for all \( D \in [0, \tilde{D}) \), the same argument as that used in the proof of in Proposition 2 applies to show that the price function \( \zeta(D) = c - \frac{\gamma_2 + 2\phi_2 D}{r} = \frac{m(D)}{r} \) satisfies the rational expectations requirement for all \( D \in [0, \tilde{D}) \). And given the price function \( \zeta(D) = \frac{m(D)}{r} \) for \( D \in [0, \tilde{D}) \), the firm’s optimal response is to use the output strategy \( g(D) \) defined by equation (A.13).
Now, since sales are gradual for all \( D \in [0, \tilde{D}) \), customers will purchase if \( D < \tilde{D} \), which occurs if and only if
\[
\lim_{D \to \tilde{D}} \zeta(D) = \zeta(\tilde{D}),
\]
i.e.
\[
\zeta(\tilde{D}) = c - \frac{1}{r} (\gamma_2 + 2\phi_2 \tilde{D}), \tag{A.15}
\]
where \( \zeta(\tilde{D}) \) denotes the expected equipment price at \( D = \tilde{D} \). Condition (A.15) is necessary to support equilibrium, because if \( \zeta(\tilde{D}) < c - \frac{1}{r} (\gamma_2 + 2\phi_2 \tilde{D}) \) then, for some small \( \varepsilon > 0 \), consumers whose type is in interval \((\tilde{\theta}, \tilde{\theta} + \varepsilon)\) would not want to buy the equipment when \( D \in (\tilde{D} - \varepsilon, \tilde{D}) \), where \( \tilde{\theta} \equiv 1 - \tilde{D} \); and if \( \zeta(\tilde{D}) > c - \frac{1}{r} (\gamma_2 + 2\phi_2 \tilde{D}) \) agents would make speculative gains by purchasing the durable good immediately before \( D \) reaches \( \tilde{D} \), and resell it an instant later.

(ii) Given (26), any deviation by the monopolist implying a discontinuous variation in \( D \) in the interval \([0, \tilde{D})\) can be ruled out as was shown in the proof of Proposition 2. Given the definition of \( \tilde{D} \), a deviation implying an upward jump from some \( D \in [0, \tilde{D}) \) to 1 is ruled out since it would yield a payoff \( \frac{(1-D)(\gamma_1 + \gamma_2 + \omega - rc) + \gamma_2 + \phi_2}{r} < \frac{\gamma_2 D + \phi_2 D^2}{r} \). (A deviation to some value in \((\tilde{D}, 1)\) is even less profitable.) Given that \( \zeta(D) \) is constant in the interval \((\tilde{D}, 1]\), an argument similar to that used in the proof of the MPE with strong AMNE shows that it is always better to jump from any \( D \) in this interval to 1 than to stop at \( D \) or to jump to any other value in \((D, 1)\). Finally, when \( D = \tilde{D} \), the monopolist is indifferent between two actions: (a) selling \( 1 - D \) immediately or (b) stopping to sell the equipment. The two actions yield by construction the same payoff to the firm since
\[
\frac{\gamma_2 D + \phi_2 D^2}{r} = \frac{(1 - \tilde{D})(\gamma_1 + \phi_1 + \omega - rc) + \gamma_2 + \phi_2}{r}.
\]
Being indifferent between actions (a) and (b), the firm can choose action (a) with probability \( \lambda \) and action (b) with probability \( 1 - \lambda \). The value of \( \lambda \) must be determined such that consumers cannot gain by arbitrage. The consumers will rationally expect an equipment
price equal to
\[ \lambda \frac{b(1)}{r} + (1 - \lambda) \frac{b(\hat{D})}{r}. \]

Using condition (A.15), we then obtain equation (27) which determines the equilibrium value of \( \lambda. \)

References


