Long-Run Market Configurations in a Dynamic Quality-Ladder Model with Heterogeneity

_Preliminary and Incomplete, Do not Cite_

Mario Samano* Marc Santugini†

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*Department of Applied Economics and CIRPÉE, HEC Montréal. Email: mario.samano@hec.ca.
†Department of Applied Economics and CIRPÉE, HEC Montréal. Email: marc.santugini@hec.ca.
Abstract

We study the long-run market configurations in a quality-ladder dynamic model. Specifically, we assume that the return to investment in quality differs across the firms. That is, for a given level of investment, one firm has a higher probability to raise the quality of the good it produces. We show that the model can generate different types of long-run market configurations (market collapse, market dominance by either firm, market dominance by the leading firm, duopoly and monopolies, duopoly). We extend the model by considering a patent release situation in which there is an externality on quality proportional to the competitor’s quality. We find the array of possible market structures that can arise from this game, including the case of dominance by the lagging firm.

Keywords: Differentiated-good markets, Quality-ladder model, Heterogeneity, Dynamic investment.

JEL Classifications: C61, C73, L13.
1 Introduction

Recently, Goettler and Gordon (2011) have estimated a dynamic quality-ladder model for the computer processors industry. They find evidence for heterogeneity in the likelihood of success of investment, which can explain differences in the levels of investment and ultimately differences in the levels of quality between the goods.\textsuperscript{1} Motivated by this finding, this paper asks the following question. What is the effect of heterogeneity in firms’ ability to invest in quality on long-run market configurations? Specifically, under what circumstances are we more likely to observe a situation in which goods are quite differentiated? To answer these questions, we adapt the quality ladder model described in Ericson and Pakes (1995) and the algorithms to numerically solve for its equilibrium such as the one described in McGuire and Pakes (1994) and in a particular case in Levhari and Mirman (1980) to the case of heterogeneous likelihood of success of investment.

Our analysis shows that the dynamic quality-ladder model can generate in the long-run five different distributions on the space of market configurations (market collapse, market dominance by either firm, market dominance by the leading firm, duopoly and monopolies, duopoly). As the investment becomes more reversible (i.e., higher depreciation rates), the long-run configurations containing only duopolies become less common in the parameter space of likelihood of investment. If the investment is highly reversible, monopoly configurations become more common but, against usual intuition, the possibility of a duopoly does not completely go away: there is a positive probability that either of the two firms dominates or they coexist. Interestingly, a high degree of heterogeneity can mitigate the effect of highly reversible investments on the probability of market collapse, giving rise to non-negligible probabilities of observing a duopoly or even dominance of the weakest firm in the case of non-symmetrical externalities.

\textsuperscript{1}In their adaptation of the Ericson-Pakes model, the source of this heterogeneity in the model is twofold: specific parameters for each firm and the quality distance between the leader and the follower. Specifically, they find a parameter value of 0.0010 for Intel, a value of 0.0019 for AMD. The estimated parameters are different for each firm, capturing the observed heterogeneity of firm dominance in their data.
We restrict attention to the quality-ladder model without entry or exit. This is not that of a strong assumption since we allow for quality levels of zero yielding zero demand, meaning that the firm producing such good has in exited the market. That however does not prohibit the same firm to become active again in the market if it achieves to increase quality to a positive level. We also note that in our motivating example in Goettler and Gordon (2011), they do not consider entry and exit since the industry they study does not exhibit such behavior during the time window of their data. In another example of the estimation of a quality ladder model, Gowrisankaran and Town (1997) consider the possibility of entry and exit, however all hospitals belong to one of two firm types, and thus if all firms of one type exit, this is equivalent to having quality zero for that type of firm in our model.

Heterogeneity in the Ericson-Pakes dynamic models has been studied in the context of capacity games. Besanko and Doraszelski (2004) conclude that asymmetries of firm size can be due to the effects of price competition leading to long run distributions that exhibit positive probabilities on the monopoly outcomes. Their analysis keeps parameters symmetric across the two firms. We also find such configurations in homogeneous cases, but those configurations can arise from other parameter combinations as well. The asymmetries in price competition in their model arise because of small asymmetries in capacity accumulation that occur accidentally which makes one firm slightly dominant over the other, making the other firm to give up if investment is highly reversible. In Borkovsky et al. (2010) and Borkovsky et al. (2012), it is shown that the dynamic quality-ladder model can exhibit multiplicity of equilibria even in the absence of entry or exit if the investment is highly permanent. We take a different approach and allow firms to have different parameters in their investment success function and study the limiting distribution over the quality space given the unique equilibrium policies.

\[\text{Goettler and Gordon (2011) pp. 1151.}\]

\[\text{Gowrisankaran and Town (1997) consider two types of hospitals, for-profits and non-for-profits and the ratio between them is endogenous in the model. The parameter governing the probability of success of investment is restricted to be the same for the two hospital types, and yet, the observed market configurations in the data are not symmetric.}\]

\[\text{This behavior was not found under quantity competition.}\]

\[\text{In Borkovsky et al. (2010) figure 5 they provide evidence on the existence of multiple}\]
Our preliminary results show that asymmetries in the likelihood of success of investment can have relevant effects on long-run market configurations that show the richness of the baseline model. Another result shows that changes in the depreciation rate can significantly affect the number and types of long-run market configurations. More specifically, higher depreciation rates increases the likelihood of market collapse and market dominance at the expense of the probability of duopoly. In our analysis of the model with externalities, we find that even though the externality may be beneficial to decrease the outside good market share, it could harm the leader and allow the lagging firm to dominate the market if the asymmetry in the externality is above certain level. We apply the model to a recent case of patent release by Tesla Motors in 2014.

The remainder of this article has the following structure. Section 2 introduces the model. In Section 3 we provide computational details and the parametriziation of the model. Section 4 and 5 presents the results.

2 Model

In this section, we extend the Ericson-Pakes dynamic quality latter model to the case in which each firm’s valuation of the good sold depends not only on its own quality level, but is also (potentially) influenced by the quality level achieved by the other firm. For instance, consider the case of the electric car industry. Here quality refers to the availability and effectiveness of the network of recharging stations. An improvement in the size or the effectiveness of this network translates into an improvement in quality and thus an increase in consumers’ valuation for electric cars. There are two possible cases to study. First, suppose that there is no compatibility among the different firms. Then, an improvement in the quality of one firm affects only consumers’ valuation for its own good. Second, suppose that there is imperfect compatibility. For instance, one firm’s electric cars can recharge equilibria for depreciation rates below 0.1. Our analysis uses depreciation rates above that level and we check for potential multiplicity of equilibria solving the game in consecutive finite time horizons versions of the model a la Levhari and Mirman (1980).
in any recharging stations. Then, an improvement in the quality of one
firm’s network of recharging stations (i.e., a higher quality) affects (although
asymmetrically) consumers’ valuation for all goods in that industry.

To study the long-run implications of such industry, we must take ac-
count of heterogeneity. There are two kinds of heterogeneity worth consider-
ing. First, the technological ability to improve quality varies across firms,
i.e., some firms are more capable than others of turning investment into a
successful upgrade in quality. The second layer of heterogeneity concerns
the link between quality and consumers’ valuation. For instance, the leading
firm might not benefit from quality improvement of the lagging firm as much
as the lagging firm would benefit from quality improvement on the part of
the leading firm.

We now provide a detailed description of the Ericson-Pakes dynamic qual-
ity latter model under the presence of quality externalities and various layers
of heterogeneity. For simplicity, we restrict attention to the case of two firms
and abstract from entry or exit.\footnote{As discussed in the introduction, one of our two empirical examples in the literature
(Goettler and Gordon (2011)) does not consider entry or exit. Moreover, we allow for
quality levels of zero and the demand function in this case becomes null, this is equivalent
to exiting the market.}

**Demand.** Specifically, we consider a differentiated-product market in
which two firms compete à la Bertrand as well as invest to improve the
quality of their products. For $j = 1, 2$, let $\omega_j \in \{0, 1, 2, ..., M\}$ be firm $j$’s
quality of the product out of $M$ possible values. Given qualities $\{\omega_1, \omega_2\}$ and
prices $\{p_1, p_2\}$, firm $j$’s demand is

$$
D(p_j, p_{3-j}; \omega_j, \omega_{3-j}) = \frac{e^{g_j(\omega_j, \omega_{3-j}) - \lambda p_j}}{1 + e^{g_j(\omega_j, \omega_{3-j}) - \lambda p_j} + e^{g_{3-j}(\omega_{3-j}, \omega_j) - \lambda p_{3-j}}} \tag{1}
$$

where $m > 0$ is the size of the market and

$$
g_j(\omega_j, \omega_{3-j}) = \begin{cases}
-\infty & \omega_j = 0 \\
\omega_j + \eta_j\omega_{3-j}, & 1 \leq \omega_j + \phi_j\omega_{3-j} < \omega^* \\
\omega^* + \ln(2 - \exp(\omega^* - \omega_j - \eta_j\omega_{3-j})), & \omega^* \leq \omega_j + \phi_j\omega_{3-j} \leq M
\end{cases} \tag{2}
$$

6
maps firm $j$’s product quality into consumer’s valuation, $\omega^* \in (0, M]$. Expression (2) introduces the first type of heterogeneity in the model via the externality in quality. That is, consumers’ valuation for good $j$ depends on the quality achieved by firm $3-j$, i.e., $\omega_{3-j}$. The parameter $\eta_j \geq 0$ governs the influence of firm $3-j$. If $\eta_1 = \eta_2 = 0$, then we go back to the baseline model without externality.\(^7\) An increase in $\eta_j$ reflects a greater positive influence of the competitor’s quality on the firm’s good. Differences between $\eta_1$ and $\eta_2$ means that one firm benefits more from the quality externality than the other one.

**Profits.** Firm $j$’s instantaneous profits are

$$\pi (p_j, p_{3-j}; \omega_j, \omega_{3-j}) = D (p_j, p_{3-j}; \omega_j, \omega_{3-j}) (p_j - c)$$

(3)

where $c > 0$ is the constant marginal cost of production. Because market competition has no effect on the dynamics, the pricing game is static. Let $\Pi (\omega_j, \omega_{3-j})$ be firm $j$’s instantaneous profit corresponding to the static Bertrand game.\(^8\)

**Investment.** Each period, firm $j$ invests an amount $x_j \geq 0$ at unit cost $d > 0$ intended to improve product quality. The process for quality is stochastic and subject to an industry-wide shock. Specifically, firm $j$’s product quality evolves stochastically as

$$\tilde{\omega}’_j | \omega_j = \min \{ \max \{ \omega_j + \tilde{\tau}_j + \tilde{\eta}, 0 \}, M \}$$

(4)

where $\tilde{\tau}_j$ is a firm-specific shock and $\tilde{\eta}$ is an industry-wide depreciation shock.\(^9\)

\(^7\)Note also that our specification in (2) is similar to Borkovsky et al. (2012) in that $\omega_j = 0$ drives firm $j$’s demand to zero. Although entry or exit are not explicitly modeled, the state $(\omega_1, \omega_2) = (0, 0)$ essentially leads to a temporary collapse of the market. We call it a temporary collapse of the market since firms can still invest to go back up. In other words, it is possible that for a particular set of parameters even if $(\omega_1, \omega_2) = (0, 0)$, firms’ optimal policy functions are positive at that state and they may go back into the game.

\(^8\)That is, for $j = 1, 2$, $\Pi (\omega_j, \omega_{3-j}) = D (p^*_j, p^*_{3-j}; \omega_j, \omega_{3-j}) (p^*_j - c)$ where the pair $(p^*_j, p^*_{3-j})$ is the Bertrand equilibrium defined as $p^*_j = \arg \max_{p_j > 0} D_j (p_j, p^*_{3-j}; \omega_j, \omega_{3-j}) (p_j - c)$. For all $\{\omega_1, \omega_2\}$, there exists a unique Bertrand-Nash equilibrium (Caplin and Nalebuff (1991)).

\(^9\)A tilde sign distinguishes a random variable from a realization whereas a prime sign indicates a variable in the subsequent period.
Each random variable is binary. The firm-specific shock has support \{0, 1\} and depends on the amount of investment, i.e.,

$$\Pr[\tilde{\tau}_j = 1|x_j] = \frac{\alpha_j x_j}{1 + \alpha_j x_j} = \phi_j(x_j)$$

(5)

is firm \(j\)'s probability of success conditional on investing \(x_j \geq 0\). Here, \(\alpha_j > 0\) is specific to firm \(j\), which is our second source of parameter heterogeneity.

The industry-wide depreciation shock has support \{-1, 0\} such that

$$\Pr[\tilde{\eta} = -1] = \delta \in [0, 1]$$

(6)

is the probability of quality depreciation.\(^{10}\)

**Value Function.** Before proceeding with the definition and characterization of the equilibrium, it is useful to write down the firm’s value function taking as given the behavior of the other firm. Specifically, for \(j = 1, 2\), given \(x_{3-j}\), firm \(j\)'s infinite-horizon value function satisfies

$$v_j(\omega_j, \omega_{3-j}) = \max_{x_j \geq 0} \{ \Pi(\omega_j, \omega_{3-j}) - dx_j + \beta \mathbb{E}[v_j(\tilde{\omega}^r_j, \tilde{\omega}^r_{3-j})|\omega_j, \omega_{3-j}, x_j, x_{3-j}] \}$$

(7)

where the expected continuation value function is written as

$$\mathbb{E}[v_j(\tilde{\omega}^r_j, \tilde{\omega}^r_{3-j})|\omega_j, \omega_{3-j}, x_j, x_{3-j}]$$

$$= \phi_j(x_j)\phi_{3-j}(x_{3-j}) \cdot (\delta v_j(\omega_j, \omega_{3-j}) + (1 - \delta) v_j(\omega_j^+, \omega_{3-j}^+))$$

$$+ \phi_j(x_j)(1 - \phi_{3-j}(x_{3-j})) \cdot (\delta v_j(\omega_j, \omega_{3-j}^-) + (1 - \delta) v_j(\omega_j^-, \omega_{3-j}^-))$$

$$+ (1 - \phi_j(x_j))\phi_{3-j}(x_{3-j}) \cdot (\delta v_j(\omega_j^-, \omega_{3-j}^-) + (1 - \delta) v_n(\omega_n, \omega_{3-j}^+))$$

$$+ (1 - \phi_j(x_j))(1 - \phi_{3-j}(x_{3-j})) \cdot (\delta v_j(\omega_j^+, \omega_{3-j}^+) + (1 - \delta) v_j(\omega_j, \omega_{3-j}))$$

(8)

\(^{10}\)The specific values for \(\alpha_j\) we use in our simulations lie well within those in the literature (Goettler and Gordon (2011), Gowrisankaran and Town (1997), Borkovsky et al. (2010)).
with

\[ \omega_j^+ \equiv \min\{\omega_j + 1, M\}, \]
\[ \omega_{3-j}^+ \equiv \min\{\omega_{3-j} + 1, M\}, \]
\[ \omega_j^- \equiv \max\{\omega_j - 1, 0\}, \]
\[ \omega_{3-j}^- \equiv \max\{\omega_{3-j} - 1, 0\}. \]  

Given an initial state \((\omega_j, \omega_{3-j})\), expression (8) summarizes all possible changes in the states corresponding to investment levels \((x_j, x_{3-j})\).

**Equilibrium.** We restrict attention to Markov-perfect equilibrium (MPE) in pure strategies. The pair \(\{X_1(\omega_1, \omega_2), X_2(\omega_2, \omega_1)\}\) is an equilibrium if, for \(j = 1, 2\), given \(X_{3-j}(\omega_{3-j}, \omega_j)\)

\[
X_j(\omega_j, \omega_{3-j}) = \arg \max_{x_j \geq 0} \{ \Pi(\omega_j, \omega_{3-j}) - dx_j \\
+ \beta \mathbb{E}[V_j(\tilde{\omega}_j', \tilde{\omega}_{3-j}'; \omega_j, \omega_{3-j}, x_j, X_{3-j}(\omega_{3-j}, \omega_j))] \}
\]  

where for any \((\omega_j, \omega_{3-j}) \in \{0, 1, ..., M\}^2\), the value function satisfies

\[
V_j(\omega_j, \omega_{3-j}) = \Pi(\omega_j, \omega_{3-j}) - dX_j(\omega_j, \omega_{3-j}) \\
+ \beta \mathbb{E}[V_j(\tilde{\omega}_j', \tilde{\omega}_{3-j}'; \omega_j, \omega_{3-j}, X_j(\omega_j, \omega_{3-j}), X_{3-j}(\omega_{3-j}, \omega_j))]
\]

where, using (9), (10), (11), and (12),

\[
\mathbb{E}[V_j(\tilde{\omega}_j', \tilde{\omega}_{3-j}'; \omega_j, \omega_{3-j}, X_j(\omega_j, \omega_{3-j}), X_{3-j}(\omega_{3-j}, \omega_j))]
\]

\[
= \phi_j(X_j(\omega_j, \omega_{3-j})) \phi_{3-j}(X_{3-j}(\omega_{3-j}, \omega_j)) \cdot (\delta V_j(\omega_j, \omega_{3-j}) + (1 - \delta)V_j(\omega_j^+, \omega_{3-j}^+)) \\
+ \phi_j(X_j(\omega_j, \omega_{3-j}))(1 - \phi_{3-j}(X_{3-j}(\omega_{3-j}, \omega_j))) \cdot (\delta V_j(\omega_j, \omega_{3-j}^-) + (1 - \delta)V_j(\omega_j^-, \omega_{3-j}^-)) \\
+ (1 - \phi_j(X_j(\omega_j, \omega_{3-j}))) \phi_{3-j}(X_{3-j}(\omega_{3-j}, \omega_j)) \cdot (\delta V_j(\omega_j^-, \omega_{3-j}^-) + (1 - \delta)V_j(\omega_j^-, \omega_{3-j}^-)) \\
+ (1 - \phi_j(X_j(\omega_j, \omega_{3-j}))(1 - \phi_{3-j}(X_{3-j}(\omega_{3-j}, \omega_j))) \cdot (\delta V_j(\omega_j^-, \omega_{3-j}^-) + (1 - \delta)V_j(\omega_j, \omega_{3-j})).
\]

The first-order condition and complementary slackness condition are used.
to characterize the equilibrium. Specifically, for $j = 1, 2$,

$$X_j(\omega_j, \omega_{3-j}) = \max \left\{ -1 + \sqrt{\frac{1}{\alpha_j} \sqrt{\frac{\beta \alpha_j}{1+\alpha_3-j X_{3-j}(\omega_{3-j}, \omega_j) \Delta_j + \Psi_j}}}, 0 \right\} \quad (15)$$

when $\alpha_{3-j} X_{3-j}(\omega_{3-j}, \omega_j) \Delta_j + \Psi_j \geq 0$ and $X_j(\omega_j, \omega_{3-j}) = 0$ otherwise. Here, using (9), (10), (11), and (12),

$$\Delta_j \equiv \delta \left[ V_j (\omega_j, \omega_{3-j}) - V_j (\omega_{3-j}, \omega_j) \right] + (1 - \delta) \left[ V_j (\omega_{3-j}, \omega_{3-j}) - V_j (\omega_j, \omega_{3-j}) \right], \quad (16)$$

$$\Psi_j \equiv \delta \left[ V_j (\omega_j, \omega_{3-j}) - V_j (\omega_{3-j}, \omega_j) \right] + (1 - \delta) \left[ V_j (\omega_{3-j}, \omega_{3-j}) - V_j (\omega_j, \omega_{3-j}) \right]. \quad (17)$$

### 3 Computation and Parametrization

We use the Pakes-McGuire algorithm to numerically solve for $\{ X_1 (\omega_1, \omega_2), X_2 (\omega_2, \omega_1) \}$ and $\{ V_1 (\omega_1, \omega_2), V_2 (\omega_2, \omega_1) \}$. Since firms are heterogeneous, i.e., $\alpha_1 \neq \alpha_2$ and $\eta_1 \neq \eta_2$, the algorithm consists of iterating on best response operators (since firms are heterogeneous) until convergence is reached. Specifically, at the initial iteration $\tau = 0$, we set

$$\{ X_1^0 (\omega_1, \omega_2), X_2^0 (\omega_2, \omega_1) \} = \{ 0, 0 \}, \quad (18)$$

with the corresponding value functions is

$$\{ V_1^0 (\omega_1, \omega_2), V_2^0 (\omega_2, \omega_1) \} = \{ \Pi (\omega_1, \omega_2), \Pi (\omega_1, \omega_2) \}. \quad (19)$$

For iteration $\tau = 1, 2, \ldots$, given $\{ X_1^{\tau-1} (\omega_1, \omega_2), X_2^{\tau-1} (\omega_2, \omega_1) \}$ and
\{V^\tau_1^{-1}(\omega_1, \omega_2), V^\tau_2^{-1}(\omega_2, \omega_1)\},

\begin{align*}
X^\tau_1(\omega_1, \omega_2) &= \max \left\{ -1 + \frac{1}{\alpha_1} \sqrt{\frac{\beta \alpha_1}{1 + \alpha_2 X^\tau_2^{-1}(\omega_2, \omega_1)}} \sqrt{\frac{\alpha_2 X^\tau_2^{-1}(\omega_2, \omega_1) \Delta^\tau_1 - \Psi^\tau_1}{\alpha_1}}, 0 \right\} \\
X^\tau_2(\omega_2, \omega_1) &= \max \left\{ -1 + \frac{1}{\alpha_2} \sqrt{\frac{\beta \alpha_2}{1 + \alpha_1 X^\tau_1^{-1}(\omega_1, \omega_2)}} \sqrt{\frac{\alpha_1 X^\tau_1^{-1}(\omega_1, \omega_2) \Delta^\tau_2 - \Psi^\tau_2}{\alpha_2}}, 0 \right\}
\end{align*}

(20, 21)

when \(\alpha_2 X^\tau_2^{-1}(\omega_2, \omega_1) \Delta^\tau_1 + \Psi^\tau_1 \geq 0\) and \(X^\tau_1(\omega_1, \omega_2) = 0\) otherwise, and

\begin{align*}
\Delta^\tau_j &= \delta \left[ V^\tau_j^{-1}(\omega_j, \omega^\tau_{3-j}) - V^\tau_j^{-1}(\omega_j^-, \omega^\tau_{3-j}) \right] \\
&\quad + (1 - \delta) \left[ V^\tau_j^{-1}(\omega^+_j, \omega^\tau_{3-j} + 1) - V^\tau_j^{-1}(\omega^+_j, \omega^\tau_{3-j}) \right] \\
\Psi^\tau_j &= \delta \left[ V^\tau_j^{-1}(\omega_j, \omega^\tau_{3-j}) - V^\tau_j^{-1}(\omega_j^-, \omega^\tau_{3-j}) \right] \\
&\quad + (1 - \delta) \left[ V^\tau_j^{-1}(\omega^+_j, \omega^\tau_{3-j}) - V^\tau_j^{-1}(\omega_j, \omega^\tau_{3-j}) \right]
\end{align*}

(22, 23)

In addition to (20) and (21), the value functions are defined by

\begin{align*}
V^\tau_1(\omega_1, \omega_2) &= \Pi(\omega_1, \omega_2) - dX^\tau_1(\omega_1, \omega_2) + \beta E[V^\tau_1^{-1}(\tilde{\omega}^+_1, \tilde{\omega}^+_2)|\omega_1, \omega_2, X^\tau_1(\omega_1, \omega_2), X^\tau_2(\omega_2, \omega_1)] \\
&\quad + \beta E[V^\tau_2^{-1}(\tilde{\omega}^-_1, \tilde{\omega}^-_2)|\omega_2, \omega_1, X^\tau_2(\omega_2, \omega_1), X^\tau_1(\omega_1, \omega_2)]
\end{align*}

(24)

\begin{align*}
V^\tau_2(\omega_2, \omega_1) &= \Pi(\omega_2, \omega_1) - dX^\tau_2(\omega_2, \omega_1) + \beta E[V^\tau_2^{-1}(\tilde{\omega}^-_1, \tilde{\omega}^-_2)|\omega_2, \omega_1, X^\tau_2(\omega_2, \omega_1), X^\tau_1(\omega_1, \omega_2)]
\end{align*}

(25)

The algorithm stops when some convergence criterion for the value functions and the policy functions are met.

In the PM algorithm, the computed levels of investment at each iteration do not constitute an equilibrium since the best responses (in terms of investment) at iteration \(\tau\) are in reaction to the investments computed at iteration \(\tau - 1\). However, stationary points of such iterations are MPEs. In addition to the PM algorithm, we also apply the algorithm suggested by Levhari and
Mirman (1980) (LM) in a resource extraction dynamic game. The algorithm consists of computing the equilibrium for any finite horizon and increasing the horizon (making use of the computation for shorter horizons) until convergence is met. Unlike the PM algorithm, the levels of investment computed under the LM algorithm at each iteration constitutes a Markov-perfect equilibrium. In our numerical analysis, we compute the equilibrium using both algorithms, which always lead to the same converged policy functions. The algorithm that computes the limit of a finite horizon game has been applied in the context of the Ericson-Pakes framework (Goettler and Gordon, 2011). A description of the LM algorithm is relegated to the appendix A. We note that the PM algorithm is much faster than the LM algorithm. However, the LM algorithm allows us to make sure that reactions function cross at most once.

The parameters we use are the same as in Borkovsky et al. (2010) except that we allow for several different values of $\alpha$ and $\eta$.

Table 1: Parameter values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$M$</th>
<th>$m$</th>
<th>$c$</th>
<th>$\omega^*$</th>
<th>$\beta$</th>
<th>$\lambda$</th>
<th>$d$</th>
<th>$\alpha_j$</th>
<th>$\eta_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>value</td>
<td>18</td>
<td>5</td>
<td>5</td>
<td>12</td>
<td>0.925</td>
<td>1</td>
<td>1</td>
<td>[0.1, 10]</td>
<td>[0, 0.3]</td>
</tr>
</tbody>
</table>

In the next two sections, we provide a numerical analysis of the effect of heterogeneity on the long-run market structures. We begin with the case of no quality externality, i.e., $\eta_1 = \eta_2 = 0$ so that an improvement of quality of firm 1 has no effect on consumers’ valuation for the good sold by firm 2. In that case, we investigate how an advantage in the investment technology changes the equilibrium, which, in turn, affects the long-run market configurations. We then proceed with the case of quality externality, i.e., $\eta_1, \eta_2 > 0$. Here, we show that the presence of quality externality may lead a leading firm to give up market dominance (vis-a-vis its competitor, the lagging firm) in order to increase the industry market share (vis-a-vis the outside option). However, in some cases, the presence of quality externality (induced by a
patent released on the part of the leading firm) might also lead the lagging firm to dominate the market if the quality externality is more favorable to the lagging firm.

4 Analysis: No Quality Externality

Suppose that $\eta_1 = \eta_2 = 0$, i.e., there is no quality externality. We consider different values for the probability of industry-wide negative shocks, i.e., $\delta \in \{0.3, 0.5, 0.8\}$. Right now, we concentrate on the case of $\delta = 0.5$. To facilitate the discussion, we parametrize the heterogeneity in $\alpha_1$ and $\alpha_2$ as follows $\alpha_1 = \mu \in [0, 10]$ and $\alpha_2 = \mu - \varepsilon$ where $\varepsilon \in [0, 10]$ measures heterogeneity between the two firms. In the figures, firm 1 and firm 2 are referred to as firm A and firm B, respectively. Firm 1 (or A) is the leading firm and firm 2 (or B) is the lagging firm.\textsuperscript{11}

Comment 1: Differences between $\alpha_1$ and $\alpha_2$ have an effect on the equilibrium investment policy functions and the corresponding probabilities of success.

Figures 5, 6, 7, and 8 provide the converged value and policy functions as well as the corresponding probabilities of success. We consider two cases. When the likelihood of success of investment is the same for both firms (Figures 5 and 7), the policy and value functions are identical. However, when this likelihood is not the same across the firms (Figures 6 and 8), the lagging firm (the one with a lower $\alpha$ value, firm B in the graph) invests more in some states (Figure 6) or in most states (Figure 8) to compensate for this lack of likelihood of success. Because of the low probability of success of increasing its product quality and the higher amount of money spent in the investment, firm B receives in the long run a lower stream of cash flows and ends up having lower values for its value function compared to firm A. This is even true when firm B sells a high quality product and firm A is absent (its quality is equal to 0). The reason for this is that the depreciation effect

\textsuperscript{11}In the figures, qualA refers to $\omega_A = \omega_1$ and qualB refers to $\omega_B = \omega_2$.
is strong enough to counteract the possibility of quality improvements, thus leading to low net discounted profits.

Having discussed briefly the policy functions, we turn our attention to the long run market configurations implied by the policy functions. We show that the Ericson-Pakes model yields many different types of market configurations. To that end, let $\pi_t = [\pi^0_t, \ldots, \pi^{(M+1)^2}_t]$ is $(M + 1)^2 \times 1$ where $\pi^s_t$ is the probability that the industry is in state $s = (\omega_j, \omega_k)$ such that $\sum_s \pi^s_t = 1$.

Let $P$ be a $(M + 1)^2 \times (M + 1)^2$ transition matrix such that each element provides the probability to transition from one industry state to the other one, i.e., $\Pr[(\hat{\omega}'_H, \hat{\omega}'_L) \mid (\omega_H, \omega_L)]$.\(^{12}\)

In general, the transient distribution satisfies

$$\pi_t = P\pi_{t-1} \quad \text{(26)}$$

or

$$\pi_t = P^t\pi_0 \quad \text{(27)}$$

given the initial condition $\pi_0$. For each set of parameters, we use the converged policy functions $x^*_H(\omega)$ to calculate $P$. In each of the cases we study, there is one eigenvalue equal to one, the limiting distribution $\pi^*$ exists and satisfies

$$\pi^* = P\pi^*. \quad \text{(28)}$$

Once we obtain the distribution $\pi^*$, we count the number of modes. Each of these modes represents the maximum probability of a specific market configuration.\(^{13}\) We show that in the long run, the distribution might be unimodal (i.e., only one configuration occurs) or bimodal (two different market configurations are possible) or tri-modal (three different market structures can arise from the same set of parameters). Specifically, the market may collapse, i.e., quality is driven to zero with probability one and firms do not

\(^{12}\)Appendix B provides a detailed derivation of the transition matrix.

\(^{13}\)We discard modes that have an associated probability of less than $10^{-3}$. This threshold is equivalent to discard duopolies that have an associated probability of less than 0.1% chance of occurring.
sell anything. It is also possible to observe a duopoly. Finally, one firm may end up dominating, i.e., one firm offers a good of positive quality, i.e., $\omega_j \neq 0$ whereas the other firm offers a good of zero quality, essentially becoming insignificant, i.e., $\omega_{3-j} = 0$. For this case, it is possible to observe a realization in which the lagging firm dominates the market.

Comment 2: The quality ladder model can exhibit five different limiting distributions depending on parameter values. Those different limiting distributions are: 1) market collapse, 2) market dominance by either firm, 3) market dominance by the leading firm, 4) duopoly and monopolies, 5) duopoly.

Figure 1 illustrates comment 2 by providing a general overview of long-run market configurations for different values of $\alpha_1$ and $\alpha_2$ such that $\alpha_1 = \mu$ and $\alpha_2 = \mu - \epsilon$. Specifically, Figure 1 summarizes all market configurations for different $\mu$, $\epsilon$ when the rate of depreciation is $\delta = 0.5$. Each point $(\mu, \epsilon)$ represents an entire probability distribution in the long run. Points on the vertical axis represent the cases where both firms are identical. Any point to the right of the vertical axis represents a case of heterogeneity in which firm 2 is lagging. The farther to the right from the vertical axis, the higher the degree of heterogeneity. The letter $A$ means that firm $A$ (or 1) dominates the market. The letter $D$ refers to duopoly. The term $A,B$ means that the limiting distribution for quality is bimodal, i.e., either firm may take over as a monopoly. Finally, the letter $C$ indicates that the market collapses.

As investment becomes more reversible (higher depreciation rate) the region for duopoly shrinks from occupying almost the entire set of parameter combinations to no presence at all. Compare Figure 1 with Figures 9 and 10 in the Appendix.

Comment 3: The rate of depreciation affects the number and types of long-run market configurations that can be observed. A higher depreciation rate increases the likelihood of market collapse and market dominance at the expense of duopoly.

---

14 Since below the diagonal the difference $\mu - \epsilon < 0$, none of those points are associated to any model specification and they are left in blank.
Finally, we investigate how an increase in heterogeneity (an increase in $\epsilon$ keeping $\mu$ constant) leads to changes in the long-run market structures.

Figure 11 shows no effect of heterogeneity when firms are overall unable to invest successfully (i.e., $\alpha_1$ and $\alpha_2$ are low). However, for more capable firms (i.e., higher values of $\alpha_1$ and $\alpha_2$), an increase in heterogeneity leads the market to change from a bimodal distribution (i.e., $\{A, B\}$) to a unimodal distribution with the leading firm dominating the market (Figure 12). The effect of heterogeneity is even stronger in Figure 13 in which $\alpha_1 = \alpha_2 = 3.5$ yields either Duopoly or market dominance by either one of the firms whereas $\alpha_1 = 3.5 > \alpha_2 = 1.4$ yields in the long-run only run market dominance for firm A. Finally, Figure 2 shows that long-run market configurations can yield market dominance to firm $A$, the leading firm, as $\epsilon$ increases.
Figure 2: From \( \{D\} \) to \( \{D, A\} \) to \( \{A\} \) as \( \varepsilon \) increases
5 Quality Externality

Having discussed the case of heterogeneity in the absence of quality externality, we now turn to the situation in which there is an externality. This present analysis is motivated by the electric vehicle (EV) market. Indeed, the quality ladder model with heterogeneity allows us to study the potential market outcomes in the fast-growing electric vehicles (EV) market. This market has recently witnessed a major change since the leading firm (Tesla Motors) has released most patents, allowing other firms to benefit from Tesla’s improvements in quality as well as to contribute to quality’s improvements.

Before proceeding with the numerical analysis, we discuss briefly the Tesla case. As opposed to hybrid vehicles, EVs function only using electricity and do not have an internal combustion engine. Their main disadvantage for consumers is that the storage of electricity is still very expensive and thus most EV models have a low range of around 50 miles per battery charge. However, an expanded network of recharging stations that are compatible with the owner’s specific EV can increase the utility of owning such car even though the battery is of low-performance. Tesla Motors is one car manufacturer that has attempted to improve upon these two aspects of the product. Its technology for electricity storage allows the Tesla S model to have a range of around 200 miles per battery charge, significantly above that of its competitors, and their technology used in their network of recharging stations allows Tesla car owners to fully recharge the battery is about one hour, much below that of its competitors. In June 2014, Elon Musk, the CEO of Tesla Motors, announced that they were releasing most of the company’s patents because he said “We believe that Tesla, other companies making electric cars, and the world would all benefit from a common, rapidly-evolving technology platform.” Patent release changes the trade-off between market dominance (of the leader releasing its patents) and overall EV industry growth (currently the market share of EVs is very small). Most likely, what the CEO of Tesla Motors expects that the industry converges to the same technologies for recharging stations so that the network expands for all EV users, which

\[15\text{http://www.teslamotors.com/en\_CA/blog/all-our-patent-are-belong-you}\]
will increase the market share of EVs, Tesla Motor’s market share included.

In our model, the release of the patent means that consumers’ valuation for each good depends on the quality levels of both goods. That is, in the presence of quality externality, consumers’ valuation for good \( j \) depends on \( \omega_j + \eta_j \omega_{3-j} \) where \( \eta_j \geq 0 \).

Our first comment is that allowing for quality externality through the release of a patent removes market dominance by allowing the lagging firm to benefit from the leading firm’s investment and vice versa. However, the loss of market dominance as shown in Figure 3 is compensated by a higher market share of EV’s (i.e., a lower market share of the outside option).

However, although the presence of market externality might be beneficial to the industry, this trade-off can be harmful to the leader if the competitors take advantage of this positive externality (the patent release) to a degree where the lagging firm ends up dominating the market. This occurs if the benefit of the lagging firm from the leading firm’s quality improvement is strong (i.e., \( \eta_B = 0.3 \)) whereas the benefit of the leading firm from the lagging firm’s quality improvement is weak (i.e., \( \eta_A = 0.1 \)). Figure 4 shows that releasing a patent from a lagging firm might lead to a total loss of market shares as the lagging firm (firm B) takes over the market.
Figure 3: The Effect of Releasing Patent, going from $\eta_A = \eta_B = 0$ to $\eta_A = \eta_B = 0.3 > 0$
Figure 4: The Effect of Releasing Patent, going from \( \eta_A = \eta_B = 0 \) to \( \eta_A = 0.1 \) and \( \eta_B = 0.3 > 0 \)
References


A LM Algorithm

In this appendix, we describe the Levhari-Mirman (1980) algorithm.

Value Function, Finite Programs. For \( j = 1, 2 \), consider firm \( j \)'s maximization problem for a horizon of \( \tau \) periods, \( \tau = 0, 1, \ldots \). For \( j = 1, 2 \), given \( x_{3-j} \geq 0 \), firm \( j \)'s value function for a \( \tau \)-period horizon is

\[
v_j^\tau (\omega_j, \omega_{3-j}) = \max_{x_j \geq 0} \{ \Pi_j (\omega_j, \omega_{3-j}) - d_j x_j + \beta_j \mathbb{E}[v_{j}^{\tau-1}(\bar{\omega}_j', \bar{\omega}'_{3-j})|\omega_j, \omega_{3-j}, x_j, x_{3-j}] \}
\]

(29)

where \( \mathbb{E}[\cdot] \) is the expectation operator with respect to \( \{\bar{\omega}_j', \bar{\omega}'_{3-j}\} \) according to (4), (5), and (6). The value function for the static game (i.e., \( \tau = 0 \)) is

\[
v_j^0 (\omega_j, \omega_{3-j}) = \max_{x_j \geq 0} \{ \Pi_j (\omega_j, \omega_{3-j}) - d_j x_j \}
\]

(30)

Consistent with (29), firm \( j \)'s value function for the infinite-period horizon is thus

\[
v_j^\infty (\omega_j, \omega_{3-j}) = \max_{x_j \geq 0} \{ \Pi_j (\omega_j, \omega_{3-j}) - d_j x_j + \beta_j \mathbb{E}[v_{j}^{\infty}(\bar{\omega}_j', \bar{\omega}'_{3-j})|\omega_j, \omega_{3-j}, x_j, x_{3-j}] \}
\]

(31)

Equilibrium. Next, we define the Markov-perfect equilibrium for a game lasting \( T + 1 \) period, i.e., a horizon of \( T \) periods, \( T = 0, 1, \ldots, \infty \). The equilibrium consists of the strategies of the two firms for every horizon from the first period (when there are \( T \) periods left) to the last period (when there is no horizon). Condition 1 defines the Nash equilibrium in the static game. Note that in fact, there is no externality since \( X_0_{3-j}(\omega_{3-j}, \omega_j) \) has no effect on the zero-period-horizon objective function for firm \( j \). Condition 2 states the equilibrium for every higher horizon of the game. For \( \tau = 1, 2, 3, \ldots, T \), expressions (34) and (35) reflect the recursive nature of the equilibrium in which the equilibrium continuation value function for a \( \tau - 1 \)-period horizon depends on the equilibrium strategies for \( \tau' \)-period horizons, \( \tau - 1 > \tau' \geq 0 \).

Definition A.1. The tuple \( \{X^T_1(\omega_1, \omega_2), X^T_2(\omega_2, \omega_1)\}_{T=0}^T \) is a Markov-perfect Nash equilibrium for a game of \( T \)-period horizons if, for all \( \{\omega_1, \omega_2\} \),
1. For $\tau = 0$, for $j = 1, 2$, given $X^0_{3-j}(\omega_{3-j}, \omega_j)$,

$$X^0_j(\omega_{3-j}, \omega_j) = \arg \max_{x_{j+1}} \{ \Pi_j(\omega_j, \omega_{3-j}) - d_jx_j \}.$$  \hspace{1cm} (32)

2. For $\tau = 1, 2, \ldots, T$, for $j = 1, 2$, given $X^\tau_{3-j}(\omega_{3-j}, \omega_j)$ and $\{X^\tau_1(\omega_1, \omega_2), X^\tau_2(\omega_2, \omega_1)\}_{t=0}^{T-1}$,

$$X^\tau_j(\omega_{3-j}, \omega_j) = \arg \max_{x_{j+1}} \{ \Pi_j(\omega_j, \omega_{3-j}) - d_jx_j$$

$$\begin{align*}
+ & \beta_j \phi_j(x_j) \phi_{3-j}(X^\tau_{3-j}(\omega_{3-j}, \omega_j)) \cdot \left( \delta V_{j}^{\tau-1}(\omega_j, \omega_{3-j}) + (1 - \delta)V_{j}^{\tau-1}(\omega_j + 1, \omega_{3-j} + 1) \right) \\
+ & \beta_j \phi_j(x_j)(1 - \phi_{3-j}(X^\tau_{3-j}(\omega_{3-j}, \omega_j))) \cdot \left( \delta V_{j}^{\tau-1}(\omega_j, \omega_{3-j} - 1) + (1 - \delta)V_{j}^{\tau-1}(\omega_j + 1, \omega_{3-j}) \right) \\
+ & \beta_j(1 - \phi_j(x_j)) \phi_{3-j}(X^\tau_{3-j}(\omega_{3-j}, \omega_j)) \cdot \left( \delta V_{j}^{\tau-1}(\omega_j + 1, \omega_{3-j} + 1) \right) \\
+ & \beta_j(1 - \phi_j(x_j))(1 - \phi_{3-j}(X^\tau_{3-j}(\omega_{3-j}, \omega_j))) \cdot \left( \delta V_{j}^{\tau-1}(\omega_j - 1, \omega_{3-j} - 1) + (1 - \delta)V_{j}^{\tau-1}(\omega_j, \omega_{3-j}) \right)
\end{align*}$$

\hspace{1cm} (33)

where, for any $y, z \in \{1, 2, \ldots, M\}$,

$$V_{j}^{\tau'-1}(y, z) = \begin{cases} 
\Pi_j(y, z) - d_jX^\tau_j(y, z) & \tau' = 1 \\
\Pi_j(y, z) - d_jX^{\tau'-1}_j(y, z) + \beta_j \cdot \Gamma^{\tau'-2}_j(X^{\tau'-1}_j(y, z), X^{\tau'-1}_{3-j}(z, y)) & \tau' = 2, 3, \ldots
\end{cases}$$

\hspace{1cm} (34)

is the value function for a $\tau' - 1$ period horizon for any state vector $(y, z)$ with

$$\Gamma^{\tau'-2}_j(X^{\tau'-1}_j(y, z), X^{\tau'-1}_{3-j}(z, y))$$

$$= \phi_j(X^{\tau'-1}_j(y, z)) \phi_{3-j}(X^{\tau'-1}_{3-j}(z, y)) \cdot \left( \delta V_{j}^{\tau'-2}(y, z) + (1 - \delta)V_{j}^{\tau'-2}(y + 1, z + 1) \right)$$

$$+ \phi_j(X^{\tau'-1}_j(y, z))(1 - \phi_{3-j}(X^{\tau'-1}_{3-j}(z, y))) \cdot \left( \delta V_{j}^{\tau'-2}(y, z - 1) + (1 - \delta)V_{j}^{\tau'-2}(y + 1, z) \right)$$

$$+ (1 - \phi_j(X^{\tau'-1}_j(y, z))) \phi_{3-j}(X^{\tau'-1}_{3-j}(z, y)) \cdot \left( \delta V_{j}^{\tau'-2}(y - 1, z) + (1 - \delta)V_{j}^{\tau'-2}(y, z + 1) \right)$$

$$+ (1 - \phi_j(X^{\tau'-1}_j(y, z)))(1 - \phi_{3-j}(X^{\tau'-1}_{3-j}(z, y))) \cdot \left( \delta V_{j}^{\tau'-2}(y - 1, z - 1) + (1 - \delta)V_{j}^{\tau'-2}(y, z) \right)$$

\hspace{1cm} (35)

is the expected continuation value function corresponding to the equilibrium for a horizon of $\tau' - 2$ periods.
Proposition states the Markov-perfect Nash equilibrium for each horizon of the game.

**Proposition A.2.** Consider a game of $T$-period horizons.

1. For $\tau = 0$,
   \[
   \{X^0_1 (\omega_1, \omega_2), X^0_2 (\omega_2, \omega_1)\} = \{0, 0\},
   \tag{36}
   \]
   with the corresponding value function is
   \[
   V^0_j (\omega_j, \omega_{3-j}) = \Pi_j (\omega_j, \omega_{3-j}).
   \tag{37}
   \]

2. For $\tau \geq 1$, given $\{V^{\tau-1}_1 (\omega_1, \omega_2), V^{\tau-1}_2 (\omega_2, \omega_1), \{X^\tau_1 (\omega_1, \omega_2), X^\tau_2 (\omega_1, \omega_2)\}$ is defined by
   \[
   X^\tau_1 (\omega_1, \omega_2) = \max \left\{ -1 + \frac{1}{\alpha_1} \sqrt{\frac{\beta_1 \alpha_1}{1 + \alpha_2 X^\tau_2 (\omega_2, \omega_1)}}, \Psi^{\tau-1}_1 \right\},
   \tag{38}
   \]
   \[
   X^\tau_2 (\omega_2, \omega_1) = \max \left\{ -1 + \frac{1}{\alpha_2} \sqrt{\frac{\beta_2 \alpha_2}{1 + \alpha_1 X^\tau_1 (\omega_1, \omega_2)}}, \Psi^{\tau-1}_2 \right\}, \tag{39}
   \]
   where for $j = 1, 2$,
   \[
   \Delta^{\tau-1}_j \equiv \delta \left[ V_j^{\tau-1} (\omega_j, \omega_{3-j}) - V_j^{\tau-1} (\omega_j - 1, \omega_{3-j}) \right]
   + (1 - \delta) \left[ V_j^{\tau-1} (\omega_j + 1, \omega_{3-j} + 1) - V_j^{\tau-1} (\omega_j, \omega_{3-j} + 1) \right], \tag{40}
   \]
   \[
   \Psi^{\tau-1}_j \equiv \delta \left[ V_j^{\tau-1} (\omega_j, \omega_{3-j} - 1) - V_j^{\tau-1} (\omega_j - 1, \omega_{3-j} - 1) \right]
   + (1 - \delta) \left[ V_j^{\tau-1} (\omega_j + 1, \omega_{3-j}) - V_j^{\tau-1} (\omega_j, \omega_{3-j}) \right]. \tag{41}
   \]
Proof. The first-order condition corresponding to (33) is

\[- d_j + \beta_j \frac{\alpha_j}{(1 + \alpha_j x_j)^2} \phi_{3-j}(X_{3-j}^r(\omega_{3-j}, \omega_j)) \cdot (\delta V_{j}^{r-1}(\omega_j, \omega_{3-j}) + (1 - \delta) V_{j}^{r-1}(\omega_j + 1, \omega_{3-j} + 1))
+ \beta_j \frac{\alpha_j}{(1 + \alpha_j x_j)^2} (1 - \phi_{3-j}(X_{3-j}^r(\omega_{3-j}, \omega_j))) \cdot (\delta V_{j}^{r-1}(\omega_j, \omega_{3-j} - 1) + (1 - \delta) V_{j}^{r-1}(\omega_j + 1, \omega_{3-j}))
- \beta_j \frac{\alpha_j}{(1 + \alpha_j x_j)^2} (1 - \phi_{3-j}(X_{3-j}^r(\omega_{3-j}, \omega_j))) \cdot (\delta V_{j}^{r-1}(\omega_j - 1, \omega_{3-j} - 1) + (1 - \delta) V_{j}^{r-1}(\omega_j, \omega_{3-j}))\]

which yields (15) and thus (39), as long as the second-order condition is satisfied, i.e., for \(j, 3 - j = 1, 2, j \neq 3 - j,\)

\[- \beta_j \frac{2 \alpha_j^2}{(1 + \alpha_j x_j)^3} \frac{\alpha_{3-j} x_{3-j}}{1 + \alpha_{3-j} x_{3-j}} \cdot (\delta V_{j}^{r-1}(\omega_j, \omega_{3-j}) + (1 - \delta) V_{j}^{r-1}(\omega_j + 1, \omega_{3-j} + 1))
- \beta_j \frac{2 \alpha_j^2}{(1 + \alpha_j x_j)^3} \frac{\alpha_{3-j} x_{3-j}}{1 + \alpha_{3-j} x_{3-j}} \cdot (\delta V_{j}^{r-1}(\omega_j, \omega_{3-j} - 1) + (1 - \delta) V_{j}^{r-1}(\omega_j + 1, \omega_{3-j}))
+ \beta_j \frac{2 \alpha_j^2}{(1 + \alpha_j x_j)^3} \frac{\alpha_{3-j} x_{3-j}}{1 + \alpha_{3-j} x_{3-j}} \cdot (\delta V_{j}^{r-1}(\omega_j - 1, \omega_{3-j} - 1) + (1 - \delta) V_{j}^{r-1}(\omega_j, \omega_{3-j} + 1))
+ \beta_j \frac{2 \alpha_j^2}{(1 + \alpha_j x_j)^3} \frac{1}{1 + \alpha_{3-j} x_{3-j}} \cdot (\delta V_{j}^{r-1}(\omega_j - 1, \omega_{3-j} - 1) + (1 - \delta) V_{j}^{r-1}(\omega_j, \omega_{3-j} + 1)) < 0.\]

(44)

Algorithm. Having described the model and define the equilibrium. We now proceed with the characterization of the MPE. Here, we solve the equilibrium recursively as in Levhari and Mirman (1980). Consider first the static game of investment, i.e., \(\tau = 0.\) Then, there is no externality, and no firm has an incentive to invest, i.e., the Markov-perfect equilibrium for a game of 0-period horizon is simply

\[\{ X_1^1(\omega_1, \omega_2), X_2^1(\omega_1, \omega_2) \} = \{0, 0\},\]

(45)
with the corresponding value function is

\[ V_j^0 (\omega_j, \omega_{3-j}) = \Pi_j (\omega_j, \omega_{3-j}). \]  

(46)

Hence, there is a unique equilibrium for the no-horizon game in which the firms do not invest and the value function is equal to the profit function corresponding to the Bertrand game.

Consistent with the solution of the equilibrium, we characterize the equilibrium for each horizon. Each iteration is an horizon with the caveat that at each iteration, the solution to the reaction function is a Markov-perfect Nash equilibrium (and not an approximation). Hence, wherever we stop, we have an equilibrium. The question remains whether we converge to the stationary Markov-perfect Nash equilibrium (in infinite horizons).

1. For \( \tau = 0 \),

\[ \{ X_1^0 (\omega_1, \omega_2), X_2^0 (\omega_2, \omega_1) \} = \{ 0, 0 \}, \]  

(47)

with the corresponding value function is

\[ V_j^0 (\omega_j, \omega_{3-j}) = \Pi_j (\omega_j, \omega_{3-j}). \]  

(48)

2. For \( \tau \geq 1 \), given \( \{ V_1^{\tau-1} (\omega_1, \omega_2), V_2^{\tau-1} (\omega_2, \omega_1) \} \), firm \( j \)'s reaction function

\[ R_1^\tau (x_2) = \max \left\{ \frac{\alpha_1}{-1 + \sqrt{\frac{1}{d_1} \sqrt{\frac{\beta_1 \alpha_1}{1 + \alpha_2 x_2} \sqrt{\alpha_2 x_2 \Delta_1^{\tau-1} + \Psi_1^{\tau-1}}}}}, 0 \right\}, \]  

(49)

\[ R_2^\tau (x_1) = \max \left\{ \frac{\alpha_2}{-1 + \sqrt{\frac{1}{d_2} \sqrt{\frac{\beta_2 \alpha_2}{1 + \alpha_1 x_1} \sqrt{\alpha_1 x_1 \Delta_2^{\tau-1} + \Psi_2^{\tau-1}}}}}, 0 \right\} \]  

(50)
where for $j, 3-j = 1, 2, j \neq 3-j,$

$$
\Delta_1^{\tau-1} \equiv \delta \left[ V_j^{\tau-1}(\omega_j, \omega_{3-j}) - V_j^{\tau-1}(\omega_j - 1, \omega_{3-j}) \right] \\
+ (1 - \delta) \left[ V_j^{\tau-1}(\omega_j + 1, \omega_{3-j} + 1) - V_j^{\tau-1}(\omega_j, \omega_{3-j} + 1) \right],
$$

(51)

$$
\Psi_1^{\tau-1} \equiv \delta \left[ V_j^{\tau-1}(\omega_j, \omega_{3-j} - 1) - V_j^{\tau-1}(\omega_j - 1, \omega_{3-j} - 1) \right] \\
+ (1 - \delta) \left[ V_j^{\tau-1}(\omega_j + 1, \omega_{3-j}) - V_j^{\tau-1}(\omega_j, \omega_{3-j}) \right]
$$

(52)

where

$$
V_j^{\tau-1}(y, z) =
\begin{cases} 
\Pi_j(y, z) - d_jX_j^{y}(y, z) & \tau = 1 \\
\Pi_j(y, z) - d_jX_j^{y-1}(y, z) + \beta_j \cdot \Gamma_j^{\tau-2}(X_j^{y-1}(y, z), X_{3-j}^{y-1}(z, y)) & \tau = 2, 3, \ldots, T
\end{cases}
$$

(53)

is the value function for a $\tau - 1$ period horizon for any state vector $(y, z)$ with

$$
\Gamma_j^{\tau-2}(X_j^{\tau-1}(y, z), X_{3-j}^{\tau-1}(z, y))
\begin{align*}
&= \phi_j(X_j^{\tau-1}(y, z))\phi_{3-j}(X_{3-j}^{\tau-1}(z, y)) \cdot \left( \delta V_j^{\tau-2}(y, z) + (1 - \delta)V_j^{\tau-2}(y + 1, z + 1) \right) \\
&\quad + \phi_j(X_j^{\tau-1}(y, z))(1 - \phi_{3-j}(X_{3-j}^{\tau-1}(z, y))) \cdot \left( \delta V_j^{\tau-2}(y, z - 1) + (1 - \delta)V_j^{\tau-2}(y + 1, z) \right) \\
&\quad + (1 - \phi_j(X_j^{\tau-1}(y, z)))(1 - \phi_{3-j}(X_{3-j}^{\tau-1}(z, y))) \cdot \left( \delta V_j^{\tau-2}(y - 1, z) + (1 - \delta)V_j^{\tau-2}(y, z + 1) \right) \\
&\quad + (1 - \phi_j(X_j^{\tau-1}(y, z)))(1 - \phi_{3-j}(X_{3-j}^{\tau-1}(z, y))) \cdot \left( \delta V_j^{\tau-2}(y - 1, z - 1) + (1 - \delta)V_j^{\tau-2}(y, z) \right)
\end{align*}
$$

(54)

B Transition Probability Matrix

Using the converged policy functions, for $j = 1, 2,$

$$
\tilde{\omega}_j | \omega_j = \min\{\max\{\omega_j + \tau_j + \eta, 1\}, M\}
$$

(55)

where $\tau_j \in \{1, 0\}$ such that $\Pr[\tau_j = 1] = \phi_j(\omega_1, \omega_2) = \frac{\alpha_jX_j(\omega_1, \omega_{3-j})}{1 + \alpha_jX_j(\omega_1, \omega_{3-j})}$ and $\eta \in \{-1, 0\}$ such that $\Pr[\eta = -1] = \delta.$

We want to calculate all transition probabilities such as $\Pr[(\omega_1', \omega_2') | (\omega_1, \omega_2)].$ We consider each case separately.
1. Suppose that \((\omega_1, \omega_2)\) is such that \(\omega_1, \omega_2 \notin \{0, M\}\). Given \((\omega_1, \omega_2)\), there are \((M + 1)^2\) conditional probabilities to calculate. All of them are zero except

\[
\Pr[(\omega_1, \omega_2) | (\omega_1, \omega_2)] = \delta \phi_1(\omega_1, \omega_2) \phi_2(\omega_2, \omega_1) + (1 - \delta)(1 - \phi_1(\omega_1, \omega_2))(1 - \phi_2(\omega_2, \omega_1)),
\]

\((56)\)

\[
\Pr[(\omega_1 + 1, \omega_2) | (\omega_1, \omega_2)] = (1 - \delta) \phi_1(\omega_1, \omega_2)(1 - \phi_2(\omega_2, \omega_1)),
\]

\((57)\)

\[
\Pr[(\omega_1 - 1, \omega_2) | (\omega_1, \omega_2)] = \delta(1 - \phi_1(\omega_1, \omega_2)) \phi_2(\omega_2, \omega_1),
\]

\((58)\)

\[
\Pr[(\omega_1, \omega_2 - 1) | (\omega_1, \omega_2)] = \delta \phi_1(\omega_1, \omega_2)(1 - \phi_2(\omega_2, \omega_1)),
\]

\((59)\)

\[
\Pr[(\omega_1 - 1, \omega_2 - 1) | (\omega_1, \omega_2)] = \delta(1 - \phi_1(\omega_1, \omega_2))(1 - \phi_2(\omega_2, \omega_1)),
\]

\((60)\)

\[
\Pr[(\omega_1, \omega_2 + 1) | (\omega_1, \omega_2)] = (1 - \delta)(1 - \phi_1(\omega_1, \omega_2)) \phi_2(\omega_2, \omega_1),
\]

\((61)\)

\[
\Pr[(\omega_1 + 1, \omega_2 + 1) | (\omega_1, \omega_2)] = (1 - \delta) \phi_1(\omega_1, \omega_2) \phi_2(\omega_2, \omega_1).
\]

\((62)\)

2. Suppose that \((\omega_1, \omega_2) = (0, 0)\). Given \((\omega_1, \omega_2)\), there are \((M + 1)^2\) conditional probabilities to calculate. All of them are zero except

\[
\Pr[(0, 0) | (0, 0)] = 1 - (1 - \delta)(\phi_1(0, 0) + \phi_2(0, 0) - \phi_1(0, 0)\phi_2(0, 0)),
\]

\((63)\)

\[
\Pr[(1, 0) | (0, 0)] = (1 - \delta)\phi_1(0, 0)(1 - \phi_2(0, 0)),
\]

\((64)\)

\[
\Pr[(0, 1) | (0, 0)] = (1 - \delta)(1 - \phi_1(0, 0)) \phi_2(0, 0),
\]

\((65)\)

\[
\Pr[(1, 1) | (0, 0)] = (1 - \delta)\phi_1(0, 0)\phi_2(0, 0).
\]

\((66)\)

3. Suppose that \((\omega_1, \omega_2) = (M, M)\). Given \((\omega_1, \omega_2)\), there are \((M + 1)^2\)
conditional probabilities to calculate. All of them are zero except

\[ \Pr[(M, M) \mid (M, M)] = 1 - \delta(1 - \phi_1(M, M)\phi_2(M, M)), \]

(67)

\[ \Pr[(M - 1, M) \mid (M, M)] = \delta(1 - \phi_1(M, M)\phi_2(M, M)), \]

(68)

\[ \Pr[(M, M - 1) \mid (M, M)] = \delta\phi_1(M, M)(1 - \phi_2(M, M)), \]

(69)

\[ \Pr[(M - 1, M - 1) \mid (M, M)] = \delta(1 - \phi_1(M, M))(1 - \phi_2(M, M)). \]

(70)

4. Suppose that \((\omega_1, \omega_2) = (0, M)\). Given \((\omega_1, \omega_2)\), there are \((M + 1)^2\) conditional probabilities to calculate. All of them are zero except

\[ \Pr[(0, M) \mid (0, M)] = 1 - (1 - \delta)\phi_1(0, M) - \delta(1 - \phi_2(0, M)), \]

(71)

\[ \Pr[(1, M) \mid (0, M)] = (1 - \delta)\phi_1(0, M), \]

(72)

\[ \Pr[(0, M - 1) \mid (0, M)] = \delta(1 - \phi_2(0, M)). \]

(73)

5. Suppose that \((\omega_1, \omega_2) = (M, 0)\). Given \((\omega_1, \omega_2)\), there are \((M + 1)^2\) conditional probabilities to calculate. All of them are zero except

\[ \Pr[(M, 0) \mid (M, 0)] = 1 - (1 - \delta)\phi_2(0, M) - \delta(1 - \phi_1(M, 0)), \]

(74)

\[ \Pr[(M, 1) \mid (M, 0)] = (1 - \delta)\phi_2(0, M), \]

(75)

\[ \Pr[(M - 1, 0) \mid (M, 0)] = \delta(1 - \phi_1(M, 0)). \]

(76)

6. Suppose that \((\omega_1, \omega_2)\) is such that \(\omega_1 = 0\) and \(\omega_2 \notin \{0, M\}\). Given \((\omega_1, \omega_2)\), there are \((M + 1)^2\) conditional probabilities to calculate. All
of them are zero except

\[
\begin{align*}
\Pr[(0, \omega_2) \mid (0, \omega_2)] &= \delta \phi_2 (\omega_2, 0) \\
&\quad + (1 - \delta) (1 - \phi_1 (0, \omega_2)) (1 - \phi_2 (\omega_2, 0)), \\
\Pr[(1, \omega_2) \mid (0, \omega_2)] &= (1 - \delta) \phi_1 (0, \omega_2) (1 - \phi_2 (\omega_2, 0)), \\
\Pr[(0, \omega_2 - 1) \mid (0, \omega_2)] &= \delta (1 - \phi_2 (\omega_2, 0)), \\
\Pr[(0, \omega_2 + 1) \mid (0, \omega_2)] &= (1 - \delta) (1 - \phi_1 (0, \omega_2)) \phi_2 (\omega_2, 0), \\
\Pr[(1, \omega_2 + 1) \mid (0, \omega_2)] &= (1 - \delta) \phi_1 (0, \omega_2) \phi_2 (\omega_2, 0).
\end{align*}
\]

(77)  (78)  (79)  (80)  (81)

7. Suppose that \((\omega_1, \omega_2)\) is such that \(\omega_1 \not\in \{0, M\}\) and \(\omega_2 = 0\). Given \((\omega_1, \omega_2)\), there are \((M + 1)^2\) conditional probabilities to calculate. All of them are zero except

\[
\begin{align*}
\Pr[(\omega_1, 0) \mid (\omega_1, 0)] &= \delta \phi_1 (\omega_1, 0) \\
&\quad + (1 - \delta) (1 - \phi_2 (0, \omega_1)) (1 - \phi_1 (\omega_1, 0)), \\
\Pr[(\omega_1, 1) \mid (\omega_1, 0)] &= (1 - \delta) \phi_2 (0, \omega_1) (1 - \phi_1 (\omega_1, 0)), \\
\Pr[(\omega_1 - 1, 0) \mid (\omega_1, 0)] &= \delta (1 - \phi_1 (\omega_1, 0)), \\
\Pr[(\omega_1 + 1, 0) \mid (\omega_1, 0)] &= (1 - \delta) (1 - \phi_2 (0, \omega_1)) \phi_1 (\omega_1, 0), \\
\Pr[(\omega_1 + 1, 1) \mid (\omega_1, 0)] &= (1 - \delta) \phi_2 (0, \omega_1) \phi_1 (\omega_1, 0).
\end{align*}
\]

(82)  (83)  (84)  (85)  (86)

8. Suppose that \((\omega_1, \omega_2)\) is such that \(\omega_1 = M\) and \(\omega_2 \not\in \{0, M\}\). Given \((\omega_1, \omega_2)\), there are \((M + 1)^2\) conditional probabilities to calculate. All
9. Suppose that \((\omega_1, \omega_2)\) is such that \(\omega_1 \notin \{0, M\}\) and \(\omega_2 = M\). Given \((\omega_1, \omega_2)\), there are \((M+1)^2\) conditional probabilities to calculate. All of them are zero except

\[
\begin{align*}
\Pr[(M, \omega_1^2) \mid (M, \omega_2^2)] &= \delta \phi_1 (M, \omega_1^2) \phi_2 (\omega_2^2, M) \\
&\quad + (1 - \delta)(1 - \phi_2 (\omega_2^2, M)) \quad (87) \\
\Pr[(M - 1, \omega_2) \mid (M, \omega_2)] &= \delta (1 - \phi_1 (M, \omega_2)) \phi_2 (\omega_2, M) \quad (88) \\
\Pr[(M, \omega_2 - 1) \mid (M, \omega_2)] &= \delta \phi_1 (M, \omega_2) (1 - \phi_2 (\omega_2, M)) \quad (89) \\
\Pr[(M - 1, \omega_2 - 1) \mid (M, \omega_2)] &= \delta (1 - \phi_1 (M, \omega_2)) (1 - \phi_2 (\omega_2, M)) \quad (90) \\
\Pr[(M, \omega_2 + 1) \mid (M, \omega_2)] &= (1 - \delta)\phi_2 (\omega_2, M) \quad (91)
\end{align*}
\]

9. Suppose that \((\omega_1, \omega_2)\) is such that \(\omega_1 \notin \{0, M\}\) and \(\omega_2 = M\). Given \((\omega_1, \omega_2)\), there are \((M+1)^2\) conditional probabilities to calculate. All of them are zero except

\[
\begin{align*}
\Pr[(M, \omega_1) \mid (\omega_1, M)] &= \delta \phi_2 (M, \omega_1) \phi_1 (\omega_1, M) \\
&\quad + (1 - \delta)(1 - \phi_1 (\omega_1, M)) \quad (92) \\
\Pr[(\omega_1, M - 1) \mid (\omega_1, M)] &= \delta (1 - \phi_2 (M, \omega_1)) \phi_1 (\omega_1, M) \quad (93) \\
\Pr[(\omega_1 - 1, M) \mid (\omega_1, M)] &= \delta \phi_2 (M, \omega_1) (1 - \phi_1 (\omega_1, M)) \quad (94) \\
\Pr[(\omega_1 - 1, M - 1) \mid (\omega_1, M)] &= \delta (1 - \phi_2 (M, \omega_1)) (1 - \phi_1 (\omega_1, M)) \quad (95) \\
\Pr[(\omega_1 + 1, M) \mid (\omega_1, M)] &= (1 - \delta)\phi_1 (\omega_1, M) \quad (96)
\end{align*}
\]
C Additional figures

![Graphs showing V_A and V_B at mu = 1.1, epsilon = 0.0, depreciation = 0.5.]

Figure 5: $\alpha_1 = \alpha_2 = 1.1$
Figure 6: $\alpha_1 = 1.1$ and $\alpha_2 = 0.1$
Figure 7: $\alpha_1 = \alpha_2 = 1.1$ and $\delta = 0.5$
Figure 8: $\alpha_1 = 1.1$ and $\alpha_2 = 0.1$ and $\delta = 0.5$
Figure 9: Long-Run Market Configurations, $\delta = 0.3$
Figure 10: Long-Run Market Configurations, $\delta = 0.8$
Figure 11: Market Collapse

$\mu = 0.4$  $\epsilon = 0.0$  $\text{depreciation} = 0.5$

$\mu = 0.4$  $\epsilon = 0.3$  $\text{depreciation} = 0.5$
Figure 12: From \{A, B\} to \{A\} as $\epsilon$ increases

\[\text{mu} = 1.1 \quad \text{epsilon} = 0.0 \quad \text{depreciation} = 0.5\]

\[\text{mu} = 1.1 \quad \text{epsilon} = 1.0 \quad \text{depreciation} = 0.5\]
Figure 13: From \{D, A, B\} to \{A\} as $\varepsilon$ increases

\[
\mu = 3.5 \quad \text{epsilon} = 0.0 \quad \text{depreciation} = 0.5
\]

\[
\mu = 3.5 \quad \text{epsilon} = 2.1 \quad \text{depreciation} = 0.5
\]