Information Acquisition in Duopoly Competition with Signaling Incentives

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Abstract

This paper investigates firms’ information acquisition incentives in duopoly competition when there are firm-specific demand shocks. Two firms compete in price in two periods. Prior to the competition in the first period, each firm acquires a signal about its own demand shifter which is fixed over time. While firms’ prices are publicly observable, their sales remain their private information. After the first-period competition, a firm learns perfectly its own demand shifter through its sales but can only infer the rival’s demand intercept through the rival’s first period price. Hence, firms can manipulate rivals’ beliefs on their demand shifters through first period prices, which creates an incentive for firms to charge high prices to signal strong demands. This signaling incentive disappears when firms agree to share first-period sales reports because each firm’s demand shifter will be perfectly revealed by its sales. We investigate how much firms benefit from more precise signals when they share and when they do not share sales reports.

Keywords: Information acquisition, duopoly competition, demand uncertainty, signaling
1 Introduction

We investigate the impact of sharing sales reports on firms’ incentives to acquire information on uncertain demand in duopoly competition. Two firms produce differentiated products and compete in price in two consecutive periods. A firm’s demand in a period is fully determined by its own demand intercept and the prices charged by both firms in that period. Firms’ demand intercepts are random variables and are independently drawn from the same distribution. A firm’s demand intercept is fixed over time and is unknown to both firms. Prior to the price competition, the firm receives a signal on its own demand intercept and can choose the precision of the signal. While the firm’s choice of the precision of its signal is public information, the realization of the signal remains the firm’s private information. At the end of the first period, firms’ prices become public information but their sales remain their private information. We investigate firms’ incentives to acquire more precise signals when they share and when they do not share the reports of first-period sales.

Sharing first-period sales reports has two opposing effects on firms’ information acquisition incentives. First, it reduces firms’ information acquisition incentives from the second period’s perspective. Since there is a one-to-one mapping between a firm’s first-period sales and its demand intercept, the firm’s demand intercept is revealed by its first-period sales. Hence, when firms share sales reports, they compete with perfect information on their demand intercepts in the subsequent period. Conditional on the realized demand intercepts, firms’ private signals do not provide any additional information relevant for the competition. Hence, their optimal second period prices and profits do not depend on their signals. This implies that a firm does not gain in the second period by improving the precision of its signal when they share sales reports.

By contrast, when firms do not share sales reports, they gain in the second period from more precise signals. We consider a fully revealing equilibrium in which firms’ private signals are fully revealed by their first period prices (i.e., firms use invertible pricing strategies). So, a firm updates the belief on its rival’s demand intercept based on the signal it infers from its rival’s first period price. When Firm $i$’s signal is not perfect, Firm $j$’s posterior belief on Firm $i$’s demand intercept is in general different from Firm $i$’s realized demand intercept which is privately revealed to $i$ after its first period sales are realized. So, when firms do not share sales report, they compete with imperfect information on each other’s demand intercept. In equilibrium, Firm $i$’s second period price not only increases in its own realized demand intercept but also increases in Firm $j$’s posterior belief about Firm $i$’s demand intercept. This is because Firm $j$ expects Firm $i$ to charge a higher price if it believes that Firm $i$ has a high demand intercept. Since the two products are substitutes, Firm $j$ will charge a higher price given its belief, which in turn will induce a higher price from Firm $i$ (strategic complementarity). Furthermore, Firm $i$’s second period profit is quadratic in Firm $j$’s posterior belief on Firm $i$’s demand intercept. When Firm $i$’s signal is more precise, the
variance of Firm $j$’s posterior belief is larger and hence Firm $i$’s expected second period profit is higher. Essentially, sharing sales reports reduces firms’ information acquisition incentives from the second-period perspective because it changes the information structure of the duopoly competition in the second period.

Next, sharing first-period sales reports increases firms’ information acquisition incentives from the first period’s perspective. First, consider that firms share sales report. At the beginning of period one, a firm observes the realization of its own signal but does not observe its own demand intercept, its rival’s demand intercept or signal. Since there is no linkage between the firm’s first period price and its second period profit, the firm chooses first-period price to maximize its first period profit and its optimal first-period price is a linear increasing function of its posterior belief on its own demand intercept. We show that the firm’s expected first-period profit is quadratic in its posterior belief on its own demand intercept, and hence the firm’s expected first-period profit increases in the precision of its signal.

When firms do not share sales report, the information structure under which they engage in price competition remains unchanged in the first period. However, a firm’s optimal first-period price is different from that when firms share sales reports because of its incentive to signal a strong demand to its competitor. To see this, when Firm $i$ charges a high price in period one, Firm $j$ infers that Firm $i$ must have received a good signal on $i$’s demand intercept and hence is likely to face a strong demand in period two. Firm $j$’s optimistic belief on Firm $i$’s demand will soften the price competition in the second period. Due to this positive linkage between Firm $i$’s first period price and its second period profit, Firm $i$’s first period price is distorted above the level it would like to charge when its first period price has no impact on its second period profit. A firm continues to benefit from a more precise signal of its own demand intercept when firms do not share sales report. However, the gain from better information is smaller than when firms share sales report. This is because when firms do not share sales reports, a firm’s first period expected profit is less convex in its posterior belief of its own demand intercept and hence the firm benefits less from a more precise signal ex-ante.

Despite the two opposing effects, we find that overall firms have a weaker incentive to acquire information when they share sales report. The extent to which sharing sales reports reduces firms’ information acquisition incentives depends on the substitutability of the two products, the cost of information acquisition and the market environment such as the mean and variance of a firm’s demand intercept.

2 The Model

Players and payoffs. Two risk neutral firms $i$ and $j$ produce differentiated products and compete in price in two consecutive periods. Firm $i$’s demand in period $t$, $t = 1, 2$, is given by $q_{it} = a_i - bp_{it} + \ldots$
Firms’ demand intercepts \( a_i, a_j \) are random variables and independently drawn from the same distribution \( N(\mu_a, \sigma_a^2) \). Denote by \( \tau_a = \frac{1}{\sigma_a^2} \) the precision of firms’ demand intercepts.

Prior to price competition, each firm can invest in information acquisition on its own demand intercept. Specifically, Firm \( i \) can acquire a signal \( s_i = a_i + \epsilon_i \), where \( \epsilon_i \sim N(0, \sigma_{\epsilon_i}^2) \), \( \epsilon_i, \epsilon_j \) are independent and they are independent of \( a_i \) and \( a_j \). Denote by \( \tau_{\epsilon_i} = \frac{1}{\sigma_{\epsilon_i}^2} \) the precision of \( \epsilon_i \). If Firm \( i \) chooses precision \( \tau_{\epsilon_i} \), the associated cost is assumed to be \( \frac{1}{2} \tau_{\epsilon_i}^2 k \), where \( k \) is a positive constant.

Both firms have the same constant marginal production cost which is normalized to zero. Each firm wishes to maximize the sum of its expected profits from the two periods subtracted by the cost of information acquisition. For simplicity, we assume that there’s no discounting.

**Timing.** The game has four stages, as described below. On can refer to Figures 3 and 4.

Stage 1 Nature draws \( a_i \) and \( a_j \) independently according to \( N(\mu_a, \sigma_a^2) \). The realizations of \( a_i \) and \( a_j \) are unknown to both firms.

Stage 2 Firms choose \( \tau_{\epsilon_i} \) and \( \tau_{\epsilon_j} \) simultaneously and their choices of precisions are publicly known. Firm \( i, i = 1, 2 \), then receives a signal \( s_i \) which is Firm \( i \)'s private information.

Stage 3 Firms engage in the first period price competition and choose \( p_{i1} \) and \( p_{j1} \) simultaneously. First period prices are publicly observable. But each firm’s realized first period sales may not be publicly known, depending on which information regime we focus on, see the following for details.

Stage 4 Firms engage in the second period price competition and choose \( p_{12} \) and \( p_{22} \) simultaneously. Sales are realized and the game end.

**Information regimes** Concerning whether firms share information about the realized first period sales, we consider two different information regimes, as mentioned in Stage 3 above. In Regime 1, it is assumed that the two firms do not share their first-period sales information at the end of Stage 3 (first period competition). In Regime 2, instead we assume firms commit to share their first-period sales reports with each other.

Whether firms share first period sales reports or not will make significant differences. The main reason is that with no information sharing, the nature of the second period competition will become a game of incomplete information, since a firm’s (say \( i \)) optimal second period pricing decision will depend on the opponent’s (Firm \( j \)'s) pricing strategy, which depends on the its true demand intercept \( (a_i) \), but this intercept is not known to the firm (Firm \( i \)). Therefore, with the

\[ cp_{ji}, \ b > c > 0. \]  

*It is well known that such linear demand functions could be derived from the utility maximization problem of a representative consumer with a quadratic preference for the two firms’ products and an additively separable numeraire \( e \). Such quadratic preference can take the following form \( u(q_i, q_j, e) = \alpha_i q_i + \alpha_j q_j - \frac{1}{2}(\beta q_i^2 + 2\gamma q_i q_j + \beta q_j^2) + e \). Given the prices \( p_i, p_j \), representative consumer maximizes his utility. The optimal consumption of the consumer satisfies \( a_i = \frac{\beta\alpha_i - \gamma}{\beta - \gamma}, b = \frac{\beta}{\beta - \gamma}, c = \frac{\gamma}{\beta - \gamma}. \) Assuming this linear demand function is common in the literature.
presence of incomplete information, each firm may have incentive to manipulate the first period price to misled the rival so as to induce the the rival firm to adopt a more favorable pricing strategy in the second period.

We characterize the Perfect Bayesian equilibrium in each of the regimes and investigate how firms’ information acquisition incentives differ in these two regimes.

3 Firms Do Not Share Sales Report

In this section, it is assumed that after the first period competition, firms learn their respective sales (hence they can infer their respective demand intercept) but keep this information private. The timing of the game is illustrated in Figure 4.

Figure 1: Timing

We consider a symmetric separating equilibrium in which a firm’s first period price is linear and strictly increasing in its own signal \(s_i\). Hence

\[
\begin{align*}
p_{j1} &= \alpha_1 s_j + \beta_1 \\
p_{i1} &= \alpha_1 s_i + \beta_1,
\end{align*}
\]

where \(\alpha_1 > 0, \beta_1 > 0\) are constants.

Stage 4 We start with firms’ problems in Stage 4. In the second period, \(p_{j1}\) and \(p_{i1}\) are both observable but not the realized sales of the rival firm. Given the first period pricing strategies (1), Firm \(i\) could infer \(s_j\) from the first period price \(p_{j1}\), similarly for Firm \(j\). Let \(\hat{s}_i(p_{i1})\), \(\hat{s}_j(p_{j1})\) denote
the inferred signals

\[ \hat{s}_j = \frac{p_{j1} - \beta_1}{\alpha_1} \]  
\[ \hat{s}_i = \frac{p_{i1} - \beta_1}{\alpha_1} \]  

(2)
(3)

So Firm i’s information at the beginning of the second period is summarized by \( a_i, \hat{s}_j, \hat{s}_i \). However, even though \( s_j \) could be inferred, Firm i still does not know the realization of \( a_j \) because \( q_{j1} \) is unobservable to \( i \). Firm i’s posterior belief on \( a_j \) is updated to \( F(a_j|\hat{s}_j) \), where \( F(\cdot) \) is the cdf of the normal distribution. Note that firm \( j \) can manipulate firm i’s belief \( \hat{s}_j \) by changing \( p_{j1} \), but in equilibrium firm i’s belief \( \hat{s}_j \) should be consistent with the true \( s_j \).

In Stage 4 (second period competition), conditional on \( (a_i, \hat{s}_j, \hat{s}_i) \), Firm i chooses \( p_{i2} \) to maximize its expected profit. That is

\[
\text{Max}_{p_{i2}} \int \left[ a_i - bp_{i2} + cp_{j2}(a_j) \right] p_{i2} dF(a_j|\hat{s}_j). 
\]

Firm i’s best response function is

\[
p_{i2} = \frac{a_i + cE_{a_j}[p_{j2}(a_j)|\hat{s}_j]}{2b} 
\]

(4)

Similarly Firm j’s best response function \( p_{j2} \) is

\[
p_{j2} = \frac{a_j + cE_{a_i}[p_{i2}(a_i)|\hat{s}_i]}{2b} 
\]

(5)

**Lemma 3.1** The second period equilibrium pricing functions are linear and are given by the following

\[
p_{i2}^*(a_i, \hat{s}_i, \hat{s}_j) = \frac{a_i}{2b} + \frac{cE_{a_j}(a_j|\hat{s}_j)}{4b^2 - c^2} + \frac{cE_{a_i}(a_i|\hat{s}_i)}{2b(4b^2 - c^2)} 
\]

(6)

\[
p_{j2}^*(a_j, \hat{s}_j, \hat{s}_i) = \frac{a_j}{2b} + \frac{cE_{a_i}(a_i|\hat{s}_i)}{4b^2 - c^2} + \frac{c^2E_{a_j}(a_j|\hat{s}_j)}{2b(4b^2 - c^2)} 
\]

(7)

The proof is relegated to the appendix.

From (2), (3) and (6) (7), we observe the dependence of the optimal second period prices on
the first period prices
\[
\frac{\partial p^*_i}{\partial p_i} = \frac{c^2}{2b(4b^2 - c^2)} \frac{\partial E_{a_i}(a_i|\hat{s}_i)}{\partial \hat{s}_i} > 0
\]
\[
\frac{\partial p^*_j}{\partial p_i} = \frac{c}{4b^2 - c^2} \frac{\partial E_{a_j}(a_j|\hat{s}_j)}{\partial \hat{s}_j} > 0.
\]

Here \( \frac{\partial E_{a_i}(a_i|\hat{s}_i)}{\partial \hat{s}_i} = t_i \) (see (9) below). We see an increase in Firm \( i \)'s first period price will drive up both firms' equilibrium second period prices. This effect can be analyzed as following: First, higher \( p_{1i} \) will drive up Firm \( j \)'s belief about \( s_i \) (namely \( \hat{s}_i \)). According to (7), higher \( \hat{s}_i \) will lead to higher \( E(a_i|\hat{s}_i) \), hence higher \( p_{2j} \). By FOC (4), Firm \( i \)'s optimal second period price is strictly increasing in rival's second period price (complementarity in prices) which leads to higher second period price \( p_{2i} \). This observation clearly exhibits firms signaling incentive in the first period.

Since we assume \( b > c > 0 \), hence from the expressions above we know \( \frac{\partial p^*_j}{\partial p_{1i}} > \frac{\partial p^*_j}{\partial p_i} \), indicating that by marginally increase the first period price, a firm's second period equilibrium price will increase more rapidly then the rival firm's second period equilibrium price. For a given \((a_i, \hat{s}_j, \hat{s}_i)\), Firm \( i \)'s expected second period equilibrium profit \( \Pi_2(p^*_i, p^*_j|a_i, \hat{s}_j, \hat{s}_i) \) is
\[
\Pi_2(p^*_i, p^*_j|a_i, \hat{s}_j, \hat{s}_i) = \int_{a_i} (a_i - bp^*_i + cp^*_j)p^*_jdF(a_j|\hat{s}_j),
\]
where \( \hat{s}_i, \hat{s}_j, p^*_i \) and \( p^*_j \) are determined by (2), (3), (6) and (7). Using the first order condition (4), \( \Pi_2(p^*_i, p^*_j|a_i, \hat{s}_j, \hat{s}_i) \) could be reduced to
\[
\Pi_2(a_i, \hat{s}_j, \hat{s}_i) = b(p^*_i)^2,
\]
which is convex in the equilibrium price \( p^*_i \).

**Stage 3** Now, we consider the price competition in the first period. In this stage, Firm \( i \) knows only the private signal \( s_i \) but not true intercept \( a_i \) or opponent’s signal \( s_j \). Based on \( s_i \), Firm \( i \) updates its belief on \( a_i \) according to Bayes rule. Since we assume \( a_i \sim N(\mu_a, \sigma_a^2) \), and \( s_i = a_i + e_i, e_i \sim N(0, \sigma_e^2) \), it can be shown that the posterior is also a normal distribution which takes the following form
\[
a_i|s_i \sim N\left(t_is_i + (1-t_i)\mu_a, \frac{1}{\tau_a + \tau_e}\right), \text{ where } t_i = \frac{\tau_e}{\tau_a + \tau_e}
\]
In particular, \( E(a_i|s_i) = t_is_i + (1-t_i)\mu_a \), and Var\( (a_i|s_i) = \frac{1}{\tau_a + \tau_e} \). Similarly for \( a_j|s_j \). Note the posterior mean \( E(a_i|s_i) \) is a weighted average of the signal \( s_i \) and the mean \( \mu_a \), and the more precise the signal is (as measured by \( \tau_e \)), the more weight will be put on the signal \( s_i \). Namely, the weight \( t_i \) is an increasing function of the precision of the signal \( \tau_e \). This is an important channel through which the
precision of the signal (which is determined in the second stage) affects the result of the game.

Given this posterior belief, Firm $i$’s problem in the first period is to choose price $p_{i1}$ to maximize the sum of profits from the two periods (we assume there’s no discounting). Denote by $E\Pi_2(a_i, \hat{s}_j, \hat{s}_i|s_i)$ Firm $i$’s expected second period profit from Stage 3’s perspective. Hence

$$E\Pi_2(a_i, \hat{s}_j, \hat{s}_i|s_i) = \int_a \int_{\hat{s}_j} \Pi_2(p_{i2}, p_{j2}|a_i, \hat{s}_j, \hat{s}_i) dF(\hat{s}_j)dF(a_i|s_i)$$

$$= \int_a \int_{\hat{s}_j} b(p_{i2}^*)^2 dF(\hat{s}_j)dF(a_i|s_i). \quad (10)$$

The second equality is due to equation (8).

Conditional on $s_i$, Firm $i$ chooses $p_{i1}$ to maximize the sum of its expected profits from Stage 3 and Stage 4:

$$\max_{p_{i1}} \left\{ \int_{s_i} \int_{a_i} (a_i - bp_{i1} + cp_{j1}(s_j))p_{i1} dF(a_i|s_i)dF(s_j) \right\}$$

$$+ \int_{a_i} \int_{\hat{s}_j} \Pi_2(p_{i2}, p_{j2}|a_i, \hat{s}_j, \hat{s}_i) dF(\hat{s}_j)dF(a_i|s_i) \right\}$$

$$= \max_{p_{i1}} \left\{ (E(a_i|s_i) - bp_{i1} + cE_{sj}(p_{j1}(s_j)))p_{i1} + E\Pi_2(a_i, \hat{s}_j, \hat{s}_i|s_i) \right\} \quad (11)$$

The first order condition is

$$E(a_i|s_i) - 2bp_{i1} + cE_{sj}(p_{j1}(s_j)) + \frac{\partial E\Pi_2(a_i, \hat{s}_j, \hat{s}_i|s_i)}{\partial p_{i1}} = 0. \quad (12)$$

Symmetrically for Firm $j$, we have a similar first order condition

$$E(a_j|s_j) - 2bp_{j1} + cE_{sj}(p_{j1}(s_j)) + \frac{\partial E\Pi_2(a_j, \hat{s}_i, \hat{s}_i|s_j)}{\partial p_{j1}} = 0. \quad (13)$$

We consider a symmetric separating equilibrium in which $p_{i1}, p_{j1}$ are linear in own signal, namely they take the form of (1). Based on the two first order conditions (12), (13) and the linear pricing functions (1), imposing consistency in beliefs (namely, the inferred signals should equal the true signals, $\hat{s}_i = s_i$, $\hat{s}_j = s_j$), we can derive the first period equilibrium pricing strategies

**Lemma 3.2** If we focus on linear first period pricing strategies, then there’s a unique equilibrium in which beliefs are consistent. The equilibrium pricing strategies are given by the following:

$$p_{i1}^*(s_i) = \frac{2bE(a_i|s_i) + c\mu_a(4b^2 + c^2)}{4b^2 - c^2} + \frac{c\mu_a(4b^2 + c^2)}{2b(2b - c)(4b^2 - c^2)} \quad (14)$$

$$p_{j1}^*(s_j) = \frac{2bE(a_j|s_j) + c\mu_a(4b^2 + c^2)}{4b^2 - c^2} + \frac{c\mu_a(4b^2 + c^2)}{2b(2b - c)(4b^2 - c^2)} \quad (15)$$
The proof is given in the appendix. Using (9), substitute out $E(a_i|s_i)$ and $E(a_j|s_j)$, the first period equilibrium pricing strategies can be elaborated as

$$p_{i1}^*(s_i) = \left(\frac{2bt_i}{4b^2-c^2}\right) s_i + \left(\frac{2b\mu_a(1-t_i)}{4b^2-c^2} + \frac{c\mu_a(4b^2+c^2)}{2b(2b-c)(4b^2-c^2)}\right)$$

$$p_{j1}^*(s_j) = \left(\frac{2bt_j}{4b^2-c^2}\right) s_j + \left(\frac{2b\mu_a(1-t_j)}{4b^2-c^2} + \frac{c\mu_a(4b^2+c^2)}{2b(2b-c)(4b^2-c^2)}\right)$$

Having derived the pricing functions, we can continue to calculate the expected profits of the two periods (from the perspective of stage 3). The F.O.C. (12) can be rearranged as

$$E(a_i|s_i) - bp_{i1}^* + cE_s(p_{j1}^*(s_j)) = bp_{i1}^* \frac{\partial \Pi_2(a_i, s_j, s_i|s_i)}{\partial p_{i1}} \quad (16)$$

Substitute (16) into (11), we can rewrite Firm $i$'s expected total gross profits from the two periods as

$$\Pi_i(s_i) = \left(bp_{i1}^* - \frac{\partial \Pi_2(a_i, s_j, s_i|s_i)}{\partial p_{i1}}\right) p_{i1}^* + \Pi_2(a_i, s_j, s_i|s_i)$$

$$= b(p_{i1}^*)^2 - \frac{\partial \Pi_2(a_i, s_j, s_i|s_i)}{\partial p_{i1}} p_{i1}^* + \Pi_2(a_i, s_j, s_i|s_i),$$

where we have defined function $h(s_i)$ as the expected first period equilibrium profit.

In the proof for Lemma 3.2 (see Appendix), we've derived the expression for $\frac{\partial \Pi_2(a_i,s_j,s_i|s_i)}{\partial p_{i1}}$. In equilibrium, using conditions $\hat{s}_i = s_i$, $\hat{s}_j = s_j$, we immediately have

$$\frac{\partial \Pi_2(a_i, s_j, s_i|s_i)}{\partial p_{i1}} = \frac{c^2t_i}{(4b^2-c^2)a_1} \left(\frac{2bE(a_i|s_i)}{4b^2-c^2} + \frac{c\mu_a}{4b^2-c^2}\right)$$

$$= \frac{2bc^2E(a_i|s_i) + c^3\mu_a}{2b(4b^2-c^2)}$$

$$= \frac{c^2}{2b} \left[p_{i1}^* - \frac{c^2\mu_a}{2b(2b-c)^2}\right]$$

Substitute this expression into $h(s_i)$, we have

$$h(s_i) = b(p_{i1}^*)^2 - \frac{c^2}{2b} \left[p_{i1}^* - \frac{c^2\mu_a}{2b(2b-c)^2}\right] p_{i1}^*$$

$$= \frac{2b^2-c^2}{2b} (p_{i1}^*)^2 + \frac{c^4\mu_a}{4b^2(2b-c)^2} p_{i1}^*,$$  \(17\)

where $\frac{2b^2-c^2}{2b} > 0$ given $b > c$.

Using (8) and (6), the expected second period profit (from the perspective of first period) can
be calculated as
\[
E_{\Pi_2}(a_i, s_j, s_i | s_i) = \int_{a_i} \int_{s_j} b(p_i^*)^2 dF(s_j) dF(a_i | s_i) \\
= b \int_{a_i} \int_{s_j} \left( \frac{a_i}{2b} + \frac{cE_{a_j}(a_j | s_j)}{4b^2 - c^2} + \frac{c^2E_{a_i}(a_i | s_i)}{2b(4b^2 - c^2)} \right)^2 dF(s_j) dF(a_i | s_i).
\]

**Stage 2** In this stage, the two firms simultaneously choose the precisions of their signals \(\tau_{\epsilon_i}, \tau_{\epsilon_j}\) to maximize their respective expected net total profits. Note for the choice of precision \(\tau_{\epsilon_i}\), Firm \(i\) has to induce a convex information acquisition cost \(\frac{1}{2} \tau_{\epsilon_i}^2 k\), which will be taken into consideration in Stage 2. Let’s denote this expected net total profit function (from the perspective of Stage 2) as \(E_{s_i}\Pi_i(s_i)\), clearly
\[
E_{s_i}\Pi_i(s_i) = Eh(s_i) + E_{s_i}\Pi_{i2}(a_i, s_j, s_i | s_i) - \frac{1}{2} \tau_{\epsilon_i}^2 k
\]

We would like to know the two firms’ optimal choices of \(\tau_{\epsilon_i}\) and \(\tau_{\epsilon_j}\). The results are summarized in the following proposition. Again, the proof is relegated to the appendix.

**Proposition 1** When firms do not share sales information, a firm’s expected first period equilibrium profit is strictly increasing in the precision of the firm’s own signal and is unaffected by the signal precision of the rival firm. More explicitly, for Firm \(i\) (similarly for Firm \(j\))
\[
\frac{\partial E( h(s_i))}{\partial \tau_{\epsilon_i}} = \frac{2b(2b^2 - c^2)}{(4b^2 - c^2)^2} \frac{1}{\left(\tau_{\epsilon_i} + \tau_a\right)^2};
\]

(18)

A firm’s expected second period equilibrium profit is strictly increasing in the precision of its own signal as well as in the rival firm’s signal precision. More explicitly, for Firm \(i\) (similarly for Firm \(j\))
\[
\frac{\partial E_{s_i}\Pi_{i2}(a_i, s_j, s_i | s_i)}{\partial \tau_{\epsilon_i}} = \frac{8b^2c^2 - c^4}{4b(4b^2 - c^2)^2} \frac{1}{\left(\tau_{\epsilon_i} + \tau_a\right)^2} \frac{1}{(4b^2 - c^2)^2} \left(\tau_{\epsilon_i} + \tau_a\right)^2;
\]

(19)
\[
\frac{\partial E_{s_j}\Pi_{i2}(a_i, s_j, s_i | s_i)}{\partial \tau_{\epsilon_j}} = \frac{bc^2}{(4b^2 - c^2)^2} \frac{1}{\left(\tau_{\epsilon_j} + \tau_a\right)^2};
\]

(20)

And the equilibrium choice of signal precision for Firm \(i\) (similarly for Firm \(j\)) is uniquely determined by the solution to
\[
(\tau_{\epsilon_i} + \tau_a)^2 = \frac{4b^2 + c^2}{4b(4b^2 - c^2)k \tau_{\epsilon_i}}.
\]

(21)

Denote the solution to (21) as \(\tau_{\epsilon_i}^*\). From this proposition, we see a firm benefits from both higher precision of its own signal and the higher precision of rival firm’s signal. Without information acquisition costs, it’s optimal for both firms to acquire perfect signal.
4 Firms Share Sales Report

Now, consider the scenario in which firms’ sales reports become public information at the end of Stage 3. All other assumptions remain the same. The timing of the game is illustrated in Figure 3.

![Figure 2: Timing](image)

With information sharing, the second period competition will be changed from a game of incomplete information to a game with complete information. Another crucial difference with the no information sharing scenario is that from the perspective of the first period, a firm no longer has signaling incentives to mislead the rival’s belief about its true signal by manipulating first period price. Therefore, by studying this alternative scenario, we can shed some light on how do the signaling incentive and the game structure affect the firms’ equilibrium behaviors. The analysis is basically using backward induction.

**Stage 4** In this stage, each firm knows perfectly both \(a_i\) and \(a_j\). Therefore, the second period competition is a standard Bertrand game. It’s easy to shown that the two firms’ equilibrium second period prices are

\[
p_{i2} = \frac{2ba_i + ca_j}{4b^2 - c^2},
\]
\[
p_{j2} = \frac{2ba_j + ca_i}{4b^2 - c^2}.
\]

Accordingly, Firm \(i\)’s second period equilibrium profit can be shown to be

\[
\Pi_{i2}^S (a_i, a_j) = \frac{b(2ba_i + ca_j)^2}{(4b^2 - c^2)^2}.
\]

Here we use the superscript “IS” to represent information sharing scenario. Symmetrically we can derive Firm \(j\)’s second period equilibrium profit.
Stage 3 Now we go backward one stage and consider the first period competition. In period 1, each firm only knows its own signal. Since equilibrium profit $\Pi_{i1}^{IS}(a_i, a_j)$ does not depend on $p_{i1}$, it’s optimal for Firm $i$ to myopically choose $p_{i1}$ to maximize its expected first period profit $\Pi_{i1}^{IS}(s_i)$. That is,

$$p_{i1}^* = \arg\max \Pi_{i1}^{IS}(s_i) = \arg\max(E(a_i|s_i) - bp_{i1} + cE_s(p_{j1}(s_j)))p_{i1}.$$  

Again, we focus on symmetric linear pricing strategy for each firm. One can verify that the equilibrium first period prices in the information sharing scenario take the following form

$$p_{i1}^*(s_i) = \frac{t_is_i}{2b} + \frac{2b(1-t_i)\mu_a + c\mu_at_i}{2b(2b-c)}$$

(23)  

$$p_{j1}^*(s_j) = \frac{t_js_j}{2b} + \frac{2b(1-t_j)\mu_a + c\mu_at_j}{2b(2b-c)}$$

Using the first order condition, we can write Firm $i$’s expected first period equilibrium profit conditional on $s_i$ as

$$\Pi_{i1}^{IS}(s_i) = b[p_{i1}^*(s_i)]^2.$$  

(24)  

Stage 2 Firm $i$’s expected profit in Stage 2 $\Pi_i^{IS}$ is the sum of the expected equilibrium profits from the two periods:

$$\Pi_i^{IS} = \int_{s_i} \Pi_{i1}^{IS}(s_i) \, dF(s_i) + \int_{a_i} \int_{a_j} \Pi_{i2}^{IS}(a_i, a_j) \, dF(a_i) \, dF(a_j) - \frac{1}{2} \tau^2 \epsilon_k.$$  

where $\Pi_{i1}^{IS}(s_i)$ and $\Pi_{i2}^{IS}(a_i, a_j)$ are given by (24) and (22) respectively. As in the no information sharing scenario, we calculate the firms’ equilibrium choices of $\tau_\epsilon, \tau_{\epsilon_j}$. The results are summarized in the following proposition. The proof can be found in the appendix.

**Proposition 2** A firm’s expected first period equilibrium profit (from the perspective of Stage 2) is strictly increasing in the precision of the firm’s own signal and is unaffected by the signal precision.
of the rival firm. Specifically,

\[
\frac{\partial E_{s_i} \left[ \Pi_{i1}^h(s_i) \right]}{\partial \tau_{\epsilon_i}} = \frac{1}{4b(\tau_a + \tau_{\epsilon_i})^2}.
\]  

(25)

A firm’s expected second period equilibrium profit (from the perspective of Stage 2) is a constant that is unaffected by both firms’ signal precisions; The equilibrium choice of \( \tau_{\epsilon_i} \) for Firm i (similarly for Firm j) is uniquely determined by the solution to

\[
(\tau_a + \tau_{\epsilon_i})^2 = \frac{1}{4bk}\tau_{\epsilon_i}.
\]  

(26)

Let’s denote the unique solution to (26) as \( \hat{\tau}_{\epsilon_i}^* \). Comparing \( \hat{\tau}_{\epsilon_i}^* \) with the equilibrium precision \( \tau_{\epsilon_i}^* \) in the no information sharing scenario, we have the following theorem

**Theorem 4.1** A firm has stronger incentive to acquire information when the two firms do not share sales reports than when they share. A firm’s marginal gain in period one from an increased signal precision is smaller when they do not share sales information than when they share sales information.

Algebraically, the theorem says \( \hat{\tau}_{\epsilon_i}^* < \tau_{\epsilon_i}^* \) and that

\[
\frac{\partial E_{s_i} \left[ \Pi_{i1}^h(s_i) \right]}{\partial \tau_{\epsilon_i}} > \frac{\partial E_{h(s_i)}}{\partial \tau_{\epsilon_i}}.
\]

Proof. The proof is straightforward. Comparing (21) with (26), we see that the RHS of (21) is strictly greater than the RHS of (26) and both are strictly decreasing. The LHS of (26) and (21) are the same function, which is strictly increasing in \( \tau_{\epsilon_i} \), so the first part of the theorem follows. To show \( \frac{\partial E_{s_i} \left[ \Pi_{i1}^h(s_i) \right]}{\partial \tau_{\epsilon_i}} > \frac{\partial E_{h(s_i)}}{\partial \tau_{\epsilon_i}} \), simply refer to Propositions 1 and 2 and compare the two expressions directly. 

This result is driven by the fact that Firm i’s expected second period profit increases in the precision of its signal when firms do not share sales information (see Proposition 1) and does not change in the precision of its signal when firms share sales report. In other words, Firm i’s information acquisition incentive is enhanced when they do not share sales information purely because it has an extra gain in the second period profit. So, signaling incentives reduce a firm’s gain from more information if the firm only cares about its first period expected profit.

5 An Auxiliary Model

Consider the following auxiliary model as a benchmark. In this model there’s a monopoly in the market. Suppose the monopolist’s demand function is given by \( q_t = a - bp_t \), \( t = 1, 2 \), \( a \sim N(\mu_a, \sigma_a^2) \) as before and \( b \) is the same as in the demand function in the standard model. The firm will receive
a signal \( s \) at the beginning of the first period. After the first period sales have been realized, it could infer the true value of \( a \), hence at the beginning of the second period, the firm knows \( a \). The timing of the game is essentially the same as the standard model. In each period, the firm’s strategy is to choose a price conditional on the information available.

Given \( a \) the second period problem is simply \( \max_{p_2} (a - bp_2)p_2 \), which yields \( p_2^* = \frac{a}{2b} \) with optimal second period profit \( \Pi_2 = \frac{a^2}{4b} \).

For the first period, the firm’s problem is \( \max_{p_1} \int_a (a - bp_1)p_1dF(a|s) \). FOC gives \( p_1 = \frac{E(a|s)}{2b} = \frac{(s + (1-t)\mu_e)}{2b} \), where \( t = \frac{\tau_e}{\tau_e + \tau_a} \), hence the optimal expected first period profit is \( \Pi_1(s) = b \left[ \frac{(s + (1-t)\mu_e)}{2b} \right]^2 \).

From Stage 2’s perspective (information acquisition stage), the expected optimal first period profit can be calculates as \( E_s \Pi_1(s) = \int_a \frac{a^2}{4b}dF(a) \). Hence in Stage 2, the firm’s problem is

\[
\max_{\tau_e} \left( \frac{\tau_e}{\tau_a(\tau_e + \tau_a)} + \mu_a^2 \right) + \int_a \frac{a^2}{2b}dF(a) - \frac{1}{2} \tau_e^2k
\]

Note the second term doesn’t involve \( \tau_e \), hence the FOC is

\[
\frac{1}{4b(\tau_e + \tau_a)^2} - \tau_e k = 0 \iff (\tau_e + \tau_a)^2 = \frac{1}{4b\tau_e k}
\]

Again, use the same reasoning in Propositions 1 and 2, we know that there’s a unique solution to the equation above. Denote this solution as \( \hat{\tau}_e \). By comparing (27) with (26), we see that \( \hat{\tau}_e = \hat{\tau}_e \).

This is because in these two cases, the signal precision doesn’t affect the second period profits, only the first period profits is affected.

6 Robustness

In this section we consider a slightly different version of our standard model to check whether our results are robust.

In this variant, we continue to assume that Firm \( i \)’s demand in period \( t \) is fully determined by the demand intercept in that period and the prices charged by the two firms. We departure from the main model by assuming that Firm \( i \)’s demand intercepts in the two periods are not identical. Specifically,

\[ q_{it} = a_{it} - bp_{it} + cp_{jt}, \text{ for } i = 1, 2 \text{ and } t = 1, 2 \]

and

\[ a_{i2} = a_{i1} + \eta_i \]

where \( \eta_i \sim N(0, \sigma^2) \). For the purpose of exposition, let’s assume \( \eta_1 \) and \( \eta_2 \) are drawn from the
same distribution. Furthermore, $\eta_i$, $\eta_j$ are independent and are independent of $a_{i1}$, $a_{i2}$, $\epsilon_i$, $\epsilon_j$.

At the beginning of period one, Firm $i$ receives a signal $s_i = a_{i1} + \epsilon_i$ and chooses the first period price based on $s_i$. At beginning of period two, Firm $i$ is perfectly informed about $a_{i1}$ and updates its belief on $a_{i2}$ accordingly. If firms share sales report, they will compete with the same posterior beliefs on $a_{i2}$ and $a_{j2}$. When firms do not share sales report, they compete with different posterior beliefs on $a_{i2}$ and $a_{j2}$. Firm $i$’s posterior belief on $a_{i2}$ is $F(a_{i2}|a_{i1})$ and Firm $j$’s posterior belief is $F(a_{i2}|s_i)$.

6.1 Firms do not share sales report

At the beginning of second period, Firm $i$ knows the realized sales $q_{i1}$, hence $a_{i1}$, $s_i$, $\hat{s}_j$. Firm $i$ doesn’t know the realization of $a_{j1}$, his posterior belief about $a_{j1}$ is $F(a_{j1}|\hat{s}_j)$. In stage 4 (second period), Firm $i$’s objective is

$$\max_{p_{i2}} \int_{a_{i2}} \int_{a_{j1}} \left[ a_{i2}(a_{i1}) - bp_{i2} + cp_{j2}(a_{j1}) \right] p_{i2} \, dF(a_{j1}|\hat{a}_j) \, dF(a_{i2}|a_{i1})$$

Notice $E(a_{i2}|a_{i1}) = a_{i1}$, since $a_{i2} = a_{i1} + \eta_i$, where $\eta_i \sim N(0, \sigma^2_{\eta})$. Hence Firm $i$’s best response function is

$$p_{i2} = \frac{a_{i1} + cE(p_{j2}(a_{j1}|\hat{s}_j))}{2b}$$

similarly Firm $j$’s best response function is

$$p_{j2} = \frac{a_{j1} + cE(p_{i2}(a_{i1}|\hat{s}_i))}{2b}$$

We characterize equilibria in which each firm adopts a linear pricing strategy in expected second period intercept $E(a_{i2}|a_{i1})$, which is $a_{i1}$ in this case, ie

$$p_{i2} = \alpha_{i2}a_{i1} + \beta_{i2}, \quad i = 1, 2$$

$$p_{j2} = \alpha_{j2}a_{j1} + \beta_{j2}$$

$$p_{i2} = \frac{a_{i1}}{2b} + \frac{cE(\alpha_{j2}a_{j1} + \beta_{j2}|\hat{s}_j)}{2b} = \frac{a_{i1}}{2b} + \frac{c\alpha_{j2}E(a_{j1}|\hat{s}_j) + c\beta_{j2}}{2b}$$

$$p_{j2} = \frac{a_{j1}}{2b} + \frac{cE(\alpha_{i2}a_{i1} + \beta_{i2}|\hat{s}_i)}{2b} = \frac{a_{j1}}{2b} + \frac{c\alpha_{i2}E(a_{i1}|\hat{s}_i) + c\beta_{i2}}{2b}$$
equilibrium conditions require that

\[ \alpha_{i2} = \alpha_{j2} = \frac{1}{2b} \]

\[ \beta_{i2} = \frac{c\alpha_{i2}E(a_{i1}|\hat{s}_j) + c\beta_{i2}}{2b} \]

\[ \beta_{j2} = \frac{c\alpha_{j2}E(a_{j1}|\hat{s}_i) + c\beta_{j2}}{2b} \]

Solving for \( \beta_{i2}, \beta_{j2} \), we obtain firms’ equilibrium pricing strategies in the second period

\[ p^*_{i2}(a_{i1}, \hat{s}_j, \hat{s}_i) = \frac{a_{i1}}{2b} + \frac{cE(a_{j1}|\hat{s}_j)}{4b^2 - c^2} + \frac{c^2E(a_{i1}|\hat{s}_i)}{2b(4b^2 - c^2)} \]

\[ p^*_{j2}(a_{j1}, \hat{s}_i, \hat{s}_j) = \frac{a_{j1}}{2b} + \frac{cE(a_{i1}|\hat{s}_i)}{4b^2 - c^2} + \frac{c^2E(a_{j1}|\hat{s}_j)}{2b(4b^2 - c^2)} \]

We see that the second period pricing strategies are the same as the standard model. For given \( a_{i1}, \hat{s}_j, \hat{s}_i \), Firm \( i \)’s expected second period profit (from the perspective of second period)

\[ \Pi_{i2}(p^*_{i2}, p^*_{j2}|a_{i1}, \hat{s}_j, \hat{s}_i) = \int_{a_{i2}} \int_{a_{j1}} \left[ a_{i2} - bp^*_{i2} + cp^*_{j2}(a_{j1}) \right] p^*_{i2} \ dF(a_{i1}|\hat{s}_i) \ dF(a_{i2}|a_{i1}) \]

\[ = a_{i1}p^*_{i2} + cE(p^*_{j2}(a_{j1})|\hat{s}_j)p^*_{i2} - (bp^*_{i2})^2 \]

Using FOC \( a_{i1} + cE(p^*_{j2}(a_{j1})|\hat{s}_j) = 2bp^*_{i2} \), we have

\[ \Pi_{i2}(p^*_{i2}, p^*_{j2}|a_{i1}, \hat{s}_j, \hat{s}_i) = b(p^*_{i2})^2 \]

which is the same as the expected second period profit in the original model. Therefore, the expected second period profit from the perspective of first period and information acquisition stage are also the same. Given this result, the first period problem and the information acquisition stage become identical as that of the standard model, thereby all previous results remain the same in this variant.

### 6.2 Firms share sales report

Focus on the second period. Firms have perfect information about \( a_{i1}, a_{j1} \). Firm \( i \)’s problem is

\[ \max_{p_{i2}} \int_{a_{i2}} \left[ a_{i2} - bp_{i2} + cp_{j2} \right] p_{i2} \ dF(a_{i2}|a_{i1}) \]

\[ \Leftrightarrow \max_{p_{i2}} a_{i1}p_{i2} - bp_{i2}^2 + cp_{j2}p_{i2} \]
Similarly for Firm $j$. FOCs

\[
p_{i2} = \frac{a_{i1} + cp_{j2}}{2b}
\]
\[
p_{j2} = \frac{a_{j1} + cp_{i2}}{2b}
\]

Solving, we have

\[
p_{i2}^* = \frac{2ba_{i1} + ca_{j1}}{4b^2 - c^2}
\]
\[
p_{j2}^* = \frac{2ba_{j1} + ca_{i1}}{4b^2 - c^2}
\]

Accordingly, Firm $i$'s expected second period profit is

\[
\Pi_{i2}(a_{i1}, a_{j1}) = b\left(\frac{2ba_{i1} + ca_{j1}}{4b^2 - c^2}\right)^2
\]

which is the same as the the second period profit in the standard model.

Now consider the first period. Since the true values of $a_{i1}$ and $a_{j1}$ are perfectly known in the second period and $a_{i2} = a_{i1} + \eta_i$, the strategies in the first period doesn’t affect the second period strategies and payoffs (no signaling incentives). The first period, hence the information acquisition stage are the same as that of the standard model, all our previous calculations go through.

To conclude, all our results remain unchanged for this variant.

7 Conclusion

A Appendix

proof for Lemma 3.1 We first provide proof for Lemma 3.1.

Proof. We characterize equilibria in which each firm adopts a pricing strategy which is linear in its true demand intercept, i.e.

\[
p_{i2}(a_i) = \alpha_{2i}a_i + \beta_{2i},
\]

for $i = 1, 2$.

Using the linear strategy $p_{j2}$, we can write

\[
p_{j2} = \frac{a_i + cE_{a_j}(a_{2j}a_j + \beta_{2j}|s_j)}{2b}
\]

\[
= \frac{a_i}{2b} + \frac{cE_{a_j}(a_j|s_j) + c\beta_{2j}}{2b}
\]
Similarly,

\[
p_{j2} = \frac{a_j}{2b} + \frac{cE_0(\alpha_{2j}a_i + \beta_{2j}\hat{s}_i)}{2b} \\
= \frac{a_j}{2b} + \frac{c\alpha_{2j}E_0(a_i|\hat{s}_i) + c\beta_{2j}}{2b}.
\]

Equilibrium condition requires

\[
\alpha_{2i} = \frac{1}{2b} \\
\beta_{2i} = \frac{c\alpha_{2j}E_0(a_i|\hat{s}_i) + c\beta_{2j}}{2b} \\
\beta_{2j} = \frac{c\alpha_{2j}E_0(a_i|\hat{s}_i) + c\beta_{2j}}{2b}.
\]

Solve for \(\beta_{2i}\) and \(\beta_{2j}\) directly we obtain firms’ equilibrium pricing strategies in period 2:

\[
p_{i2}^*(a_i, \hat{s}_i, \hat{s}_j) = \frac{a_i}{2b} + \frac{cE_0(a_i|\hat{s}_j)}{4b^2 - c^2} + \frac{c^2E_0(a_i|\hat{s}_i)}{2b(4b^2 - c^2)} \\
p_{j2}^*(a_j, \hat{s}_j, \hat{s}_i) = \frac{a_j}{2b} + \frac{cE_0(a_i|\hat{s}_i)}{4b^2 - c^2} + \frac{c^2E_0(a_i|\hat{s}_j)}{2b(4b^2 - c^2)}.
\]

which are the expressions in Lemma 3.1. ■

proof for Lemma 3.2

Proof. So, the derivative

\[
\frac{\partial E\Pi_{i2}(a_i, \hat{s}_j, \hat{s}_i|s_i)}{\partial p_{i1}} = \int_{a_i} \int_{s_j} 2bp_{i2}^* \frac{\partial p_{i2}^*}{\partial p_{i1}} dF(\hat{s}_j)dF(a_i|s_i)
\]

\[
= \frac{c^2}{(4b^2 - c^2)} \int_{a_i} \int_{s_j} p_{i2}^* \frac{\partial E_0(a_i|\hat{s}_i)}{\partial p_{i1}} dF(\hat{s}_j)dF(a_i|s_i)
\]

\[
= \frac{c^2t_i}{(4b^2 - c^2)a_1} \int_{a_i} \int_{s_j} p_{i2}^* dF(\hat{s}_j)dF(a_i|s_i)
\]

\[
= \frac{c^2t_i}{(4b^2 - c^2)a_1} \left[ \frac{E(a_i|s_i)}{2b} + \frac{cE_0(a_i|\hat{s}_j)}{4b^2 - c^2} + \frac{c^2E_0(a_i|\hat{s}_i)}{2b(4b^2 - c^2)} \right],
\]

where the third equality follow from

\[
\frac{\partial E_0(a_i|\hat{s}_i)}{\partial p_{i1}} = \frac{\partial E_0(a_i|\hat{s}_i)}{\partial \hat{s}_i} \frac{\partial \hat{s}_i}{\partial p_{i1}} = \frac{t_i}{a_1}.
\]

Firms’ beliefs should be consistent with strategies in equilibrium, namely \(\hat{s}_i = s_i, \hat{s}_j = s_j\). Substitute
\[ \frac{\partial E\Pi t 2(a_i, i, j, s_i)}{\partial p_i} \]

\[ E_{s_j}(p_{s_j}(s_j)) = \alpha_1 E(s_j) + \beta_1 \]

\[ E(a_i s_i) = t_i s_i + (1 - t_i) \mu_a \]

\[ E(s_j) = \mu_a \]

\[ E_{s_j} E_{a_j}(a_j s_j) = \mu_a \]

into (12) and let \( \hat{s}_i = s_i \) and \( \hat{s}_j = s_j \). The F.O.C. (12) boils down to

\[ p_{li} = \left[ t_i s_i + (1 - t_i) \mu_a \right] \left[ \frac{\alpha_1 (4b^2 - c^2)^2 + 2bc^2 t_i}{2b\alpha_1 (4b^2 - c^2)^2} \right] + \left[ \frac{c(\alpha_1 \mu_a + \beta_1)}{2b} + \frac{c^3 \mu_a}{2b} \right] \]

\[ = t_i \left[ \frac{\alpha_1 (4b^2 - c^2)^2 + 2bc^2 t_i}{2b\alpha_1 (4b^2 - c^2)^2} \right] s_i + (1 - t_i) \mu_a \left[ \frac{\alpha_1 (4b^2 - c^2)^2 + 2bc^2 t_i}{2b\alpha_1 (4b^2 - c^2)^2} \right] + \frac{c(\alpha_1 \mu_a + \beta_1)}{2b} + \frac{c^3 \mu_a}{2b\alpha_1 (4b^2 - c^2)^2} \]

Equilibrium conditions require that

\[ \alpha_1 = t_i \left\{ \frac{\alpha_1 (4b^2 - c^2)^2 + 2bc^2 t_i}{2b\alpha_1 (4b^2 - c^2)^2} \right\} \]

\[ = t_i \left\{ \frac{1}{2b} + \frac{c^2 t_i}{\alpha_1 (4b^2 - c^2)^2} \right\} \]

\[ \beta_1 = (1 - t_i) \mu_a \left\{ \frac{\alpha_1 (4b^2 - c^2)^2 + 2bc^2 t_i}{2b\alpha_1 (4b^2 - c^2)^2} \right\} + \frac{c(\alpha_1 \mu_a + \beta_1)}{2b} + \frac{c^3 \mu_a}{2b\alpha_1 (4b^2 - c^2)^2} \]

\[ = (1 - t_i) \mu_a + \frac{c(\alpha_1 \mu_a + \beta_1)}{2b} + \frac{2b(1 - t_i) + c^2 t_i \mu_a}{2b\alpha_1 (4b^2 - c^2)^2} \]

We solve for \( \alpha_1 \) and \( \beta_1 \) from (28) and (29):

\[ \alpha_1 = \frac{2bt_i}{4b^2 - c^2} \]  

\[ \beta_1 = \frac{2b \mu_a (1 - t_i)}{4b^2 - c^2} + \frac{c \mu_a (4b^2 + c^2)}{2b(2b - c)(4b^2 - c^2)} \]

Hence, Firm \( i^{'}s \) equilibrium strategy in Stage 3 is

\[ p_{li}^*(s_i) = \left( \frac{2bt_i}{4b^2 - c^2} \right) s_i + \left( \frac{2b \mu_a (1 - t_i)}{4b^2 - c^2} + \frac{c \mu_a (4b^2 + c^2)}{2b(2b - c)(4b^2 - c^2)} \right) \]

\[ = \frac{2b E(a_i s_i)}{4b^2 - c^2} + \frac{c \mu_a (4b^2 + c^2)}{2b(2b - c)(4b^2 - c^2)} \]

Symmetrically, one can easily obtain the expression for \( p_{ji}^*(s_j) \).

**proof for Proposition 1**

**Proof.** This proposition is proved in a few steps.
STEP 1 We prove the first part of the proposition. We show that

\[
\frac{\partial E(h(s_i))}{\partial \tau_{e_i}} = \frac{2b(2b^2 - c^2)}{(4b^2 - c^2)^2} \frac{1}{(\tau_{e_i} + \tau_{a_i})^2};
\]

Using (17),

\[
E(h(s_i)) = \frac{2b^2 - c^2}{2b} [\text{Var}(p_{11}^*) + E^2(p_{11}^*)] + \frac{c^4 \mu_a}{4b^2(2b - c)^2} E(p_{11}^*)
\]

where

\[
E(p_{11}^*) = \frac{2b \mu_a}{4b^2 - c^2} + \frac{c \mu_a(4b^2 + c^2)}{2b(2b - c)(4b^2 - c^2)}
\]

\[
\text{Var}(p_{11}^*) = \frac{4b^2 \tau_{11}^2}{(4b^2 - c^2)^2} \text{Var}(s_i) = \frac{4b^2 \tau_{11}^2}{(4b^2 - c^2)^2} \left( \frac{1}{\tau_{e_i}} + \frac{1}{\tau_{a_i}} \right)
\]

hence

\[
E(h(s_i)) = \frac{2b^2 - c^2}{2b} \frac{4b^2 \tau_{11}^2}{(4b^2 - c^2)^2} \left( \frac{1}{\tau_{e_i}} + \frac{1}{\tau_{a_i}} \right) + \frac{2b^2 - c^2}{2b} \left( \frac{2b \mu_a}{4b^2 - c^2} + \frac{c \mu_a(4b^2 + c^2)}{2b(2b - c)(4b^2 - c^2)} \right) ^2
\]

The last two terms don’t depend on \( t_i \) (hence \( \tau_{e_i} \)). Therefore,

\[
\frac{\partial E(h(s_i))}{\partial \tau_{e_i}} = \frac{2b(2b^2 - c^2)}{(4b^2 - c^2)^2} \left( \frac{1}{\tau_{e_i}} + \frac{1}{\tau_{a_i}} \right)^2 2t_i \frac{\tau_{a_i}}{(\tau_{e_i} + \tau_{a_i})^2} - \frac{2b(2b^2 - c^2)}{(4b^2 - c^2)^2} \frac{t_i^2}{\tau_{e_i}}
\]

Substituting in \( t_i \), one can obtain

\[
\frac{\partial E(h(s_i))}{\partial \tau_{e_i}} = \frac{2b(2b^2 - c^2)}{(4b^2 - c^2)^2} \frac{1}{(\tau_{e_i} + \tau_{a_i})^2}
\]

which is desired result. The expression above is clearly positive given \( b > c \), indicating the expected (from the perspective of the information acquisition stage) a firm’s period 1 profit is strictly increasing in the precision of the firm’s own signal \( \tau_{e_i} \). Also notice that the expected period-1 profit doesn’t depend on the rival firm’s precision of signal \( \tau_{e_i} \).

STEP 2 Now let’s focus on the expected second period profit \( E_{s_i} E \Pi_2(a_i, s_j, s_i | s_i) \). We would like to see how the precision of signals affect this expected second period profit. The following will be used as reference in the following calculation

\[
E_{s_i} [E(a_i | s_i)] = \mu_a
\]

\[
E_{s_i} [E(a_i^2 | s_i)] = \frac{1}{\tau_{a_i}} + \mu_a^2
\]

\[
E_{s_i} [E(a_i | s_i)^2] = \frac{\tau_{e_i}}{(\tau_{e_i} + \tau_{a_i}) \tau_{a_i}} + \mu_a^2
\]
Now the expected second period profit from Stage 2’s perspective can be calculated as following

\[
E_s E \pi_{12}(a_i, s_j, s_i | s_i)
= \int_{s_i} \int_{s_j} \int_{s_i} b \left( \frac{p_{12}^2}{a_i} \right)^2 dF(s_j) dF(a_i | s_i) ds_i
\]

\[
= b \int_{s_i} \int_{s_j} \int_{s_i} \left( \frac{a_i}{2b} + \frac{c^2 E_{a_i}(a_i | s_i)}{2b(4b^2 - c^2)} + \frac{c E_{a_i}(a_i | s_i)}{4b^2 - c^2} \right)^2 dF(s_j) dF(a_i | s_i) ds_i
\]

\[
= b \int_{s_i} \int_{a_i} \left( \frac{a_i}{2b} + \frac{c^2 E_{a_i}(a_i | s_i)}{2b(4b^2 - c^2)} \right)^2 + \frac{2c \mu_k}{4b^2 - c^2} \left( \frac{a_i}{2b} + \frac{c^2 E_{a_i}(a_i | s_i)}{2b(4b^2 - c^2)} \right) + \frac{c^2}{4b^2 - c^2} \left( \frac{\tau_j (\tau_j + \tau_0) + \mu_k^2}{\tau_j (\tau_j + \tau_0) + \mu_k^2} \right) \right) dF(a_i | s_i) ds_i
\]

\[
= b \int_{s_i} \int_{a_i} \left( \frac{E(a_i | s_i)^2 + \frac{c^2}{4b^2 - c^2}(a_i | s_i)^2}{4b^2(4b^2 - c^2)} + \frac{c^2 E_{a_i}(a_i | s_i)^2}{b(4b^2 - c^2)} + \frac{c \mu_k}{4b^2 - c^2} \left( \frac{a_i}{b(4b^2 - c^2)} + \frac{c^2}{b(4b^2 - c^2)} \right) + \frac{c^2}{4b^2 - c^2} \left( \frac{\tau_j (\tau_j + \tau_0) + \mu_k^2}{\tau_j (\tau_j + \tau_0) + \mu_k^2} \right) \right) dF(a_i | s_i) ds_i
\]

\[
= b \left( \frac{1}{\tau_0} + \frac{\mu_k^2}{4b^2} \right) \left( \frac{8b^2 c^2 - c^4}{4b(4b^2 - c^2)^2} + \frac{c \mu_k^2}{4b(4b^2 - c^2)^2} \right) + \frac{c^2}{4b^2 - c^2} \left( \frac{\tau_j (\tau_j + \tau_0) + \mu_k^2}{\tau_j (\tau_j + \tau_0) + \mu_k^2} \right)
\]

Only the second term involves \( \tau_{e_i} \), hence

\[
\frac{\partial E_s E \pi_{12}(a_i, s_j, s_i | s_i)}{\partial \tau_{e_i}} = \frac{8b^2 c^2 - c^4}{4b(4b^2 - c^2)^2} \frac{1}{(\tau_{e_i} + \tau_d)^2} > 0
\]

The results indicates that given the precision of the rival firm \( \tau_{e_j} \), firm \( i \)'s optimal expected second period profit is strictly increasing in the precision of its own signal.

**Step 3** Now we can calculate Firm \( i \)'s optimal precision:

\[
\frac{\partial E_s \pi_i(s_i)}{\partial \tau_{e_i}} = 0
\]

\[
\frac{2b(2b^2 - c^2)}{(4b^2 - c^2)^2} \frac{1}{(\tau_{e_i} + \tau_d)^2} + \frac{8b^2 c^2 - c^4}{4b(4b^2 - c^2)^2} \frac{1}{(\tau_{e_i} + \tau_d)^2} = \tau_{e_i} k
\]

\[
\frac{4b^2 + c^2}{4b(4b^2 - c^2)(\tau_{e_i} + \tau_d)^2} = \tau_{e_i} k
\]

\[
(\tau_{e_i} + \tau_d)^2 = \frac{4b^2 + c^2}{4b(4b^2 - c^2)k \tau_{e_i}}.
\]

The optimal precision \( \tau_{e_i} \) the equation above. This last equation has a unique solution in \((0, \infty)\). The RHS is a strictly decreasing and continuous function of \( \tau_{e_i} \), with \( \lim_{\tau_{e_i} \to 0} = \infty \) and \( \lim_{\tau_{e_i} \to \infty} = 0 \), while the RHS is a strictly increasing and continuous function in \( \tau_{e_i} \) that is strictly positive. Hence there’s a
unique solution to this equation. This ends the proof of Proposition 1.

\[ \text{proof for Proposition 2} \]

\text{Proof.} Firm \( i \)'s expected profit in Stage 2 \( \Pi_{IS}^{IS} \) is the sum of the expected equilibrium profits from the two periods:

\[
\Pi_{IS}^{IS} = \int_{s_i} \Pi_{IS}^{IS}(s_i) \, dF(s_i) + \int_{a_i} \int_{a_j} \Pi_{IS}^{IS}(a_i, a_j) \, dF(a_i) \, dF(a_j) - \frac{1}{2} \tau_{a_i}^2 k.
\]

where

\[
E_{s_i} \left[ \Pi_{IS}^{IS}(s_i) \right] = \frac{E_{s_i}E^2(a_j|s_j)}{4b} + \frac{c \mu_a E_{s_j}E(a_j|s_j)}{2b(2b - c)} + \frac{c^2 \mu_a^2}{4b(2b - c)^2}.
\]

Substituting

\[
E_{s_i}E^2(a_j|s_j) = E_{s_i}[(t_i s_i + (1 - t_i) \mu_a)^2]
\]

\[
= E_{s_i}[t_i^2 s_i^2 + (1 - t_i)^2 \mu_a^2 + 2 t_i s_i (1 - t_i) \mu_a]
\]

\[
= t_i^2 \text{var}(s_i) + t_i^2 \mu_a^2 + (1 - t_i)^2 \mu_a^2 + 2 t_i (1 - t_i) \mu_a^2
\]

\[
= t_i^2 \text{var}(s_i) + \mu_a^2
\]

\[
E_{s_j}E(a_j|s_j) = \mu_a
\]

into \( E_{s_i} \left[ \Pi_{IS}^{IS}(s_i) \right] \), it boils down to

\[
E_{s_i} \left[ \Pi_{IS}^{IS}(s_i) \right] = \frac{t_i^2 (\frac{1}{\tau_c} + \frac{1}{\tau_a}) + \mu_a^2}{4b} + \frac{c \mu_a^2}{2b(2b - c)} + \frac{c^2 \mu_a^2}{4b(2b - c)^2}
\]

\[
= \frac{1}{4b (\tau_{ai} + \tau_{ei})^2} \frac{\tau_c + \tau_d}{\tau_c \tau_d} + \frac{\mu_a^2}{4b} + \frac{c \mu_a^2}{2b(2b - c)} + \frac{c^2 \mu_a^2}{4b(2b - c)^2}
\]

\[
= \frac{1}{4b (\tau_{ai} + \tau_{ei}) \tau_a + \mu_a^2}{(2b - c)^2}
\]

Accordingly,

\[
\frac{\partial E_{s_i} \left[ \Pi_{IS}^{IS}(s_i) \right]}{\partial \tau_{ei}} = \frac{1}{4b (\tau_{ai} + \tau_{ei})^2}.
\]

Notice that the expected second period profit does not depend on the precision of signals, since at the beginning of the second period, each firm perfectly knows the true values of \( a_i \) and \( a_j \), hence \( \frac{\partial E_{s_i} \left[ \Pi_{IS}^{IS}(a_i,a_j) \right]}{\partial \tau_{ei}} = 0 \).
The optimal precision $\hat{\tau}_{e_i}$ satisfies
\[
\frac{\partial \Pi_l^{IS}(\tau_{e_i})}{\partial \tau_{e_i}} = 0
\]
\[
\Leftrightarrow \frac{\partial E_s \Pi_{l1}^{IS}(s_i)}{\partial \tau_{e_i}} = \tau_{e_i} k
\]
\[
\Leftrightarrow (\tau_a + \tau_{e_i})^2 = \frac{1}{4bk\tau_{e_i}k}.
\]
This last equation has a unique solution, the reason is exactly the same as in the proof for Proposition 1. This ends the proof for Proposition 2. ■