The Endgame

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March 30, 2014

Abstract

On December 1st, 2009 President Obama announced that the U.S. troops would have started leaving Afghanistan on July 2011. Rather than simply waiting for the U.S. troops to withdraw, the Taliban forces responded to the announcement with a surge in attacks followed by a decline as the withdrawal date approached. In order to better understand these, at first, counter-intuitive phenomena, this paper addresses the question of how knowing versus concealing the exact length of a strategic interaction changes the optimal equilibrium strategy by studying a two-player, zero-sum game of known and unknown duration. We show why the equilibrium dynamics is non-stationary under known duration and stationary otherwise. We show that when the duration is known the performance and effort might be increased or impaired depending on the length itself and on the nature of the interaction. We then test the model by using data available for soccer matches in the major European leagues. Most importantly, we exploit the change in rule adopted by FIFA in 1998 requiring referees to publicly disclose the length of the added time at the end of the 90 minutes of play. We study how the change in rule has affected the probability of scoring both over time and across teams' relative performance and find that the rule's change led to a 28% increase of goals during the added time.

Keywords: conflict resolution, information, stopping time games, soccer.
JEL codes: D74, D83, C72, L83

1. Introduction

In many instances actors and observers recognize that knowing the exact length of a game-strategic interaction matters, independently of the length itself. By fixing the duration, the parties not only

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[†]Giulio Seccia would like to thank Kyoto Institute for Economic Research (Kyoto University) for the hospitality during the academic year 2012-2013. We would like to thank José Apesteguia, Alessandro Lizzeri and Ignacio Palacios-Huerta for valuable comments.

know that the game will end at a certain point in time, but they also know that it will *not* end before then. This is in contrast to the case where the game might be over at *any* point in time. This paper shows that players' equilibrium behaviour in a game with a known fixed duration is qualitatively different than in a game with unknown duration.

An army involved in a foreign country intervention is a case in point. The duration of an armed conflict might be either uncertain or fixed. The uncertainty might be simply due to lack of information about how long it would take to resolve the conflict or lack of public or political support. Alternatively, the length of the involvement might be exogenously fixed, *e.g.,* by the budgetary decision of a political body.[1] Regardless of the need and/or motivation for fixing the length of the involvement, the parties usually recognize that whether the duration is fixed or unknown affects the equilibrium behaviour of all parties involved. The Iraq and Afghanistan wars are good examples to demonstrate this point. In both cases the American high command and politicians alike were very much aware of the implications of announcing a definite withdrawal date, as fixing the troops' repatriation essentially fixes the conflict's length, thereby changing the nature of the game from unknown to known duration.[2] In anticipation of a subsequent change in both parties' strategies, the U.S. withdrawal announcement was either preceded by or made contemporary to a surge in troops deployment. Specifically, in the case of Iraq, in preparation of the agreement to hand over to the new Iraqi forces the control of the territory,[3] President Bush ordered a surge in troops in June, 15th 2007. In the case of Afghanistan, President Obama insisted that the announcement of both troops surge (30,000 troops) as well as the beginning of the withdrawal (July 2011) would occur at the same time.[4] Indeed, both announcements were made during the same speech at West Point on December 1st, 2009 (White House (2009)).[5]

Informed observers of the Afghanistan conflict have noticed a discontinuous change in the strategy of the Taliban army in response to the announcement fixing the duration of the involvement of the

[1] In this paper we will not analyze the case where the duration is part of players' optimal choice.

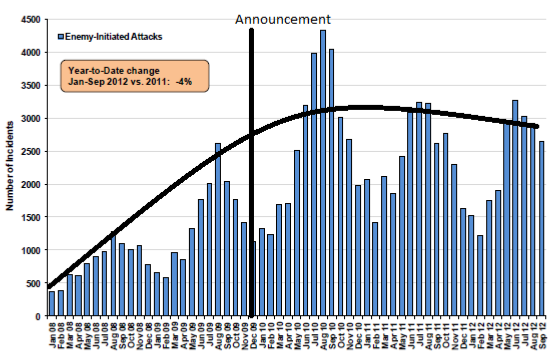
[2] One of the main point of Senator Obama's first presidential campaign was the setting of a date for troops' withdrawal from Iraq. See: http://www.washingtonpost.com/wp-dyn/content/article/2007/01/30/AR2007013001586.html.

[3] This was later named the U.S. - Iraq Status of Forces Agreement which fixed the U.S. complete withdrawal to December 31, 2011. This date was later on postponed. For a timeline of the events see http://www.reuters.com/article/2011/12/15/us-iraq-usa-pullout-idUSTRE7BE0EL20111215.

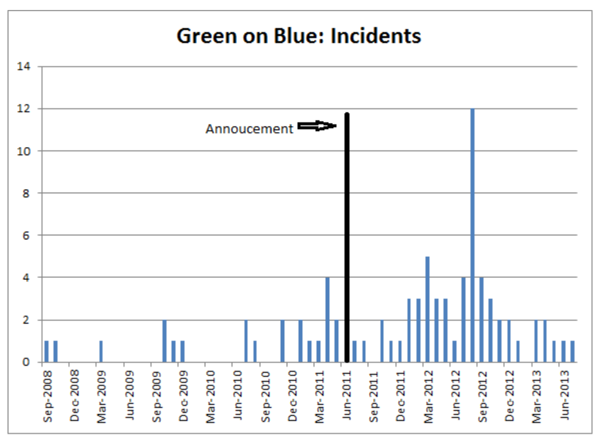
[4] "The military was told to come up with a plan to send troops quickly and then begin bringing them home quickly," Baker (2009).

[5] This major surge was preceded by an increase in troops of minor entity in February 17, 2009 (17000 troops) and in March 27th, 2009 (4000 troops).

U.S. forces. The two plots in Figure 1 provide some evidence for these claims. Figure 1(a), published by the NATO's Afghanistan Assessment Group, plots the "Enemy Initiated Attacks" (EIA) by Taliban forces across the period January 2008 - September 2012. Abstracting from seasonality due to the Afghan winter, the figure shows an increase in attacks after the first announcement of troops withdrawals made in November 2009 (Afghanistan Assessment Group (2012))[6] followed by a modest decrease in the number of incidents. Figure 1(b), presents the number of attacks on coalition forces by Afghan forces - the so-called "Green-on-Blue" attacks - for the period of September 2008 to June 2013 and including the date of a second announcement by President Obama. This period covers a second announcement made by President Obama (June 22nd, 2011) when Obama confirmed the withdrawal was to start in the following month. As Roggio and Lundquist (2012) claim, the number of this type of attacks "[...] began spiking in 2011, just after President Barack Obama announced the plan to pull the surge forces, end combat operations in 2014, and shift security to Afghan forces. The Taliban also have claimed to have stepped up efforts at infiltrating the Afghan National Security Forces."[7] Interestingly, in both cases the attacks on the U.S. forces following both announcements have an inverted U-shaped curve.



(a) Enemy Initiated Attacks Afghanistan



(b) Green-on-Blue Attacks Afghanistan

Figure 1: Number of Attacks over Time

These reactions might appear at first counter intuitive. In particular, why did the Taliban initiated more attacks following the announcement rather than simply wait for the U.S. troops to withdraw

[6] The withdrawal date was later postponed till 2014.

[7] This claim appears in *The Long War Journal,* among the most comprehensive, available data collection on the attacks conceded by the U.S. troops in the Afghan war.

and increase its attacks against the much weaker Afghan Army forces left behind? Similarly, why did President Obama announce a surge in troops concurrently to fixing the duration of the involvement? More generally, why does the information about the end of the game result in such a discontinuous change in players' behavior?

Armed conflicts are inherently complex. Consequently, we will not attempt a comprehensive rationalization of such intricate events. Rather, we will try to explain why and how knowing versus not knowing the duration of a strategic interaction affects equilibrium actions. We start by modeling a two-player, zero-sum game where players' actions can be classified according to their governance as "attack" or "defense." We assume that all actions require the same level of effort but differ in their probability of success and in the probability of counteracting a successful action by the rival. We study the game under two alternative settings: i) fixed known duration; and ii) unknown duration with a strictly positive probability that the game will end in the next period. We characterize the players' optimal strategies which determine the probability of successful actions for both players and then study how these change over time and across players' positions (in terms of number of successful actions).

We find that given players' relative position, if the duration of the game is fixed and known, the equilibrium dynamics of the probability of a successful action is non-stationary, either monotonic or non-monotonic, depending on the form of the probability function. In contrast, when the duration is unknown, the dynamics of the probability of a successful action is stationary. Finally, given the point of time of play, knowing or not knowing the exact duration of the game does not affect the behaviour of the probability of a successful action across players' relative positions.

In order to validate the model we exploit the unique opportunity for a natural experiment offered by a change in the rules of the soccer game introduced in 1998 and concerning the time added to the game in order to recoup any lost minutes during the play. Contrary to the previous regulation, where players did not know with certainty the length of the added time till the referee's whistle was blown, the new rule requires the referee to make this time publicly known to players and spectators alike at the end of each half. In doing so the endgame duration becomes common knowledge. We study players' behavior under unknown duration (pre-1998 seasons) and known duration (post-1998 seasons). Moreover, we exploit the availability of the minute-by-minute data for the matches' regular time (a known duration game) to further validate our model' s predictions for a known duration game.

As predicted by the theoretical model, our empirical analysis shows that during the regular time the probability of scoring a goal is non-stationary over time. Analyzing the added time we find that the dynamics of the probability of scoring pre and post-1998 seasons is significantly different and lead to an increase of 28% in the number of goals scored during the added time after the rule change.

Our analysis is related to the literature on deadlines (see e.g., Spear, (1998) and Yildiz, (2011)). Notice, however, that knowing the duration of the game is not equivalent to having a deadline. A deadline does not necessarily imply that the duration of the game is known, as it does not prevent the game from ending beforehand. Similarly, even absent a deadline, the players might know that the game will not be over before a certain time has elapsed and hence, up to that time, their optimal strategy will be identical to the strategy adopted under known duration. Consequently, we prefer to adopt the terminology of duration and clearly separate our results from deadline effects found in the literature. Our analysis is also related to the literature on repeated games, see *e.g.*, Aumann and Shapley (1994) and Rubinstein (1979). In contrast to the zero-sum nature of our game, this literature focuses on the incentives to coordinate in order to improve both players' outcome.

2. The game

Consider a game played by two players A and B, over time $t = 1, 2, \ldots$. At each t, each player contemporarily chooses an action, $a \in \boldsymbol{A} = [0, 1]$ by player A and $b \in \boldsymbol{B} = [0, 1]$ by B. We interpret higher values of the action as offensive play and lower values as defensive play.[8] The action of, at most, one player can be successful at each t and this in turn determines the realization $x = -1, 0, 1$ of the random variable X. Specifically, X takes the value of 1 if player A's action has been successful, 0 if neither players' action has been successful and -1 if player B's action has been successful. The probability p_x associated with the realization x is a function of both players' actions and is defined as:

$$p_x : \boldsymbol{A} \times \boldsymbol{B} \rightarrow [0, 1], \quad x = -1, 1. \tag{1}$$

[8]The values an action can take are simply labels and they are not meant to be interpreted as real values, *e.g.*, action $a = 0$ does not mean inaction.

All actions require the same level of effort and bear the same direct costs or disutility. Equivalently, assume they bear no cost.

Assumption 1. *1. In the interior on the action set, the probability of each player's successful action is increasing in both a and b,* i.e., *for $(a, b) \in Int(\boldsymbol{A} \times \boldsymbol{B})$:*

$$\partial_a p_x(a,b) > 0 \text{ and } \partial_b p_x(a,b) > 0,\ x = -1, 1. \tag{2}$$

2. At the boundaries of the action set, the marginal probability of a successful action by players A and B is such that:

$$\partial_a p_1(1,b) = \partial_a p_{-1}(0,b) = 0; \tag{3}$$

$$\partial_b p_1(a,0) = \partial_b p_{-1}(a,1) = 0. \tag{4}$$

3. The probability of a successful action is concave in each player's own action and convex in the opponent's action, i.e.,

$$\partial_a^2 p_1(a,b) < 0 \text{ and } \partial_b^2 p_{-1}(a,b) < 0; \tag{5}$$

$$\partial_a^2 p_{-1}(a,b) > 0 \text{ and } \partial_b^2 p_1(a,b) > 0, \tag{6}$$

and the cross derivatives $\partial_{ab}^2 p_x$ and $\partial_{ba}^2 p_x$ are different than 0.

Assumption 1.1 implies that a player choosing a higher action increases both his own and the opponent's probability of a successful action. The latter represents an implicit cost of increasing the action. Assumption 1.2 provides sufficient conditions for obtaining an equilibrium of the game. Assumption 1.3 implies that the marginal probability of a successful action decreases in the player's own action level and increases in the opponent's action level. These assumptions are meant to capture circumstances where a more offensive action increases the odds of success but weakens the defense level. This trade-off between offense and defense is typical in conflictual situation where one can identify actions of attack or defense. In armed conflicts, for example, an offensive action increases both the chances of inflicting casualties to the enemy and of suffering casualties. In soccer, playing in attack implies the increasing chance of scoring as well as conceding a goal by a counter-attack. Moreover, the convexity of the probability of success in the other player action guarantees the concavity of the players objective functions.

Let d denote the difference in the number of successful actions from player A's perspective. At the end of the game, player A's value from the game is +1 if the difference in successful actions is positive, 0 if nil and -1 if negative, *i.e.,* player A receives a value:

$$I\{d \geq 0\} - I\{d \leq 0\}, \tag{7}$$

while player B receives a value:

$$-I\{d \geq 0\} + I\{d \leq 0\}, \tag{8}$$

where I denotes the indicator function.[9] The above values imply that at the last stage of play the game is zero-sum.

We are now in a position to analyze the game when the duration is known, *i.e.,* fixed stopping time (next section) and subsequently when the duration is unknown and the game can end at any point in time with positive probability, *i.e.*, random stopping time (Section 2.2). All proofs are in the Appendix.

2.1 Known duration: fixed stopping time

Suppose that both players know that the game will be of duration T, *i.e.,* it will end at $t = T$ and not before. Denote by $V_T^A(T,d)$ and by $V_T^B(T,d)$ the values in (7) and (8), respectively, where the subscripts refer to a game of known duration T and (T,d) is the state variable identifying the time of play and the difference in the number of successful actions.

For any $t < T$, the players' actions are determined by the solution to the following system of equations:

$$V_T^A(t,d) = \max_a \sum_{x=-1}^{1} p_x(a, b_T^*(t,d)) V_T^A(t+1, d+x); \tag{9}$$

$$V_T^B(t,d) = \max_b \sum_{x=-1}^{1} p_x\left(a_T^*(t,d), b\right) V_T^B(t+1, d+x), \tag{10}$$

[9] These values are for simplicity only and without loss of generality. The results hold for any increasing function in d that is symmetric around zero.

where $a^*_T(t,d)$ and $b^*_T(t,d)$ are the *argmax* of (9) and (10), respectively. The sequence $(a^*_T(t,d),\ b^*_T(t,d) : t = 0,\ldots,T,\ d \in \mathbb{N})$ represents the (Markov perfect) equilibrium of the game of length T. In the following analysis we fix the length of the game and drop the subscript T to simplify notation.[10] Given the symmetry in $p_1(a,b)$ and $p_{-1}(a,b)$, it is easy to show the symmetry in $V^A(a,b)$ and $V^B(a,b)$.

Lemma 1. *At any given* (t,d) *the value of the game for player* A *and* B *is such that:*

$$V^B(t,d) = -V^A(t,d). \tag{11}$$

Lemma 1 implies that it suffices to characterize the game by $V^A(t,d)$ only, simplify notation by dropping the superscript and refer to $V(t,d)$ as the *value of the game*.

Lemma 2. *An equilibrium for the game exists and is unique.*

We can now show that the value of the game is monotonic in the difference in the number of successful actions, *i.e.,*

Lemma 3. *At any given* t *the value function is monotonically increasing in* d, i.e., $V(t,d+1) \geq V(t,d)$.

Given the action pair (a,b) let us define player A's *relative elasticity of success* as the following ratio:[11]

$$\epsilon^A(a,b) = \frac{\partial_a p_1(a,b)/p_1(a,b)}{\partial_a p_{-1}(a,b)/p_{-1}(a,b)}. \tag{12}$$

The value $\epsilon^A(a,b)$ gives the player's odds of achieving a successful action relative to conceding one. $\epsilon^A(a,b) > 1$ thus implies that an *increase* in the action by player A improves the player's ratio of the odds of achieving a successful action relative to conceding one. Similarly, $\epsilon^A(a,b) < 1$ implies that *decreasing* the level of A's action improves the ratio of the player's odds of conceding a successful action relative to the odds of achieving one. Consistently with this observation we will say that player A has a *relative advantage in attacking (defending)* if $\epsilon^A(a,b) > 1$ ($\epsilon^A(a,b) < 1$). Similarly for player B.

[10] We will need the explicit reference to the game length in the empirical analysis.

[11] Accordingly, one could define the term $\partial_a p_1(a,b)\frac{a}{p_1(a,b)}$ player A's elasticity of a successful action and $\partial_a p_{-1}(a,b)\frac{a}{p_{-1}(a,b)}$ the elasticity of conceding a successful action.

Lemma 4. *At equilibrium the relative elasticities of success are such that:*

$$\epsilon^{A*}(t,d) = \frac{1}{\epsilon^{B*}(t,d)}. \tag{13}$$

In general, a relative advantage in attacking may elicit an offensive or defensive response by the opponent. However, a straightforward implication of Lemma 4 is that at any given point on the equilibrium trajectory both players cannot have a relative advantage in attacking.

Specifically, let us denote by $\mathcal{A}_+(b)$ the set of player A's actions such that, given b, player A has a relative advantage in attacking, *i.e.,* $\mathcal{A}_+(b) = \{a : \epsilon^A(a,b) > 1\}$. Let $\mathcal{A}_-(b)$ denote the complementary set. Similarly, given action a, one can define the set of player B's actions where he has relative advantage in attacking, *i.e.,* $\mathcal{B}_+(a)$ and $\mathcal{B}_-(a)$. Let $\mathcal{A}^*_+(t,d) = \mathcal{A}_+(b^*(t,d))$ and similarly define $\mathcal{A}^*_-(t,d)$, $\mathcal{B}^*_+(t,d)$, and $\mathcal{B}^*_-(t,d)$. From Lemma 4 it follows that at any given (t,d), $a^*(t,d) \in \mathcal{A}^*_+(t,d)$ if and only if $b^*(t,d) \in \mathcal{B}^*_-(t,d)$. Hence a relative advantage in attacking (defending) is always met by the countering strategy.

In order to see how the value of the game changes over t given d, we look at the marginal value of time. This can be nicely written as a function of the marginal value of successful action and the player's relative advantage. Let $p^*_x(t,d) = p_x(a^*(t,d), b^*(t,d))$. From the first order condition of player A's maximization problem one can show the following relationship between the marginal value of time, the player's relative advantage and the marginal value of a successful action:

$$\overbrace{V(t,d) - V(t-1,d)}^{\text{marginal value of time}} = p_1(t,d)\partial_a p^*_{-1}(t,d) \underbrace{[\epsilon^{A*}(t,d) - 1]]}_{\text{relative advantage}} \overbrace{[V(t,d) - V(t,d-1)]}^{\text{marginal value of successful action}} . \tag{14}$$

Equation (14) is helpful in understanding the mechanism at play. On the left hand side we have player A's marginal value at $t-1$ of moving forward one extra period toward the endgame. On the right hand side we have the term determining the relative advantage at t multiplied by the marginal value of one extra successful action at t. Hence, equation (14) establishes that moving forward toward the endgame has a positive value for the player with the relative advantage in attacking at period t. Correspondingly, the marginal value of time is negative for the player with a relative advantage in defending. The next lemma summarizes this result.

Lemma 5. *At any given (t, d), if $a^*(t, d) \in \mathcal{A}^*_+(t, d)$ then the marginal value of time is positive at t. The opposite dynamics holds if $a^*(t, d) \in \mathcal{A}^*_-(t, d)$.*

We can now characterize the players' action across different d's, given t, and over t, given d.

Proposition 1. *1. Player A's equilibrium action decreases in d at any given t.*
2. If $a^*(t, d) \in \mathcal{A}^*_+(t, d)$, then player A's equilibrium action decreases in t for any given d. The opposite holds if $a^*(t, d) \in \mathcal{A}^*_-(t, d)$.
Player B's equilibrium action behaves symmetrically.

The first part of the proposition follows from the concavity of the probability of success in the player's own action.[12] The second part states that a player with an advantage in attacking starts with a relatively high action and decreases it as it approaches the endgame. This latter result might appear at first counterintuitive. The equilibrium behaviour of the optimal action is determined by the value of the game and, due to the concavity of the probability function, the two are inversely related. Since according to Lemma 5 and Lemma 1 the value of the game is increasing both in d for any given (t, d), and in t for players with advantage in attacking, the resulting equilibrium action is decreasing over t and d.

2.2 Unknown duration: random stopping time

Suppose now that the players do not know the exact duration of the game and at each time t assign a probability π_t that the game might continue to the next period. Equivalently let $1 - \pi_t$ be the probability that the game does not continue to $t + 1$. We assume that the continuation probability is strictly less than 1 (*i.e.,* players assume that at each t there is a strictly positive probability that the game might end at $t + 1$). Notice that π_t, being t dependent, needs not be constant and might be decreasing overtime.

Assumption 2. *The continuation probability is strictly less than 1,* i.e., *at any t, the probability that the game might stop is strictly positive*

$$\sup \pi_t < 1. \tag{15}$$

[12]The result is consistent with the empirical observation in the soccer context by Garicano and Palacio-Huerta (2014) where they observe that "[...] when a team is ahead it deploys a strategy aiming at conserving the score relative to the possibility of scoring more goals.

Since the stopping time random, the game is an infinite horizon stochastic zero-sum game (*e.g.,* see Parthasarathy and Raghavan (1971)). Let $W^A(t, d, \pi_t)$ ($W^B(t, d, \pi_t)$) denote the value of the game for player A (B) in state (t, d) with a continuation probability of π_t. Also, given d, let $W^A(d)$ be the value for player A when the game stops. As in the known duration case, when the game ends the value of the game for player B is $-W^A(d)$. We can then recursively write the value function as:

$$W^A(t-1, d, \pi_t) = \pi_t \max_a \{ \sum_{x=-1}^{1} p_x \left(a, \tilde{b}(t, d, \pi_t) \right) W(t, d+x, \pi_{t+1}) \} + (1-\pi_t) W^A(d); \qquad (16)$$

$$W^B(t-1, d, \pi_t) = \pi_t \max_b \{ \sum_{x=-1}^{1} p_x(\tilde{a}(t, d, \pi_t), b) W^B(t, d+x, \pi_{t+1}) \} - (1-\pi_t) W^A(d), \qquad (17)$$

where the values $\tilde{a}(t, d, \pi_t)$ and $\tilde{b}(t, d, \pi_t)$ are the equilibrium solutions to (16) and (17) (the tilde distinguishes them from the solutions to (9) and (10)). The following proposition characterizes the equilibrium actions.

Proposition 2. *1. Player A's equilibrium action decreases in d at any given t.*
2. The equilibrium actions are stationary and π_t*-independent, i.e.,* $\tilde{a}(t, d, \pi_t) = \tilde{a}(d)$ *and* $\tilde{b}(t, d, \pi_t) = \tilde{b}(d)$.
3. The equilibrium actions at any t *for the game with random stopping time equal the* $T-1$ *equilibrium strategies of the game with fixed stopping time,* i.e., $\tilde{a}(d) = a^*(T-1, d)$, *and* $\tilde{b}(d) = b^*(T-1, d)$.

An important implication of Proposition 2 is that the dynamics of the unknown duration game does not converge to the dynamics of the known duration game as π_t approaches 1. According to Proposition 2 when the duration of the game is stochastic, players employ a stationary strategy by acting at each point in time as if it was the last of the game. This holds regardless of the value of π_t. This result is central to the findings of the paper and suggests that removing the certainty about the duration of the game changes players' optimal behavior qualitatively and discontinuously: qualitatively because their behavior becomes stationary; discontinuously because it is independent of the level of uncertainty, *i.e.,* the stopping probability π_t. Mathematically, unlike the known duration case, when the duration is uncertain the contraction mapping theorem holds, and thus allows us to identify a stationary and t-independent solution.

Part 3 of the proposition follows from the fact that the unknown duration game includes the case where $\pi_t = 0$. Since for $\pi_t = 0$ the problem is equivalent to the $T-1$ problem in the known duration game, it follows that player's equilibrium actions in the unknown duration game are identical to the actions at $T-1$ in the known duration game. From part 3 of the proposition it also follows that all the equilibrium properties holding at $t = T-1$ for the known duration game, in particular Lemma 4, must also hold for all t in the unknown duration game.

2.2.1 Some intuition of the results so far

Before proceeding to the characterization of the probability of a successful action, let us provide an intuition that is central to the results in Proposition 1 and 2. This is based on well known properties of two dynamic processes. The first is a unit root process (in t and d), *e.g.,*

$$z_{t-1,d} = \sum_{x=-1}^{1} q_x z_{t,d+x}; \tag{18}$$

where $\sum_{x=-1}^{1} q_x = 1$ and the terminal condition is given by $z_{T,d} = d$. The second is a mixed process, *e.g.*,

$$\tilde{z}_{t-1,d} = \pi_t \sum_{x=-1}^{1} q_x \tilde{z}_{t,d+x} + (1-\pi_t)d. \tag{19}$$

By backward induction the unit root process (18) has a non-stationary solution for $q_1 \neq q_{-1}$ given by:

$$z^*_{t,d} = (q_1 - q_{-1})(T-t) + d, \tag{20}$$

whereas by contraction mapping (assuming $\sup \pi_t < 1$) the mixed process (19) has the stationary solution $\tilde{z}_d$ given by:

$$\tilde{z}^*_d = (q_1 - q_{-1}) + d. \tag{21}$$

The solutions $z^*_{t,d}$ and $\tilde{z}^*_d$ coincide at $t = T-1$ and if $q_1 = q_{-1}$ this holds for all t.

The link to the essential dynamics in the results so far is easy to identify. We note that $V(t,d)$ in (14) is a unit root process in t and d as in equation (18) with t-dependent $q_1 - q_{-1}$. Its sign depends on which agent has the relative advantage. Moreover, the value of the game at T plays the role of d in (18). Similarly, equation (16) describes a mixed process in t as in (19) with d given by the payoff $W(d)$. The stationarity or non-stationarity of the dynamics of the solutions to the processes (9) and (16) simply follows from the above observation.

2.3 Characterizing the probability of successful actions

The previous section has provided a characterization of players' equilibrium actions overtime and across goal differences. However in many instances, including our empirical analysis, only the outcomes of actions, not the actions themselves, are observable. These outcomes can, nevertheless be used to recover the probability of a successful action. The aim of this section is to show how our model can provide a characterization of these probabilities. We start by looking at the changes in $p_x(a,b)$ over t and across d. These can be computed as:

$$\frac{dp_1^*(t,d)}{dt} = \frac{dp_1(a^*(t,d))}{da^*(t,d)}\frac{da^*(t,d)}{dt}; \tag{22}$$

$$\frac{dp_1^*(t,d)}{dd} = \frac{dp_1(a^*(t,d))}{da^*(t,d)}\frac{da^*(t,d)}{dd}. \tag{23}$$

First, notice that the difference between the two derivatives in equations (22) and (23) lies solely in the change in the equilibrium action, as a function of t in equation (22) and as a function of d in equation (23). Moreover for unknown duration games, changes in t follow immediately from Proposition 2 independently of the functional form taken by $p_x(a,b)$:

Proposition 3. *In a game of unknown duration the probability of a successful action is stationary.*

Proposition 1, Proposition 3 together with equations (22) and (23) generate a set of implications on the probability of a successful action under alternative endgame rules which will form the basis of our hypothesis testing in the empirical analysis. Notice that they do not require further restrictions beyond Assumption 1.

Implication 1. *In games of known duration, $p_x(t,d)$ is non-stationary in both t and d.*

Implication 2. *In games of unknown duration:*

1. *$p_x(t,d)$ across d must have the same trajectory as in the known duration game;*

2. *$p_x(t,d)$ over t must have a different trajectory than in the known duration game.*

In order to fully characterize the probability of a successful action further restrictions on its functional form are necessary, as the same actions can deliver very different behavior for different forms of p_x. To this end, we study the following class of exponential wrapping log-convex functions, which given its simplicity is convenient to study the behavior of p_x :

$$p_1(a, b) = \exp(C_a a - f(a) + f(b)), \tag{24}$$

$$p_{-1}(a, b) = \exp(C_b b - f(b) + f(a)), \tag{25}$$

where C_a and $C_b \in \mathbb{R}_+,$ f is such that $0 \leq f' \leq \min\{C_b, C_a\}$ with $f'' > 0$ and $[f']^2 > f''$ so that Assumption 1 is satisfied. We also assume that the parameters are such that $p_1(a, b) + p_{-1}(a, b) < 1$ for any (a, b).[13]

Lemma 4 and equation (24) identify the relationship between the two players' equilibrium actions:

Lemma 6. *In equilibrium, the relationship between the two players' actions is given by:*

$$b(a^*) = [f']^{-1}\left[\frac{C_b}{C_a}[C_a - f'(a^*)]\right]. \tag{26}$$

Lemma 6 thus implies that we can let $p_x(a^*, b(a^*)) = p_x(a^*)$ and $p_x(a^*(t, d), b(a^*(t, d)) = p_x(a^*(t, d))$. The following lemma identifies two important values in the action space of player A. The first is the (t, d)-independent value a_+ that partitions the action space into two subsets over which player A has or does not have the relative advantage in attacking. With the log-convexity of p_x this implies that the set $\mathcal{A}^*_+(t, d)$ is connected and, being (t, d)-independent, can be written as $\mathcal{A}^*_+$ (similarly for the complementary set). The second interesting value, denoted by a_p, determines the turning point of $p_1(a^*(t, d))$. In particular, the following lemma studies the behavior of $p_1(a^*(t, d))$ over $a^*(t, d)$ and shows that when the probability of a successful action is non-monotonic then $p_1(a^*(t, d))$ is either U-shaped or inverted U-shaped, depending on the sign of f''' at the action level a_p:

Lemma 7. *1. There exists a unique action level a_+ such that $a^*(t, d) \in \mathcal{A}^*_+(t, d) = \mathcal{A}^*_+$ if and only if $a^*(t, d) < a_+$.*
2. There exists a unique turning point a_p such that $p_1(a^*(t, d))$ is locally concave (convex) at a_p if and only if $f'''(a) > 0$ ($f'''(a) < 0$).

We now have sufficient elements to characterize the behaviour of p_x over t and across d. Propositions 1 and 2 describe the terms $\frac{da^*(t,d)}{dt}$ and $\frac{da^*(t,d)}{dd}$ for both the known and unknown duration cases

[13] Alternatively, we could pre-multiply the two functions by a constant small enough to satisfy the inequality.

while Lemma 7.2 describes how p_x changes with the players' actions. We combine these results to obtain $\frac{dp_1^*(t,d)}{dt}$ (Section 2.3.1) and $\frac{dp_1^*(t,d)}{dd}$ (Section 2.3.2).

2.3.1 The probability of a successful action over t

For games of unknown duration the dynamics of the probability of a successful action is characterized by Proposition 3 and is independent of the functional form of $f(\cdot)$. For games of known duration with $p_x(\cdot)$ described by (24) and (25) the dynamics of the probability of a successful action depends on two factors. The first are the values a_+ and a_p that are determined by the relative magnitude of the parameters C_b and C_a. These values identify both the sets characterizing the behaviour of $a^*(t,d)$ overtime (the set $\mathcal{A}_+^*$ and $\mathcal{A}_-^*$) and of $p_1(a^*(t,d))$ across $a^*(t,d)$. The second factor is the sign of f''' that determines whether the function $p_1(a^*(t,d))$ is concave or convex at the action level a_p.

Proposition 4. *Suppose the duration of the game is known. 1. Let* $f''' > 0$*. Then, given* d*,* $p_1(a^*(t,d))$ *is inverted-U shaped over* t *on* $\mathcal{A}_+^*$ *if and only if* $C_b > C_a$ *or on* $\mathcal{A}_-^*$ *if and only if* $C_b < C_a$*.* $p_1(a^*(t,d))$ *is monotonically decreasing in the complementary sets.*
2. Let $f''' < 0$*. Then, given* d*,* $p_1(a^*(t,d))$ *is U-shaped over* t *on* $\mathcal{A}_+^*$ *if and only if* $C_b > C_a$ *or on* $\mathcal{A}_-^*$ *if and only if* $C_b < C_a$*.* $p_1(a^*(t,d))$ *is monotonically increasing in the complementary sets.*[14]

Figure 2 shows all possible configurations of $p_1(a^*(t,d))$ that can occur according to Proposition 4. The first, second and third rows present the case where $f''' > 0$, $f''' < 0$ and $f''' = 0$, respectively. The first, second and third columns present the case where $C_a < C_b$, $C_a > C_b$ and $C_a = C_b$, respectively. In all plots, the shaded area presents the set $\mathcal{A}_+^*$ and the arrows show the dynamics of $p_1(a^*(t,d))$. The arrows to the right of a_+ show the dynamics over time when the player has a relative advantage in defending, while arrows to the left of a_+ show the dynamics over time when the player has a relative advantage in attacking. Consider, for example, plot (a). This represents the case where $f''' > 0$, implying that $p_1(a^*(t,d))$ is inverted-U shaped with a turning point at a_p, and $C_a < C_b$, implying that $a_p < a_+$. That is, a_p is to the left of the turning point and thus belongs to the set of equilibrium actions $\mathcal{A}_+^*$. Recall that $a^*(t,d)$ decreases overtime in $\mathcal{A}_+^*$ and increases otherwise. Hence, when $a^*(t,d) \in [0; a_p)$ the action decreases overtime. Since $p_1(a^*(t,d))$ increases with the action, the total effect on $p_1(a^*(t,d))$ is to decrease overtime. When $a^*(t,d) \in (a_p; a+]$

[14] Since $p_{-1}(a,b) = p_1(b,a)$, it is possible to compute $p_{-1}(a^*(t,d))$ in a similar way.

the optimal action still decreases overtime but $p_1(a^*(t,d))$ decreases with $a^*(t,d)$. Hence, the total effect on $p_1(a^*(t,d))$ is to be increasing on this interval values. Finally, for $a^*(t,d) > a_+$ the optimal action $a^*(t,d)$ increases overtime and the equilibrium probability $p_1(a^*(t,d))$ decreases with the action resulting in a decreasing probability of a successful action. The same logic applies to the other plots.[15]

In the empirical part we compute the equilibrium probability of a successful action and show the data are consistent with the inverted-U shaped plots (a) and (b) in Figure 2.

2.3.2 The probability of a successful action across d

Given t, it is now relatively straightforward to show how $p_1(a^*(t,d))$ changes across d. Interestingly the behaviour is independent of the type of game, *i.e.,* known or unknown duration:

Proposition 5. *Irrespective of the rules governing the endgame, if* $f''' > 0$ *(*$f''' < 0$*) then* $p_1(a^*(t,d))$ *is inverted-U shaped (U shaped) across* d.

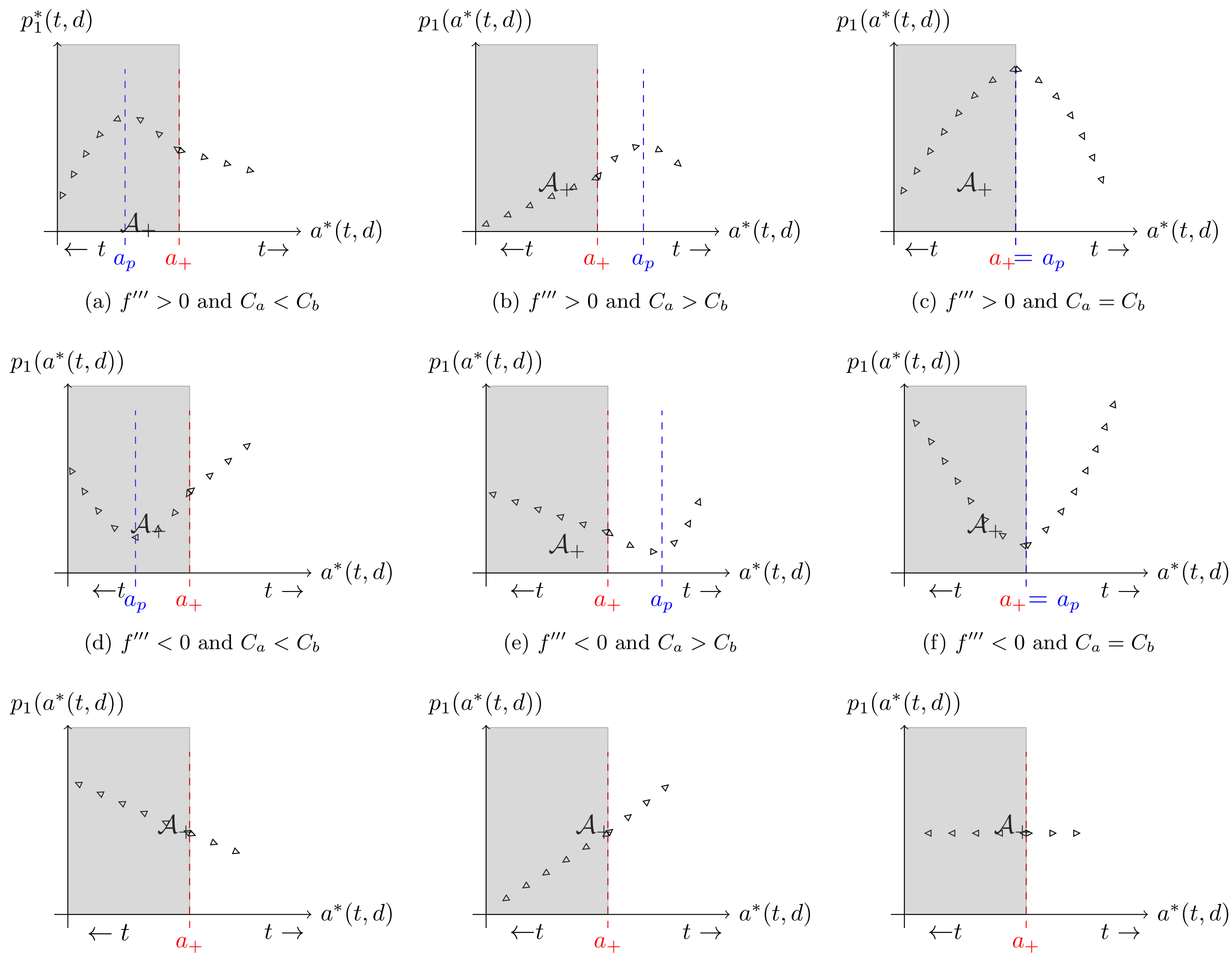
3 A natural experiment: the endgame in soccer

The soccer game represents a strategic conflict that is very close to the representation of the theoretical model. It is a relatively simple environment to study and the available data are reliable and less prone to errors than many other potential applications, like economic or political competitions or armed conflicts. Most importantly, a change in rule of the game that occurred in the 1998 season presents a unique opportunity for running a natural experiment to test the theoretical model.

To this end, we have collected data on primary league matches starting with the 1995-1996 season and ending with the 2003-2004 season for England, Germany, Ireland, Italy, Scotland and Spain. For each match, we recorded the total number of goals scored and how far into the game each goal was scored. The data for the analysis were compiled from individual game's box scores, largely obtained from Soccerbot.com, an online site reporting results and standings for soccer leagues around the world.

In the top division, the English, Italian, German and Spanish leagues have, on average, 20 teams while the Scottish league has an average of 10 teams. In all the leagues studied, each team plays on average 38 games per season, resulting in about 1,500 observations per season.

[15] In the third raw, as $f''' = 0$, there is no turning value a_p.



(a) $f''' > 0$ and $C_a < C_b$

(b) $f''' > 0$ and $C_a > C_b$

(c) $f''' > 0$ and $C_a = C_b$

(d) $f''' < 0$ and $C_a < C_b$

(e) $f''' < 0$ and $C_a > C_b$

(f) $f''' < 0$ and $C_a = C_b$

(g) $f''' = 0$ and $C_a < C_b$

(h) $f''' = 0$ and $C_a > C_b$

(i) $f''' = 0$ and $C_a = C_b$

Figure 2: Equilibrium Probability over Time

Unlike other sports, in soccer the game clock is always running and the referee does not pause it for fouls, injuries, penalty kicks, etc. In order to decrease the incentive the leading team might have to waste time so as to stay in the lead, FIFA instructs its referees to add at the end of each half the estimated amount of time lost. The duration of the added or stoppage time is at the sole discretion of the referee, and he/she alone decides when the match is officially over. Following the characterisation of the theoretical part, in a regular soccer match one can identify two sub-games. The first, analyzed in section 3.1, is composed of the regular time (hereafter RT), *i.e.,* the initial 90 minutes of play, and constitutes a known duration game. The second, analyzed in 3.2, is given by the added time (hereafter AT). This last part of the game can be identified as either of known duration or unknown duration depending on whether the referee is required to make the AT length known to the players at the end of the RT subgame. Specifically, the AT subgame was of unknown duration in all seasons prior 1998, as referees were not required to announce the duration of AT. The AT became a known duration subgame in all following seasons when FIFA required referees to announce the number of minutes they intend to add to the game.

In our dataset all goals during AT are recorded as scored at the 90th minute, with no information regarding the length of the AT.[16] Table 3 summarizes the number of matches observed, the average total number of goals scored during a game, and the average number of goals scored during AT over the 1995-2003 period.

3.1 The regular time subgame

The RT represents a game of known duration. All the results of the theoretical model with known duration should thus apply, though the value of the game at the last minutes of play might differ depending on the rule governing the AT. To see this, it suffices to replace (9) with value function:

$$V(89, d) = \max_a \Sigma_x p_x(a, b^*(89, d))V(90, d + x). \tag{27}$$

If the referee is required to communicate the AT duration at minute 89 of the RT then:

[16]Discussions with sports commentators suggest that during the period we study, it was possible to have some reporting differences where goals scored during AT would later be reported as scored at minute 89, if the goal was not scored in the last minute of AT. That is, if the game lasted $90 + z$ minutes, then all goals scored between the minute 90 and minute $90 + (z - 1)$ were reported as having been scored during minute 89. Thus for the empirical analysis, we take the first 88 minutes of each game as RT and the game from minute 90 onward as AT.

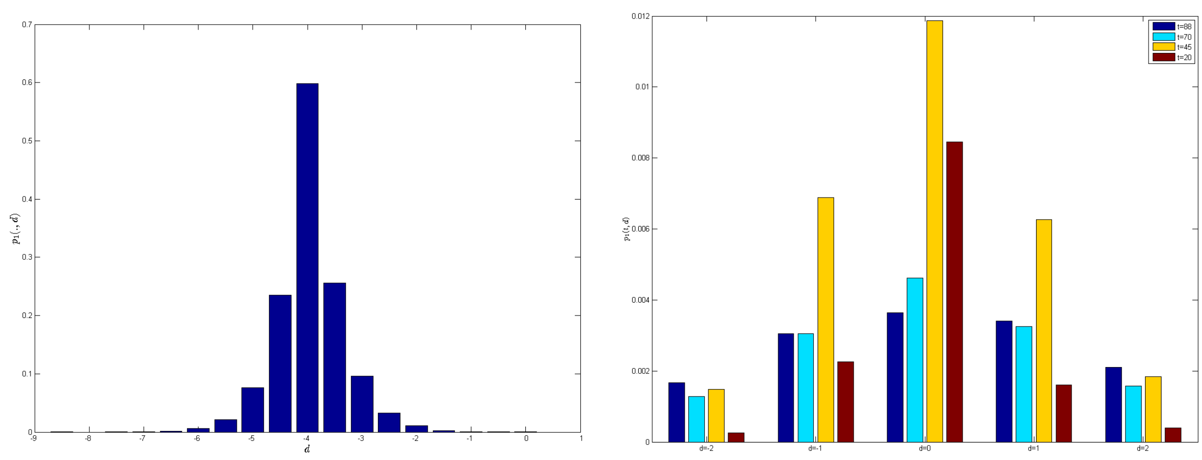
$$V(90, d) = \Sigma_{T=1}^{\infty} \pi_T^{AT} V_T(90, d) \tag{28}$$

The value function in equation (28) represents the expected continuation value where the expectations are taken over all possible durations of the AT that could be announced. The value π_T^{AT} denotes the probability that the referee might announce an AT of length T. Although π_T^{AT} depends on the game, in the empirical section we assume that π_T^{AT} is drawn from the same distribution so that in expectation these are the same across games.

If the referee is not required to communicate the AT duration at minute 89 of the RT then, by Proposition 2, at $t = 90$ the value function is constant and equals $W(d)$. Hence the equilibrium value of the game at minute 90 becomes:

$$V(90, d) = W(d).$$

Given these values, the RT game is solved by backward induction as in the theoretical part. Figure 3(a) plots the expected unconditional probability of scoring a goal across d. Figure 3(b) presents the probability of scoring a goal across d for t=20, 45, 70 and 88. Both figures clearly show the non-stationarity, specifically an inverted-U shape relationship, of the probability of scoring with respect to d.



(a) $p_1(., d)$ (b) $p_1(t, d)$, t=20,45,70 and 88

Figure 3: Distribution of estimated probability of scoring across d: (a) for all t and (b) at t=20, 45, 70, 88.

In order to study the unconditional probability of scoring with respect to both t and d, we estimate

	Specification I		Specification II		Specification III	
	coeff	z-stat	coeff	z-stat	coeff	z-stat
d	0.0542	2.46	0.00825	0.13		
d^2	-0.519	-27.97	-0.5206	-27.9		
d^3			0.01369	0.77		
t	0.053	9.91	0.0648	4.57	0.010032	5.42
t^2	-0.00037	-6.6	-0.00069	-1.95	-7.99E-05	-3.97
t^3			2.30E-06	0.9		
constant	-6.76	-61.4	-6.863	-43.45	-4.53649	-127.18
R^2	0.7		0.7		0.3997	
F	241.5***		161***		28.29***	
Turning point	71				63	

Table 1: Estimation results of the probability ratio for the regular time

the following model of scoring a goal as a function of time, t, and the goal difference, d :

$$\log(\hat{p_1}(t,d)/(1-\hat{p_1}(t,d)) = \beta_0 + \delta_1 d + \delta_2 d^2 + \beta_1 t + \beta_2 t^2 + \epsilon \qquad (29)$$

where:

$$\hat{p_1}(t,d) = \frac{\#\,(\text{a team scores a goal at } t \text{ when leading by } d)}{\#\text{matches}}.$$

Notice that whenever a goal is scored, the teams' relative position changes and the game moves to a new trajectory.

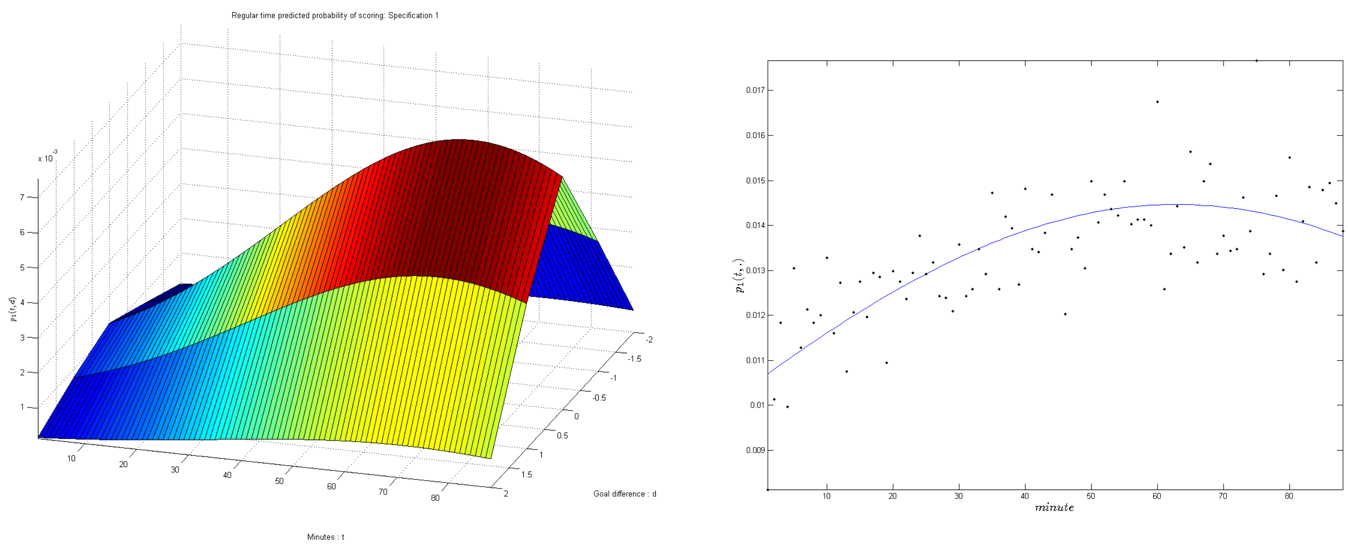
The first implication of the theoretical analysis can now be reformulated for model (29) as follows:

Hypothesis 1. *Non-stationarity in t and d: $\delta_i \neq 0,\ i = 1,2$ and $\beta_i \neq 0,\ i = 1,2$.*

Table 1 presents the estimation results of model (29) along with other specifications to test for robustness. In particular, we estimate the model with cubic terms to check for parsimony (Specification II), and estimate a simple probability of scoring over time without controlling for d (Specification I). The table shows that we fail to reject Hypothesis 1. Moreover, it shows that the probability of scoring has an inverted-U shape in t.

In general, there are three possible trajectories over time, namely: $\beta_1 < 0$ and $\beta_2 > 0$ which implies a U shaped curve; $\beta_1 > 0$ and $\beta_2 < 0$ which implies an inverted-U shaped curve and the case where both coefficients have the same sign. In both cases where the coefficients have opposite sign, the turning point in t occurs at $t_p = -\frac{\beta_1}{2\beta_2}$. If both coefficients have the same sign the function is monotonic, either increasing or decreasing. Since in our results the coefficient on t is positive and significant while the coefficient on t^2 is negative and significant, the table confirms that the probability of scoring has an inverted-U shape in t. The turning point for Specification I is at $t_p = 71$. Specification II in Table 1 rejects a cubic relationship, while Specification III shows that the inverted-U relationship remains even when we do not control for goal differences.

The behaviour across d is similar to the behavior over t with a turning point at $d_p = -\frac{\delta_1}{2\delta_2} = 0.52$, so between the values of 0 and 1.



(a) Specification I

(b) Specification III

Figure 4: (a) Dynamics of the probability of scoring: (a) Specification I and (b) Specification III

3.2 The added time subgame

Originally the duration of the AT was not made public as the game neared halftime or the end of the game, and referees would only blow the whistle to announce the end of each half. This changed in September 1998 when the International Football Association Board (IFAB) required FIFA referees to publicly announce at the end of minute 44 and minute 89 of play the time he/she intends the add to the first and second half, respectively.[17] Specifically, starting with the 1998-99 season, the referee must communicate the duration of the AT to the fourth official who in turn makes it common knowledge to players and spectators by holding up a board reporting the number

[17]See Rule 7 in the FIFA's "Laws of The Game," *http://www.fifa.com.*

of minutes to be added to the game.[18]

The intention of the change of rule was to make the game more exciting as well as to let players and spectators know that all of the time spent on injuries and other lost time are indeed added back to the game. For the purpose of our analysis, however, the most important effect of the new rule is that it turned the AT from a game of unknown duration, pre-98 seasons, to one of known duration, post-98 seasons.

Unlike the RT subgame where goals are reported minute by minute, especially during the early years, data on goals scored at AT report only the aggregate scoring; i.e., the total number of goals scored during the entire AT. Although this makes running a direct comparison of the probability of scoring pre and post-98 impossible, the data are still valuable for testing other important implications of our model. To this end we combine all matches played before the 1998-1999 season into one set–*Pre98*–and all matches starting from the1998-1999 season into the *Post98* set. We use these sets to empirically test Implication 1(a) and Implication 2(b).

3.2.1 The probability of scoring across score differences (d) pre and post-1998

According to Implication 2.1, the rule change should have no qualitative effect on the behaviour of the probability of scoring across d. In order to test this prediction, we estimate a logit model of the probability of scoring a goal during AT pre and post the rule change, conditional on d. We use the following logit model, analogous to the regular time model (29):

$$y_i = f\left(\gamma_0 + \gamma_{dum} dum98 + \gamma_1 d + \gamma_2 d^2\right) + \epsilon_i, \qquad (30)$$

where y_i equals 1 if a goal was scored during the AT in game i, and zero otherwise. The dummy variable $dum98$ equals 1 if game i is played after the 1997-98 season and 0 otherwise. The variables d and d^2 are the goal difference and square of the goal difference by which the team scoring the goal leads or lags.

Implication 2(a) can then be reformulated for the logit model in (30) as follows:

Hypothesis 2. *The sign on the coefficients* $\gamma_i,\ i = 1, 2$ *and* $\delta_i,\ i = 1, 2$ *must be the same.*

Table 2 presents the results of the estimation of equation (29). The results in the table show that

[18] Starting with the 2011 season the fourth official must keep a written record of the game's interruptions (see http://www.bbc.co.uk/sport/0/football/20159223).

the coefficients on γ_1 is positive and γ_2 is negative; the same signs as δ_1 and δ_2 in the estimation of equation (29). This implies that the model in (30) is an inverted-U shaped curve in d; confirming Hypothesis 2. Consistent with Proposition 2.1 and Implication 2, the results in specification II show that there is no difference between known and unknown duration games (*i.e.,* pre- and post-1998 seasons) with respect to the behaviour of the probability of scoring across d. This result is further reinforced by the shape of the histograms in Figure 3 and 5.

3.2.2 The probability of scoring over time pre and post-1998

The last testable implication of the model is that changing the information on the duration of the game changes the players' optimal strategies and in turn the dynamics of the probability of scoring (Implication 1). Hence, *ceteris paribus,* we should observe *the total number of goals scored during the AT in the pre-98 and in the post-98 seasons to be significantly different.* A first confirmation that this is the case is provided by the descriptive statistics in Table 3.

In terms of the estimation model in (30) this change in the dynamics of scoring due to a change of rule on the disclosure of the duration can be expressed as follows:

Hypothesis 3. *The coefficient on γ_{dum} is statistically significant.*

Indeed, in both specifications in Table 2, the z-stat value confirms that the coefficient on γ_{dum} is statistically significant. Thus we accept Hypothesis 3. Most interestingly, the disclosure of the game duration has led to a substantial increase of 30% in the of goals scored during AT, and a 28% in the probability of scoring from 0.101 to 0.128 (see Table 3).

Why an increase in goals in the post-98 seasons?

Table 3 reports the statistics for the average total number of goals throughout the game across seasons. It is interesting to notice that the communication of time left till the end of the game has not actually affected the average total number of goals. Rather, it has significantly reallocated goals from the regular time subgame toward the added time subgame. In this paper we will not try to address the reason for the scoring reallocation as this will be the objective of future work. However, the fact that the total number of goals did not increase over time has very important implications as it suggests that the increase in the number of goals scored during AT cannot be simply attributed to an increase in the average length of the AT (of which, moreover, there is no evidence) as this would have definitely affected the total number of goals scored in a game.

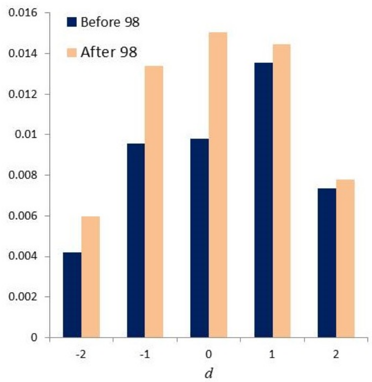


Figure 5: Probability of scoring during AT pre and post-1998

	Specification I		Specification II	
	coeff	z-stat	coeff	z-stat
d	0.0969	6.4	0.143	4.5
d^2	-0.0159	-2.87	-0.1845	-1.7
$dum98$	0.256	4.52	0.264	3.99
$d * dum98$			-0.06	-1.67
$d^2 * dum98$			0.0029	0.23
constant	-2.9	-56.1	-2.91	-50.3
Log likelihood	-6907.5		-6906.1	
Chi-square	69.2***		72.1***	

Table 2: Logit Estimation for AT

Season	Observations	Avg. Total Goals	Avg RT Goals	Avg AT Goals	Range
1995-1996	1319	2.73	2.65	0.081	0-11
1996-1997	1618	2.7	2.58	0.117	0-9
1997-1998	1349	2.7	2.6	0.099	0-10
Total pre-98	**4286**	**2.71**	**2.6**	**0.101**	
1998-1999	2402	2.55	2.43	0.121	0-10
1999-2000	2427	2.66	2.54	0.118	0-11
2000-2001	1597	2.76	2.63	0.135	0-9
2001-2002	1588	2.66	2.54	0.127	0-9
2002-2003	1594	2.67	2.53	0.149	0-9
2003-2004	1359	2.71	2.58	0.128	0-9
Total post-98	**10967**	**2.66**	**2.53**	**0.128**	
Total All	**15253**	**2.67**	**2.55**	**0.120**	

Table 3: Descriptive statistics

Having ruled out that possibility, the next question is whether an increase in the number of goals scored during AT in the post-98 seasons is consistent with our theoretical model. Let us first look at the scoring dynamics post-98 seasons. Although only aggregated (rather than minute-by-minute) data are available for the AT games the scoring dynamics of the AT in the post-98 season can still be inferred from the dynamics of the last minutes of the RT game. To clarify the point, notice that the value function in (28) is the expected value at $t = 90$ replaced by the actual value once the AT starts. While these expectations might over- or under-estimate the actual time added, these differences are "averaged out" by empirical estimation. This implies that, on average, players correctly anticipate the referee's announcement on the AT's length. Hence, in the case of the post-98 games, the AT is simply a continuation of the RT. Since the dynamics in the last minutes of the RT is decreasing, post-98 the dynamics in the AT must be decreasing.

Let us now look at the pre-98 season. Being an unknown duration game, Proposition 2.2 tells us that in the pre-98 seasons the probability of scoring a goal throughout the AT must be stationary. Furthermore, according to Proposition 2.3 this stationary probability must be equal to the probability of scoring during the last minute of the equivalent known duration game. That is, the probability of scoring a goal in the last minute of the AT should be the same for both the post-98 and pre-98 seasons. This together with the result that the probability of scoring during AT in the post-98 seasons is decreasing imply that, with the exception of the last minute of the AT, the probability of scoring a goal in the post-98 seasons must always be above the corresponding probability in the pre-98 seasons.

Figure 6 illustrates the argument for both increasing and decreasing dynamics. The plot on the left

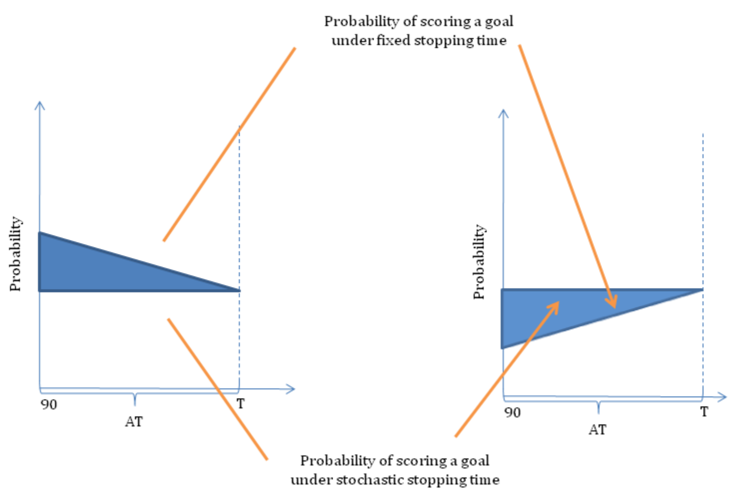


Figure 6: Possible Scoring Dynamics During AT

presents the case where the probability in the known duration game is decreasing while the right hand side plot presents the case in which it is increasing. The flat line in both cases represents the probability of scoring when the duration is stochastic. The results in the previous section indicate that the probability of scoring during the last minutes of the RT game is decreasing, implying that the probability of scoring during AT in the post-98 seasons must be consistent with the plot on the left-hand side and thus results in an increase in the number of goals scored during AT following the rule change.

4. Concluding remarks

This paper presents a theoretical analysis of a zero-sum game where the duration is either known or unknown to players. We test the model's predictions by exploiting the 1998 rule change imposed by FIFA mandating soccer referees to announce the duration of the added time. The 2008 U.S. presidential campaign -where the setting of a withdrawal date from Afghanistan was central to the debate- is evidence of policymakers' awareness of the potential implications of disclosing the duration of a conflict, both as a response to public opinion pressure and as a strategic commitment. A better understanding of these issues would also help in the optimal management of international peace-keeping missions, especially when considering the optimal allocation of troops across multiple fronts.

As a final note, in our theoretical model and in the empirical application, neither the game's duration nor the communication of the duration is part of the players' strategies, as both are taken to be exogenous. An interesting extension of the model would consider the case where agents can unilaterally fix the duration and hence decide whether to release this information or keep it private. Notice that having abstracted from this case does not detract from the interest of the analysis as in many situations the duration of the game is not part of the players actions' set. For example, in the case of an armed conflict or peace mission, budgetary and political considerations often determine the length of the involvement, which is then only communicated to the actors on the field.

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Appendix

For the following proofs, it will be convenient to let $p^*(t,d) = p(a^*(t,d), b^*(t,d))$.

Proof of Lemma 1: For $t = T$ this is simply given by (8). For $t = T-1$ and a given d the value of the game for player B is given by:

$$V^B(T-1,d) = \sum_{x=-1}^{1} p_x(T,d) V^B(T,d)$$
$$= \sum_{x=-1}^{1} p_x(T,d)(-V^A(T,d)) = -V^A(T-1,d).$$

The result for $t < T-2$ follows by backward induction. □

Proof of Lemma 2: Consider the following function of $(a,b) \in \mathbf{A} \times \mathbf{B}$:

$$U(a,b;\mathbf{V}) = \sum_{x=-1}^{1} p_x(a,b) V_x, \tag{A1}$$

where $\mathbf{V} = (V_x : x = -1, 0, 1 \text{ and } V_1 > V_0 > V_{-1})$ represents a vector of parameters. Let:

$$a^* = \arg\max_a U(a, b^*; \mathbf{V}) \tag{A2}$$

$$b^* = \arg\min_b U(a^*, b; \mathbf{V}). \tag{A3}$$

We will later interpret the solutions to the above problems a^* and b^* as $a^*(t,d)$ and $b^*(t,d)$, respectively.

A necessary condition for an equilibrium is that for any given b:

$$\partial_a U(a,b;\mathbf{V}) = 0. \tag{A4}$$

Similarly, for any given a:

$$\partial_b U(a,b;\mathbf{V}) = 0. \tag{A5}$$

For all $\mathbf{V}$ by Assumption 1.2:

$$\partial_a U(0,b;\mathbf{V}) = \partial_a p_1(0,b)(V_1 - V_0) - \partial_a p_{-1}(0,b)(V_0 - V_{-1}) = \partial_a p_1(0,b)(V_1 - V_0)$$
$$\partial_a U(1,b;\mathbf{V}) = \partial_a p_1(1,b)(V_1 - V_0) - \partial_a p_{-1}(1,b)(V_0 - V_{-1}) = -\partial_a p_{-1}(1,b)(V_0 - V_{-1})$$

Since $\partial_a^2 p_1 < 0$ we have $\partial_a p_1(0,b) > \partial_a p_1(1,b) = 0$. Also, since $\partial_a^2 p_{-1} > 0$ we have $\partial_a p_{-1}(1,b) > \partial_a p_{-1}(0,b) = 0$. Thus $\partial_a U(0,b;\mathbf{V}) > 0$ and $\partial_a U(1,b;\mathbf{V}) < 0$ hence by the intermediate value theorem for each given b there exists a value $a^*(b)$ such that $\partial U(a^*(b), b; \mathbf{V}) = 0$. A similar argument proves that for each given a there exists a value $b^*(a)$ such that $\partial U(a, b^*(a); \mathbf{V}) = 0$.

In order to show uniqueness, notice that by implicit function theorem, the slope of the reaction function is given by:

$$\frac{da^*(b)}{db} = -\frac{\partial^2 U(a,b;\mathbf{V})}{\partial a \partial b}\left(\frac{\partial^2 U(a,b;\mathbf{V})}{\partial a^2}\right)^{-1}.$$

Similarly, again by the implicit function theorem on the first order condition of the dual problem $\min_b U(a,b;\mathbf{V})$ for any given a obtain:

$$\frac{db^*(a)}{da} = -\frac{\partial^2 U(a,b;\mathbf{V})}{\partial a \partial b}\left(\frac{\partial^2 U(a,b;\mathbf{V})}{\partial b^2}\right)^{-1}.$$

Notice that by assumption $\frac{\partial^2 U(a,b;\mathbf{V})}{\partial a^2} < 0$ and $\frac{\partial^2 U(a,b;\mathbf{V})}{\partial b^2} > 0$ and since the numerators have the same signs, the slopes of the reaction functions $\frac{da^*(b)}{db}$ and $\frac{db^*(a)}{da}$ have opposite signs. Thus the value at which the reaction functions cross is unique. □

Proof of Lemma 3: The statement is true at $t = T$, *i.e.*, $V(T, d+1) \geq V(T, d)$ for any d. It follows that this will hold at $t = T - 1$ as well since:

$$\begin{aligned} V(T-1, d+1) &= \sum_{x=-1}^{1} p_x^*(a^*(t,d), b^*(t,d))V(T, d+1+x) \qquad \text{(A6)} \\ &\geq \sum_{x=-1}^{1} p_x^*(a^*(t,d), b^*(t,d))V(T, d+x) = V(T-1, d). \end{aligned}$$

The result for all $t < T - 1$ follows by backward induction. □

Proof of Lemma 4: From the first order conditions of (9) and (10), computing the derivatives at $a = a^*(t,d)$ and $b = b^*(t,d)$, and using Lemma 1 and the fact that $p_0 = 1 - p_1 - p_{-1}$ obtain:

$$\partial_a p_1(a,b)[V(t+1,d+1) - V(t+1,d)] = \partial_a p_{-1}(a,b)[V(t+1,d) - V(t+1,d-1)],$$

$$\partial_b p_1(a,b)[V(t+1,d+1) - V(t+1,d)] = \partial_b p_{-1}(a,b)[V(t+1,d) - V(t+1,d-1)].$$

Dividing the equations above and multiplying both sides by $\frac{p_{-1}}{p_1}$ gives the result. □

Proof of Lemma 5: From player A's maximization problem and using the fact that $p_0 = 1 - p_1 - p_{-1}$ obtain:

$$V(t-1,d) - V(t,d) = p_1^*(t,d)[V(t,d+1) - V(t,d)] - p_{-1}^*(t,d)[V(t,d) - V(t,d-1)].$$

Using the first order conditions of the problem at $t - 1$, *i.e.*,

$$\sum_x \partial_a p_x^*(t, d+x) V(t, d+x) = 0 \tag{A7}$$

obtain equation (14).

Since $V(t,d) > V(t,d-1)$ by Lemma 3 and since $a^*(t,d) \in \mathcal{A}_+^*(t,d)$ it follows that:

$$p_{1,t}^*(t,d) \frac{\partial_a p_{-1,t}^*(t,d)}{\partial_a p_{1,t}^*(t,d)} - p_{-1,t}^*(t,d) < 0,$$

that implies

$$V(t,d) > V(t-1,d).$$

□

Proof of Proposition 1: Consider the first order condition of problem (A1):

$$\frac{\partial U(a, b^*; \mathbf{V})}{\partial a} = \sum_{x=-1}^{1} \frac{\partial p_x(a, b^*)}{\partial a} V_x = 0. \tag{A8}$$

Letting $U(a^*, b^*; \mathbf{V}) = \max_a U(a, b^*; \mathbf{V})$ then by the envelope theorem it follows that:

$$dU(a^*, b^*; \mathbf{V}) = \sum_{x=-1}^{1} p_x(a^*, b^*) \partial V_x.$$

Holding the value of $U(a^*, b^*; \mathbf{V})$ constant, *i.e.*, $dU(a^*, b^*; \mathbf{V}) = 0$ as well as $\partial V_{-1} = 0$, obtain:

$$\frac{\partial V_1}{\partial V_0} = -\frac{p_0(a^*, b^*)}{p_1(a^*, b^*)}.$$

Similarly holding $\partial V_1 = 0$, obtain:

$$\frac{\partial V_{-1}}{\partial V_0} = -\frac{p_0(a^*, b^*)}{p_{-1}(a^*, b^*)}.$$

Using the implicit function theorem on the first order condition it follows that:

$$\frac{da^*}{dV_0} = -\frac{\partial^2 U(a, b^*; \mathbf{V})}{\partial a \partial V_0} \left(\frac{\partial^2 U(a, b^*; \mathbf{V})}{\partial a^2} \right)^{-1}.$$

Since:

$$\begin{aligned}
\frac{\partial^2 U(a, b^*; \mathbf{V})}{\partial a \partial V_0} &= \partial_a p_1^* \frac{\partial V_1}{\partial V_0} + \partial_a p_{-1}^* \frac{\partial V_{-1}}{\partial V_0} + \partial_a p_0^* \frac{\partial V_0}{\partial V_0} \\
&= -\partial_a p_1^* \frac{p_0^*}{p_1^*} - \partial_a p_{-1}^* \frac{p_0^*}{p_{-1}^*} - \partial_a p_1^* - \partial_a p_{-1}^* < 0,
\end{aligned}$$

where the inequality follows by Assumption 1.1. Since by Assumption 1.3 U is concave, $\frac{\partial^2 U(a^*(\mathbf{V}),b^*(\mathbf{V}))}{\partial a^2} < 0$, then it follows that $\frac{da^*(\mathbf{V})}{dV_0} < 0$.

A similar argument shows that: $\frac{db^*(\mathbf{V})}{dV_0} > 0$.

The above static formulation of the problem is convenient as now we can prove the two parts of the proposition by appropriately reinterpreting the values V_x. It now suffices to interpret $V_x = V(t, d+x)$ for any (t, d) and $x = -1, 0, 1$ to obtain the results.

1. Let V_0 and V_0' such that $V_0 = V(t+1, d+1)$ and $V_0' = V(t+1, d)$. By Lemma 3, $V_0 = V(t+1, d+1) > V(t+1, d) = V_0'$ it follows that:

$$a^*(t, d+1) < a^*(t, d) \text{ and } b^*(t, d+1) > b^*(t, d).$$

2. Let now V_0 and V_0' such that $V_0 = V(t+1, d)$ and $V_0' = V(t, d)$. By Lemma 5, if $a^* \in \mathcal{A}_+(t, d)$, $V(t+1, d) > V(t, d)$. It follows that:

$$a^*(t, d) < a^*(t-1, d) \text{ and } b^*(t, d) > b^*(t-1, d).$$

The symmetric behaviour of player's B action follows from Lemma 1. □

Proof of Proposition 2: 1. Consider the following function of $(a, b) \in \mathbf{A} \times \mathbf{B}$:

$$U(a, b; \mathbf{W}) = \pi \sum_{x=-1}^{1} p_x(a, b) W_x + (1-\pi) W_0, \tag{A9}$$

where $\mathbf{W} = (W_x : x = -1, 0, 1 \text{ and } W_1 > W_0 > W_{-1})$ represents a vector of parameters. Notice that we can set $W_x = W(t, d+x)$ for some (t, d) and $x = -1, 0, 1$. Let:

$$a^* = \arg\max_a U(a, b^*; \mathbf{W}) \tag{A10}$$

$$b^* = \arg\min_b U(a^*, b; \mathbf{W}). \tag{A11}$$

The remaining part of the proof proceeds along the lines of the proof of Proposition 1.1[19].

2. Consider the functional $\Phi : \Upsilon \to \Upsilon$ where Υ is the space of bounded functions such that:

$$\Phi(W(t-1, d, \pi_t)) = \sum_{x=-1}^{1} \pi_t p_x(\tilde{a}(t, d, \pi_t), \tilde{b}(t, d, \pi_t)) W(t, d+x, \pi_{t+1}) + (1-\pi_t) W(d).$$

The proof consists in showing that Φ is a contraction mapping. Let $W(t-1, d, \pi_t)$ and $W'(t-1, d, \pi_t)$

[19] The full arguments is available upon request.

two functions in Υ. It follows that:

$$\Phi(W(t-1,d,\pi_t)) - \Phi(W'(t-1,d,\pi_t)) =$$
$$\pi_t \left[\sum_{x=-1}^{1} p_x(\tilde{a}(t,d,\pi_t),\tilde{b}(t,d,\pi_t))(W(t,d+x,\pi_{t+1}) - W'(t,d+x,\pi_{t+1}))\right].$$

Since $\sup_t \pi_t < 1$, we can choose a $\delta < 1$ such that $\sup_t \pi_t < \delta < 1$. Then:

$$\| \Phi(W(t-1,d,\pi_t)) - \Phi(W'(t-1,d,\pi_t)) \| \le \delta \| W(t-1,d,\pi_t) - W'(t-1,d,\pi_t) \| .$$

By the contraction mapping theorem it follows that $W(t-1,d,\pi_t) = W(d)$ and hence that $\tilde{a}(t,d,\pi_t) = \tilde{a}(d)$. A similar argument holds for the player B's optimal strategy $\tilde{b}_t(d,\pi_t)$.

3. Consider problem (16) at time $T-1$:

$$\begin{aligned}
\tilde{a}(d) &= \arg\max_a \{ \sum_{x=-1}^{1} \pi_T p_x\left(a,\tilde{b}(d)\right) W(T,d+x) + (1-\pi_T) W(T,d) \} \\
&= \arg\max_a \left\{ \begin{array}{c} \pi_T p_1\left(a,\tilde{b}(d)\right)[W(T,d+1) - W(T,d)] \\ -\pi_T p_{-1}\left(a,\tilde{b}(d)\right)[W(T,d) - W(T,d-1)] + W(T,d) \end{array} \right\} \\
&= \arg\max_a \left\{ \begin{array}{c} p_1\left(a,\tilde{b}(d)\right)[W(T,d+1) - W(T,d)] \\ -p_{-1}\left(a,\tilde{b}(d)\right)[W(T,d) - W(T,d-1)] \end{array} \right\}.
\end{aligned}$$

Notice that the above problem is the same as problem (9) at $T-1$. It suffices to replace $W(T,d+x)$, $x = -1,0,1$ with $V(T,d+x)$, $x = -1,0,1$ and impose $\tilde{b}(d) = b_T^*(t,d)$ *i.e.*,

$$\begin{aligned}
\tilde{a}(d) &= \arg\max_a \left\{ \begin{array}{c} p_1\left(a,\tilde{b}(d)\right)[V(T,d+1) - V(T,d)] \\ -p_{-1}\left(a,\tilde{b}(d)\right)[V(T,d) - V(T,d-1)] \end{array} \right\} \\
&= \arg\max_a \sum_{x=-1}^{1} p_x(a,b_T^*(t,d)) V_T^A(T,d+x) \\
&= a^*(T-1,d).
\end{aligned}$$

The proof for the optimal strategy of player B is similar. □

Proof of Proposition 3 : This simply follows from Proposition 2.2. □

Proof of Lemma 6: From the equilibrium condition (13) and the functional forms specified in

(24) and (25) obtain:

$$\frac{f'(a^*)\,p_{-1}(a^*,b^*)}{[C_a - f'(a^*)]\,p_1(a^*,b)} = \frac{[C_b - f'(b(a^*))]\,p_{-1}(a^*,b^*)}{f'(b^*(a^*))\,p_1(a^*,b^*)},$$

or:

$$f'(b) = \frac{C_b}{C_a}\left[C_a - f'(a^*))\right], \tag{A12}$$

and hence equation (26). □

Proof of Lemma 7: 1. Denoting:

$$G(a) = \frac{\partial_a p_1(a,b)}{p_1(a,b)} - \frac{\partial_a p_{-1}(a,b)}{p_{-1}(a,b)},$$

from (24) and (25) obtain:

$$G(a) = \frac{[C_a - f'(a)]\,p_1(a,b(a))}{p_1(a,b(a))} - \frac{f'(a)\,p_{-1}(a,b)}{p_{-1}(a,b)} = C_a - 2f'(a). \tag{A13}$$

Let a_+ such $G(a_+) = 0$ then, being $f' > 0$, $a \in \mathcal{A}_+(b)$ if $a < a_+$ where:

$$a_+ = \left[f'\right]^{-1}(C_a/2). \tag{A14}$$

Note that a_+ is unique since:

$$\frac{dG(a)}{da} = -2f''(a) < 0, \text{ for all } a.$$

2. Changes in $p_1(a,b)$ with respect to action a are obtained by solving:

$$\frac{dp_1(a,b)}{da} = \partial_a p_1(a,b) + \partial_b p_1(a,b)\frac{db}{da}. \tag{A15}$$

Let $\left.\frac{dp_1(a,b^*(a))}{da}\right|_{a=a^*(t,d)} = dp_1(a^*)$. Being $\partial_a p_1(a,b) = [C_a - f'(a)]p_1(a,b)$ and $\partial_b p_1(a,b) = f'(b)p_1(a,b)$, imposing the equilibrium condition in (A12) it is straightforward to show that:

$$dp_1^*(a) = \left(C_a - f'(a^*)\right)\left(1 + \frac{C_b}{C_a}b'(a^*)\right)p_1^*(a,b(a)). \tag{A16}$$

Since $C_a > f'(a)$ then at equilibrium $\frac{dp_1(a^*,b(a^*))}{da} = 0$ if:

$$b'(a) = -\frac{C_b}{C_a}. \tag{A17}$$

Computing the first derivative of $b(a^*)$ in (26) and rearranging obtain:

$$b'(a^*) = -\frac{C_b}{C_a}\frac{f''(a^*)}{f''(b(a^*))}. \tag{A18}$$

that with equation (A17) and (A18) implicitly defines a_p as

$$\frac{C_b}{C_a}\frac{f''(a^*)}{f''(b(a^*))} = \frac{C_b}{C_a} \tag{A19}$$

Let us now show that the function is locally concave at a_p. Taking the second derivative of $p_1^*(a)$ obtain:

$$\begin{aligned}
d^2p_1^*(a) &= d\left[\left(C_a - f'(a)\right)\left(1 + \frac{C_b}{C_a}b'(a)\right)p_1^*(a)\right] \\
&= -f''(a)\left(1 + \frac{C_b}{C_a}b'(a)\right)p_1^*(a) + \frac{C_b}{C_a}b''(a)\left(C_a - f'(a)\right)p_1^*(a) \\
&+ \left(C_a - f'(a)\right)\left(1 + \frac{C_b}{C_a}b'(a)\right)dp_1^*(a).
\end{aligned}$$

$$\begin{aligned}
&= -f''(a)\left(1 + \frac{C_b}{C_a}b'(a)\right)p_1^*(a) + b''(a)\left(C_a - f'(a)\right)p_1^*(a) \\
&+ \left(C_a - f'(a)\right)^2\left(1 + \frac{C_b}{C_a}b'(a)\right)^2 p_1^*(a).
\end{aligned}$$

Since from equation (A17) $1 + \frac{C_b}{C_a}b'(a_p) = 0$ it follows that:

$$d^2p_1^*(a_p) = b''(a_p)\left(C_a - f'(a_p)\right)p_1^*(a).$$

The latter is negative if and only if $b''(a_p) < 0$. Computing $b''(a_p)$ obtain:

$$\begin{aligned}
b''(a_p) &= -\frac{C_b}{C_a}\frac{f'''(a_p)f''(b(a_p)) - b'(a_p)f'''(b(a_p))f''(a)}{\left[f''(b(a_p))\right]^2} \\
&= -\frac{C_b}{C_a}\frac{f'''(a_p)\left[f''(b(a_p))\right]^2 + f'''(b(a_p))\left[f''(a_p)\right]^2}{\left[f''(b(a_p))\right]^3}.
\end{aligned}$$

It follows that $b''(a_p) < 0$ if and only $f''' > 0$ that is the necessary and sufficient condition for the local concavity of the equilibrium $p_1(a^*(t,d))$ at a_p. □

Proof of Proposition 4: We first show that $a_p \in \mathcal{A}^*_-$ if and only if $C_b > C_a$. Notice that at equilibrium from (A19):

$$[\frac{C_b}{C_a}]^2 f''(a_p) = f''(b\,(a_p))$$

If $\frac{C_b}{C_a} < 1$ then $f''(a_p) < f''(b\,(a_p))$ or $a_p < b\,(a_p)$ since $f'' > 0$. If $\frac{C_b}{C_a} > 1$ the reverse holds. Computing the first order condition of the agent's problem obtain:

$$f'\,(b\,(a_p)) - f'\,(a_p) = \frac{C_b}{C_a}\left[C_a - f'\,(a_p)\right] - f'\,(a_p)$$

since being $f'' > 0$:

$$0 > f'\,(b\,(a_p)) - f'\,(a_p) = \frac{C_b}{C_a}\left[C_a - f'\,(a_p)\right] - f'\,(a_p)$$
$$0 > \frac{C_b}{C_a}\left[C_a - f'\,(a_p)\right] - \frac{C_b}{C_a} f'\,(a_p) > \left[C_a - 2f'\,(a_p)\right]$$

therefore $a_p \in \mathcal{A}^*_-(b)$.

So if $C_b < C_a$ then $a_p \in \mathcal{A}^*_-$. From Lemma 7.2, $\frac{dp_1(a,b)}{da} < 0$ for $a < a_p$ and $\frac{dp_1(a,b)}{da} > 0$ for $a > a_p$. Since $\frac{da(t,d)}{dt} > 0$ for $a \in \mathcal{A}^*_-$ it follows that $\frac{dp_1(a,b)}{dt} < 0$ if $a < a_p$ and $\frac{dp_1(a,b)}{dt} > 0$ if $a > a_p$. On the complementary set $\mathcal{A}^*_+$ the function is monotonically decreasing overtime as $\frac{dp_1(a,b)}{da} > 0$ but $\frac{da(t,d)}{dt} < 0$.

The case $C_b > C_a$ as well as part 2 of the proposition can be proved in a similar way. □

Proof of Proposition 5: For a game of known duration this follows from Proposition 1 and Lemma 7.2 and for a game with random stopping from Proposition 2.1 □