Dynamic price competition in aftermarkets with network effects

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Abstract

This paper studies dynamic price competition between two firms that sell horizontally differentiated durable goods and, subsequently, provide complementary goods and services to their consumer bases. The paper analyzes how optimal pricing strategies are affected by the existence of network effects based on the size of consumers base for the durable good. The interaction is thoroughly analyzed as a continuous time linear quadratic differential game. We provide a necessary and sufficient condition for the existence of a unique duopoly equilibrium in linear strategies. When this condition holds, we show that optimal pricing strategies crucially depend on the nature of network effects.

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1 Introduction

The value of a durable good / equipment often depends on the subsequent consumption of complementary goods and services (CGS) sold in aftermarkets, e.g. smartphones/ tablets and applications; cell phone network subscriptions and phone calls, coffee machines and coffee capsules, printers and ink cartridges, cars and maintenance/repairing services,...

Shapiro (1995) has identified three key features of aftermarkets: *(i)* goods/ services traded in aftermarkets aim at complementing a durable good, whose value considerably depends on CGS consumption; *(ii)* equipment choices precede consumers' purchases of CGS; and *(iii)* consumers are significantly "locked-in" to their equipment, meaning that the switch from one equipment brand to another is highly unlikely.

In some aftermarkets the value of CGS depends on the number of consumers owning a similar variant of the durable good, generating *"network effects"*.[1] For example, the number and the quality of the applications made available to a certain smartphone/ tablet model depend on the number of individuals owning such a model. Similarly, the net utility of making phone calls depends on the number of potential contacts within the same cell phone network. The present paper considers the possibility of direct and/or indirect network effects.[2] Direct network effects take place when the value of each equipment is increasing in the size of the consumer base of the corresponding producer. Indirect network effects take place when the consumer base for the durable good subsequently determines the value of the CGS provided in the corresponding segment of the aftermarket. For example, the enlargement of the base of iPad users might increase the value of iPad itself (due to a conspicuous consumption effect, for example). This generates a direct network effect in the primary market. However, such enlargement may also increase the number and quality of applications available for the iPad in the aftermarket, creating indirect network effects.

The paper analyzes a theoretical model of dynamic price competition between two proprietary networks with an aftermarket. Firms produce horizontally differentiated equipment (in the primary market) and they also supply CGS to locked-in consumers (in the aftermarket). Consumers are forward-looking. When arriving in the primary market, they choose the equipment that maximizes their expected lifetime utility.

The interaction between firms is analyzed as a continuous time differential game with linear transitions and quadratic payoffs, in addition to linear expectation rules for the

[1] Network effects take place when consumers valuation of the good depend not only on its intrinsic characteristics but also on the consumption choices made by other consumers in the market (see the seminal papers by Rohlfs (1974), Katz and Shapiro (1985), Grilo *et al.* (2001); or, more recently, Amir and Lazzati (2011) and Griva and Vettas (2011)).

[2] According to Belleflamme and Peitz (2011), network effects are direct when the more agents are present on a network, the larger are the communication opportunities, and the greater are the incentives for other agents to join this network. Indirect network effects arise when the positive effect of an increase in the number of users is mediated by the availability of applications.

forward-looking aspect of the (Markov-perfect) equilibrium under consideration. [3] We provide a necessary and sufficient condition for the existence of a unique Linear Markov Perfect Equilibrium (LMPE)in which both firms have non-negative market shares. When this condition holds, instantaneous equilibrium prices in the primary market depend on the degree of differentiation between the durable goods; the profitability of the corresponding aftermarket; and the relative size of firms' consumer bases in the primary market.

Regarding the last aspect, we conclude that the equilibrium trajectories of the equipment prices may be increasing or decreasing in the size of firms' base of consumers in the primary market. In particular, the shape of these trajectories depends on the intensity of direct network effects vis-à-vis the intensity of indirect network effects. Indirect network effects induce increasing marginal returns of total profits with respect to firms' market shares in the primary market. In light of this effect, equilibrium pricing strategies would be decreasing in firms' consumer base. Conversely, direct network effects increase the attractiveness of the equipment with the larger base of consumers. Therefore, in light of this effect, equilibrium pricing strategies would be increasing in firms' consumer base. Considering the two effects, we obtain that prices are decreasing (increasing) in firms' market shares in the primary market, when indirect (direct) network effects are dominant.

In the LMPE steady state, we find that firms share the market evenly. The speed of convergence never exceeds the rate of consumers' exit/entry and the LMPE steady state price is always bellow the Hotelling benchmark price. Concerning equilibrium trajectories, we find that industry profits and new customers' average expected life time utility are decreasing with time. The average equipment price is increasing (resp. decreasing) with time when the intensity of indirect network effects is sufficiently stronger (weaker) than the intensity of direct network effects.

This paper relates to two strands of literature. First, it contributes to the recent literature dealing with dynamic price competition in network industries. Following the seminal papers by Rohlfs (1974) and Katz and Shapiro (1985), a considerable number of works have been devoted to understand static price competition in network industries. More recently, some work has been devoted to understand dynamic price competition in network industries.[4] This recent literature includes both computational models (see Markovich (2008), Markovich and Moenius (2009) and Chen *et al.* (2009)) and more analytical studies (see, for example, Doganoglu (2003), Laussel *et al.* (2004), Mitchell and Skrzypacz (2006), Driskill (2007)[5] and Cabral (2011)[6]). Our paper also follows an

[3] A book treatment account of the theory and applications of dynamic games in continuous time is Dockner *et al.* (2000). For a broad survey of applications of the discrete time case, see Amir (1996, 2003).

[4] See Garcia and Resende (2011) for a more detailed survey about this literature.

[5] Driskill (2007) focuses on the oligopoly supply of a homogeneous good and therefore, differently from our model, in his model all firms quote a similar price, regardless of the size of their networks.

[6] To be more precise, Cabral (2011) combines analytical and computational techniques.

analytical approach.

Doganoglu (2003) and Mitchell and Skrzypacz (2006) study dynamic price competition in markets with delayed network effects. Differently, we deal with forward-looking agents.

Laussel *et al.* (2004) study dynamic price competition between two horizontally differentiated "profit maximizing clubs" with congestion effects. Focusing on LMPE, they find that access prices are always decreasing with the size of clubs' memberships (due to the negative network effects). This is the closest paper to our work. However, Laussel *et al.* (2004), like most of the papers on price or quantity competition in network industries, focuses on consumers' preferences with additively separable network effects. Hence, they are not able to fully capture neither the demand-side increasing returns produced by network effects (as the ones arising in this paper[7]) nor the viability problem. A remarkable exception is the recent work by Amir and Lazzati (2011) who analyze general demand functions with non-separable network effects in a static setting.

Cabral (2011) also considers dynamic competition between two proprietary networks. He proposes a discrete-time model with idiosyncratic preferences that generate stochastic dynamics, whereas we concentrate on a deterministic continuous time model. While Cabral's model is admittedly more general than ours, his "theoretical results apply only to extreme values of key parameters" , as he puts himself (3.1., §1,page 97). By contrast, our approach, though less general, provides analytical results which are valid outside "restricted sets of parameter values"[8]

This paper is also related to the literature on aftermarkets (see, for example, Borentstein *et al.* (2000), Shapiro (1995), Morita and Waldman (2004) or Chen and Ross (1993, 1998)). We add to this literature by investigating the interplay between primary markets and the aftermarket when network effects take place.

The rest of the paper is organized as follows. Section 2 presents the basic ingredients of the model. Section 3 describes the LMPE of the dynamic game. The strategic implications of firms' equilibrium behavior are studied in Section 4. Finally, Section 5 concludes.

2 The model

This section describes a model of dynamic price competition between two firms that sell horizontally differentiated goods (in the primary market) and provide CGS to their base

[7]In the context of our model, the demand-side increasing returns are crucial when indirect network effects are sufficiently stronger than direct network effects.

[8]Page 93, §3, line 4. For instance, both papers show that in equilibrium larger firms might charge either higher or lower prices than the smaller firm. But ours provides necessary and sufficient conditions for each case to occur in terms of the relative importance of direct and indirect network effects, while Cabral's gives only very restrictive sufficient conditions. We also establish that "weak market dominance" holds generally in any unique LMPE and not only for low values of the discount factor.

of consumers (in the aftermarket). At each instant of time, there is an inflow of new consumers (arriving at rate μ) and an identitical outflow of old consumers. There are network effects associated with firms' consumer base for the durable good. With this respect, the model encompasses the possibility of direct and/or indirect network effects. Direct network effects take place when the value of each equipment is increasing in the size of the consumer base of the corresponding producer. Indirect network effects take place when the consumer base for the durable good subsequently determines the value of the CGS provided in the corresponding segment of the aftermarket. In what follows, we provide a detailed description of the main ingredients of the model.

In the primary market firms produce infinitely-lived and horizontally differentiated durable goods (Equipment 1 and Equipment 2) at zero marginal cost. Horizontal differentiation is modelled à la Hotelling with quadratic transportation costs (see d' Aspremont et al., 1979). The equipment are located at the opposite extremes of the Hotelling line $[0,1]$: Equipment 1 is located at point $x_1 = 0$, while Equipment 2 is located at point $x_2 = 1$. Potential equipment buyers are uniformly distributed on the Hotelling line. Switching costs[9] (learning costs, habit formation,...) keep consumers locked-in to one supplier after their initial purchase of the durable good.

The expected lifetime utility obtained by a consumer located at $x \in [0,1]$ in the Hotelling line who buys durable good i at instant t, is denoted by $V_i(x,t)$, $i = 1,2$, equal to:

$$\int_t^{\infty} \left[\vartheta - \tau (x - x_i)^2 + \omega D_i(t)\right] e^{-(r+\mu)(s-t)} ds + \int_t^{\infty} u_i^e(s) e^{-(r+\mu)(s-t)} ds - p_i(t). \qquad (1)$$

The term $\vartheta - \tau (x - x_i)^2 + \omega D_i(t)$ represents the instantaneous intrinsic gross utility[10] delivered by equipment i. $\vartheta > 0$ stands for the intrinsic utility of the "ideal" equipment variant; τ is the Hotelling unit transportation cost; and $D_i(t)$ is firm i's market share in the primary market at instant t, corresponding to the mass of consumers that are locked-in to this equipment at instant t. The term $\omega D_i(t)$ corresponds to the benefit obtained from direct network effects arising in the primary market (e.g. the benefits derived from conspicuous consumption or communication benefits). The constant ϑ is considered to be large enough so that all newborn consumers are able to find a durable good for which $V_i(x,t)$ is positive at equilibrium, leading to full market coverage in the primary market. The parameter r stands for the conventional discount rate and μ represents the instantaneous probability of exiting the market. The term $u_i^e(s)$ corresponds to the expected lifetime utility obtained from CGS consumption at instant $s > t$, while $p_i(t)$ denotes the price of

[9] To learn more on problems related to switching costs see, for example, Klemperer (1987) or Farrell and Shapiro (1988).

[10] By intrinsic utiliy, we mean the equipment value that does not depend on the consumption of CGS.

durable good i at instant t.

It is worth noting that, given the deterministic structure of our model, the expected utility $u_i^e(s)$ does not refer to the usual probabilistic expectation. Due to network effects, agents need to anticipate firms' future market shares in the primary market. As we assume forward-looking agents,both firms and consumers correctly anticipate future market outcomes and agents' beliefs about future market shares are confirmed in equilibrium, i.e., $u_i^e(s) = u_i(s)$, since $D_i^e(s) = D_i(s)$, where $D_i^e(s)$ stands for the expected market share[11] of firm i at instant s.

In the aftermarket, the two firms provide perishable CGS[12]. Consumers are assumed to be fully locked-in to their equipment suppliers, as each firm is the exclusive provider of the CGS for its equipment.[13] As a result, the aftermarket can be decomposed in two mutually exclusive monopoly markets.

In segment i of the aftermarket, the instantaneous utility derived from CGS consumption by a consumer who is locked-in to equipment i is given by:

$$u_i(t) = [\gamma + \phi D_i(t)] [k_i(t) - \tfrac{1}{2}k_i^2(t)] - \rho_i(t)k_i(t), \tag{2}$$

where the term $\gamma[k_i(t) - \frac{1}{2}k_i^2(t)]$ corresponds to the "stand alone value of CGS"[14]. $\gamma > 0$ is a parameter that measures the marginal utility of CGS consumption (in the absence of network effects), $k_i(t)$ denotes the instantanous level of consumption of CGS_i. The term $\phi D_i(t)[k_i(t) - \frac{1}{2}k_i^2(t)]$ corresponds to the network benefit associated with CGS consumption: the value of CGS_i is an increasing function of the base of consumers locked-in to equipment i. The intensity of these indirect network effect is measured by the parameter $\phi > 0$. Finally, $\rho_i(t)$ denotes the unit price of CGS_i at instant t.

We restrict our analysis to the consumption levels $k_i(t)$ for which the utility $u_i(t)$ is positive and non-decreasing with $k_i(t)$. A necessary (not-sufficient) condition for that to be the case is $0 \leq k_i(t) \leq 1$. Under such condition, the utility function $u_i(t)$ verifies the following additional properties: (i) $\frac{\partial u_i(t)}{\partial D_i(t)} \geq 0$, due to positive indirect network effects,[15]; (ii) $\frac{\partial^2 u_i(t)}{\partial D_i^2(t)} = 0$; (iii) $\frac{\partial^2 u_i(t)}{\partial k_i^2(t)} \leq 0$, implying that the marginal utility of CGS is decreasing with $k_i(t)$ [16]; and (iv) $\frac{\partial^2 u_i(t)}{\partial k_i(t)\partial D_i(t)} \geq 0$, meaning that the marginal utility of CGS is increasing in

[11] In the primary market.

[12] Examples of perishable CGS include repairing services, maintenance services, and so on.

[13] In other words, the CGS supplied by firm i, $i = 1, 2$, are only compatible with its equipment, and they do not provide any value when consumed with the other equipment. Recall that consumers' lock-in is one of the main features of aftermarkets as described by Shapiro (1995).

[14] In the literature about network effects, the "stand alone value" refers to the value of a good that is independent of its network size (see, for example, Katz and Shapiro (1985)).

[15] $\frac{\partial u_i(t)}{\partial D_i(t)} = \phi[k_i(t) - \frac{1}{2}k_i^2(t)] \geq 0 \ \forall 0 \leq k_i(t) \leq 1$.

[16] $\frac{\partial^2 u_i(t)}{\partial k_i^2(t)} = -\gamma - \phi D_i(t)$ .For example, in the case of tablets/ applications, the marginal utility of buying an application tends to be decreasing on the number of applications that the consumer has previously

$D_i\,(t)$.[17] Regarding the last property, note that the marginal utility of CGS consumption may be increasing in the size of the firm's base of equipment customers in several situations. For example, when $D_i\,(t)$ has a positive impact on the quality[18] of the CGS$_i$, an increase in $D_i\,(t)$, by increasing the quality of CGS$_i$ might increase the marginal utility of CGS consumption.[19]

Our analysis of firms' interaction in the primary market and the aftermarket is based on a continuous time differential game with the following structure. In the primary market, at each instant of time t, there is an inflow of new consumers (arriving in the industry at a rate of μ) and an outflow of old consumers (exiting the industry at the same rate μ). At instant t, each firm sets the price of its equipment, anticipating its effect on firms' current and future profits. Newborn consumers observe these prices and make a lifetime choice between the two durable goods. Once decisions in the primary market are made, interaction in the aftermarkets takes place: each firm sets the price of the corresponding CGS; and the (locked-in) equipment owners choose the consumption level of CGS. We assume that CGS suppliers cannot commit to future CGS prices. Accordingly, the dynamics of the market shares in the primary market only affect outcomes in the aftermarket through indirect network effects.

The previous assumption allows us to study interaction in the aftermarket as a static problem embedded into the dynamic pricing game arising in the primary market. This "static-dynamic" breakdown is quite frequent in the context of the applied dynamic analysis in Industrial Organization[20] and it allows us to treat the instantaneous profit function in the aftermarket as a primitive of the dynamic pricing game. The equilibrium analysis is performed in the following section.

3 Equilibrium analysis

In this section the interaction between firms is througoutly analyzed as a continuous time differential game with linear transitions and quadratic payoffs, in addition to linear expec-

bought. This is consistent with property (ii).

[17] $\frac{\partial^2 u_i(t)}{\partial k_i(t)\partial D_i(t)} = \phi[1 - k_i(t)] \geq 0 \; \forall 0 \leq k_i(t) \leq 1.$

[18] In fact, in the context of a model in which indirect network effects are translated into the quality level of the CGS, the term $\gamma + \phi D_i\,(t)$, in the utility specification $u_i\,(t)$,could be reinterpreted as a quality index. In that case, ϕ would measure the value of the quality improvement associated with an enlargement of the firm's consumer base and γ would capture all the other quality determinants besides $D_i\,(t)$.

[19] Another example could be the case in which some form of conspicous consumption arises the aftermarket. Also in that case, an increase in $D_i\,(t)$ might increase the marginal utility of CGS consumption (since more people are now consuming similar CGS).

[20] See, for example, Ericson and Pakes, 1995; or, more recently, Doraszelski and Satterthwaite, 2010. These models study a dynamic game with endogenous entry, exit and investment decisions. Once these decisions are made, product market competition takes place. In their work, the "static-dynamic" breakdown is associated with the assumption that outcomes in the product market do not affect the dynamics of the industry.

tation formation rules for the forward-looking aspect of the (Markov perfect) equilibrium under consideration (the LMPE[21]). As pointed out in the end of the previous section, under the assumptions that (i) consumers are fully locked-in to their equipment, and (ii) CGS suppliers cannot commit to future CGS prices, the instantaneous profit function in the aftermarket can be conceived as a primitive of the dynamic pricing game. Accordingly, to solve the dynamic pricing game, we start by computing outcomes in the aftermarket as a function of the one dimensional state variable $D_i(t)$.[22] The equilibrium outcomes in the aftermarket will then be a primitive of the dynamic pricing game, whose linear Markov perfect equilibrium is studied in section 3.2.

3.1 Aftermarket

At each instant of time, consumers endowed with equipment i choose the amount of CGS$_i$ they want to consume (the owners of equipment i do not have any interest in CGS$_j$ as their equipment does not deliver any additional value when combined with such CGS). Consumers' problem in the aftermarket can be formulated as follows:

$$\max_{k_i(t)} u_i(t), \tag{3}$$

where $u_i(t)$ is given by (2). The solution[23] to problem (3) corresponds to the consumption level $k_i(\rho_i(t))$ for which the marginal benefit entailed by the consumption of CGS$_i$, $[\gamma + \phi D_i(t)][1 - k_i(t)]$, is perfectly balanced by its marginal cost, $\rho_i(t)$, yielding:

$$k_i(\rho_i(t)) = 1 - \frac{\rho_i(t)}{\gamma + \phi D_i(t)}.$$

The market demand[24] for CGS$_i$ is equal to $Q_i(\rho_i(t)) = k_i(\rho_i(t)) D_i(t)$. Assuming that firms produce CGS at zero marginal cost[25], the instantaneous profit made by firm i in its aftermarket is given by:

$$\pi_i^A(\rho_i(t); D_i(t)) = \left(\rho_i(t) - \frac{\rho_i^2(t)}{\gamma + \phi D_i(t)}\right) D_i(t). \tag{4}$$

[21] A formal definition of Linear Markov Perfect Equilibrium is provided in Section 3.2.

[22] In fact, instantaneous equilibrium outcomes in the aftermarket (including both segment 1 and segment 2) should depend on the vector of market shares $(D_1(t), D_2(t))$. However, full market coverage in the primary market implies $D_2(t) = 1 - D_1(t)$. Accordingly, the one dimensional state variable $(D_i(t))$ conveys all the payoff relevant information.

[23] The interior solution to problem (3) is obtained for $\frac{\partial u_i(t)}{\partial k_i(t)} = 0$ and $\frac{\partial^2 u_i(t)}{\partial k_i^2(t)} < 0$, for $0 < k_i(t) < 1$

[24] Recall that *(i)* only equipment i owners buy CGS$_i$; and *(ii)* all equipment i owners have the same individual demand for CGS$_i$.

[25] Without loss of generality, the qualitative nature of our results would not change under constant and symmetric marginal costs.

At each instant of time, each monopolist provider of CGS will choose the unit price $\rho_i(t)$ that maximizes its instantaneous aftermarket profit (4), yielding:

$$\rho_i(D_i(t)) = \frac{\gamma + \phi D_i(t)}{2}.$$

In equilibrium, firm i's profit in the aftermarket conditional on the state variable is equal to:

$$\pi_\iota^A(D_i(t)) = \frac{\gamma D_i(t) + \phi D_i^2(t)}{4}. \tag{5}$$

It is convex in the network size, due to the multiplicative character of the indirect network effects. A trivial consequence is that the industry aftermarket profits are increasing in the difference between the firms' market shares in the primary market.

The instantaneous utility of CGS as a function of the state variable $D_i(t)$ for a consumer locked-in to equipment i is:

$$u_i(D_i(t)) = \frac{\gamma + \phi D_i(t)}{8}. \tag{6}$$

From the previous results, it follows that the equilibrium CGS price, equilibrium aftermarket profits, and equilibrium CGS instantaneous utility are all increasing in $D_i(t)$, with:

$$\begin{aligned}
\frac{\partial \rho_i(D_i(t))}{\partial D_i(t)} &= \frac{\phi}{2} > 0; \\
\frac{\partial \pi_\iota^A(D_i(t))}{\partial D_i(t)} &= \frac{\phi}{2} D_i(t) + \frac{\gamma}{4} > 0; \\
\frac{\partial u_i(D_i(t))}{\partial D_i(t)} &= \frac{\phi}{8} > 0.
\end{aligned} \tag{7}$$

At this point, we already determined the equilibrium instantaneous aftermarket profits and instantaneous utilities as a function of the state variable $D_i(t)$. Thus, we can use $\pi_\iota^A(D_i(t))$and $u_i(D_i(t))$ as primitives of the dynamic pricing game taking place in the primary market.

3.2 Primary market

At instant t, only newborn consumers (who arrive in this industry at a rate of μ) have to choose between the available durable goods. Given his/her preferences, each newborn consumer compares $V_1(x,t)$ versus $V_2(x,t)$ and he/she makes a lifetime equipment choice. When computing $V_i(x,t)$, forward-looking consumers anticipate that $u_i^e(s) = u_i(s)$. Plugging equation (6) in (1), we obtain:

$$\int_t^\infty \left[\vartheta - \tau\left(x - x_i\right)^2 + \omega D_i(t)\right] e^{-(r+\mu)(s-t)} ds + \int_t^\infty \left[\frac{\gamma + \phi D_i(s)}{8}\right] e^{-(r+\mu)(s-t)} ds - p_i\left(t\right),$$

which simplifies to:

$$V_i\left(x,t\right) = \frac{\vartheta - \tau\left(x - x_i\right)^2 + \gamma/8}{r+\mu} + \left(\omega + \frac{\phi}{8}\right) \int_t^\infty D_i(s) e^{-(r+\mu)(s-t)} ds - p_i\left(t\right). \quad (8)$$

For more parsimonious notation, from now on we denote by $\Lambda_i\left(t\right)$ agents' (correct) beliefs, at moment t, regarding firm i's accumulated market shares from that period on, i.e.

$$\Lambda_i\left(t\right) = \int_t^\infty D_i^e(s) e^{-(r+\mu)(s-t)} ds = \int_t^\infty D_i(s) e^{-(r+\mu)(s-t)} ds,$$

where $D_i^e\left(s\right)$ stands for the expected market share of firm i, in the primary market, at instant $s \geq t$, and, $D_i^e\left(s\right) = D_i\left(s\right)$ since consumers are forward-looking. Accordingly, equation (8) is obtained from equation (1) when *(i)* for more parsimonious notation, we introduce $\Lambda_i\left(t\right)$ to denote agents' (correct) beliefs, at moment t, regarding firm i's accumulated market shares from that period on; *(ii)* we incorporate the equilibrium instantaneous utilities of CGS consumption as a function of $D_i\left(t\right)$, (using (6)) into the dynamic problem,and *(iii)* we integrate everything that is independent of time.

Note that $\Lambda_i(t)$ must verify two properties. First,

$$\Lambda_i(t) \in \left[0, \frac{1}{r+\mu}\right] \quad (9)$$

since $D_i(t) \in [0,1]$. Second,

$$\Lambda_1(t) + \Lambda_2(t) = \frac{1}{r+\mu}$$

due to full market coverage in the primary market.

At instant t, for given $p_1\left(t\right)$, $p_2\left(t\right)$, $\Lambda_1(t)$ and $\Lambda_2(t)$, it is possible to identify the position $\widetilde{x}\left(t\right) = \widetilde{x}(p_1(t),\ p_2(t),\ \Lambda_1(t), \Lambda_2(t))$ of the newborn consumer who is indifferent between buying equipment 1 or equipment 2, i.e., $V_1\left(\widetilde{x},t\right) = V_2\left(\widetilde{x},t\right)$, with $\widetilde{x}(t)$ defined by:

$$\widetilde{x}(t) = \frac{1}{2} + \frac{(r+\mu)\left[p_2\left(t\right) - p_1\left(t\right) + 2(\omega + \frac{\phi}{8})\Lambda_1(t)\right] - (\omega + \frac{\phi}{8})}{2\tau}. \quad (10)$$

When $\widetilde{x}(t) \in (0,1)$, newborn consumers located at the left of the indifferent consumer (i.e. $x\left(t\right) < \widetilde{x}\left(t\right)$) prefer equipment 1 over equipment 2, while newborn consumers located at the right of the indifferent consumer (i.e. $x\left(t\right) > \widetilde{x}\left(t\right)$), prefer to buy equipment 2.

When $\widetilde{x}(t) < 0$ (resp. $\widetilde{x}(t) > 1$), at instant t, the entire set of newborn consumers prefer equipment 2 to equipment 1 (resp. equipment 1 to equipment 2). We focus on duopoly equilibrium outcomes, in which both firms keep non-negative market shares. Thereby, we rule out eviction cases, assuming that

$$-\tau \leq (r+\mu)\left[p_2(t) - p_1(t) + 2(\omega + \frac{\phi}{8})\Lambda_1(t)\right] - (\omega + \frac{\phi}{8}) \leq \tau$$

Under the last condition, the instantaneous demands for equipment 1 and 2 are given by:

$$d_1 = \mu\widetilde{x}(t) \text{ and } d_2(t) = \mu\left[1 - \widetilde{x}(t)\right], \tag{11}$$

where $\widetilde{x}(t)$ is given by (10). Note that $d_1(t) + d_2(t) = \mu$.

Firms' instantaneous profits in the primary market are equal to

$$\pi_i^{PM}(t) = d_i(t)\, p_i(t),$$

where $d_i(t)$ is given by (11) and $p_i(t)$ represents the price of equipment i at instant t.

In equilibrium, firms optimally choose their pricing strategies in the primary market. They are assumed to use Markov pricing strategies. A pricing strategy for firm i is said to be Markovian if it is a rule $G_i(.)$ which tells firm i what price to charge at time t, based only on the knowledge of its current market share $D_i(t)$.[26] It is a best reply to its rival's pricing strategy $G_j(.)$ and the agents' expectations rules when this pricing strategy yields a time path $p_i^*(t)$ which maximizes firm i's total discounted profits for all starting (date, state) pairs $(t, D_i(t))$

$$\Pi_i(t) = \int_t^{\infty} \left[\pi_i^{PM}(s) + \pi_\iota^A(s)\right] e^{-r(s-t)} ds, \tag{12}$$

taking as given the rival's equipment pricing strategy $G_j(.)$ and the agents' expectation rules, which are both functions of the state variable. In equilibrium, newborn consumers optimally choose their equipment as they are forward-looking agents whose equipment choices are based on (correct) beliefs about firms' future market shares.

Without any further assumption, this dynamic problem has no closed-form solution. To make the problem tractable, in line with other papers studying dynamic pricing in network industries (e.g. Laussel et al. (2004) or Doganoglu (2003)), we concentrate our analysis on the class of expectation rules and price strategies that are linear on the state variable $D_i(t)$. As far as pricing strategies go, Linear Markov pricing strategies arise when the price strategy functions $G_i(.)$ are linear on the state variable $D_i(t)$:

[26] The state variable is therefore $D_i(t)$. Due to full market coverage in the primary market we have $D_j(t) = 1 - D_i(t)$.

$$G_i(D_i(t)) = \eta_i + \sigma_i D_i(t) \tag{13}$$

Regarding our exclusive focus on linear price strategies, note that our dynamic game corresponds to a linear-quadratic differential game: the instantaneous equipment demands are linear in the control variables $(p_i(t), p_j(t))$; the equation of motion of $D_i(t)$ is linear in the state variable; and firms' total profits are quadratic in the control variables and linear in the state variable (see equation (12)). Our differential game being linear-quadratic, it is natural to search for the simplest strategies that might-solve this category of differential games: the linear strategies[27] (see Dockner et al. (2000), Long, N.V. and Leonard, D. (1992) and Laussel et al. (2004)). In addition, as far as strategies go, it is well known that for a finite horizon linear quadratic game, there is a unique Markov equilibrium in linear strategies. The limit of this equilibrium as the horizon tends to infinity is always an equilibrium of the infinite horizon game, although there may be other (non-linear equilibria).[28]

Regarding expectation rules, given the linear-quadratic structure of the game and the linear structure of equilibrium prices, linear expectation rules seem to be a natural class of expectations to consider. However, differently from the case of linear pricing strategies (in which linearity is essentially not an assumption, given the structure of the game), in the case of expectations, linearity is indeed an assumption in view of the fact that other, more general, classes of expectations could be formulated and confirmed at equilibrium.

Definition 1 (Linear Markov expectation rules) *Linear Markov expectation rules correspond to a pair of affine functions* $(F_1(\cdot), F_2(\cdot))$ *that maps any observed value of* $D_i(t)$, $0 < D_i(t) < 1$ *to a vector* $(\Lambda_1(t), \Lambda_2(t))$ *such that*

$$F_i(\cdot) = \delta_i + \beta_i D_i(t);$$

with $i = 1, 2$.

From Definition 1 follows that $\Lambda_i(t) = F_i(\cdot) = \delta_i + \beta_i D_i(t)$.

Concentrating our attention on Linear Markov pricing strategies, under the assumption of Linear Markov expectation rules, we have that a linear Markovian pricing strategy, $G_i(D_i)$, consitutes the best reply of firm i to the competitor's price strategy, $G_j(\cdot)$, and consumers' expectation rules, $(F_i(\cdot), F_j(\cdot))$, if this pricing strategy yields a time path $p_i^*(t)$ that maximizes firm i's total discounted profits (12), given $G_j(\cdot)$ and $(F_i(\cdot), F_j(\cdot))$.

Definition 2 provides a formal definition of a Linear Markov Perfect Equilibrium.

Definition 2 (Linear Markov Perfect Equilibrium) *The LMPE of the dynamic game corresponds to a quadruple of linear functions* $\{G_1(.), G_2(.), F_1(.), F_2(.)\}$, *such that:*

[27] In the context of linear-quadratic games, the best reply to linear strategies is also linear.

[28] Despite the last point, applications often concentrate on the linear equilibrium, only.

(i) $G_i(.)$ is firm i's best reply to the pricing strategy $G_j(.)$, given the expectation rules $(F_i(.), F_j(.))$, for $i, j = 1, 2, i \neq j$, and
(ii) expectations are confirmed at equilibrium for $i = 1, 2$,

$$F_i(t) = \Lambda_i(t) = \int_t^{\infty} [D_i(s)] e^{-(r+\mu)(s-t)} ds. \tag{14}$$

In light of the previous definitions, firm i's LMPE price strategy $G_i(.)$ yields a time path $p_i^*(t)$ that solves the following dynamic optimization problem:

$$\max_{p_i(t)} \Pi_i(t) \tag{15a}$$

s.t.

$$\frac{dD_i(t)}{dt} = d_i(t) - \mu D_i(t)$$

$$d_i(t) = \frac{\mu}{2} + \frac{\mu(r+\mu)\left[p_j(t) - p_i(t) + 2(\omega + \frac{\phi}{8})F_i(t)\right] - \mu(\omega + \frac{\phi}{8})}{2\tau} \tag{15b}$$

$$\Lambda_i(t) = \int_t^{\infty} [D_i(s)] e^{-(r+\mu)(s-t)} ds = F_i(t) = \delta_i + \beta_i D_i(t), \tag{15c}$$

$$\Pi_i(t) \geq 0 \text{ and } 0 < D_i(t) < 1,$$

where $p_i(t)$ is the control variable , $D_i(t)$ is the state variable[29] and $\Pi_i(t)$ corresponds to firm i's total discounted profits (12), equal to

$$\Pi_i(t) = \int_0^{\infty} e^{-rt}\left[d_i(t) p_i(t) + \frac{\gamma D_i(t) + \phi D_i^2(t)}{4}\right] dt,$$

with $d_i(t)$ given by (15b).

The profit maximizing problem just described corresponds to an infinite horizon optimal control problem with a linear-quadratic structure. It can be analytically solved (see, Long, N.V. and Leonard,D. (1992), Chapter 9, for theorems stating necessary and sufficient conditions) allowing us to characterize firms' equilibrium pricing strategies.

In light of the symmetry of the primitives of our model (recall firms are similar with every respect except for their initial market shares in the primary market), when we focus on duopoly equilibrium outcomes, not surprisingly the unique equilibrium of the symmetric game is symmetric,[30] implying $p_i^*(t) = \delta^* + \beta^* D_i(t)$ and $G_i^*(t) = \eta^* + \sigma^* D_i(t)$, $i = 1, 2$.

[29] To obtain (15b), introduce (10) in (11), taking into consideration the notation used for the (linear) pricing strategy, $G_i(t)$ and for the (linear) expectation rule, $F_i(t)$.

[30] If one of the firms had an exogenous advantage over its rival (e.g. a quality advantage or a more favorable location on the Hotelling line), asymmetric price strategies and expectation rules would arise in the LMPE (see, for example, Argenziano (2008) for a model of static price competition in a duopoly with

Proposition 1 identifies the necessary and sufficient conditions for the existence of a unique LMPE.

Proposition 1 (Existence and uniqueness of the LMPE) *A unique LMPE exists if and only if the intensities of direct and indirect network effects, ω and ϕ are relatively weak:*

$$\tau \geq \overline{\tau} = \frac{\omega(r+2\mu) + \frac{\phi}{8}(5r+6\mu)}{3r+2\mu} \tag{16}$$

Proof. See the Appendix. ■

According to Proposition 1, a unique LMPE exists only when the degree of differentiation between equipment is above the threshold $\overline{\tau}$. When the equipment become close substitutes (low τ), everything else the same, a duopoly equilibrium outcome is less likely to occur as firms compete more aggressively in the primary market. The critical threshold $\overline{\tau}$ depends on the parameters ω, ϕ, r, and μ. Regarding the intensity of network effects, Proposition 1 shows that $\overline{\tau}$ is increasing both in ω and ϕ. Accordingly, the unique LMPE equilibrium exists when network effects (direct and/or indirect) are not too strong. Otherwise, market tipping phenomena are likely to occur and firms may be interested in moving away from symmetric pricing strategies to evict the rival firm. In addition, the threshold $\overline{\tau}$ is increasing in μ and decreasing in r. Note that in the LMPE, both an increase μ and an increase in r raise the fraction of new consumers who are willing to purchase the initially dominant equipment.[31] Therefore, market tipping in favor of the initially dominant firm becomes more likely (i.e. the possibility of strong market dominance would be stronger). While this is the only effect of an increase in μ (yielding $\frac{\partial \overline{\tau}}{\partial \mu} > 0$), an increase in r yields an additional effect. As r increases, firms put less weight on the future profitability of the aftermarket sales and thereby they are less eager to adopt aggressive demand-enhancing strategies in the primary market leading to the eviction of the rival firm. Since $\frac{\partial \overline{\tau}}{\partial r} < 0$, we have that firms' lower-discount rate effect dominates the first one and therefore the existence condition becomes easier to satisfy when r inceases.

In the following Section, we assume condition (16) is met and we investigate further properties of the equilibrium price strategies.

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vertical and horizontal differentiation together with network effects). Similarly, our focus on the duopoly outcomes rules out the possibility of market tipping (which would lead to asymmetric pricing strategies). The proof of the symmetry of the LMPE is available from the authors upon request.

[31] To confirm this effect, compute $\widetilde{x}_i(t)$ for the optimal values of the parameters $\sigma(\beta^*)$ and $\delta(\beta^*)$. Introducing (27) and (28) in (10), we obtain: $\widetilde{x}(\beta,t) = \frac{2(\beta(r+2\mu)-1)D_1(t)+1-\beta(r+\mu)}{2\beta\mu}$, with $\frac{\partial \widetilde{x}(\beta,t)}{\partial \mu} > 0$ and $\frac{\partial \widetilde{x}(\beta,t)}{\partial r} > 0$ for $D_1(t) > \frac{1}{2}$.

4 Equilibrium Analysis

4.1 Market Dominance

In a model with positive network effects, whether direct or indirect, everything else the same, the large network should be more attractive than the small one. When the large network attracts a new customer with higher probability than the small network, there is "weak market dominance" (Cabral, 2011). Proposition 2 shows that, in our model, weak market dominance holds unconditionnally at the LMPE[32].

Proposition 2 (Weak market dominance) *In the LMPE, the large network attracts a larger proportion of new consumers than the small one.*

Proof. See the Appendix. ∎

However, the observence of weak market dominance does not translate into an increase in the size of the larger network as time goes by. Strong market dominance, arising when the large network tends to increase in size (Cabral, 2011) is ruled out by condition (16).

4.2 Equilibrium pricing strategies

The equilibrium pricing strategies $p_i^*(t) = \eta^* + \sigma^* D_i(t)$ include two terms. The term η^* captures the pricing behavior that is independent of the evolution of the state. The second term: $\sigma^* D_i(t)$ captures the effect of the state variable, $D_i(t)$ on the optimal equipment price. Accordingly, the equilibrium $\sigma - value$ can be interpreted as a "network mark-up" (when $\sigma^* > 0$) or a "network discount" (when $\sigma^* < 0$)[33].

Proposition 3 (Pricing strategies) *Along the LMPE price trajectories,* $p_i^*(t) - p_j^*(t) = \sigma^*[D_i(t) - D_j(t)]$. *The equipment benefiting from a larger installed base of consumers is less (resp. more) expensive than its rival, i.e.* $\sigma^* < 0$ *(resp.* $\sigma^* > 0$*), iff indirect (resp. direct) network effects are prevalent, i.e. iff* $\frac{\phi}{8} > (<) \omega \left(1 - \omega \frac{4\mu(r+\mu)}{\tau(r+2\mu)^2}\right)$.

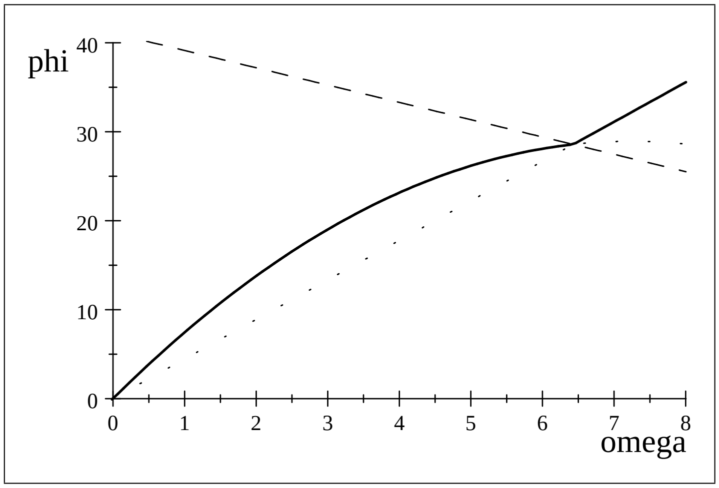
Proof. See the Appendix. ∎

The figure below depicts,[34] in the space (ω, ϕ), the two areas for which we obtain the two opposite pricing strategies. The existence condition restricts the (ω, ϕ) space to the area below the dashed line. Under this restriction, the area below the solid line corresponds to $\sigma^* > 0$, whereas the area above the solid line yields $\sigma^* < 0$.

[32] Cabral (2011) derives this result in his Proposition 4 for a low value of the discount factor.

[33] A *"network mark-up"* (resp. *"network discount"*) exists when the equipment benefiting from a larger base of consumers quotes a higher (lower) price than the other one.

[34] The diagram is drawn for $\mu = 0.02$, $r = 0.05$ and $\tau = 10$



Proposition 3 shows that the nature of the optimal pricing strategies crucially depends on the relative intensity of direct and indirect network effects.

On the one hand, due to indirect network effects, firms' profits in the aftermarket exhibit increasing marginal returns in relation to firms' customer base in the primary market[35], with $\frac{\partial^2 \pi_i^A(D_i(t))}{\partial D_i^2(t)} = \frac{\phi}{2} > 0$. The larger equipment producer is therefore able to raise the price of CGS on a larger set of consumers and today's benefit from gaining a new customer (in terms of changes in future aftermarket profits) is smaller for the firm with the narrower base of locked-in consumers.[36] In light of this, we have that indirect network effects alone would lead the larger firm to set a lower equipment price than the smaller firm, i.e. $\sigma^* < 0$.

On the other hand, due to direct network effects, everything else the same, the equipment producer with the largest customer base offers a better equipment than its rival. In a duopoly setting, one would expect this qualitative advantage to be translated into higher equipment prices, leading to $\sigma^* > 0$. This result is in line with the idea of a "network mark up" pointed out in Laussel et al. (2004), Doganoglu (2003) and most of the static literature studying pricing in network industries with horizontally differentiated goods.

According to Proposition 3, when indirect network effects are dominant, the first effect dominates the second one and, therefore the path of optimal equiment prices is decreasing with $D_i(t)$. Conversely, when direct network effects are dominant, the profitability of the aftermarket becomes less relevant and the path of optimal equiment prices is increasing in $D_i(t)$.

[35] Recall that the marginal utility entailed by CGS consumption is increasing on the number of consumers who are locked in to the corresponding durable good, $D_i(t)$, and so are the prices of CGS (see equation (7)). These two effects are on the basis of the marginal increasing returns of aftermarket profits with respect to firms' market shares in the primary market.

[36] Within our model, such dynamic mechanism is conveyed by the co-state variable $\lambda_i(t)$. This variable can be interpreted as the expected total (current and future) extra-profits associated with a marginal increase in firms' instantaneous market shares (the shadow price of the state variable). From equation (21), it follows that the equilibrium value of $\lambda^*(t)$ depends positively on $D_i(t)$.

Proposition 3 provides a general necessary and sufficient condition for equipment prices to be increasing or decreasing in the network size, extending the theoretical results by Cabral (2011) who derives only sufficient conditions in more restrictive settings.[37] Cabral (2011) has established, for low values of the discount factor, that (a) if equilibrium aftermarket profits are linear in the network size and consumers' equilibrium aftermarket utility is increasing in the network size, then the equipment price is increasing in the network size, and (b) if consumer's equilibrium aftermarket utility is independent of the market size and equilibrium aftermarket profits are increasing at an increasing rate with the market size, then the equipment price is decreasing in the network size. He also shows that (c) if consumer's equilibrium aftermarket utility is linearly increasing at a bounded rate with the network size and equilibrium aftermarket profits are independent of the market size, the equipment price is increasing in the network size. Differently from Cabral (2011), our analysis applies to high values of the discount factor r (in light of the existence condition). In addition, we can recover his results from our model by appropriately fixing the values of the parameters defining the intensities of network effects. When we fix $\phi = 0$ and $\omega > 0$, result (a) holds. When we fix $\omega + \phi = 0$ and $\phi > 0$, result (b) holds. And finally, when we assume that $\phi = 0$ and ω is bounded above by some $\overline{\omega}$, we recover result (c).

It is also worth noting that the effect of $D_i(t)$ on firms' optimal pricing policies in the primary market is exclusively due to the dynamic aspects of price competition generated by network effects. When we do not account for such dynamics, the following result holds.

Lemma 1 *In a static setting, firms would set an identical equipment price, with*

$$\lim_{r\to\infty} \sigma(\beta^*) = 0.$$

Proof. In appendix. ■

To provide a full characterization of firms' optimal pricing strategy, it remains to study the pricing behavior that is independent of the evolution of the state, given by η^*.

Proposition 4 *In the LMPE, the value of $\eta^* = \eta(\beta^*)$ depends on (a) the degree of differentiation between the equipment, (b) the profitability of the aftermarkets in the absence of network effects, and (c) the intensity of network effects:*

$$\eta(\beta^*) = \underbrace{\frac{\tau}{(r+\mu)}}_{(a)} - \underbrace{\frac{\gamma}{4(r+\mu)}}_{(b)} + \underbrace{\tau\left[\beta^*(r+\mu) - 2\right]\frac{\beta^*(r+2\mu) - 1}{\beta^{*2}\mu(r+\mu)^2}}_{(c)}. \tag{17}$$

Proof. See the Appendix. ■

[37] He notices himself (Subsection 3.1, page 97, $1) that his theoretical results apply to extreme values of key parameters.

Proposition 4 shows that η^* is the sum of three terms: (a) the discounted "Hotelling price", $\frac{\tau}{r+\mu}$, that would be charged if firms were not involved in the provision of CGS; (b) a price discount resulting from the linkages between the aftermarket and the primary market, and (c) a demand-enhancing price discount associated with network effects (either direct or indirect)

In the absence of network effects, $(\phi = 0,\ \omega = 0)$, we have $\beta^* = \frac{1}{r+2\mu}$ and term (c) is equal to zero. Network effects boost competition in the primary market.

Lemma 2 *In the LMPE:*

$$\frac{\tau}{r+\mu} - \frac{\gamma}{4\left(r+\mu\right)} - \frac{1}{8}\tau\frac{\left(r+3\mu\right)^2}{\mu\left(r+\mu\right)^2} \leq \eta(\beta^*) \leq \frac{\tau}{r+\mu} - \frac{\gamma}{4\left(r+\mu\right)}.$$

Proof. See the Appendix. ■

From the previous lemma follows that η^* can be positive or negative, depending on the values of the parameters. When the degree of differentiation τ is relatively low, i.e. $\tau < \frac{\gamma}{4}$, η^* is always negative regardless of the intensity of network effects. When indirect effects are dominant, $\sigma^* < 0$. Hence, for $\tau < \frac{\gamma}{4}$, equipment prices are always fixed below the marginal cost. When direct effects are dominant and equipment are sufficiently differentiated, dumping in the primary market (understood as pricing below marginal cost) may no longer be an optimal strategy.

4.3 LMPE trajectories and Steady state

In this subsection we analyze the LMPE steady state and the LMPE trajectories of firms' instantaneous market shares $D_i\left(t\right)$, average equipment prices, industry profits and new customers' avarage expected utility.

By definition, in the steady state equilibrium $d_i\left(t\right) = \mu D_i\left(t\right)$, or equivalently

$$\frac{dD_i\left(t\right)}{dt} = 0, \tag{18}$$

yielding[38] $\overline{D}_i = \frac{1}{2}$,where $\overline{D}_i$ stands for firm i's LMPE steady state market share, $i = 1, 2$. Not surprisingly, given the symmetry of the primitives of the model, we obtain symmetric steady state market shares in the primary market.

The speed of convergence to the symmetric steady state depends on the characteristics of the market.

[38] To obtain the steady state market share, in (18), substitute $d_i\left(t\right)$ by the expression (15b). Then, introduce $\delta = \frac{1-\beta(r+\mu)}{2(r+\mu)}$ (see (28)) and account for the linear Markov price strategies of firm i. Finally, solve equation (18), obtaining $\overline{D}_i = \frac{1}{2}$.

Proposition 5 *In the LMPE, market shares converge to the symmetric steady state equilibrium at a rate equal to*

$$\psi = \frac{1}{\beta^*} - (r + \mu).$$

Proof. See the Appendix. ■

Corollary 1 *In the LMPE, the speed of convergence to the symmetric steady state is such that* $0 \leq \psi \leq \mu$.

Proof. Follows directly from Proposition 5 and the fact that $\frac{1}{r+2\mu} \leq \beta^* \leq \frac{1}{r+\mu}$. ■

Proposition 5 shows that the speed of convergence to the symmetric steady state depends on the effective discount rate $(r + \mu)$ and on the equilibrium value of β^*. This result is in line with Doganoglu (2003) and Laussel et al. (2004). As long as the turnaround in the population of equipment owners is sufficiently limited (small μ), the convergence to the steady state LMPE is always slow, regardless of the intensity of network effects and the level of product differentiation.

Lemma 3 *In the steady state LMPE, firms charge identical equipment prices* $p_i = p_j = \overline{p}$, *with:*

$$\overline{p} \leq \frac{\tau}{r + \mu}.$$

Proof. See the Appendix. ■

The steady state equipment prices are always below the discounted "Hotelling price", $\frac{\tau}{r+\mu}$. Depending on the parameters of the model, namely the strength of indirect network effects, the degree of differentiation between equipment, and consumers' willingness to pay for CGS, equilibrium steady state prices may be negative. Despite making negative profits in the primary market (where firms set prices below marginal cost), this type of pricing strategy allows firms to get more profits in the aftermarket.

Proposition 6 (Dynamics) *In the LMPE, if* $D_i(0) \neq \frac{1}{2}$, *as market shares converge to the symmetric steady state equilibrium,*
(i) the average equipment price is increasing (resp. decreasing) with time if

$$\phi/8 > (<)\omega - \omega^2 \frac{4\mu(r + \mu)}{\tau(r + 2\mu)^2},$$

(ii) the industry profits and the new customers' average expected life time utility are decreasing with time.

Proof. See the Appendix. ■

The average equipment price is increasing (resp. decreasing) through time whenever the equilibrium equipment price is decreasing (resp. increasing) in the network size. In order to get the intuition of this result, let us, without loss of generality, consider the case $\sigma(\beta^*) < 0$[39]. Here, as the trajectories converge to the steady state, the difference between market shares becomes smaller and smaller. As a result, the price of the dominant equipment increases, more than compensating the reduction in the small network's price. At the same the market share of the large network, with the smaller price, decreases. The two combined effects lead to an increase in the average equipment price.

Regarding industry profits, first notice that industry profits in the primary market are equal to the average equipment price times the entry/ exit rate μ. Hence, the results obtained for the average equipment price are also valid for the industry profits in the primary market. Since industry profits in the aftermarket are unambiguously decreasing through time, the same directly holds for aggregate industry profits when $\sigma^* > 0$, given that, in that case, also industry profits in the primary market are decreasing with time. In the reverse case, however, industry profits in the two markets evolve in opposite directions. Proposition 6 shows that, in this case, the reductiom of aftermarket profits outweights the increase of primary market profits. This holds true as long as there are strictly positive network effects.

Finally, concerning the evolution of the new consumers' average expected lifetime utility, Proposition 6 shows that as the size of firms' networks becomes more similar, the average benefit of network effects to consumers decreases and this always outweighs the effects of a smaller average distance from the "ideal equipment" and, possibly, of a lower average equipment price (if $\sigma^* > 0$).

5 Conclusion

In this paper, we propose a theoretical model of dynamic competition in primary markets and aftermarkets. In line with the previous literature, our model encompasses the key elements to an aftermarket: (i) the complementarity between durable goods and CGS; (ii) the existence of a time lag between equipment purchases and CGS consumption; and (iii) consumers' lock-in. In addition, we consider the possibility of direct and indirect network effects.

The strategic interaction between firms is thoroughly analyzed as a continuous time differential game with linear transitions and quadratic payoffs, in addition to linear expectation formation rules for the forward-looking aspect of the (Markov-perfect) equilibrium

[39] A reverse but similar argument holds in the opposite case.

under consideration. We search for the unique LMPE in which both firms have non-negative market shares and we provide a necessary and sufficient condition for the existence of such equilibrium.

When the existence and uniqueness condition is fulfilled, we conclude that equipment prices' equilibrium trajectories may be increasing or decreasing in firms' base of consumers in the primary market depending on the intensity of direct network effects vis-à-vis the intensity of indirect network effects. Indirect network effects induce increasing marginal returns of total profits with respect to firms' market shares in the primary market. In light of this effect, equilibrium pricing strategies would be decreasing in $D_i(t)$ as larger firms benefit more than smaller firms from price reductions in the primary market. Conversely, direct network effects increase the attractiveness of the equipment with the larger base of consumers, yielding increasing pricing strategies in $D_i(t)$. Considering the two effects, we obtain that prices are decreasing (increasing) in firms' market shares in the primary market, when indirect (direct) network effects prevail.

The paper also characterizes the LMPE steady state and the trajectories of $D_i(t)$, avarage equipment prices, industry profits and the expected lifetime surplus of new consumers. We find that in the LMPE steady state, firms share the market evenly. The LMPE steady state price is always bellow the Hotelling benchmark price. When the equipment are sufficiently close substitutes, the marginal utility of CGS is sufficiently high and/or indirect network effects are sufficently strong, the optimal steady state pricing might involve dumping in the primary market (with firms quoting prices below the marginal cost).

Along the equilibrium trajectories, the market share of the smaller network is increasing with time, whereas the one of the larger firm is decreasing. The speed of convergence never exceeds the rate of consumers' exit/entry.

We find that industry profits and new customers' average expected life time utility are decreasing with time. The average equipment price is increasing (resp. decreasing) with time when the intensity of indirect network effects is sufficiently stronger than the intensity of direct network effects.

In our future research, we intend to investigate to which extent network effects resulting from the interplay between primary markets and aftermarkets may lead to eviction outcomes: when network effects are very strong in relation to product differentiation, equipment producers may have incentives to adopt aggressive pricing policies in the initial periods, in order to evict the rival equipment producer. Other extensions of the model are worthwhile as well. A natural extension would be to introduce the possibility of price commitment in the aftermarket (while price commitment does not seem to be relevant in industries such as printers, software,... in other industries this commitment is very important, e.g. in the case of mobile telecommunications).

6 Appendix

Proof of Proposition 1

The proof is organized as follows. First, we identify the LMPE candidate in which both firms remain active in the market, deriving the equilibrium conditions for δ, β, η and σ. Afterwards, a sufficient and necessary condition for existence and uniqueness is derived.

To identify the LMPE candidate, that solves the optimal control problem (15a), we introduce the current-value co-state variable, $\lambda_i(t)$, and we define the current-value Hamiltonian for firm i as follows:

$$H_i(t) = [p_i(t) + \lambda_i(t)]\, d_i(t) + \frac{\gamma D_i(t) + \phi D_i^2(t)}{4} - \mu \lambda_i(t)\, D_i(t),$$

where $d_i(t)$ is given by (15b), under linear Markovian expectation rules and linear Markovian price strategies.

The necessary conditions to guarantee the optimality of firm i's equipment price strategies include[40]:

$$\frac{\partial H_i(t)}{\partial p_i(t)} = 0 \tag{19}$$

and

$$\frac{d\lambda_i(t)}{dt} = r\lambda_i(t) - \frac{\partial H_i(t)}{\partial D_i(t)}, \tag{20}$$

with $i = 1, 2$.

From the first order conditions (19) and (20), we obtain:

$$\lambda_i(t) = \frac{\tau - (\omega + \frac{\phi}{8})}{(r+\mu)} + \eta + \sigma[1 - D_i(t)] - 2p_i(t) + 2(\omega + \frac{\phi}{8})[\delta + \beta D_i(t)], \tag{21}$$

and

$$\frac{d\lambda_i(t)}{dt} = \lambda_i(t)(\mu + r) - \frac{[p_i(t) + \lambda_i(t)]\,\mu(r+\mu)}{2\tau}\left[2\beta(\omega + \frac{\phi}{8}) - \sigma_j\right] - \frac{1}{4}[\gamma + 2\phi D_i(t)] \tag{22}$$

with $i = 1, 2$.

In condition (22), replace $\lambda_i(t)$ for the RHS of condition (21) and introduce the linear Markov price strategy of firm i, $p_i(t) = \eta + \sigma D_i(t)$, obtaining $\frac{d\lambda_i(t)}{dt} = A_i + B_i D_i(t)$, where the polynomials A_i and B_i depend on the parameters of the model as well as on the values of β, δ, η, and σ. Then, introduce firm i's linear Markov pricing strategy in condition (21) and differentiate it with respect to time, obtaining

$$\frac{d\lambda_i(t)}{dt} = \left(2\beta(\omega + \frac{\phi}{8}) - 3\sigma\right)\frac{dD_i}{dt}, \tag{23}$$

[40] See Dockner *et al.* (2000), Long N.V. and Leonard, D (1992) and Laussel *et al.* (2004).

where $\frac{dD_i(t)}{dt}$ is determined by the motion equation when we account for the linear Markov price strategy of firm i.

Introducing the resulting expression for $\frac{dD_i(t)}{dt}$ in (23) and putting $D_i(t)$ on evidence, one obtains $\frac{d\lambda_i(t)}{dt} = T_i + V_i D_i(t)$,where again T_i and V_i depend on the parameters of the model as well as on the values of β, δ, η, and σ.

In the LMPE of the game, it must be the case that, at each instant $t \geq 0$, and for $i = 1, 2$, $A_i + B_i \; D_i(t) = T_i + V_i \; D_i(t) \; \forall \; D_i(t)$. Thus, it follows that $A_i = T_i$ and $B_i = V_i$, for $i = 1, 2$.

The condition $B_i = V_i$ implies that:

$$\frac{\mu (r+\mu)}{\tau} \left(-2\sigma + 2\beta(\frac{\phi}{8} + \omega) \right)^2 + (r + 2\mu) \left(3\sigma - 2\beta(\frac{\phi}{8} + \omega) \right) + \frac{\phi}{2} = 0. \tag{24}$$

In addition, the assumption of linear Markov expectation rules implies that $\delta + \beta D_i(t) = \Lambda_i(t)$, where $\Lambda_i(t)$ is given by (15c). Differentiating both sides of this equality with respect to time, we obtain

$$\beta \frac{dD_i(t)}{dt} = (r+\mu) \left[\delta + \beta D_i(t) \right] - D_i(t) \tag{25}$$

Replacing in (25) the law of motion $\frac{dD_i(t)}{dt}$ and considering $\sigma_i = \sigma_j = \sigma$, it follows the condition $MD_i(t) + N_i = 0$, where M is a function of β, σ and the parameters of the model, while N_i is a function of the parameters of the model as well as on the values of β, σ, η and δ. Since condition $MD_i(t) + N_i = 0$ must hold for all values of $D_i \in [0, 1]$, it follows that $M = 0$, or equivalently

$$1 - \beta\mu \left(\frac{1}{\tau} (r+\mu) \left(\sigma - \beta(\omega + \frac{\phi}{8}) \right) + 1 \right) - \beta (r+\mu) = 0 \tag{26}$$

It also follows that $N_i = 0$, yielding

$$\delta (\beta, \sigma) = \beta\mu \frac{\frac{1}{2\tau} \left(-(\omega + \frac{\phi}{8}) + (r+\mu)\sigma \right) + \frac{1}{2}}{(r+\mu) \left(1 - \beta \frac{1}{\tau} \mu (\omega + \frac{\phi}{8}) \right)}.$$

Now we analyze under which conditions the candidate LMPE is indeed an equilibrium of the dynamic game. To obtain such conditions, we start by solving equation (26) with respect to σ, obtaining $\sigma(\beta)$ equal to:

$$\sigma(\beta) = \frac{1}{\beta} \frac{\tau}{\mu (r+\mu)} \left(\frac{1}{\tau} \mu (\omega + \frac{\phi}{8}) (r+\mu) \beta^2 - (r + 2\mu) \beta + 1 \right). \tag{27}$$

Replacing $\sigma(\beta)$ in $\delta(\beta,\sigma)$, we obtain:

$$\delta(\beta) = \frac{1-\beta(r+\mu)}{2(r+\mu)} \tag{28}$$

Then, in condition (24) replace σ for its value in equation (27), obtaining the LMPE equilibrium condition

$$X(\beta) = 0 \tag{29}$$

that implicitly expresses the equilibrium value of β, denoted by β^*, as a function of the parameters of the model, with $X(\beta)$ being equal to

$$\beta^3\mu(r+\mu)(r+2\mu)(\omega+\frac{\phi}{8})+\beta^2((r+2\mu)^2\tau+4\mu(r+\mu)\frac{\phi}{8})-5\beta(r+2\mu)\tau+4\tau$$

However, not all the values of β that solve (29) constitute a LMPE. The linear expectation rules for the forward-looking aspect of the (Marvok perfect) equilibrium under consideration impose the following additional restrictions:

$$0 < \beta^* \leq \frac{1}{r+\mu}. \tag{30}$$

To obtain that $\beta^* > 0$, notice that equation (25) corresponds to a differential equation. Upon integration, we get:

$$D_i(s) = K + (D_i(t) - K)\, e^{\frac{(r+\mu)\beta-1}{\beta}(s-t)}, \tag{31}$$

with $K = \delta\frac{r+\mu}{1-\beta(\mu+r)}$.

Finally, substituting (31) in $\Lambda_i(t)$ in (14), we obtain:

$$\Lambda_i(t) = \tfrac{\delta}{1-\beta(\mu+r)} + [D_i(t) - K]\int_t^\infty e^{-\frac{(s-t)}{\beta}}ds.$$

The previous expression can only be equal to $F(\cdot) = \delta + \beta D_i(t)$ if β^* is strictly positive.

Regarding the condition $\beta^* \leq \frac{1}{r+\mu}$, notice that condition (9) requires:

$$0 \leq \delta + \beta^* D_i(t) \leq \frac{1}{r+\mu}\ \forall\ D_i(t) \in [0,1]. \tag{32}$$

In particular, for $D_i(t) = 0$, condition (32) implies $0 \leq \delta \leq \frac{1}{r+\mu}$. Similarly, for $D_i(t) =$

1, condition (32) implies $-\beta \leq \delta \leq \frac{1}{r+\mu} - \beta$. Therefore

$$0 \leq \delta \leq \frac{1}{r+\mu} - \beta. \tag{33}$$

To guarantee that the set of $\delta-$values in (33) is non-empty, β^* must be small enough. More precisely $\beta^* \leq \frac{1}{r+\mu}$.

In this context, the LMPE value of β^* corresponds to the root(s) of the cubic polynomial in (29) that are included in the interval $0 < \beta^* \leq \frac{1}{r+\mu}$

To conclude the proof, we investigate under which conditions the polynomial in condition (29), hereafter denoted by $X(\beta)$ has only one root such that $\beta^* \in \left(0, \frac{1}{r+\mu}\right]$. In this case, the LMPE exists and it is unique. To start, notice that $X(\beta)$ is a third-degree polynomial. Thus, it has, at most, three roots. Given that $\lim_{\beta\to-\infty} X(\beta) = -\infty$ and $X(0) = 4\tau > 0$; one of the roots of $X(\beta)$ is necessarily negative, violating condition (30). Hence, at most two roots verify condition (30). It is worth noting that

$$X\left(\frac{1}{r+\mu}\right) = \mu \frac{(5r+6\mu)\frac{\phi}{8} + \omega(r+2\mu) - \tau(2r+3\mu)}{(r+\mu)^2} \lesseqgtr 0,$$

and $\lim_{b\to+\infty} X(b) = +\infty$.

Therefore, a sufficient and necessary condition for the existence of a unique root of $X(\beta)$ satisfying condition (30) is $X\left(\frac{1}{r+\mu}\right) \leq 0$, or equivalently

$$\tau \geq \frac{\omega(r+2\mu) + \frac{\phi}{8}(5r+6\mu)}{3r+2\mu}$$

When $X\left(\frac{1}{r+\mu}\right) > 0$, either the polynomial $X(\beta)$ has two roots in the interval $\left(0, \frac{1}{r+\mu}\right]$, or it doesn't have any root in this interval. Thus, condition (16) is a necessary and sufficient condition for the existence of a unique LMPE.■

Proof of Proposition 2

We start from equation (10) which defines $\widetilde{x}(t)$ and we replace $p_1(t)$, $p_2(t)$ and $\Lambda_1(t)$ by their values as functions of $D_1(t)$ from (13) and Definition 1. Then replace σ by $\sigma(\beta^*)$ using (27) and δ by $\delta(\beta^*)$, using (28). We obtain

$$\widetilde{x}(\beta^*, t) = \frac{1}{2} + \frac{(2D_1(t)-1)(\beta^*(r+2\mu)-1)}{2\beta^*\mu}. \tag{34}$$

Now notice that $\beta^* > \frac{1}{r+2\mu}$ since $X\left(\frac{1}{r+2\mu}\right) = \frac{\mu(r+\mu)(\omega+5\frac{\phi}{8})}{(r+2\mu)^2} > 0$. Accordingly, if $D_1(t) > \frac{1}{2} \Rightarrow \widetilde{x}(D_1(t)) > \frac{1}{2}$, implying weak market dominance: the large network attracts more new consumers than the small network. ■

Proof of Proposition 3

The result $p_i^*(t) - p_j^*(t) = \sigma^*[D_i(t) - D_j(t)]$ follows directly from the definition of Linear Markov Price Strategies together with the symmetry of the LMPE. From (27), we can obtain the equilibrium value of σ^* as a function of the optimal β^*, with $\sigma^* = \sigma(\beta^*)$. Then, we solve the equilibrium condition (29) for τ, obtaining

$$\tau = -\frac{(\beta^*)^2 \mu(r+\mu)(4\frac{\phi}{8} + \beta^*(r+2\mu)(\omega+\frac{\phi}{8}))}{4 - 5\beta^*(r+2\mu) + (\beta^*)^2 (r+2\mu)^2}, \tag{35}$$

where β^* denotes the equilibrium value of β. Introducing (35) in (27), the value of $\sigma(\beta^*)$ simplifies to:

$$\sigma(\beta^*) = \frac{2\beta^* \left[2\omega - \beta^*(r+2\mu)\left(\omega + \frac{\phi}{8}\right)\right]}{4 - \beta^*(r+2\mu)}. \tag{36}$$

Notice that the sign of $\sigma(\beta^*)$ is the sign of the bracketed term which is decreasing in β^* and equals 0 iff $\beta^* = \overline{\beta} = \frac{2\omega}{(r+2\mu)(\omega+\frac{\phi}{8})}$. It follows that $\sigma(\beta^*) > (<)\ 0 \Leftrightarrow \beta^* < (>)\overline{\beta}$. Recall that $1 - \beta^*(2\mu + r) < 0$.

Consequently, when the existence and uniqueness condition (16) is met, it must be the case that the unique LMPE is obtained for $\frac{1}{r+2\mu} < \beta^* < \frac{1}{r+\mu}$. Since we focus on the unique LMPE, for which $X(\beta)$ has only one solution in the interval $\left(\frac{1}{r+2\mu}, \frac{1}{r+\mu}\right)$, it must be the case that $X(\beta) > 0$ for all $\beta \in [\frac{1}{r+2\mu}, \beta^*)$ and $X(\beta) < 0$ for all $\beta \in (\beta^*, \frac{1}{r+\mu}]$, since $X(\frac{1}{r+2\mu}) > 0$ and $X(\frac{1}{r+\mu}) < 0$.

Consequently:

(i) $\beta^* > \overline{\beta}$, (or, equivalently, $\sigma^*(\beta^*) < 0$), iff $X(\overline{\beta}) > 0$ and $\overline{\beta} < \frac{1}{r+\mu}$,

(ii) $\beta^* < \overline{\beta}$, (or, equivalently, $\sigma^*(\beta^*) > 0$), iff $X(\overline{\beta}) < 0$ or $\overline{\beta} \geq \frac{1}{r+\mu}$.

Notice that $X(\overline{\beta}) > (<)\ 0$ iff $\frac{\phi}{8} > (<)\ \omega - \omega^2 \frac{4\mu(r+\mu)}{\tau(r+2\mu)^2}$ and $\overline{\beta} < (\geq)\ \frac{1}{r+\mu}$ iff $\frac{\phi}{8} > (\leq)$ $\omega\frac{r}{r+2\mu}$. Therefore: $\sigma^*(\beta^*) < (>)\, 0$ iif $\frac{\phi}{8} > (<) \max\left\{\omega - \omega^2 \frac{4\mu(r+\mu)}{\tau(r+2\mu)^2}, \omega\frac{r}{r+2\mu}\right\}$. To complete the proof notice that $\omega - \omega^2\frac{4\mu(r+\mu)}{\tau(r+2\mu)^2} = \omega\frac{r}{r+2\mu}$ if $\omega = \frac{1}{2}\tau\frac{r+2\mu}{r+\mu}$ or $\omega = 0$. For $0 < \omega < \frac{1}{2}\tau\frac{r+2\mu}{r+\mu}$, $\omega - \omega^2\frac{4\mu(r+\mu)}{\tau(r+2\mu)^2} > \omega\frac{r}{r+2\mu}$ whereas for $\omega > \frac{1}{2}\tau\frac{r+2\mu}{r+\mu}$, the opposite holds. However, for $\omega > \frac{1}{2}\tau\frac{r+2\mu}{r+\mu}$, the condition $\frac{\phi}{8} > \omega\frac{r}{r+2\mu}$, cannot hold since the existence condition (16) is violated. Accordingly, under the existence condition, we have $\sigma^*(\beta^*) < (>)\, 0$ iif $\frac{\phi}{8} > (<)$ $\omega - \omega^2\frac{4\mu(r+\mu)}{\tau(r+2\mu)^2}$. ■

Proof of Lemma 1

From (36), $\frac{d\sigma(\beta)}{d\beta} < 0$, so that $\sigma^*(\frac{1}{r+\mu}) < \sigma^*(b^*) < \sigma^*(\frac{1}{r+2\mu})$. Moreover, $\lim_{r\to\infty}\sigma\left(\frac{1}{r+\mu}\right) = \lim_{r\to\infty}\sigma\left(\frac{1}{r+2\mu}\right) = 0$ so that $\lim_{r\to\infty}\sigma(\beta^*) = 0$.■

Proof of Proposition 4

Equilibrium conditions (21) and (22) lead to the equation $A_i = F_i$, $i = 2$. Solving this condition for η, after introducing $\delta = \frac{1-\beta(r+\mu)}{2(r+\mu)}$ and condition (27), we get (17). ■

Proof of Lemma 2

From (17) we have $\frac{\partial\eta(\beta^*)}{\partial\beta^*} = \tau\frac{5\beta\mu+3\beta r-4}{\beta^3\mu(r+\mu)^2}$. For $\beta < \frac{4}{5\mu+3r}$, we have that $\frac{\partial\eta(\beta^*)}{\partial\beta^*} < 0$, whereas $\frac{\partial\eta(\beta^*)}{\partial\beta^*} > 0$ for $\beta > \frac{4}{5\mu+3r}$, we have that . Accordingly, the minimum value of $\eta^*(\beta)$ occurs for $\beta = \frac{4}{5\mu+3r}$, yielding

$$\eta^*\left(\frac{4}{5\mu+3r}\right) = \frac{\tau}{r+\mu} - \frac{\gamma}{4(r+\mu)} - \frac{1}{8}\tau\frac{(r+3\mu)^2}{\mu(r+\mu)^2}$$

that corresponds to the lower threshold in Lemma 2. The behavior of $\frac{\partial\eta(\beta^*)}{\partial\beta^*}$ together with the fact that $\frac{1}{r+2\mu} \leq \beta^* \leq \frac{1}{r+\mu}$ guarantees that the maximum value of $\eta^*(\beta)$ occurs at one of the extremes of the feasible values for β^*. Evaluating (17) at the two extremes, we obtain

$$\eta^*\left(\frac{1}{r+\mu}\right) < \eta^*\left(\frac{1}{r+2\mu}\right),$$

since $-\frac{\gamma}{4(r+\mu)} < \frac{\tau}{r+\mu} - \frac{\gamma}{4(r+\mu)}$.

Hence, the upper threshold in Lemma 2 is given by $\eta^*\left(\frac{1}{\mu+2r}\right)$ that is equal to $\frac{\tau}{(r+\mu)} - \frac{\gamma}{4(r+\mu)}$.■

Proof of Proposition 5

Consider the motion law $\frac{dD_i(t)}{dt} = d_i(t) - \mu D_i(t)$. Then, substitute $d_i(t)$ by expression (15b), after introducing $\delta = \frac{1-\beta(r+\mu)}{2(r+\mu)}$ (see (28)), accounting for the linear Markov price strategies of firm i and substituting σ for expression (27). The resulting expression is a first-order differential equation, whose close solution is given by

$$D_i(t) = \frac{1}{2} + \left(D_i(0) - \frac{1}{2}\right)e^{-\left(\frac{1}{\beta}-(r+\mu)\right)t}$$

and the speed of convergence at the LMPE is therefore equal to $\frac{1}{\beta^*} - (r+\mu)$.■

Proof of Lemma 3

In the steady state LMPE, firms share the equipment market evenly and, therefore, price equipment is given by $\eta + \frac{\sigma}{2}$. Considering the equilibrium values of η^* and σ^*, one obtains, after substituting for τ its value from (35) in the third term of the RHS:

$$\overline{p} = \frac{\tau}{(r+\mu)} - \frac{\gamma/4}{r+\mu} - \frac{2\beta^*\mu\omega + \frac{\phi}{4}(4-\beta^* r)}{(r+\mu)(4-\beta^*(r+2\mu))}.$$

Since $\beta^* < \frac{1}{r+\mu}$, we have that $4 - \beta^* r > 0$. Therefore the two last terms in the previous expression are negative and the steady state equilibrium price $\overline{p}$ is below the discounted

Hotelling price $\frac{\tau}{(r+\mu)}$.■

Proof of Proposition 6

Average equipment price: The average equipment price at instant t equals $p_1(t)\widetilde{x}(t)+p_2(t)(1-\widetilde{x}(t))$ and it can be computed as follows:

$[\eta(\beta^*)+\sigma(\beta^*)D_1(t)]\widetilde{x}(t)+[\eta(\beta^*)+\sigma(\beta^*)(1-D_1(t))](1-\widetilde{x}(t))$. Substituting for $\widetilde{x}(t)$ its value from equation (34), we obtain the following second-order polynomial in $D_1(t)$:

$$\eta(\beta^*)+\frac{\sigma(\beta^*)}{2\beta^*\mu}\left[-1+\beta^*(r+3\mu)+4(\beta^*(r+2\mu)-1)D_1(t)(D_1(t)-1)\right].$$

Since $\beta^*>\frac{1}{r+2\mu}$, this polynomial is convex (resp. concave) in the market share $D_1(t)$ if $\sigma(\beta^*)$ is positive (resp. negative), taking its minimum (resp. maximum) value at $D_1(t)=\frac{1}{2}$. From Proposition 3, we know that $\sigma(\beta^*)$ is negative (resp. positive) iff $\phi/8>(<)\omega-\omega^2\frac{4\mu(r+\mu)}{\tau(r+2\mu)^2}$.

Industry profits: Note that, at instant t, industry profits in the primary market are given by $p_1(t)d_1(t)+p_2(t)d_2(t)$, which is equal to $p_1(t)\mu\widetilde{x}(t)+p_2(t)\mu(1-\widetilde{x}(t))$. Therefore, industry profits in the primary market are equal to the average equipment price times the birth/ death rate μ.

Furthermore, from equation (5) follows that industry profits in the aftermarket are equal to

$$\frac{2\phi\left[[D_1(t)]^2-D_1(t)\right]+\gamma+\phi}{4}.$$

This is unambiguously a convex function in market share $D_1(t)$ which takes its minimum value at $D_1(t)=\frac{1}{2}$. Summing the industry profits in the two markets, then substituting for $\sigma(\beta^*)$ its value from equation (36), we obtain a second order polynomial in market share $D_1(t)$, $a_0+a_1\left[[D_1(t)]^2-D_1(t)\right]$ where

$$a_1=\frac{4(2-\beta^*(r+2\mu))\left((\beta^*(r+2\mu)-1)\omega+(2+\beta^*(r+2\mu))\frac{\phi}{8}\right)}{4-\beta^*(r+2\mu)}.$$

From $\frac{1}{r+2\mu}<\beta^*<\frac{1}{r+\mu}$, we conclude that $a_1>0$. Industry profits are accordingly a convex function of $D_1(t)$ taking its minimum value at $D_1(t)=\frac{1}{2}$. From equation (**??**), it follows that they are always decreasing through time iff $D_1(0)\neq\frac{1}{2}$.

Expected lifetime utility of new consumers: The expected lifetime utility delivered by equipment i $(i=1,2)$ to a new customer located at $x(t)$ is given by equation (8) with $x_1=0$ and $x_2=1$. The average life-time utility of new customers equals

$$\int_0^{\widetilde{x}(t)}V_1(x,t)dx+\int_{\widetilde{x}(t)}^1 V_2(x,t)dx,$$

where $\widetilde{x}(t)$ follows from equation (34). Substituting for $p_1(t), p_2(t)$ and $\Lambda_i(t)$ their values as functions of $D_1(t)$ using equation (13) and Definition 1, and then replacing σ by its value from equation (27) and δ by its value from equation (28), we obtain the new customers average expected lifetime utility as the following convex second-order polynomial in $D_1(t)$:

$$\frac{\tau(1-\beta^*(r+2\mu))^2}{(\beta^*)^2\,\mu^2(r+\mu)}\left[[D_1(t)]^2 - D_1(t)\right] + b_0$$

where b_0 is a constant depending on parameter values. The expression above is always decreasing through time iff $D_1(0) \neq \frac{1}{2}$.■

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