Product Differentiation and Pricing Behavior on a Hotelling Hypercube

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February 1, 2013

Abstract

Models of oligopoly through spatial competition have traditionally limited themselves to one dimensional consumer spaces. By treating products as combinations of characteristics and letting firms compete over the space that this generates, I generalize a Hotelling framework to an n-dimensional hypercube. My main finding is that the dimension of the hypercube has a large, negative impact on prices. Additionally, I allow firms to use research and development to determine the space over which they compete to see that firms prefer hypercubes to hyperrectangles, and firms may have an incentive to invent new characteristics in order to make room for numerous competitors.

JEL codes D43, L10 and L40

1 Introduction and Relationship to the Literature

In 1929, Hotelling[3] introduced an intuitively appealing and easily computable model of duopoly. By locating firms at a distance from one another and allowing them to choose their prices in an attempt to capture physical space, he demonstrated how firms could exercise their oligopoly power without either colluding or restricting their output. Since Hotelling's initial paper, there has been a large literature dedicated to exploring and refining the idea of spatial competition. First is the important correction issued by d'Aspermont, Gabszewicz, and Thisse[2]. While Hotelling had concluded that duopolists would choose to minimize the difference between their two products, d'Aspermont *et al* showed that his framework was flawed and the equilibrium he predicted did not exist. Then, using a slight alteration to the original model, they showed the existence of an equilibrium in which firms do exactly the opposite, namely maximize differentiation.

In the next major development, Salop[6] took the model from a model of duopoly to a model of oligopoly with his celebrated "circular road" framework. While the circular road certainly poses a number of interesting questions, especially concerning the existence of equilibrium, it seems somewhat hollow as a model of oligopoly since each firm only competes with two other firms.

In order to correct this problem, Von Ungern-Sternberg[7] created a model of oligopolistic spatial competition in which each firm competes directly with each other firm. To do this for more than three firms, it was necessary for him to create an n-dimensional hyperpyramid. However, his analysis suffers somewhat in that he, like the circular road, restricts consumers to a one dimensional subset of the overall space. In the case where all vertices of the hyperpyramid are occupied, his model is not substantively different from the circular road in firm behavior.

On the other hand, there is another entire literature owing to Chamberlin[1] which considers monopolistic competition over various goods by modeling their substitutability explicitly. In this literature, the prices charged by firms are not linked to any form of distance between them, but instead by the number of firms in the market and the degree of substitutability between goods. Lancaster's[5] key insight that single goods can be viewed as bundles of characteristics offers and interesting way to bridge the two literatures. By letting the space over which firms compete be the space of consumer's preferences over possible goods, it is possible to use the insights of Hotelling styel spatial competition to address issues of product differentiation and research and development.

Irmen and Thisse[4] tested the generality of the d'Aspermonte et al's prediction by imbedding the Lancasterian notion of products as bundles of characteristics into a model of spatial competition similar to Hotelling. The natural "characteristic space" created by this interpretation took the form of a hypercube. The question then became whether or not two firms competing over this hypercube would choose to maximize their differentiation by locating at opposite corners. Irmen and Thisse found that this was not the case, but instead firms would choose to maximally differentiate in one characteristic and match each other exactly in the remaining n-1 characteristics.

While duopoly is a natural object of study on a line, it is less clearly fundamental when firms compete over a hypercube. In fact, in an n-dimensional hypercube, Immen and Thisse's positioning, given by minimal differentiation in n-1 dimensions and maximal differentiation in one dimension, can be replicated with a full 2^n firms in a hypercube by placing firms at the corners. Furthermore, since Irmen and Thisse basically show that the behavior of a duopoly on a line does not generalize into higher dimensions, it is reasonable to ask if oligopoly results are similarly distorted by the restriction to a single dimension.

The model that I introduce places the firms in a Lancasterian characteristic space in the form of an n-dimensional hypercube. I locate the firms on the vertices of the hypercube and allow them to compete over prices. My main finding from this case is that the dimension of the hypercube has a strong and negative effect on prices and profits. After basic behavior in the model is established, I allow firms to use research and development to endogenize the space over which they compete. I find that firms, in general, prefer hypercubes to hyperrectangles and that dimension has an even more pronounced negative effect on prices and profits in this setting. Lastly, I find that, despite the strong negative impact of dimension on prices, firms may still have an incentive to increase dimension to accomodate a large number of competitiors.

The rest of the paper is organized as follows. Section 2 lays out the model formally and provides an example case to clarify. Section 3 solves the model for arbitrary dimensions in the absence mergers. Section 4 extends the model from hypercubes to more general hyperrectangles and addresses some natural questions in that framework. Section 5 concludes.

2 The Model

There exists a product with n "characteristics". As an example, consider pizza with its characteristics "Thin crust vs. deep dish" and "greasy vs. non-greasy". Each of these can be normalized to a spectrum [0,s] and varied independently of each other. Then, every possible type of pizza can be represented by a square. As such, I look at an n-dimensional hypercube defined by $X_{i=1}^{n}[0, s]$. There are 2^n firms in the model, located at the vertices of the hypercube. Order the firms arbitrarily and label them firm 0 through firm $2^n - 1$.

Let x_k denote the k^{th} coordinate of the vector x. Distance is measured on the hypercube by $d(x, y) = \sum_{k=1}^{n} |x_k - y_k|$, or the L^1 distance. While the Euclidean distance is usually favored, there isn't a strong reason to prefer it on vector spaces that do not actually represent spatial location. In a certain mathematical sense, one distrance function is as good as another. I choose the L^1 distance because it yields the cleanest calculations and results [1].

There are a continuum of consumers distributed uniformly across the interior of the hypercube, with their location giving the exact configuration of the product that they most prefer. Each agent purchases exactly one good from one of the firms in the market. Agents have a linear travel cost t, such that an agent located at x receives utility $u_x(y, p) = -td(x, y) - p$ utility[2] from purchasing a product described by point y for price p. Then, an agent at x will prefer firm A to firm B if and only if $p_A + td(x, A) \leq p_B + td(x, B)$.

Let firm i be located at point z^i and charge price p^i. Define

$$S^i := \{y|\ p^i + d(y, z^i) \leq p^j + d(y, z^j)\ \forall_j\}$$

This is the set of consumers who weakly prefer firm i to any other firm. Then $\mu(S^i)$ denotes the quantity sold by firm i, where μ is the Lebesgue measure. Note that it does not matter whether or not we define S^i through weak or strict preference, as the set of indifferent consumers has a measure zero. Let $\mathbf{p}$ denote the vector of prices $(p^0, p^1, p^2...)$. I assume that firms have a constant, uniform marginal cost, m, so their profit function is given by $\pi^i(\mathbf{p}) = (p^i - m)\mu(S^i(\mathbf{p}))$. Then the firm chooses their price to maximize profits.

[1] The model is identical to one which uses the Euclidean distance but treats transport costs as quadratic. For a proof, see the appendix

[2] Agents do not have an outside option because the center of the hypercube becomes arbitrarily far away from any of the vertices as the dimensionality of the hypercube increases to infinity. That means that, for any outside option, there exists a hypercube such that some consumers will wish to excercise it. This needlessly complicates the model.

Definition 1. *A* competitive equilibrium *is a vector of prices* $(p^0, p^1, p^2...)$ *and a map* $F : \prod_{i=1}^{n}[0, s] \rightarrow \{0, 1, 2...\}$ *such that:*

1. *F(x)=i only if* $\forall_j, p^i + td(x, z^i) \leq p^j + td(x, z^j)$
2. *For each i,* p^i *maximizes* $(p^i - m)\mu(S^i(\mathbf{p}))$

Here the mapping F sends consumers to the firms they patronize and ensures that they are maximizing their utility[3]. The second condition is a familiar profit maximization condition. In order to simplify the analysis greatly, I wish to restrict attention to symmetric equilibria. Since all firms are in equivalent positions with respect to the hypercube, there should exist an equilibrium in which all firms charge identical prices.

Definition 2. *A* symmetric competitive equilibrium *is a vector of prices* $(p^0, p^1, p^2...)$ *and a map* $F : \prod_{i=1}^{n}[0, s] \rightarrow \{0, 1, 2...\}$ *such that:*

1. *F(x)=i only if* $\forall_j, p^i + td(x, z^i) \leq p^j + td(x, z^j)$
2. *For each i,* p^i *maximizes* $(p^i - m)\mu(S^i(\mathbf{p}))$
3. $p^i = p^j \ \forall_{i,j}$.

The choice to fix the firm's location that the vertices of the hypercube comes in part from its keeping with Irmen and Thisse's characterization of the optimal position for two firms in a hypercube. Note that each firm follows their characterization for each firm to which it is adjacent. However, there is an additional reason for firms to locate in this way. It will turn out, as shown in section three, that a symmetric equilibrium given by firms locating at the vertices of the hypercube is the symmetric equilibrium that maximizes profits for the firms. That is, if firms are allowed to collude on location but not on pricing, firms would opt to locate on the vertices and compete in prices, which is the situation studied in this paper.

2.1 An Example

To see how this all works, it's best to solve for the equilibrium of a simple example. Take n=2, s=t=1. There are four firms. Since I'll be looking for symmetric equilibria, it suffices to solve the problem of a single firm.

To do so, note that it is safe to assume that all other firms are choosing the same price. So, without loss of generality, consider firm zero at the point (0,0). The set of consumers indifferent between firm 0 and firm 1, located at point (1,0), is given by the equation (1), which simplifies to (2).

$$p^0 + d(x, z^0) = p + d(x, z^1) \tag{1}$$

$$x_1 = \frac{1 - p^0 + p}{2} \tag{2}$$

[3]Here each equilibrium actually represents an equivalence class of equilibria based on what exactly F has the indifferent consumers do. Since they have a measure zero, this issue has no effect on the behavior of the model.

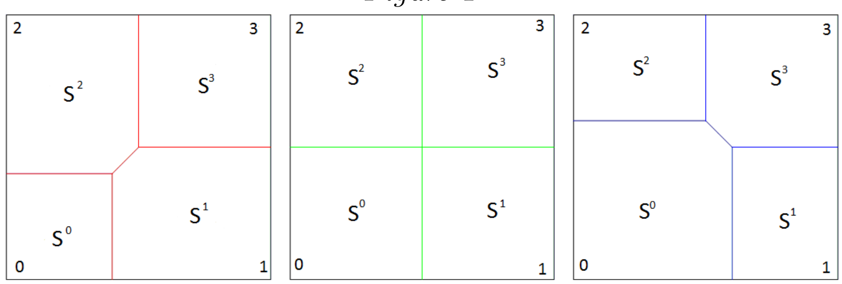
Following that line of logic, the set $S^0(\mathbf{p})$ is given by the set of x that satisfies the following constraints:

$$\begin{aligned} x_1 &\leq \frac{1-p^0+p}{2} \\ x_2 &\leq \frac{1-p^0+p}{2} \\ x_1 &\leq \frac{2-p^0+p}{2} - x_2 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

To get a better notion of what this means, some figures will be useful.

There are three general configurations for S^0 that are possible. Represented from left to right are the cases in which firm one charges a higher, equal, or lower price relative to the other three.

Figure 1



As you can see, the formula determining $\mu(S^0)$ as a function of prices depends on whether or not $p^0 < p$.

$$\begin{aligned} \text{If } p^0 < p: \quad \mu(S^0) &= \frac{(1-p^0+p)^2}{4} - \frac{(p-p^0)^2}{8} \\ \text{If } p^0 \geq p: \quad \mu(S^0) &= \frac{(1-p^0+p)^2}{4} \end{aligned}$$

Recall that firm 0's profit is given by $\pi^0(\mathbf{p}) = (p^0 - m)\mu(S^0(\mathbf{p}))$. Since the expressions are the same when $p = p^0$, the profit function is certainly continuous, but it is possibly non-differentiable at $p = p^0$, which must hold in equilibrium. For now, assume that the function is differentiable. In order to solve for p^0, I take the derivative of the profit function, set it equal to zero, and then impose that $p = p^0$. Although there may be different expressions for the right and left hand derivatives at $p = p^0$, differentiability assures us that they are equivalent. I use the right hand derivative for our calculations since it is rather simpler. Following this process, in a symmetric equilibrium $p = m + \frac{1}{2}$.

3 Solving the Model

Consider a product space with n characteristics. There are 2^n firms. In a symmetric equilibrium, all firms will charge the same equilibrium price and capture the $\frac{s}{2}$x$\frac{s}{2}$x$\frac{s}{2}$... hypercube that borders their location.

Theorem 1. *The profit function, $\pi^j(\mathbf{p})$ is everywhere differentiable.*

Proof. See Appendix B. □

I'd like to take a moment to discuss the general strategy of finding equilibria in this environment. First, it is always easiest to normalize the firm whose problem is being considered to be firm 0 located at the origin. This amounts to a rotation of the hypercube, and does not change anything essential about the model. Secondly, due to the differentiability of the profit function, I always assume that the price of the firm whose problem is being considered is greater than or equal to the price of other firms, as this greatly simplifies the expressions for the quantity captured by firms.

The most difficult part of finding the equilibrium price in this model is finding an expression for $\mu(S^0)$. Recall that S^0 is a region defined by the constraints $0 \leq x_i$ for each i and also, for each j, (3), which can be simplified to (4).

$$p^0 + t\sum_{k=1}^{n} x_k \leq p^j + t\sum_{k=1}^{n}(1 - z_k^j)x_k + z_k^j(s - x_k) \tag{3}$$

$$0 \leq \frac{st < z^j, z^j > -p^0 + p^j}{2t} - < x, z^j > \tag{4}$$

In order to find the measure of S^0, it is necessary to find out which of those constraints are redundant and which bind. Define a type j constraint to be a constraint imposed by a firm k such that $\sum_{i=1}^{n} z_i^k = j$. Geometrically speaking, type j firms differ from the origin in j coordinates. Only type one constraints bind. For proof, see Appendix A. Thus, for $p \geq p'$, the profit of firm 1 is given by equation (5).

$$(p - m)(\frac{st - p + p'}{2t})^n \tag{5}$$

Differentiating this expression, setting it equal to zero, and then setting p'=p, I find that the equilibrium is given by (6).

$$p = m + \frac{st}{n} \tag{6}$$

As mentioned previously, if each firm charges the same price, every firm will capture a hypercube of side length $\frac{s}{2}$. Thus, in a merger free n-dimensional environment, each firm will enjoy profits according to:

$$\Pi^i = \frac{ts^{n+1}}{n2^n}$$

Suppose that for any s or t chosen that the mass of consumers in the market, M, is held constant. In this new framework, firm profit functions are given by equation (7) rather than (5).

$$\pi^i(\mathbf{p}) = (p - m)\frac{\mu_i}{s^n}M \tag{7}$$

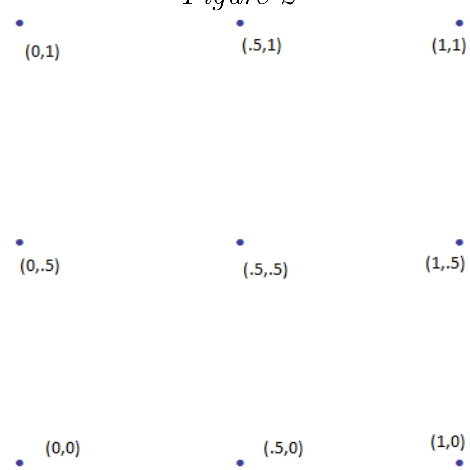
Using this new framework, it's clear that $p = m + \frac{st}{n}$ and $\Pi^i = \frac{st}{n2^n}M$. Any change in s is isomorphic to a change in t and that these parameters do not represent meaningfully distinct concepts, but rather an issue of measurment units. In keeping with the notion of the scaling of the hypercube as a normalization, fix s=1 and let t vary.

The work done here is slightly more general than it first appears. Rather than restiricting firm's location choice solely to the vertices of the hypercube, it is possible to obtain similar results from allowing them to position on the interior in a systematic way. Consider $g_k = \{0, \frac{1}{k}, \frac{2}{k}, ...1\}$.

Definition 3. *A* (k,n)-grid *is* $X_{i=1}^n g_k$ *for any natural numbers k and n.*

Figure 2 shows a simple (2,2)-grid.

Figure 2



Proposition 1. *Consider a hypercube of length 1 with travel cost t. If firms occupy a (k,n)-grid, then there exists a symmetric equilibrium in which all firms choose* $p = m + \frac{t}{kn}$.

Proof. See Appendix A. □

Consider any fully symmetric distribution of firms in the hypercube. Positioning 2^n firms symmetrically on an n-dimensional hypercube can only be accomplished by placing each firm some fixed proportion α along lines connecting vertices to the center. At $\alpha = 0$, firms are located on the vertices and at

$\alpha = 1$, all firms are located identically at the center of the hypercube. Mathematically, it is trivial to extend the work outlined in this section to show that the equilibrium in this symmetric game is given by each firm charging $p = m + \frac{t\delta}{n}$, where δ is the distance between two adjacent firms. Then, since δ decreases as α increases, it is clear that prices fall as α rises. Then, since each firm must capture $\frac{1}{2^n}$ of the consumers in the market in a symmetric equilibrium, profits are highest whenever prices are. Thus, the most profitable symmetric equilibrium for firms is the one that occurs at $\alpha = 0$, or where firms are located at the corners of the hypercube. This confirms the claim I made at the end of the model set up concerning locational collusion.

As a last note of bookkeeping, it is the case that, for all specifications of the model, prices will of the form $p = m + \hat{p}$, where $\hat{p}$ does not depend on m. Then little is lost by assuming m=0 and calling $p = \hat{p}$.

3.1 Dimension

We know that, for all firms in equilibrium, $p = \frac{t}{n}$ and $\Pi = \frac{s}{n2^n}M$ from above. Unsurprisingly, as we let n go to infinity prices and profits go to zero. This is not a shocking result of cramming an infinity of firms into a region of fixed measure, dropping the number of consumers per firm to zero. However, letting $M = 2^n$, such that the mass of consumers that each firm captures is always 1, doesn't stop the prices from dropping to zero and, of course, profits with them.

Consider an a simple circular road model where the circumference of the circle, C is always equal to m, the number of firms. Then each firm will seek to maximize the following profit function:

$$p(\frac{1 + p' - p}{2} + \frac{1 + \hat{p} - p}{2})$$

Where p is the price the firm charges and p' and $\hat{p}$ are the prices it expects its neighbors to charge. Then, in a symmetric equilibrium it will be the case that $p = p' = \hat{p} = 1$, independent of n. In a circular road, it is perhaps, unsurprising that adding more firms and consumers leaves the equilibrium unchanged, since doing so does not actually add any more firms for a particular firm to compete with, nor does it change the distance between any two firms and its competitors. A more interesting question is what happens to the hyperpyramidal structure propsed by Von Ungern-Sternberg.

In a hyperpyramid, each firm is adjacent to every other firm and consumers live on the one dimensional edges that connect firms. Then, if we set t=1, then the profit function looks rather similar to the above, given by:

$$p(\sum_{i=1}^{N-1} \frac{1 + p^i - p}{2})$$

Or, in a symmetric equilibrium:

$$(N-1)p\frac{1 + p' - p}{2}$$

Since the only way N enters the profit function is as a flat multiplier, the maximand remains independent of N, and so yields the same result as a circular road model.

The last model I wish to use for comparison is a hypercube similar to the one I have used thus far, but with consumers only living on the edges between firms as in Von Ungern-Sternberg. Amusingly enough, this yields the same profit function, scaled by $\frac{N}{N-1}$ and, as such, exhibits essentially the same solution as the circular road.

3.2 Lessons from Cubes

There are some basic comparative statics to be performed on the three parameters of the model, t, M, and n. The least interesting is the travel cost or size of the hypercube, t. Prices and profits increase linearly with t. This is not unexpected and fits easily with previous results in the literature: the more differentiated products are, the higher the profits they earn. Of middling interest is the mass of consumers in the market M. M has no effect whatsoever on prices, but has a simple linear effect on profits, similar to other models of spatial competition.

The most interesting parameter here is the dimension of the characteristic space, n. It has a strong, negative effect on prices and an even stronger negative effect on profits. This parameter doesn't have an analogue in much of the literature, with the exception of Von-Ungern Sternberg. However, despite the analogue existing, the number of dimensions in Sternberg's model has no effect on prices and even a positive effect on profits. The fact that the dimensionality of the consumer space plays such an important role in my model, the only one thus far to examine it, suggests that the previous literature's restriction of consumers to a one dimensional space makes them miss important aspects of firm behavior in spatial competition.

4 Hyperrectangles

Given a complete and relatively easy characterization of firm behavior on hypercubes, it is a fairly natural extension to examine behavior on other shapes. While the size of the region in question may be scaled arbitrarily using travel cost, there is no reason necessarily to expect that each of the characteristics can be varied exactly the same amount. That is, the natural object of study is the more general hyperrectangle, rather than a hypercube.

Suppose that there are n characteristics and the space of competition is given by $X_{i=1}^{n}[0, s_i]$. Once more, the firms are positioned at the vertices of the hyperrectangle and can be treated symmetrically. As before, assume that the mass of consumers is fixed and independent of the size of the hyperrectangle. Then, the primary difference in these set ups is that, in a hyperrectangle with symmetric behavior, a firm's profit function is given by (8).

$$\pi(\mathbf{p}) = p\frac{M}{\Pi_{i=1}^{n}s_i}\Pi_{i=1}^{n}\frac{s_i + p' - p}{2} \tag{8}$$

The profit function in (8) gives the first order condition (9).

$$\frac{\Pi_{i=1}^{n}(s_i + p' - p)}{2^n} = p\frac{\sum_{i=1}^{n}\Pi_{j\neq i}(s_j + p' - p)}{2^n} \tag{9}$$

Substituting p=p' and simplifying, the first order condition, (9), yields the equilibrium (10).

$$p = \frac{1}{\sum_{i=1}^{n}\frac{1}{s_i}} \tag{10}$$

Parallel to the cubic case, $p = \frac{hm(\{s_i\})}{n}$ where hm(.) is the harmonic mean. Of course, when all s_i are the same, the harmonic mean is equal to all of the elements, confirming that behavior in the cube is given as a special case of behavior in the hyperrectangle.

4.1 Endogeneity

Suppose that the firms may determine the size and shape of the hyperrectangle over which they compete by spending money on research and development of all of the various characteristics of the product. In particular, let firm i's contribution to dimension j be denoted by x_j^i. Then let the ultimate side length, s_j be given by $s_j = \sum_{i=0}^{2^n-1} x_j^i$. Thus, the firms are playing a Nash contribution game but, due to their symmetry, it will be efficient. Let the cost of research and development be given by eqaution (11).

$$c(\mathbf{x^i}) = \sum_{j=1}^{n}\frac{(x_j^i)^2}{2} \tag{11}$$

Each firm knows a priori that it will be located on a vertex of the hyperrectangle. Each firm chooses its contribution to research and development using backwards induction on its ultimate pricing behavior and profits. Let x_j be the sum of every other firms contribution to dimension j. Then each firm maximizes:

$$\pi^i(\mathbf{p}) = p\frac{M}{\Pi_{j=1}^{n}(x_j + x_j^i)}\frac{\Pi_{j=1}^{n}((x_j + x_j^i) + p^j - p^i)}{2^n} - \sum_{j=1}^{n}\frac{(x_j^i)^2}{2}$$

Using backwards induction, the firm may eliminate price from its profit function and simply maximize the following:

$$\pi^i(\mathbf{p}) = \frac{1}{\sum_{i=1}^{n}\frac{1}{x_i + x_i^j}}\frac{M}{2^n} - \sum_{i=1}^{n}\frac{(x_j^i)^2}{2}$$

This profit function yields the first order condition in (12).

$$x_j^i = \frac{M}{2^n(\sum_{k=1}^n \frac{x_j+x_j^i}{x_k+x_k^i})^2} \tag{12}$$

$$x_j^i = \frac{M}{n^2 2^n} \tag{13}$$

$$s_i = \frac{M}{n^2}\forall_i \tag{14}$$

(12) simplifies to (13) by noting that (12) holds for all firms simultaneously, so the sum in the denominator must simplify to n^2. Further, since there are 2^n firms, (14) follows from (13).

So, despite allowing firms to compete in a hyperrectangle, we see that they would prefer to make it a hypercube when given the choice. However, this result rings a little bit hollow. By associating the same research cost to each dimension, the equilibrium is forced, by construction, to be either symmetric or a corner solution. However, a corner solution in this context would mean a line segment. Then, having more than two firms occupy the vertices of a line segment would involve having two or more firms at literally the same place. Basic Bertrand competition reasoning tells us that this is an undesirable position for the firms.

The result here about desiring a hypercube is stronger than the proof demonstrates. Consider some off equilibrium behavior where other firms have chosen to favor some dimensions over others. For some k, let $x_k = x$ and for $j \neq k$, let $x_j = z < x$.

The the system of first order conditions gives us:

$$x_j^i = \frac{M}{2^n((\sum_{m\neq k} \frac{z+x_j^i}{z+x_m^i}) + \frac{z+x_j^i}{x+x_k^i})^2}$$

$$x_k^i = \frac{M}{2^n((\sum_{m\neq k} \frac{z+x_j^i}{z+x_m^i}) + 1)^2}$$

We can observe that the expression for all $j \neq k$ is the same, and thus the choice will be the same. With that in mind, the previous system simplifies to the following:

$$x_j^i = \frac{M}{2^n(n-1+\frac{z+x_j^i}{x+x_k^i})^2}$$

$$x_k^i = \frac{M}{2^n n^2}$$

It must be the case that $x_j^i > x_k^i$. For proof, see the Appendix. So the firm's best response is to contribute least to the side that has already been favored. This indicates that the firms have a strong interest in smoothing the sides of a hyperrectangle towards a hypercube. It is too costly to smooth perfectly, but, given the choice, firms will try to create a cube.

From this behavior we can conclude that an n-dimensonal market containing 2^n firms will tend towards an n-dimensional hypercube. Despite the side lengths of the hyperrectangle being endogenized, this construction still treats the dimension, n, as exogenous. The next natural question is how best to represent firm's choice over the dimensionality of the space in which they compete.

4.2 Dimensions Revisited

Consider, as an example, the question of whether or not four firms would prefer to occupy a line segment or a square. To make this issue non trivial, we will assume that, instead of positioning firms at the corners of the line segment, they are spaced evenly about its length. While firms are no longer treated equally in this set up, they will choose the same prices, from theorem 2.

The firms will be placed at $0, \frac{s}{3}, \frac{2s}{3}$, and s. The firms at 0 and s are symmetrically positioned, as are the two in the interior. The firms on the border and interior will maximize equations (15) and (16) respectively.

$$\frac{x + x^i}{3}\frac{M}{6} - \frac{(x^i)^2}{2} \tag{15}$$

$$\frac{x + x^i}{3}\frac{M}{3} - \frac{(x^i)^2}{2} \tag{16}$$

Then firms on the border will choose $x^i = \frac{M}{18}$, whil interior firms will choose $x^i = \frac{M}{9}$. Then $s = \frac{M}{3}$. The profits of interior firms are given by $\frac{M^2}{27} - \frac{M^2}{162}$, while the boundary firms enjoy $\frac{M^2}{54} - \frac{M^2}{648}$.

From the earlier work, we know that the profits of firms when n=2 is given by $\frac{M}{4}\frac{M}{4} - \frac{M^2}{256}$. Comparing, we see that the profits of interior firms are greater than those in the square are greater than those on the boundary. In this case, at least two firms would want to defect to a square, which is all that would be needed. Alternatively, if firms did not know ahead of time whether they would be interior or boundary, firms would prefer the square in expected value. To rephrase, the square has a higher total welfare, from the firm's perspective, than the line segment.

5 Conclusions and Extensions

In this paper I have introduced an easily computable and reasonably general model of spatial competition over a higher dimensional product space. Higher dimensional product spaces are important because they allow each consumer to choose from more than two firms, whereas the prior one dimensional models only allow a binary choice. Appealing to a Lancasterian notion of the nature of goods, the natural space of competition here is a hyperrectangle. By analyzing firm behavior in this environment, it is clear that if firms can jointly control the space over which they compete, they will choose a hypercube.

From analyzing firm behavior on the hypercube, my main finding is that the dimension of the space over which firms compete has a strong negative effect on both the pricing behavior and profits of firms, even when separated from conflating factors like the mass of consumers per firm. As this is the first model to consider a product space in which consumers occupy a higher-than-one dimensional subset of the overall space, I believe that this is an important contribution to the literature.

To develop this contribution further, I feel that a natural next step is to examine the effects of mergers on the pricing behavior and profits of firms in a hypercube set up. If individual firms behave in a qualitatively different way on a hypercube than they do on a circular road, then it makes sense that coalitions would as well. Additionally, unlike the circular road, a hypercube framework allows for large mergers that treat every merging firm symmetrically.

Following much of the work done on the circular road framework., I feel that it would be valuable to endogenize firm location decisions, likely through a simultaneous entry game. I suspect that the force of firm's attraction to the center would be inversely proportional to the dimensionality of the market.

By doing both extensions at once, it would be possible to analyze the effect of allowing merged firms to locate strategically, as opposed to forcing them to locate, and afterwards allowing them to merge.

As a final note with respect to the importance of dimension in this model, it leaves us with some interesting implications. Products with a small n are predicted to be the most profitable. These products are simpler than products with large n. This makes sense at first, considering that many of the simpler products on the market are simple because they are new, and new products are generally profitable. However, this does not explain the process by which new products become old products. That is: if simple products are most profitable, why is it that firms devote so much time and energy to making their products more complicated? The example from 4.2 with the line segment and square provides a partial answer to this quandry. It suggests that each additional firm crowds the market less in higher dimensional spaces, and so products are made to be more complex in order to make room for many competitors.

A Proofs

A.1 Proof of equivalence of metrics

Proof. I claim that the model described in this paper is identical to one in which preferences are given as in equation (17), with $\hat{d}$ denoting the Euclidean distance.

$$u_x(y,p) = -t\hat{d}(x,y)^2 - p \tag{17}$$

It is sufficient to show that a consumer prefers firm i to firm j under one preference specfication if and only if it prefers firm i to firm j under the other preference specification.

Normalize firm i to firm zero located at the origin. A consumer located at x prefers firm zero to firm j under the L^1 specification if and only if equation (18) is satisfied.

$$p^0 + t\sum_{k=1}^{n} x_k \leq p^j + t\sum_{k=1}^{n}(1 - z_k^j)x_k + z_k^j(1 - x_k) \tag{18}$$

$$p^0 \leq p^j + t\sum_{k\in K_j}(1 - 2x_k) \tag{19}$$

It is possible to simplify from equation (18) to equation (19) by subtracting the sum across and letting $K_j = \{k|z_k^j = 1\}$.

Keeping that in mind, let us repeat the process for the alternative specification. Under the Euclidean specification, a consumer at x prefers firm zero to firm j if and only if equation (20) is satisfied.

$$p^0 + t\sum_{k=1}^{n} x_k^2 \leq p^j + t\sum_{k=1}^{n}(1 - z_k^j)x_k^2 + z_k^j(1 - x_k)^2 \tag{20}$$

$$p^0 \leq p^j + t\sum_{k\in K_j}(1 - 2x_k) \tag{21}$$

With K_j defined as above, equation (21) simplifies into equation () by simply expanding the exponents and subtracting the sum of squares from each side. Since the two set ups produce the same conditions on preferences, they produce identical models.

□

A.2 Proof of Theorem 1

Theorem 1: $\pi^i(\mathbf{p})$ is differentiable at $\forall_i, p_i = p$.

Proof. I want to show that, if a firm expects all other firms to charge a uniform price, that firm's profit function is everywhere differentiable in its own price. First note that $\pi(\mathbf{p}) = (p^0 - m)\mu(S^0)$. Since the product of two differentiable functions is differentiable, then it suffices to show that $\mu(S^0(\mathbf{p}))$ is differentiable in p^0. Then S^0 is equal to the intersection of the hypercube and the set of consumers that prefer firm 0 to firm j, for each j. Suppose that all other firms charge a uniform price p. The region S^0 is given by the set of x satisfying constraints (22) and (23).

$$\forall_i, 0 \leq x_i \tag{22}$$

$$\forall_{j\geq 1}, 0 \leq \frac{< z^j, z^j > -p^0 + p}{2} - < z^j, x > \tag{23}$$

Define $\alpha = p - p^0$. Suppose that $p^0 > p$ and $\alpha < 0$. We know from a following proof that only type one constraints bind. Then, S^0 is merely a hypercube with side length $\frac{1+\alpha}{2}$.

$$\text{If } \alpha < 0, \quad \mu(S^0(\mathbf{p}) = (\frac{1+\alpha}{2})^n := F(\alpha)$$

$F(\alpha)$ is clearly differentiable in both alpha and p^0. For the other case, suppose that $p^0 < p$. Every type of constraint will bind in this case. Once more consider the hypercube defined by the type 1 constraints. Let

$$A^j := \{x \in \mathbb{R}^n | x_i \leq \frac{1+\alpha}{2} \forall_i \text{ and } 0 \geq \frac{< z^j, z^j > +\alpha}{2} - < z^j, x >\}$$

This is the set of points excluded from the hypercube by constraint j. Consider a constraint from firm j of type k, $0 \leq \frac{k+\alpha}{2} - \sum_{=1}^{k} x_i^j$. To find the measure of A^j, set up the following integral:

$$\int_0^{\frac{1+\alpha}{2}} \int_0^{\frac{1+\alpha}{2}} \cdots \int_{\frac{1-(k-2)\alpha}{2}}^{\frac{1+\alpha}{2}} \int_{\frac{2-(k-3)\alpha}{2} - x_k}^{\frac{1+\alpha}{2}} \int_{\frac{3-(k-4)\alpha}{2} - x_k - x_{k-1}}^{\frac{1+\alpha}{2}} \cdots \int_{\frac{k+\alpha}{2} - \sum_{j=2}^{k} x_j}^{\frac{1+\alpha}{2}} 1 \, \mathrm{d}x_1 \, \mathrm{d}x_2 \ldots \mathrm{d}x_n$$

Since neither the constant function being integrated nor any of the bounds of integration depend on x_{k+1} through x_n, their only contribution to the integral will be to multiply by a constant. So the above reduces to

$$(\frac{1+\alpha}{2})^{n-k} (\int_{\frac{1-(k-2)\alpha}{2}}^{\frac{1+\alpha}{2}} \int_{\frac{2-(k-3)\alpha}{2} - x_k}^{\frac{1+\alpha}{2}} \int_{\frac{3-(k-4)\alpha}{2} - x_k - x_{k-1}}^{\frac{1+\alpha}{2}} \cdots \int_{\frac{k+\alpha}{2} - \sum_{j=2}^{k} x_j}^{\frac{1+\alpha}{2}} 1 \, \mathrm{d}x_1 \, \mathrm{d}x_2 \ldots \mathrm{d}x_k)$$

The region A^j is, geometrically speaking, a hyperpyramidal hyperprism. That is, it is a k dimensional hyperpyramid that has been prismed into n-k other dimensions. Picturing the regions A^j for a three dimensional cube may make it clearer why this must be the case, and why such a shape would lead to the above integral. Evaluating the integrals yields a much simpler expression:

$$(\frac{1+\alpha}{2})^{n-k} \frac{(k-1)^k \alpha^k}{2^k k!}$$

It should be noted that $\mu(A^j) = \mu(A^i)$ if firms i and j are both type k. This is true because the hypercube may be rotated to move firm i into the position of firm j while preserving firm zero at the origin. Lastly, note that there are $\binom{n}{k}$ type k firms. Consider the function

$$H(\alpha) := \sum_{j=1}^{2^n-1} \mu(A^j) = \sum_{k=2}^{n} \binom{n}{k} (\frac{1+\alpha}{2})^{n-k} \frac{(k-1)^k \alpha^k}{2^k k!}$$

$H'(\alpha) \geq 0$ by inspection. If $\alpha > 0$, then $\mu(S^0(\mathbf{p})) = F(\alpha) - \mu(\bigcup_{j=1}^{2^n-1} A^j)$. If all of the A^j were pairwise disjoint, then the above would be equal to $F(\mathbf{p}) - H(\alpha)$.

However, since the sets are not disjoint, construct the following sets.

$$E_2 := \{x \in \mathbb{R}^n | \exists_{i,j} \text{ such that } x \in A^i \cap A^j\}$$
$$E_3 := \{x \in \mathbb{R}^n | \exists_{i,j,k} \text{ such that } x \in A^i \cap A^j \cap A^k\}$$
$$\vdots$$

In general, let E_i be the set of x such that x is in the intersection of at least i A^j. Define $G(\alpha) := \sum_{k=2}^{2^n-1} \mu(E_k)$. Then:

$$\mu(S_0(\mathbf{p})) = F(\alpha) - H(\alpha) + G(\alpha)$$

$G'(\alpha) \geq 0$, since each A^j is strictly increasing in α, it must also the case that each E_i is also strictly increasing in α. Looking at the functional form of $H(\alpha)$, it is obvious that $H'(\alpha) \geq 0$. Since $H(\alpha)$ is a polynomial in α where each term is of degree at least two, $H'(\alpha)|_{\alpha=0} = 0$. Now, note that $G(\alpha)$ would grow fastest relative to $H(\alpha)$ in α if all of the new volume were in the intersection of every A^j. Suppose, for a moment, that that is the case. $H'(\alpha) = (2^n - 1)\frac{\partial}{\partial \alpha}A^j$, while $G(\alpha) = (2^n - 2)\frac{\partial}{\partial \alpha}A^j$. So even in this most ideal case, $G'(\alpha) \leq H'(\alpha)$. Finally, the chain rule tells us that $\frac{\partial}{\partial p^0}F(\alpha) = F'(\alpha)$, and likewise for $H(\alpha)$. From the above, $\frac{\partial}{\partial p^0}\mu(S^0(\mathbf{p}))$ is bounded between $\frac{\partial}{\partial p^0}F(\alpha)$ and $\frac{\partial}{\partial p^0}(F(\alpha) - H(\alpha))$. Since these derivatives are the same when $\alpha = 0$, the squeeze theorem implies that the righthand derivative of $\mu(S^0(\mathbf{p}))$ at $p^0 = p$ is equal to $\frac{\partial}{\partial p^0}F(\alpha)$ at $\alpha = 0$. This is also the left hand derivative of $\mu(S^0(\mathbf{p}))$ at $p^0 = p$, and so $\mu(S^0(\mathbf{p}))$ is differentiable at $p^0 = p$. It is differentiable everywhere else by inspection. □

A.3 Proof That Only Type One Constraints Bind

Proof. We may assume that the price charged by the firm we are considering, p, is weakly larger than that it expects iits competiors to charge, $\hat{p}$. This is safe due to the differentiability of the profit function and the fact that, in equilibrium, they wil all charge the same price. lI wish to show that it is impossible for a vector to satisfy $x_j \leq \frac{st-p+\hat{p}}{2t}$ for all j, but violate a constraint of the form $x_1 \leq \frac{kst-p+\hat{p}}{2t} - x_2 - x_3 ... x_k$. It suffices to show that, for x_2 through x_k as large as they can be, the latter constraint is still looser than the type one constraint for x_1. Plugging these in, the type k constraint reduces to $x_1 \leq \frac{st+(k-2)(p-\hat{p})}{2t}$, whereas the type one is given by $x_1 \leq \frac{st-p+\hat{p}}{2t}$. Since $p \geq \hat{p}$ by assumption and in equilibrium, this holds and the type k constraint is, in fact, redundant. □

A.4 Proof of Proposition 1

If firms are located on a (k,n)-grid, then there exists a symmetric equilibrium in which firms choose $p = m + \frac{t}{kn}$.

Proof. In a (k,n)-grid, each firm is $\frac{1}{k}$ from its nearest neighbor. If a firm expects other firms to charge a uniform price, $\hat{p}$, then interior firms will face a profit function given by equation (24).

$$(p - m)\frac{(\frac{t}{k} - p + \hat{p})^n}{t} \tag{24}$$

Firms that border one side of the hypercube will face the same profit function, but scaled by $\frac{1}{2}$. Similarly, firms that border two sides of the hypercube will face the interior profit function, but scaled by $\frac{1}{4}$. This pattern will continue, and every firm will face a scalar multiple of the interior firm's profit function. Therefore their maximands will all be the same, $m + \frac{t}{kn}$ and the equilibrium will be symmetric. □

A.5 Proof of inequality

I claim that $x_j^i \geq x_k^i$.

Proof. Recall our work in section 4.2.

$$x_k^i = \frac{M}{2^n n^2}$$

$$x_j^i = \frac{M}{2^n(n - 1 + \frac{z + x_j^i}{x + x_k^i})^2}$$

Define the following:

$$f(x) = x$$

$$g(x) = \frac{M}{2^n(n - 1 + \frac{z + x_j^i}{x + x_k^i})^2}$$

Clearly, x_j^i will be such that $f(x_j^i) = g(x_j^i)$. f(x) is increasing, continuous, and zero at zero. g(x) is decreasing, continuous, and positive at zero. Then, if $f(x) < g(x) \implies x > x_j^i$ and $g(x) < f(x) \implies x < x_j^i$. Consider $x^* = x - z + \frac{M}{2^n n^2}$. $f(x*) = x - z + \frac{M}{2^n n^2} > \frac{M}{2^n n^2} = g(x^*)$. Then x_j^i exists and is less than x^*. Consider $\hat{x} = \frac{M}{2^n n^2}$. Note that $g(\hat{x}) < f(\hat{x})$, so it must be the case that $\frac{M}{2^n n^2} < x_j^i < x - z + \frac{M}{2^n n^2}$. Since $x_k^i = \frac{M}{2^n n^2}$, this concludes the proof. □

References

[1] Chamberlin, Edward H. (1933 and 1962), The Theory of Monopolistic Competition. Cambridge, MA: Harvard University Press.

[2] C. d'Aspremont, J. J. Gabszewicz, and J.-F. Thisse, On Hotelling's "stability in competition", Econometrica 47 (1979), 1145-1150.

[3] Hotelling, H., 1929, Stability in Competition, The Economic Journal, 153(39), 41-57.

[4] Irmen A. and J.-F. Thisse, 1998. Competition in multi-characteristics spaces: Hotelling was almost right. Journal of Economic Theory 78, 76-102.

[5] Lancaster, K.J. (1966), A New Approach to Consumer Theory, Journal of Political Economy, 74: 132-157.

[6] Salop, S.C. Monopolistic Competition with Outside Goods. Bell Journal of Economics, Vol. 10 (1979), pp. 141-156.

[7] Von Ungern-Sternberg, T., 1991, Monopolistic competition in the pyramid, The Journal of Industrial Economics, XXXIX (4), 355-368