First-Price Auctions with Speculative Resale

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Abstract

In this paper, we investigate the model of speculative resale in auctions. Resale (or secondary trade) is allowed for each bidder, but in equilibrium we show that there is only speculative resale. We consider a standard first-price auction with symmetric independent private values (IPV) among N bidders and several speculators. In the presence of resale, the winner in the first-price auction uses an optimal mechanism to resell the object to the losers. We establish a single crossing property of the model. We show the bidding function of the regular bidders to be symmetric, and is the maximum of two simple functions. We characterize the conditions under which speculators are active in equilibrium, and the bidding intervals in which they are active. Necessary and sufficient conditions are given for the speculators to be active in equilibrium. General welfare and efficiency consequences of speculative resale are possible. Resale enhances the seller revenue when speculators are active in equilibrium, but there is a loss of ex post efficiency. We show the equilibrium to be essentially unique (except for the individual speculators). For many use value distribution functions, speculators are inactive in equilibrium, and so are they when there are many regular buyers. In general speculators are active in disjoint bidding intervals, a phenomenon that we expect to hold in more general auctions with resale.

1 Introduction

Whenever there is asymmetry between bidders, winners of the auction will be interested in resale. When we allow resale, many interesting phenomenon occur. For example, if you start with private value auctions, but allow resale, then it introduces value dependence between the bidders as the resale revenue is interdependent. This means that the standard benchmark model with private values no longer serves as a good guidance in the analysis of auctions. In auctions with resale, the typical case involves a mixture of private-value and common-value elements. The dependence between valuation among the bidders introduced by resale is probably the most interesting and important type of value dependence in the real world. However it has not received sufficient attention in the literature. Hazlett and Oh (forthcoming) have argued that secondary trade or resale is more important than exact regulation on spectrum rights to avoid the inefficiency from harmful interference. Some of the recent discussions of incentive auctions (Hazlett, Porter and Smith (2012)) have information revelation issues that are also relevant in equilibrium with secondary trade. Resale also is important in explaining many problems of information aggregation or herding behavior when valuation becomes dependent through resale.

In this paper we deal with many fundamental issues of auctions with resale. Because of the complexity of analyzing auctions with resale, we will focus on the important case of speculative resale. There are two types of bidders, speculators who are in the auction only for resale, and N regular bidders who have use value for the single object for sale. The analysis provides many inspirations for the more general case. Although this case has been analyzed by Garratt and Troger (2006, supp), and we have learned a lot from their work, the analysis provided there is far from complete. We will refer to the supplement as GT (supp). We should provide a comprehensive treatment of the speculative behavior in auctions, and how speculators affect the equilibrium outcome. We provide a unique and simple characterization of the equilibrium with speculators. The equilibrium bidding function of a regular bidder is the maximum of two bidding functions, one is based on the equilibrium bidding function $B^*(v)$ of the model without resale subject to different initial conditions. Another one is $B(v)$ defined by the optimal resale revenue when a speculator sells to buyers with use value bounded by v. We also give a simple formula for the joint bid distribution $H(.)$ of the speculators using the inverse $\eta(b)$ of the bidding function $B(v)$. We offer necessary and sufficient conditions under which speculators are active in equilibrium (bidding above 0). When speculators are active in equilibrium, we give a simple characterization of the intervals of active bids which in general is a disjoint union of open intervals. To describe the active bidding intervals, we define a function $B^c(v)$ which plays a key role in the relationship between the two functiolns $B(.), B^*(.)$. The two functions can be viewed as revenue (from the resale auction after winning) and cost (of winning the first stage auction) functions of the speculator respectively. It is an important factor in the derivative of the speculator's profit function (or $B(.)-B^*(.)$). The union of all active bidding intervals of the speculators is simply the range of $B(v)$ when $B^c(v) > B(v) > B^*(v)$. In addition, we give a clear picture of how regular bidders are affected by the presence of speculators, and a welfare analysis of the regular bidders and the original seller.

Speculators are said to be inactive in equilibrium if $H(0) = 1$. In this case speculators always bid 0, and has no effect on the bidding behavior of the model. It is as if there are no speculators. When there are no speculators in the auction with resale model, one consequence of our result is that the equilibrium is unique and must be the same as the well-known equilibrium in the symmetric auction without resale. This result, although quite intuitive, and implicity used by many, has not been shown before, and must be dealt with in our analysis. When you allow resale, it is by no means obvious that the only equilibrium possible is the one without resale. Even though there is no resale in equilibrium, resale may occur out-of-equilibrium. It may also be possible that there is a non-symmetric equilibrium with resale between the bidders. You need to rule out the possibility that some bidders may be able to bid lower and buy back later during resale in equilibrium. In fact its proof is almost as involved as the more general case when we allow speculators.

Another major contribution we offer is a proof of the symmetry property that says that all regular bidders in equilibrium bid the same way. This is important for us to claim that the model has only speculative resale in equilibrium. We also prove the single crossing property for the auction with resale model, an important property for the analysis of auctions. The single crossing property is proved under very general conditions without requiring any symmetry of the bidding strategies or value distributions.

A speculator is said to be active at a bid $b > 0$, if b is in the support of $H(.)$. We said that a speculator is fully active if $H(0) = 0$. Our result shows that a speculator's bid distribution has an atom at 0, which also means that a speculator cannot be fully active in equilibrium. Furthermore, speculators are active over a disjoint union of intervals. We expect this phenomenon of disjoint active intervals to hold in more general auctions with resale model. Surprisingly, equilibrium with inactive speculators can occur quite often. We show that when the distribution function $F(v)$ is a power function $F(v) = v^a, a > 0$, a quadratic function, or a concave function, a speculator must be inactive in equilibrium. When there are many regular bidders, speculators must be inactive. We also give a more general condition under which speculators must be inactive in equilibrium. In section 4, necessary and sufficient conditions are given for the speculators to be active in equilibrium. GT (supp) has shown that if speculators are active in equilibrium, the original seller (the auctioneer) has higher revenue from the presence of the speculators. Since speculators are often inactive, it is important to know under what condtiions they will be active. We provide many answers to this question.

When there is only one regular bidder, a speculator must be fully active with a bid distribution given by $F(\eta(b))$, where $\eta(.)$ is the inverse function of $B(.), N = 1$. Hence there is a major difference in equilibrium behavior with one or more than one regular bidders. The equilibrium of the one-bidder model is closely related to the Wilson Drainage Tract model. Bidding behaviors are different in the two models. The latter model has a bidding function $B^*(v), N = 2$, while the former has the bidding function $B(v), N = 1$. In section 5, we give an example of a two-bidder model in which the speculator is active. The value distribution is piecewise linear. The graph of the two bidding functions show that the equilibrium bidding strategy $\bar{B}(v) = \max\{B^*(v), B(v)\}$ has a jump in derivatives at the point of intersection, so that we expected piecewise smooth bidding functions in this model. The active bidding interval is shown in the graph.

Speculative resale is an interesting and important model to study the effects of resale on the seller's revenue and the efficiency of auctions. This model gives us an intuitive (though somewhat special) understanding of the resale effect on such matters. Hafalir and Krishna (2009) show that, in a two-bidder model, allowing resale increases the auctioneer's revenue for three types of value distributions that are more easily analyzed in the literature. They also give an example to illustrate the possible loss of efficiency due to resale. Our model is somewhat special as it only has speculative resale in equilibrium. However the results are very elegant. Resale always increases revenue without special assumptions on the value distribution of the regular buyers when speculators are active in equilibrium. The auction without resale is efficient, but there is a loss of ex post efficiency due to the fact that the speculator may win in the first-stage auction, and then fail to resell to regular buyers in the second stage optimal auction. In other words, the resale makes the first-stage auction allocation less efficient, and that fails to be corrected during resale, as explained by Hafalir and Krishna (2009). GT (supp) is the first to show the revenue enhancing effect of speculative resale. We offer more comprehensive comparisons in the paper, including conditions under which speculators are active or not.

Resale fails to correct efficiency, due to the trade-off between revenue and ex post efficiency in the resale stage when the speculators try to maximize their revenue from resale. If the seller chooses to maximize revenue by using reservation prices, then the whole question needs to be addressed in a different manner. The question then is: what is the optimal mechanism of selling an object when resale cannot be forbidden? How would it affect the efficiency or revenue? These are questions first studied by Zheng (2002), a question we will turn to in a separate paper

We assume no discounting in our analysis, although the analysis can be easily adopted for this case. We also do not consider a reservation price for the original seller. A speculator is assumed to be more skillful in knowing how to choose the optimal reservation price from their knowledge about the distribution $F(.)$. We can consider a reservation price for the original seller, and we expect the analysis to be quite similar. However, the introduction of a reservation price for the regular bidders raises the issue of the optimal choice of such reservation price. We also need to allow different reservation prices for the speculators and regular buyers in finding the optimal auctions with resale, issues beyond the scope of the paper. Virag (2011) considered auctions with resale with two types of regular buyers. The bid distribution of the two types of buyers are not the same. Ours can be considered a special case of theirs and similar results hold. However, they did not show the symmetry property, and equilibrium characterization is much more simpler here.

The single crossing property and the symmetry property are given in section 3. Their proofs require complicated notations, and are given in section 7. There is no symmetry property for the speculators. In fact speculators can bid differently in equilibrium, but its joint bid distribution is uniquely determined. This is the reason we use the qualifier "essentially" unique. We also show that an auction with many speculators can be reduced to an auction with a single speculator with the same equilibrium outcome.

The comparison of equilibrium with and without active speculators is given in section 4. Regular bidders with a use value that corresponds to an active speculator bid are most affected by the presence of the speculators. They bid more aggressively to eliminate the profit of the speculators, otherwise they may have to pay more during resale compared to what they are paying in the first stage. The original seller benefits from the presence of the speculators when they are active in equilibrium. We also rasie questiona about ex ante efficiency from speculative resale.

In section 4.1, we give the main results on inactive speculators. We identify conditions under which speculators become inactive so that they have no impact on the equilibrium outcome. It turns out that many well-known distributions lead to inactive speculators in equilibrium. Hence we are interested in knowing exactly under what conditions this can happen. In section 6, we deal with more general assumptions, extending the results obtained in the main text under simplifying assumptions. In particular, the case of infinitely many active bidding intervals is treated. In section 7, we provide the more complicated notations and proofs.

2 A Model of First-price Auction with Speculative Resale

There are two types of buyers: regular buyers and speculators. Regular buyers have use value for the object, while speculators have no use value for the object and participate in the auction only for resale. There are N regular symmetric buyers and p speculators bidding for one object sold by the auctioneer (sometimes called the original seller to distinguish from the seller in the resale market). A regular buyer has a use value distribution $F(v)$ over $[0, \beta]$, which is assumed to be C^2 smooth. Let $f(v)$ denote the density function of $F(.)$. We assume $f(0) > 0$ to simplify the exposition in the main text. More general assumptions are allowed in section 6[1]. It is convenient to use the notation $F(.|v) = \frac{F(.)}{F(v)}$ for the conditional distribution of $F(.)$ when the use value has an upper bound v. When the underlying upper bound is clearly understood, we may use the simpler notation $\tilde{F}(.)$. The speculators will be indexed by s, while the regular buyers are indexed by $i = 1, 2, ..., N$. Although it may seem more general to assume more than one speculator, we will show that indeed a model with one speculator will be sufficient for the study of the effect of speculation on bidding behavior in this paper. The multi-speculator model is equivalent to a single speculator model. So readers may want to limit their attention to this case.

The first-price auction with resale is a two-stage game. We limit ourselves to only one- resale opportunity even though it may still be desirable for the holder of the object to resell again at the end of the second stage. The first stage auction is a first-price auction with no reservation price. A winner of the auction in the first stage may sell to the losers in the second stage when it is profitable. The seller in the resale stage chooses an optimal mechanism. By the revelation principle, we can assume that the resale mechanism is a second-price auction with optimally chosen reservation prices. At the end of the first-stage auction and before the beginning of the resale stage, the winning price (the highest bid) is announced[2]. We will assume no discounting between the first stage and the second stage.

The winner of the first-stage auction, i.e. the seller in the resale auction, will revise his or her belief about the losers' value distribution in the resale stage. Let $b_i(v), i = 1, 2, ..., N$ be the strictly increasing C^1 smooth bidding strategy[3] of a regular buyer i in the first stage auction, and let $\phi_i(.) = b_i^{-1}(.)$ be the inverse bidding function. The revised belief is described by the conditional distribution $F(.|\phi_i(b))$ for the losing buyer i, when b is the winning bid. Since $\phi_i(.)$ are not necessarily the same for all i, in general, the seller in the resale auction faces an asymmetric auction environment. Such revision of beliefs is common knowledge as the winning bid is public information. A speculator uses a mixed strategy of bidding, represented by a cumulative bid distribution function $H_s(b)$. We allow $H_s(.)$ to be degenerate at 0, or has an atom at 0, but is atomless at any $b > 0$. We also assume that $H_s(b)$ are weakly increasing and C^1 smooth. Let $H(b) = \prod_{s=1}^{p} H_s(b)$. A speculator my become inactive over some interval in which $H(b)$ is a constant. We will show later that a speculator typically is active over disjoint intervals. In other words, the support of $H(b)$ may not be connected. Since the game in the resale auction is well-understood, we will summarize the result of the resale auction by a profit function without specifying the second period strategies, so that we can focus on the first-stage bidding behavior anticipating the optimal resale outcome in the second stage.

To define an equilibrium strategy of the auction with resale, let σ denote the strategy profile $b_i(.), H_s(.), i = 1, 2, ...N, s = 1, 2, .., p$. Let σ_{-s} be the profile without $H_s(.)$, and $\pi_s(b, \sigma_{-s})$ denote the optimal resale profit (or payoff) during the resale stage. The speculator s chooses b to maximize the overall profit

$$u_s(b, \sigma_{-s}) = \pi_s(b, \sigma_{-s}) - b \prod_{k \neq i} H_k(b) \prod_{i=1}^{N} F(\phi_i(b)).$$

Given σ_{-s}, we say that b is an optimal bid for the speculator s if b is an optimal solution of the above maximization problem. Since a speculator uses a mixed strategy, all bids in the support of the bid distribution $H_s(b)$ must yield the same payoff to the speculator s.

[1] We can also allow piecewise smooth value distributions. This is done in the extension section. Our main example in the paper uses a piecewise linear value distribution function $F(.)$. We can also allow all concave $F(.)$, and all the power functions $F(v) = v^a, a > 0$, which has ∞ derivative at 0, when $a < 1$, and zero derivative at 0 when $a > 1$. All our results apply for such cases.

[2] Haile (2000,2001,2003) did the pioneering works on auctions with resale. However, in his models, resale arises out of new information. Our models are fashioned along the lines of Hafalir and Krishna (2008), Garratt and Troger (2006), Cheng and Tan (2010) and Lebrun (2010), in which there is an incentive for resale due to asymmetry between buyers in the first stage auction. Second-price auctions with resale are studied in Garratt, Tröger, and Zheng (2009).

[3] We will show the single crossing property for the auctions with resale model here. Therefore there is no loss of generality in restricting to the increasing bidding strategy. If it is not strictly increasing, there will be atoms above 0 in bid distribution which is not compatible with equilibrium conditions. Strictly speaking, the bidding strategies are only smooth in each of the bidding intervals we shall consider. At the connection point of the intervals, we may have kinks which has no effect on the equilibrium property. These more general specifications can be allowed, and are discussed in section 6.

After a regular buyer i with use value v_i submits a bid b, and wins the auction, he or she updates the belief regarding the other regular buyers the same way as the speculators, and assume that buyer j is has the value distribution $F(.|\phi_j(b)]$. If b is not in the range of $b_j(.)$, we assume that there is no change in beliefs regarding buyer j. Buyer i may sell it to buyer j during the resale stage if $\phi_j(b) > v_i$. If buyer i loses the auction, and the winner is a speculator or some regular buyer j with $\phi_j(b) < v_i$, the buyer may bid for the object and buy it from the winner during resale. When there is no resale, bidder i keeps the object.

Let $\pi_{wi}(v_i, b, \sigma_{-i})$ be the expected payoff of the seller i in the resale market after winning, and $\pi_{li}(v_i, b, \sigma)$ the corresponding amount after losing. Then the overall payoff from bidding b is

$$u_i(v_i, b, \sigma) = (v_i - b)H(b)\prod_{j\neq i} F(\phi_j(b)) + \pi_{wi}(v_i, b, \sigma_{-i}) + \pi_{li}(v_i, b, \sigma). \tag{1}$$

We say that b is an optimal or equilibrium bid for the regular buyer i with use value v_i if it maximizes $u_i(v_i, b, \sigma)$. We say that σ is a perfect Bayesian equilibrium of the auction with resale if (i) for each speculator s, any bid b in the support of $H_s(b)$ is an optimal bid for the speculator, (ii) for each regular buyer i, $b_i(v_i)$ is an optimal bid maximizing (1).

A perfect Bayesian equilibrium of the auction with resale should describe strategies in both stages of the game. For convenience, we shall abuse the language somewhat and refer to σ as a perfect Bayesian equilibrium. This is due to the fact that the resale game is easily understood. In equilibrium, if $b_i(v) = b_j(v)$, for all $i, j = 1, 2, .\, .., N$, we say that the equilibrium has symmetry property. Symmetry property for the speculators in equilibrium does not hold in general, and will not be needed for our analysis. When the symmetry property holds, the resale auction by the speculator is a symmetric auction with a single reservation price. It is convenient for the presentation to assume that the virtual value is increasing, so that there is a unique optimal reservation price[4]. Let

$$J(x, w) = x - \frac{F(w) - F(x)}{f(x)}$$

denote the conditional virtual value of x when the buyer value upper bound is w. If $J(x, \beta)$ is strictly increasing in x, then $J(x, w)$ is also strictly increasing for all w. If the seller has use value v, the optimal reservation price $r(w, v)$ conditioned on the upper bound w is determined by the solution of the following equation

$$J(x, w) = v,$$

We let $v = 0$ when the seller is a speculator. The increasing virtual value property of $J(x, \beta)$ is made to insure the uniqueness of the optimal reserve price. When the seller is the speculator, we may also use the simpler notation $r(w)$. This optimal reservation price is independent of the number of buyers.

3 Equilibrium Characterization

Due to the importance of the single-crossing property in the analysis of auctions, the first main result we want to establish is this property in the auctions with resale model. Given any profile of possibly non-symmetric bidding strategies σ, we can define the payoff function $u_i(v_i, b, \sigma)$ as in (1). By the single crossing property, we mean that the following holds for all regular bidder i with use value v_i, and bid $b > 0, H(b) > 0$:

$$\frac{\partial}{\partial v_i}\frac{\partial}{\partial b}u(v_i, b, \sigma) > 0. \tag{2}$$

A discrete version of this property has been shown in Zheng (2012), Proposition 3, without speculators.

[4]For our arguments, it is sufficient to have an increasing selection $r(w, v_0)$ as w varies, which is piecewise C^1 smooth in w (but also allowing discontinuity in the function $r(., v_0)$).

Theorem 1 *Given any profile of strategies σ, the single-crossing property (2) holds for the model of first-price auction with resale and speculators.*

Even though we assume symmetric regular bidders in the model, Theorem 1 holds more generally with asymmetric bidders. The proof we provide does not require $F(.)$ to be the same for all buyers. A proof of this theorem requires many notations which will be introduced later. The proof is in section 7.

The next result is to show the symmetry property of the model. We prove it first for the model without speculators but allowing for resale (or equilibrium with inactive speculators). We then extend it to the case with active speculators. The arguments are presented in section 7.

Lemma 1 *The equilibrium in the auction with resale model without speculators must have the symmetry property for all regular bidders.*

Lemma 2 *The equilibrium in the auction with resale model with speculators must have the symmetry property for all regular bidders.*

3.1 Equilibrium with Inactive Speculators

From Lemma 1, we can easily establish the following result.

Theorem 2 *If there are no speculators, the equilibrium with resale is unique and is the same as the one without resale.*

The first function we need for the analysis is the "cost" function $B_0(.)$. It is simply the equilibrium bidding function of the regular bidders when there is no speculator and no resale. The function is given by well-known formula in the symmetric auctions

$$B_0(v) = v - \int_0^v F^{N-1}(x|v)dx = \int_0^v x dF^{N-1}(x|v).$$

We can think of $B_0(v)$ as the bid (and cost) needed for the speculator to win and sell to the buyers with value distribution $F(.|v)$.

Lemma 3 *When $N > 1$, the function $B_0(v)$ has the following properties: (i) $B_0(0) = 0$; (ii) $B_0(v)$ is strictly increasing; (iii) The derivative is given by*

$$B_0'(v) = \frac{(N-1)f(v)}{F(v)}(v - B_0(v));$$

(iv) It is continuously differentiable at 0 with $B_0'(0) = \frac{N-1}{N}$.

Using the symmetry property, we can give a simple characterization of the equilibrium in auctions with resale when there are speculators. We will first introduce several important functions for our analysis. Let

$$B(v)=\int_{r(v)}^{v} J(x,v)dF^{N}(x|v). \tag{3}$$

The function is not defined at $v=0$, but we can let $B(0)=0$, which will make $B(v)$ a continuous function over $[0,\beta]$. The value of this function is the expected revenue of a speculator in the optimal auction of selling to N regular buyers, each having the use value distribution $F(.|v)$. The inverse function of $B(.)$ will be denoted by $\eta(.)$.

For $v\leq w$, let

$$B^{t}(v,w)=v-\int_{r(w)}^{v} F^{N-1}(x|w)dx.$$

When $N=1, B^{t}(v,w)=r(w)$. We use a simpler notation when $v=w$:

$$B^{t}(v)=v-\int_{r(v)}^{v} F^{N-1}(x|v)dx.$$

The function $B^{t}(v)$ is the amount of payment to a speculator by the "top" type when a speculator sells to N symmetric buyers with the use value distribution $F(.|v)$. We have the following useful alternative formula for $B(v)$:

$$B(v)=\int_{r(v)}^{v} B^{t}(x,v)dF^{N}(x|v). \tag{4}$$

In Lemma 4, we will show that $B^{t}(v)>B(v)$ for all $v>0$. This is trivial when $N=1$, but it holds for all N. Lemma 5 gives some basic properties of the function $B(v)$[5] and its derivative.

Lemma 4 *We have*

$$B^{t}(v)>B(v) \text{ for all } v\in[0,\beta].$$

Lemma 5 *The function $B(v)$ in (3) has the following properties: (i) It is continuous at 0, (ii) It is strictly increasing and the derivative is given by*

$$B^{\prime}(v)=\frac{Nf(v)}{F(v)}\left[B^{t}(v)-B(v)\right]>0,\ \text{for } v>0. \tag{5}$$

We need to establish two lemmata. The second one gives us the derivative of a regular bidder's payoff function.

Lemma 6 *In any equilibrium with speculators, we have $H(b)>0$ for all $b>0$. All speculators make zero profit.*

Lemma 7 *Given any symmetric strategy profile $H(.),\phi(.)$, let $u(v,b)$ be the payoff of a regular buyer with use value v bidding b. Then we have*

$$\frac{\partial u(v,b)}{\partial b}=H^{'}(b)F^{N-1}(\phi(b))[B^{t}(v,\phi(b))-b]+H(b)\frac{d\left[F^{N-1}(\phi(b))(v-b)\right]}{db}. \tag{6}$$

[5] Some of the properties have been established in Lemma 2 of GT (supp). The main difference here is that we offer the formula for the derivative of $B(v)$ and shows it to be positive. We make use of the envelop theorem for our derivation.

It may occur quite often that speculators are not active in equilibrium as documented later in section 4. When speculators are not active in equilibrium, they have no influence on the outcome of the auction, and regular bidders bid as if there is no speculator. We now give a condition that tells us precisely when speculators are active in equilibrium.

Let $\pi^0(b) = B(\phi_0(b)) - b$ be the profit function for the speculator when the regular bidders use the bidding function $B_0(.)$. Intuitively, for any possible entry of a speculator in the bidding, there should be a bid $b > 0$ at which a speculator can make a profit.

Theorem 3 *If $\pi_0(b) \leq 0$ for all $b \leq B_0(\beta)$, then in equilibrium speculators must be inactive, and regular bidders bid $B_0(v)$. The equilibrium is unique, and is identical to the one without speculators and without resale.*

The condition $\pi_0(b) \leq 0$ for all $b \leq B_0(\beta)$ is not only sufficient, but also necessary for the speculators to be inactive. In other words, if there exists $b > 0$, such that $\pi_0(b) > 0$, then a speculator must be active in equilibrium. This will become clear from the theorems to follow. An example with an active bidding interval using this idea is given in section 5

3.2 Equilibrium Property in the Initial Interval

When speculators are active in equilibrium, in general they are active over several bidding intervals. In each interval, the equilibrium property is similar. Here we will focus on the initial bidding interval. Once this case is understood, the rest is a simple generalization.

Another function we will find extremely useful is

$$B^c(v) = v - N \int_{r(v)}^{v} F^{N-1}(x|v)dx.$$

When $N = 1, B^c(v) = r(v)$. Note that, by definition, we have

$$v - B^c(v) = N(v - B^t(v))$$

The following function plays a key role in determining the active bidding intervals of the speculators. Let

$$\Delta(v) = r(v)F^N(r(v)) - (N-1)\int_{r(v)}^{v} F^N(x)dx \quad (7)$$

$$K(v) = NF^{N-1}(r(v)|v)r'(v) - (N-1).$$

Lemma 8 *We have the following properties of $B^c(v)$: (i) It is continuous at 0; (ii) The derivative is given by*

$$B^{c\prime}(v) = K(v) + \frac{(N-1)f(v)}{F(v)}(v - B^c(v));$$

(iii) it is continuously differentiable at 0 with $B^{c\prime}(0) = 2^{-N}$; (iv) $K(0) = \frac{N}{2^N} - (N-1) < 0$ when $N > 1$.

The following relationship between $B(.)$ and $B^c(.)$ is a key element in our analysis.

Lemma 9 *We have*

$$B^c(v) - B(v) = \frac{\Delta(v)}{F^N(v)} = \int_0^v K(x) F^N(x|v) dx, \tag{8}$$

$$B^{c\prime}(v) - B'(v) = K(v) - \frac{Nf(v)}{F(v)} (B^c(v) - B(v)), \tag{9}$$

$$B^{c\prime}(v) - B'(v) = K(v) \text{ when } B^c(v) = B(v), v > 0. \tag{10}$$

When $v = 0$, *we have*

$$B^{c\prime}(0) - B'(0) = \frac{1}{N+1} K(0). \tag{11}$$

Hence $B^{c\prime}(0) - B(0) < 0$ *when* $N > 1$.

The following lemma provides an important relationship between the three functions B^c, B, B_0.

Lemma 10 *We have*

$$B^c(v) - B_0(v) = \int_0^{r(v)} F^{N-1}(x|v) dx - (N-1) \int_{r(v)}^v F^{N-1}(x|v) dx, \tag{12}$$

$$B'(v) - B_0'(v) = \frac{Nf(v)}{F(v)} \left[\frac{1}{N} B^c(v) + \frac{N-1}{N} B_0(v) - B(v) \right] \tag{13}$$

Lemma 11 *We have*

$$B^c(v) - B(v) = \int_0^v K(x) \tilde{F}(x)^N dx, \tag{14}$$

$$B^c(v) - B_0(v) = \int_0^v K(x) \tilde{F}(x)^{N-1} dx, \tag{15}$$

and

$$B(v) - B_0(v) = \int_0^v K(x) \tilde{F}(x)^{N-1} (1 - \tilde{F}(x)) dx.$$

Lemma 12 *If* $N > 1$, *we have* $B^{c\prime}(0) < B'(0) < B_0'(0)$.

We say that a function $g_1(v)$ crosses another $g_2(v)$ from below at $v' \in (0, \beta)$ if $g_1(v') = g_2(v'), g_1(v) \leq g_2(v)$ in some neighborhood $(v' - \varepsilon', v')$ and for any $\varepsilon > 0$, there is some $v'' \in (v' + \varepsilon)$ such that $g_1(v'') > g_2(v'')$. Similarly, $g_1(v)$ crosses $g_2(v)$ from above at $v' \in (0, \beta)$ if there exist $\varepsilon' > 0$, such that $g_1(v) \geq g_2(v)$ for all $v \in (v' - \varepsilon', v'], g_1(v') = g_2(v')$, and for any $\varepsilon > 0$, there is some $v'' \in (v', v' + \varepsilon)$ such that $g_1(v'') < g_2(v'')$. When $g_1(.)$ crosses $g_2(.)$ from below, then $g_2(.)$ also crosses $g_1(.)$ from above, and vice versa.

Since $B^c(.)$ is initially below $B(.)$, it may stay below over the range $[0, \beta]$. In this case, it can be shown that we have an equilibrium with inactive speculators. We now assume that $B^c(.)$ crosses $B(.)$ from below, and after the first crossing, we also assume that $B^c(.)$ crosses $B(.)$ from above. Let v_1 be the first crossing point of $B^c(.)$ from above, and assume $B^{c\prime}(v_1) < B'(v_1)$. We shall focus on the equilibrium bidding behavior for regular bidders with $v \in [0, v_1]$. The equilibrium property established in this initial interval will be repeated over and over again for later bidding intervals. The following lemma gives us the implication of the $B^c(.)$ crossing of $B(.)$ from above.

Lemma 13 *There is a neighborhood* $(v_1-\varepsilon, v_1+\varepsilon)$ *such that (i)* $B^c(x) < B(x) < B_0(x)$ *for all* $x \in (v_1, v_1+\varepsilon)$*; and (ii)* $B^c(x) > B(x) > B_0(x)$ *for all* $x \in (v_1 - \varepsilon, v_1)$*.*

The following lemma provides a clear picture of the graphs of the three functions $B^c(.), B(.)$, and $B_0(.)$.

Lemma 14 *There exists a unique* $z_1 \in (0, v_1)$ *which is the unique crossing of* $B(.)$ *and* $B_0(.)$ *in the interval* $(0, v_1)$*, such that*

$$B(v) \leq B_0(v) \text{ for all } v \in [0, z_1],$$

$$B^c(v) > B(v) > B_0(v) \text{ for all } v \in (z_1, v_1).$$

Define $a_1 = B(z_1), b_1 = B(v_1)$, where z_1 is the one given in Lemma 14. Let

$$L(y) = \frac{1 - (N-1)(\eta(y) - y)\frac{f(\eta(y))}{F(\eta(y))}\eta'(y)}{B^t(\eta(y)) - y}, y \in (a_1, b_1). \tag{16}$$

Otherwise let $L(y) = 0$. The denominator $B^t(\eta(y)) - y > 0$, hence $L(y)$ is well-defined.

The following lemma shows that $L(y) > 0$ on (a_1, b_1).

Lemma 15 *The condition* $L(y) > 0$ *is equivalent to* $B^c(v) > B(v)$ *for* $v = \eta(y)$*.*

We can now demonstrate the equilibrium property in the interval $[0, v_1]$.

Lemma 16 *Speculators are not active in* $(0, a_1)$*, but active in* $[a_1, b_1]$*. Regular bidders bid* $B_0(.)$ *over* $[0, z_1]$*, and* $B(.)$ *over* $[z_1, v_1]$*. The bidding functions are optimal, if the joint bid distribution of the speculators is given by* $H(b) = H_0 \exp(-\int_b^{b_1} L(y)dy)$ *over the interval* $[0, b_1]$*, where* $H_0 > 0$ *is a constant.*

3.3 Equilibrium with Active Speculators

For a simpler exposition, we make a generic assumption that will insure that there are only finitely many active bidding intervals in equilibrium. Recall that the definition of $\Delta(v)$ from (7).

Generic Assumption:

$$\Delta'(v) \neq 0 \text{ whenever } \Delta(v) = 0.$$

The assumption is equivalent to saying

$$K(v) \neq 0 \text{ whenever } B^c(v) - B(v) = 0,$$

or

$$B^{c\prime}(v) - B'(v) \neq 0 \text{ whenever } B^c(v) - B(v) = 0.$$

This generic assumption is only used to simplify the exposition, and the general case is dealt with in section 6. With this assumption, we have a finite number of intersections of the two functions $B^c(.), B(.)$. over $[0, \beta]$. Let A be the set of all $v > 0$ such that $B^c(.)$ crosses $B(.)$ from above. Arrange the points of A in increasing order $v_1 < v_2 < ... < v_s$. The largest element in A is v_s. It is convenient to let $v_0 = 0$.

At each v_k, define the following function over $[v_k, \beta]$:

$$B_k^*(v) = v - (v_k - B(v_k))F^{N-1}(v_k|v) - \int_{v_k}^{v} F^{N-1}(x|v)dx. \quad (17)$$

This formula comes from the solution of a differential equation subject to the initial condition $B_k^*(v_k) = B(v_k)$. The differential equation is the one satisfied by a regular bidder's bidding function in the auction **without resale** with N regular bidders each having the use value distribution $F(.|v)$. This is a well-known differential equation in the theory of symmetric auctions. It is usually written for the inverse bidding function. In terms of the bidding function, the differential equation is

$$y'(v) = (N-1)(v - y(v))\frac{f(v)}{F(v)}. \quad (18)$$

Thus the function $B_k^*(.)$ is just the equilibrium bidding function without speculators subject to the initial condition that it is equal to the function $B(.)$ at v_k. We have $B_k^*(v) \leq B_{k+1}^*(v)$ at each v in the common domain. If $B(.)$ crosses $B_k^*(.)$ from below, then it crosses $B_{k-1}^*(v)$ from below as well. It is possible that $B(.)$ never crosses any $B_k^*(.)$. If it does, let m be the largest one such that $B(.)$ crosses $B_m^*(.)$ from below. If $m = s$, we let $v_{m+1} = \beta$, and $A^* = A \cup \{\beta\}$. If $m < x$, we let $A^* = \{v_k : 0 \leq k \leq m+1\}$.

Define $B^*(.)$ as follows

$$B^*(v) = B_k^*(v) \text{ over } [v_k, v_{k+1}) \text{ for } k = 0, 1, ..., m.$$

When $\beta \in A^*$, we also let $B^*(v_{m+1}) = B(\beta)$. This uniquely defines a function $B^*(.)$ over $[0, \beta]$. Intuitively, we may regard $B^*(v)$ as the cost of the speculator for the opportunity to sell to buyers with the value distribution $F(.|v)$.

At the initial interval $[v_0, v_1], v_0 = 0$, we have $B^*(v) = B_0(v)$. When $N = 1$, $B^*(v) = B_0(v) = 0$. When $N > 1$, the function $B^*(.)$ is strictly increasing. It has a up-jump at each $v_k, 0 < k \leq m+1$. Let $\phi^*(b) = B^{*-1}(b)$ be the inverse function when $N > 1$.

Remark 1 *In defining $B^*(.)$, we make use of the condition $K(v) < 0$ in selecting the intervals. This is only correct under our generic condition. If $F(.)$ is analytic, we can use higher-order derivative conditions to replace it. More generally, we need to use the crossing property to determine the intervals in defining $B^*(.)$. These aspects are treated in the extension section. The situation is similar to the first-order condition or higher order conditions in verifying an optimal property. However, when we verify an optimal property, we need to insure that there is no crossing (either maximum or minimum). In our model, we need the opposite kind of condition, the one that guarantees a crossing point, rather than a non-crossing point.*

Many properties of $B_0(.)$ apply to $B^*(.)$ as well with the same proof.

Lemma 17 *When $N > 1$, the function $B^*(v)$ has the following properties: (i) $B^*(v)$ is strictly increasing; (iii) The derivative is given by*

$$B^{*\prime}(v) = \frac{(N-1)f(v)}{F(v)}(v - B^*(v));$$

Lemma 18 *We have*

$$B'(v) - B^{*\prime}(v) = \frac{Nf(v)}{F(v)}\left[\frac{1}{N}B^c(v) + \frac{N-1}{N}B^*(v) - B(v)\right].$$

Lemma 19 *We have*

$$B^c(v) - B^*(v) = \int_{v_{k-1}}^{v} K(x)\tilde{F}(x)^{N-1}dx,$$

$$B(v) - B^*(v) = \int_{v_{k-1}}^{v} K(x)\tilde{F}(x)^{N-1}(1-\tilde{F}(x))dx. \tag{19}$$

The following lemma is a straightforward generalization of Lemma 13.

Lemma 20 *At each $v_k > 0$, there is a neighborhood $(v_k - \varepsilon, v_k + \varepsilon)$ such that (i) $B^c(x) < B(x) < B^*(x)$ for all $x \in (v_k, v_k + \varepsilon)$; and (ii) $B^c(x) > B(x) > B^*(x)$ for all $x \in (v_k - \varepsilon, v_k)$. The same applies to $v_k = 0$, except that (ii) is not relevant.*

The proof of Lemma 14 can be applied to each interval with some minor changes for the last interval.

Lemma 21 *In any interval $[v_k, v_{k+1}], k \le m$, there exists a unique $z_k \in (v_k, v_{k+1})$ which is the unique crossing of $B(.)$ and $B^*(.)$ in the interval (v_k, v_{k+1}), such that*

$$B(v) \le B^*(v) \text{ for all } v \in [v_k, z_k],$$

$$B^c(v) > B(v) > B^*(v) \text{ for all } v \in (z_k, v_{k+1}).$$

Define $a_k = B(z_k), b_k = B(v_k)$, where z_k is the one given in Lemma 21. Let

$$L(y) = \frac{1 - (N-1)(\eta(y) - y)\frac{f(\eta(y))}{F(\eta(y))}\eta'(y)}{B^t(\eta(y)) - y}, y \in (a_k, b_k). \tag{20}$$

Otherwise let $L(y) = 0$. The denominator $B^t(\eta(y)) - y > 0$, hence $L(y)$ is well-defined. It is useful to express $L(y)$ in terms of $v = \eta(y)$:

$$\mathcal{L}(v) = \frac{1 - (N-1)(v - B(v))\frac{f(v)}{F(v)B'(v)}}{B^t(v) - B(v)}. \tag{21}$$

From Lemma 5, we know

$$\frac{1}{B^t(\eta(y)) - y} = \frac{d}{dy}\ln F^N(\eta(y)),$$

hence an alternative expression for $L(y)$ is

$$L(y) = \left[1 - (N-1)(\eta(y) - y)\frac{f(\eta(y))}{F(\eta(y))}\eta'(y)\right]\frac{d}{dy}\ln F^N(\eta(y)).$$

When $N = 1$, we have $L(y) = \frac{d}{dy}\ln F^N(\eta(y))$.

We are now ready to state the next main result of the paper. First the equilibrium bidding strategy of a regular bidder is particularly simple to describe. It is the maximum of two bidding functions, given by

$$\bar{B}(v) = \max\{B(v), B^*(v)\}, v \in [0, \beta].$$

Since $B^*(v)$ is strictly increasing, and $B(v)$ is also strictly increasing by Lemma 5, this is a strictly increasing bidding strategy. Although $B^*(.)$ is not continuous, it will be shown that $\bar{B}(v)$ is continuous We shall adopt the convention that the uniqueness of equilibrium refers to the uniqueness of $H(b)$ for the speculators, and the uniqueness of the bidding strategy of the regular buyers. When there is only one speculator, his/her bid distribution is uniquely determined. We say that the speculators are not active in equilibrium if $H(0) = 1$. Otherwise, we say that speculators are active in equilibrium. We say that a speculator is active at b if b is in the support of his bid distribution. When $H'(b) > 0$, some speculator is active at b.

Theorem 4 *There is a unique equilibrium. The bidding strategy of a regular buyer is given by*

$$\bar{B}(v) = \max\{B(v), B^*(v)\}, v \in [0, \beta].$$

The joint bid distribution $H(b)$ of the speculators is given by

$$H(b) = \exp(-\int_b^{b_s^*} L(y)dy), b \in [0, b_s^*] \tag{22}$$

where b_s^* is the maximum bid of the speculators. In each interval $[v_k, v_{k+1}], k \leq m$, speculators are not active

in (b_{k-1}, a_k) and are active in $[a_k, b_k]$. Regular bidders bid $B^*(v)$ for $v \in [v_{k-1}, z_k]$, and $B(v)$ for $v \in [z_k, v_k]$. If $\beta \notin A^*$, then the maximum equilibrium bid of the speculator is $v_{m+1} < B(\beta)$. If $\beta \in A^*$, then all players have the same maximum equilibrium bid $B(\beta)$.

Remark 2 *The major difference between the equilibrium characterization here and that of GT(supp) is that we offer a simple solution to the regular bidders" equilibrium bidding strategies and the joint bid distribution of the speculators. Lemma 6, 4 are first proved in their Lemma 7, 3 respectively. The derivative relationships of three functions which play an important part of our analysis are not present in their paper. This is one major reason they did not get the characterization we have here. They instead focus on the revenue implication of speculative resale. They show that auctioneer revenue is higher if the speculator is active. However without knowing whether the speculator is active, such results are not useful. We complement this result with our various results on conditions for active or inactive speculators in section 4. Moreover, with our equilibrium characterization, the revenue implication is a simple corollary.*

Corollary 5 *With the condition $\pi_0(b) > 0$ for some $b > 0$, A^* is not empty. Hence the speculators are active*

in equilibrium. The condition is necessary and sufficient for the speculators to be active in equilibrium.

Corollary 6 *When $N > 1$, we must have $H(0) > 0$. When speculators are inactive in equilibrium, we have $\bar{B}(v) = B_0(v)$, $H(.) = 1$. When $N = 1$, we have $\bar{B}(v) = B(v), H(b) = F(\eta(b))$. When $N = 1$, we have $B^*(v) = 0, B^c(v) = B^t(v) = r(v)$ and*

$$B(v) = r(v)[1 - \frac{F(r(v))}{F(v)}].$$

The speculator is active at all bids below the maximum bid of the regular bidders. They have the same bidding interval, and the same bid distribution. The optimal resale monopoly revenue is given by[6] *$B(v)$.*

An example of the active bidding interval is offered in section 5.

Note that if the speculators are not active in equilibrium, 0 is the maximum speculator bid, and it satisfies $K(0) < 0$. If the generic condition is not satisfied, it is possible that $K(v) = 0$, and in this case,

[6] An interesting comparison can be made between this model one specculator and one regular buyer and the equilibrium in the Wilson's Drainage Tract Common Value Model in which there is one neighbor firm and one non-neighbor firm. The equilibrium bidding of our model is similar to that of a common value model with the common value defined by the resale revenue. In the Wilson's Drainage Tract model, the neighbor firm bids instead the amount $\frac{1}{F(v)}\int_0^v xdF(x)$.

higher order derivatives can be used to determine b_s^* , a situation similar to the first-order and higher order conditions. This will be treated in the extension section. Moreover, the condition $K(v) < 0$ is also important in determining the active bidding intervals in the next section.

If $B^c(.)$ crosses $B(.)$ from above at v_k, we may have $K(v_k) = 0$ without the generic assumption. In this case, if we have order derivatives (such as analytic functions), we can replace the condition $K(v_k) < 0$ by

$$K^{(n)}(v_k) < 0, \text{ if } K^{(n)}(v_k) \neq 0, K^{(j)}(v_k) = 0 \text{ for all } j < n.$$

This is similar to checking higher order conditions when verifying an optimal property. Instead of non-crossing, we need the crossing condition. Analytic functions also allow only finitely many crossing. Hence, we can only have finitely many active bidding intervals.

More generally, we can replace $K(v_k) < 0$ by a crossing condition. However, we may have infinitely many bidding intervals in this case. We need to modify the construction of the active bidding intervals somewhat. In particular v_k need not be a crossing point. We need to allow "merging points" at which the three functions $B^c(.), B(.), B^*(.)$ merge together, which occur whenever $B^c(.)$ and $B(.)$ merge together. Other than that, we have the same equilibrium property within each interval $[v_j, v_k]$, where v_j, v_k are two neighboring points of A^*. We may have infinitely many active bidding intervals.

The proofs of Theorem 4 will be given in section 7. More general versions of the two theorems are given in section 6.

Another important intuition is that $b > 0$ can be an active bid of a speculator only if $L(b) > 0$. From Lemma 7, we know that in equilibrium a regular bidder's optimal bid must satisfy

$$H^{'}(b)F^{N-1}(\eta(b))[B^t(\eta(b)) - b] + H(b)\frac{d\left[F^{N-1}(\eta(b))(v-b)\right]}{db} = 0.$$

The first term is always > 0, when the speculator is active at b, and $L(b)$ has the opposite sign of $\frac{d\left[F^{N-1}(\eta(b))(v-b)\right]}{db}$. If $L(b) > 0$ does not hold, the first order condition cannot be satisfied with an active speculator, and hence in equilibrium, b cannot be an active bid.

Assume that initially there is no speculator, and according to the standard first-price symmetric auction, bidders are bidding $B_0(.)$ in equilibrium. When a speculator contempts entry, it is not profitable to bid a small amount, as $\pi_0(b) = B(\phi_0(b)) - b < 0$ when b is small. If the revenue $B(.)$ exceeds the cost $B_0(.)$ at some v, then $\pi_0(b) > 0$ for some $b > 0$. The speculator will enter, to which the regular bidders respond with the bid $B(.)$ to reduce the profit of the speculator profit to 0. Thus, intuitively, speculators don't enter if $B(.) < B_0(.)$ and are active when $B(.) > B_0(.)$. If there is no active speculator, regular bidders bid $B_0(v) = \max(B(v), B_0(v))$. If speculators are active, they bid $B(v) = \max(B(v), B_0(v))$ instead. This is the informal intuition of our result in the initial interval. This intuition also applies to the later intervals. Regular bidders bid $B(v_{k-1})$ at the beginning of the interval. Beginning at this point, $B^c(.)$ drops below $B(.)$ as $K(v_{k-1}) < 0$. This will force $B(.)$ to drop below $B^*(.)$, and the speculator becomes inactive, until $B^c(.)$ crosses $B(.)$ from below again. After that $B(.)$ may cross $B^*(.)$ from below again. When it does, we have a second interval of active bidding. This in fact is how we prove the two Theorems in the next section.

The function $B^c(.)$ plays many important roles: it cuts the graph of $B(.)$ into different components. The cutting points correspond to the endpoints of active bidding intervals. The beginning of the active bidding intervals are formed by the crossing points of $B(.)$ and $B^*(.)$. It also has an intriguing way of influencing the relationship between $B(.), B^*(.)$ which represent the revenue function and the cost function,respectively, of a speculator. In particular, $B^c(.)$ must cross $B(.)$ from below first, before $B(.)$ can cross $B^*(.)$ from below within each interval $[v_{k-1}, v_k]$. Furthermore, in each interval, $B^c(.)$ crosses $B(.)$ only once (from below) inside the interval, and $B(.)$ crosses $B^*(.)$ only once (from below) inside the interval. These properties will be shown in the next section. The end point of each active bidding intervals is determined by the $B^c(.)$ crossing of $B(.)$ from above, while the starting point is determined by the $B(.)$ crossing of $B^*(.)$ from below.

4 Welfare Implications

For the rest of the main text (except section 7), we use the notations $H(.), \eta(b)$ specifically for the equilibrium defined in Theorem 4, unless otherwise specified.

Theorem 7 *When speculators are active, we have (i) $H(b) > F^N(\eta(b))$ whenever $H(b) < 1$; (ii) $\frac{H(b)}{F^N(\eta(b))}$ is strictly decreasing.*

Proof. At any interior of the support of $H(b)$,assume H is differentiable at b. By Theorem 4, we have $\eta(b) = \bar{B}^{-1}(b)$. From Lemma 5, at $v = \eta(b)$, from the first-order condition in Proposition 7, we have

$$\frac{H^{'}(b)}{H(b)} < \frac{Nf(\eta(b))\eta'(b)}{F(\eta(b))} = \frac{d\left[\ln F^N(\eta(b))\right]}{db}. \tag{23}$$

Outside the support of $H(.)$, we have $H'(0) = 0$, hence (23) is true also. By integration, we have

$$H(b) > F^N(\eta(b)) \text{ when } H(b) < 1.$$

Since

$$\frac{d}{db}\left[\ln H(b) - \ln F^N(\eta(b))\right] < 0,$$

we know that $\frac{H(b)}{F^N(\eta(b))}$ is strictly decreasing. ■

We now compare the equilibrium outcome with speculators and that of the auction without speculators and see how the bidders and the auctioneer are affected by the presence of speculators. In this comparison, we assume that there are active speculators. The following results compare the welfare of the seller and the regular buyers, as well as efficiency questions. The speculators make zero profit in equilibrium, so nothing more can be said of the welfare of speculators.

The following says that the bidding behavior of a regular bidder with the use value $v \geq z_1$ becomes more aggressive. Let $Q(v)$ be the total winning probability (including winning in the first stage as well as the second stage) of a regular buyer with the use value v. Let $P(v)$ be the total expected payment (including payment to the seller as well as the speculators) after winning for the buyer. We will show that $Q(v)$ is higher in the first-order strict stochastic dominance. We show that $P(v)$ is higher than $B_0(v)$, which is the expected payment after winning without speculators. Thus the payoff of a regular buyer with any use value $v > 0$ is lower.

Theorem 8 *Assume that speculators are active in the equilibrium of the auction with resale and speculators, we have (i) $\bar{B}(v) > B_0(v)$ for all $v > z_1$; (ii) $Q(v) < F^{N-1}(v)$ for all $v > 0$; (iii) $P(v) > B_0(v)$ for all $v > 0$; (iv) The payoff of a regular bidder with any use value $v > 0$ is lower;*

Proof. Property (i) has been shown before. For a regular bidder with a use value $v > 0$, (ii) is apparently true, as the winning probability in the first stage is lower, because a speculator may win, and the winning probability in the second stage (conditional on losing in the first stage) is less than 1. Hence the total winning probability (including winning in the first and second stage) $Q(v)$ is strictly lower than that $F^{N-1}(v)$. For $v > z_1$, a regular buyer pays more conditional on winning. This is because the equilibrium bid is higher, and if the object is bought back from the speculator, the expected payment is higher than the equilibrium bid (Lemma). Hence $P(v) > B_0(v)$. For $v \leq z_1$, the payment after winning is the same as $\bar{B}(v) = B_0(v)$. The seller revenue from the regular bidders in the first stage is

$$\int_0^\beta \bar{B}(v)d[F^N(v)H(v)] > \int_0^\beta B_0(v)dF^N(v). \tag{24}$$

The seller revenue is the first term in (24) plus the revenue from the speculators. ■

Hafalir and Krishna (2009) studied the revenue and efficiency effects of resale in a similar framework. There are only two bidders in their model. They showed that with three special classes of value distributions, resale enhances the original seller's revenue. It is often claimed in the literature that resale can correct inefficiency in the first-stage auction. They gave an example to show that this may not be true in general, because resale changes the bidding behavior in the first stage auction. If the value distributions between the two bidders differ a great deal, the additional inefficiency created by resale can lead to a less efficiency outcome than the case without resale.

Here we revisit this issue in the model of speculative resale. Revenue is indeed higher if the speculators are active in equilibrium, as shown in GT(suup). We can offer more in this paper with our simple characterization. We have a formula for computing the revenue with resale, which can be compared with the revenue without resale so that quantitative comparisons, rather than qualitative differences, can be made. The original seller's equilibrium revenue has two components: revenue from regular buyers, and revenue from the speculators. Since the speculators make zero profit in equilibrium, their revenue contribution to the original seller is also equal to the revenue generated during the resale stage.

Theorem 9 *If speculators are active in equilibrium, the original seller's revenue is strictly higher, and is given by*

$$\int_0^{b_s^*} bd\left[H(b)F^N(\eta(b))\right] + \int_0^{\beta} \max(B^*(v), B(v))d\left[H(B(v))F^N(v)\right]$$

$$= \int_0^{b_s^*} bd\left[H(b)F^N(\eta(b))\right] + \sum_{k=1}^{m+1}\left(\int_{v_{k-1}}^{z_k} B^*(v)d\left[H(B(v))F^N(v)\right] + \int_{z_k}^{v_k} B(v)d\left[H(B(v))F^N(v)\right]\right).$$

Note that the contribution of the regular bidders to the original seller's revenue is already higher than the revenue without resale, because $B(v) > B^*(v) > B_0(v)$ over (z_k, v_k), when speculators are active in equilibrium. This higher revenue is further boosted by the revenue contribution from the speculators.

There is another caveat we should mention for the higher revenue result with active speculators. We assume that the original seller does not use any reservation price. If the original seller attempts to maximize revenue with the possible participation of the speculators, it can be shown that an optimal reservation price is the one used by the original seller when there is no resale and no speculators. When the original seller uses the optimal reservation price without resale, the speculators will find it unprofitable to participate in bidding. They become inactive. The revenue obtained, however, is the highest possible for the original seller. This issue of optimal mechanism with resale cannot be treated here because of limited space, and will be dealt with in a separate paper. The equilibrium characterization we offer here mostly will carry over when there is a reservation price by the original seller. So we have an intriguing phenomenon here: By preventing speculators from bidding with a higher reservation price, the original seller maximizes the revenue, even though speculators can raise revenue when they are active in equilibrium.

Although the original seller's revenue is higher, this occurs at the expense of ex post efficiency. By proving the symmetry property, we show that the allocation among the regular buyers is efficient. However, inefficiency arises when the speculators win the first-stage auction, and then fail to sell to the regular buyers. So the inefficiency caused by resale in the Hafalir and Krishna (2009) example is actually a general property in our model. It is true for all $F(.)$ and N. The trade-off between revenue and ex post efficiency here is due to the trade-off in auctions without resale when the seller chooses a reservation price. The original seller does not set a reservation price in our model, but the speculators do set reservation prices, and this creates inefficiency at the same time.

A regular buyer with use value $v \in (z_k, v_k)$ has the probability of winning the object $F^{N-1}(v)$ in the auction without resale. In the auction without resale, the winning probability in the first stage auction is reduced to $H(B(v))F^{N-1}(v)$. The buyer can buyer it back from the speculators. Let b_v be the unique solution of $r(\eta(b)) = v$. This is the highest bid of the winning speculator from whom the buyer can back the object. The probability of winning in the resale stage is given by

$$\int_{B(v)}^{b_v} F^{N-1}(v|\eta(b))d(H(b)F^{N-1}(\eta(b)).$$

Hence the total winning probability of a regular buyer with use value v is given by

$$Q(v) = H(B(v))F^{N-1}(v) + \int_{B(v)}^{b_v} F^{N-1}(v|\eta(b))d(H(b)F^{N-1}(\eta(b)).$$

The loss of efficiency for this buyer is

$$\int_{B(v)}^{b_v} F^{N-1}(v|\eta(b))d(H(b)F^{N-1}(\eta(b)) - (1 - H(b))F^{N-1}(v).$$

This loss of efficiency is across the board, i.e. it occurs to each buyer of all value type. However, there is also a shift of winning probabilities from lower value types to higher value types. To determine the ex ante efficiency of speculative resale, we need to combine this welfare gain with the loss of efficiency due to unrealized gains of trade. The total measure on ex ante efficiency is given by

$$\int_0^\beta vQ(v)dF^N(v).$$

it remains to be seen there is a loss of ex ante efficiency due to speculative resale.

4.1 Active and Inactive Speculators

Whether speculators are active or inactive have many welfare implications as demonstrated above. In this subsection we give more results on conditions for active or inactive speculators. We also document many situations in which speculators are inactive in equilibrium.

We give one useful condition for determining whether speculators are active in equilibrium. The usefulness is due to the fact that it is based on a simple integral of the function $F(.)$. The condition can often be verified.

Theorem 10 *Let $N > 1$. The speculators are inactive in equilibrium if $B^c(.)$ crosses $B_0(.)$ from below, or*

$$B_0(v) - B^c(v) = N \int_{r(v)}^{v} F^{N-1}(x)dx - \int_0^v F^{N-1}(x)dx \geq 0 \text{ for all } v \leq \beta. \tag{25}$$

If $B_0(v) < B^c(v)$ at some $v > 0$ with $B_0(v) = B(v)$, then speculators are active in equilibrium.

Proof. Assume that $B^c(.)$ never crosses $B_0(.)$, so that we have $B^c(v) \leq B_0(v)$ for all $v \leq \beta$. We want to show that $B(.)$ never crosses $B_0(.)$. By Theorem 3. we know that speculators must be inactive in equilibrium. Assume that $B(.)$ crosses $B_0(.)$ from below, want to show a contradiction. Let the first crossing point be denoted by $z \in (0, \beta)$. We can find $v > z$, and close to z, so that $B(v) > B_0(v)$, and $B'(v) > B_0'(v)$. By Lemma 10, this means that $B'(v) - B_0'(v) < 0$. This is a contradiction. Now suppose $B_0(v) < B^c(v) < 0$ at some $v > 0$ with $B_0(v) = B(v)$. By Lemma 10, we have

$$\begin{aligned} B'(v) - B_0'(v) &= \frac{Nf(v)}{F(v)} \left[\frac{1}{N} B^c(v) + \frac{N-1}{N} B_0(v) - B(v) \right] \\ &= \frac{f(v)}{F(v)} \left[B^c(v) - B_0(v) \right] > 0. \end{aligned}$$

This implies that $B(.)$ crosses $B_0(.)$ at v. By Theorem 4, the speculators are active in equilibrium. ■

The following expresses the idea that there is a tendency for speculators to be inactive when there are more regular bidders.

Theorem 11 *If (25) holds when there are $N_0 \geq 2$ regular bidders, then it also holds when $N > N_0$.*

Proof. Assume that (25) holds, then for all v, we have

$$N \int_{r(v)}^{v} F^N(x)dx > (N_0 - 1)F(r(v)) \int_{r(v)}^{v} F^{N_0-1}(x)dx \geq F(r(v)) \int_0^{r(v)} F^{N_0-1}(x)dx$$
$$> \int_0^{r(v)} F^{N_0-1}(x)dx.$$

Hence (25) holds when $N > N_0$. ■

Corollary 12 *Let $N \geq 2$. The speculators are inactive in equilibrium if either (i) $F(v) = v^a, a > 0$, or (ii) $F(.)$ is concave.*

Proof. Consider $N = 2$. For (i) the optimal reservation price corresponding to $[0, v]$ is $r(v) = \frac{v}{(a+1)^{1/a}}$. We have

$$\int_{r(v)}^{v} x^a dx - \int_0^{r(v)} x^a dx = \frac{v^{a+1}}{a+1}(1 - \frac{2}{a+1}(a+1)^{-\frac{1}{a}}) \to 1 - 2e^{-1} > 0.$$

Since the limit is the infimum, (25) holds for all v. By Theorem 10, the result holds for $N = 2$. By Theorem 11, it holds for all $N \geq 2$. Hence speculators are inactive. For (ii), first we want to show that there is an optimal reservation price $r \leq 0.5v$. By concavity we have

$$f(0.5v) > \frac{F(v) - F(0.5v)}{0.5v}.$$

Hence we have $rf(r) + F(r) > F(v)$ when $r = 0.5v$. By concavity, we also have $xf(x) \leq F(x)$. Hence

$$xf(x) + F(x) \leq 2F(x) < F(v),$$

when x is small. Therefore we can choose $r < 0.5v$. We have

$$\int_r^v F(x)dx - \int_0^r F(x)dx > F(r)(v - 2r) > 0.$$

Thus (25) holds, and for all $N \geq 2$, and speculators are inactive. ■

Remark 3 *A stronger property is satisfied by both cases. It can be shown that $\pi_0'(b) < 0$.*

The following says that when there are many regular buyers, the speculators are inactive in equilibrium.

Theorem 13 *Given $F(.)$, there exists $N_0 \geq 2$ such that for all $N > N_0$, the speculators are inactive in equilibrium.*

Proof. Let $f_{\max}(v) = \max\{f(x) : 0 \le x \le \beta\}, f_{\min} = \min\{f(x) : 0 \le x \le \beta\}$. We want to show that (25) holds.

$$\frac{N-1}{F^N(v)} \int_{r(v)}^{v} F^{N-1}(x)dx \ge \frac{N-1}{N f_{\max} F^N(v)} \int_{r(v)}^{v} N F^{N-1}(x) f(x) dx = \frac{N-1}{N f_{\max}} [1 - \frac{F^N(r(v))}{F^N(v)}] \qquad (26)$$

When $N \to \infty$, the last term in (26) converges to $\frac{1}{f_{\max}} > 0$. Hence there is a constant $c > 0$, and N_1 such that

$$\frac{N-1}{F^N(v)} \int_{r(v)}^{v} F^{N-1}(x)dx \ge c \text{ for all } x \le \beta, N \ge N_1.$$

However

$$\frac{1}{F^N(v)} \int_0^{r(v)} F^{N-1}(x)dx \le \frac{1}{N f_{\min} F^N(v)} \int_0^{r(v)} N F^{N-1}(x) f(x) dx = \frac{F^N(r(v))}{N f_{\min} F^N(v)} \to 0 \text{ as } N \to \infty.$$

Furthermore, this convergence is uniform over $[0, \beta]$. Hence there exists N_2 such that we have

$$\frac{1}{F^N(v)} \int_0^{r(v)} F^{N-1}(x)dx \le \frac{c}{2} \text{ for all } x \le \beta, N \ge N_2.$$

Let $N_0 = \max\{N_1, N_2\}$. Then for $N \ge N_0$, we have

$$\frac{N-1}{F^N(v)} \int_{r(v)}^{v} F^{N-1}(x)dx > \frac{1}{F^N(v)} \int_0^{v} F^{N-1}(x)dx,$$

and (25) follows immediately for all $v \le \beta$, and the proof is complete. ■

The following is a useful result for the extension beyond power function cases to other convex distributions.

Theorem 14 *Let $F_1(.), F_2(.)$ be convex, and $N = 2$. Assume $r_1(v) < r_2(v)$, and (25) holds for $F_2(.)$. Furthermore, assume that (25) also holds for $F_1(.)$ when $r_1(v)$ is replaced by $r_2(v)$. Then (25) holds for any convex combination $F(.) = (1-s)F_1(.) + sF_2(.)$.*

Proof. For buyer one, we have

$$\int_{r_2(v)}^{v} ((1-s)F_1(x) + sF_2(x))\, dx > (1-s) \int_0^{r_2(v)} F_1(x)dx + s \int_0^{r_2(v)} F_2(x)dx$$

$$= \int_0^{r_2(v)} ((1-s)F_1(x) + sF_2(x))\, dx.$$

If $r(v) < r_2(v)$, then we have

$$\int_{r(v)}^{v} F(x)dx = \int_0^{r(v)} F(x)dx,$$

and we are done. To show $r(v) < r_2(v)$. Note that by convexity,

$$r_2(v) f_1(r_2(v)) + F_1(r_2(v)) > F_1(v)$$

$$r_2(v) f_2(r_2(v)) + F_2(r_2(v)) = F_2(v)$$

Taking the convex combination, we get

$$r_2(v) f(r_2(v)) + F(r_2(v)) > F(v).$$

By the increasing virtual value property, we have $r(v) < r_2(v)$. The proof is complete. ■

We apply Theorem 14 to the quadratic case.

Theorem 15 *Let $N \geq 2$. Speculators are inactive when $F(v) = cv + v^2$.*

Proof. Let $F_1(v) = v, F_2(v) = v^2$. We have $r_1(v) = 0.5v < \frac{1}{\sqrt{3}}v = r_2(v)$. We also have

$$\int_{\frac{v}{\sqrt{3}}}^{v} xdx = \frac{1}{3}v^2 > \frac{1}{6}v^2 = \int_0^{\frac{v}{\sqrt{3}}} xdx.$$

Condition (25) is satisfied for $F_2(.)$. Hence we conclude that $F(v) = (1-s)x + sx^2$. All other quadratic cases are known to hold as well. ■

5 Example of An Active Bidding Interval

GT (supp) has an example of an active speculator. However, they do not analyze the determination of the active bidding interval. Here we give an example to illustrates how the active and inactive bidding intervals are determined. There are two regular bidders and one speculator. The distribution of the use value is given by the following piecewise linear convex function

$$\begin{aligned} F(v) &= tv, \text{ for } v \leq 1 \\ &= t(8v-7) \text{ for } v > 1. \end{aligned}$$

The scale parameter t is used to scale the value of F below 1. Since t does not affect $B^*(v), B(v)$, we can ignore the specification of the upper bound of the use value β. We have

$$\begin{aligned} J(x,v) &= t(2x-v) \text{ for } v < 1, x < 1 \\ &= t(2x-8v+7) \text{ for } v > 1, x < 1 \\ &= t(2x-v) \text{ for } v > 1, x > 1 \end{aligned}$$

$$\begin{aligned} r(v) &= 0.5v \text{ for } v \leq 1 \\ &= 4v - 3.5 \text{ for } v \in [1, 1.125] \\ &= 1 \text{ for } v \in [1.125, 2] \\ &= 0.5v \text{ for } v \geq 2 \end{aligned}$$

In this example, there are only two intervals. We will focus on the first interval, in which we have $B^{**}(v) = B^*(v)$. The equilibrium bidding function without speculators is given by

$$\begin{aligned} B_0(v) &= \frac{1}{v}\int_0^v xdx = 0.5v, \text{ for } v \leq 1 \\ &= \frac{4v^2 - 3.5}{8v-7} \quad \text{for } v > 1. \end{aligned}$$

The function $B(v)$ can be easily computed. We have

$$\begin{aligned} B(v) &= 0.416\,667\,v, \text{ for } v \leq 1 \\ &= \frac{2}{(8v-7)^2}\left(32v^3 - 56v^2 + 21v + 3.\,208\,33\right), \text{ for } v \in [1, 1.125] \\ &= \frac{16}{(8v-7)^2}\left(\frac{4}{3}v^3 - 3v + \frac{5}{3}\right), \text{ for } v \in (1.125, 2] \\ &= \frac{16v^2}{(8v-7)^2}\left(\frac{4}{3}v - 1.\,75\right), \text{ for } v \geq 2. \end{aligned}$$

We also have, for $v \in [1.125, 2]$,

$$\begin{aligned} B^c(v) &= 0.25v, v \le 1 \\ &= \frac{64v^2 - 84v + 21}{32v - 28}, v \in [1, 1.125] \\ &= v - \frac{2}{8v-7}\int_1^v (8x-7)dx = \frac{7v-6}{8v-7}, v \in [1.125, 2] \end{aligned}$$

and for $v \in [1, 1.125]$, we have

$$B^c(v) = v - \frac{2}{8v-7}\left(\int_{4v-3.5}^1 xdx + \int_1^v (8x-7)dx\right) = \frac{64v^2 - 84v + 21}{32v - 28}$$

The end point v_1 of the first interval is determined by unique solution of following equation

$$B(v) = B^c(v), v > 0.$$

We get $v_1 = 1.2405$. In the interval $[0, v_1]$, the two curves $B(v), B^*(v)$ have two intersections at $v_1' = 1.1633, v_2' = 1.3305$. We have $b_1 = B(v_1) = 0.91775 = b_x^*$. The interval is divided into two parts by $z_1 = v_1'$. The speculator is inactive in the lower part $[0, z_1]$, while active in the upper part $[z_1, v_1]$. The bidding function of the regular bidder is $B_0(v)$ in the lower part, and $B(v)$ in the upper part, i.e.

$$\begin{aligned} \bar{B}(v) &= B_0(v), v \in [0, 1.1633]; \\ &= B(v), v \in [1.1633, 1.2405]. \end{aligned}$$

The graph of the two curves together is shaped like a fish when the two end points $(v_1, B(v)), (v_1, B_0(v))$ are connected vertically. This vertical section of the fish tail is also the place $B^*(v)$ has an up-jump. The mouth of the fish (0,0) is usually pointed (except this first one which may not be pointed). The second interval $[v_1, \beta]$ is also the last interval in this example. The graph is only part of the fish with the rear part cut off. In this example, we have $B^*(v)$ defined by the initial condition $B^*(v_1) = B(v_1)$ and the differential equation

$$B^{*\prime}(v) = (N-1)(v - B(v))\frac{f(v)}{F(v)}.$$

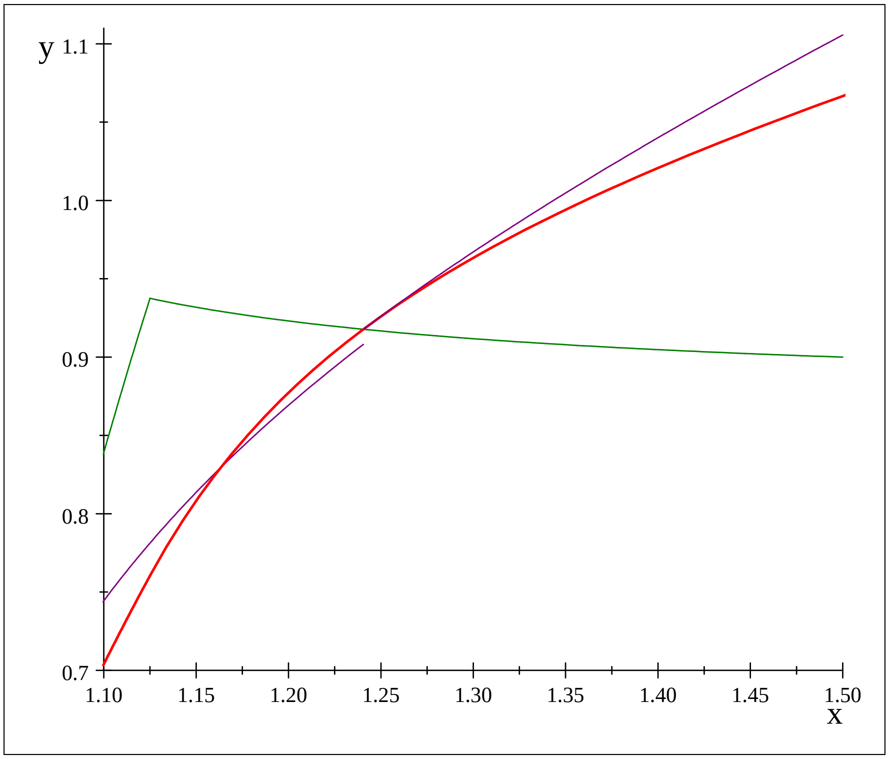
The regular bidder's bidding function is $\bar{B}(v) = B^*(v)$, with the maximum bid $b^* = B^*(\beta)$. The speculator is inactive in this last interval. More generally, in the last interval the speculator may be active when the fish shape is cut off at the tail section. Here the fish is cut off at the body section, and in this case the speculator is not active. The (partial) fish shape in the second interval has a pointed mouth. This is always the case except possibly for the first interval. The fish end (or the mouth of the following one) is determined by the function $B^c(v)$. The speculator is only active in the fish tail part, not active in the body part.

We plot the curves $B_0(v)$ (in black) and $B(v)$ (in red), and the curve $B^c(v)$ (in green). The two curves meet at $z_1 = 1.16326$, and the first fish ends at the vertical cut at $v_1 = 1.2405$. The minimum speculator positive bid is given by $b_1 = B^*(z_1) = 0.829414$. The speculator maximum bid is given $b_1 = B(v_1) = 0.91775$. When we take $\beta = 1.5$, with a properly chosen $t = \frac{1}{8*1.5-7} = 0.2$, the maximum bid of the regular bidders is given by $B^*(1.5) = 1.1$. The speculator is active in $\{0\} \cup [0.829, 0.91775]$. For $v \ge 1.2405$, we have

$$B^*(v) = v - (1.2405 - 0.91775)\frac{8*1.2405-7}{8v-7} - \int_1^v \frac{8x-7}{8v-7}dx = \frac{4v^2 - 3.9437}{8v-7}.$$

Speculators can only be active on the red segment between the green line and the purple line. The upper purple line is $B^*(v)$, and the lower purple line is $B_0(v)$.

These two potentially long strands entwine like vines in a way that is called "antiparallel". A third strand $B^c(v)$ serves to determine where to slice $B(v)$ and then graft $B^*(.)$ to the cutoff point (v_1, b_1) on the graph of $B(.)$.


y
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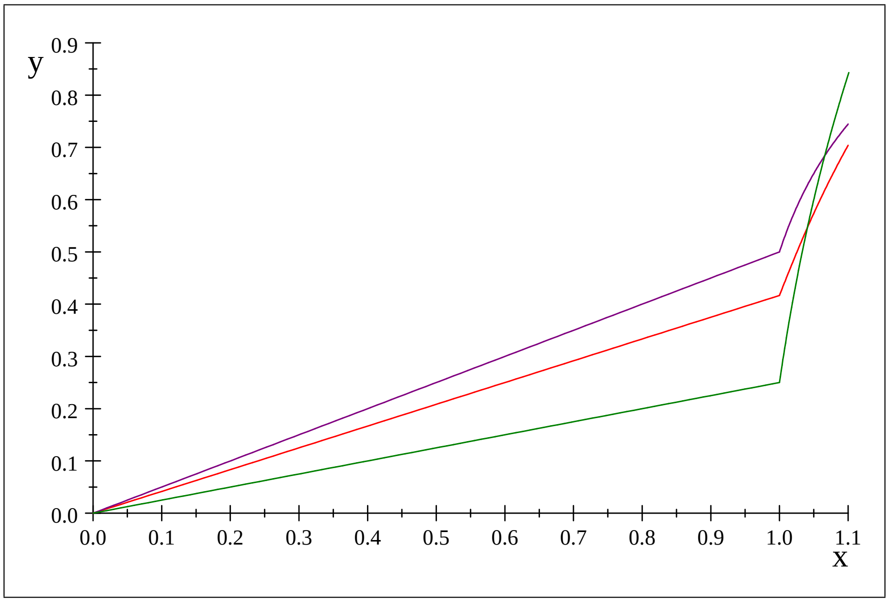
1.0
0.9
0.8
0.7
1.10
1.15
1.20
1.25
1.30
1.35
1.40
1.45
1.50
x


y
0.9
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1.0
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x

6 Extensions

Since the equilibrium bidding functions are only piecewise differentiable, it is natural to allow $F(.)$ to be only piecewise differentiable. For example, in our example, we take $F(.)$ to be a piecewise linear function. A typical example of a piecewise C^2 smooth function is the maximum or minimum of two smooth C^2 functions. Formally a function is called piecewise C^2 smooth if (i) it is C^2 except at a countable number of points D. (ii) At a point in D, the left and right derivatives (up to the second order) exist and are continuous from the right and left respectively. Note that we must have $0 \notin D$ and so is true for β . Although D may be different among different buyers, we can take the union of all such sets and use a single D to denote the set of all potential points of discontinuity of the derivatives. With proper modifications, using derivative arguments outside D, we can establish the same equilibrium characterization result.

We shall focus on two types of extensions in this section. In the section 6.1, we let $F(.)$ to be analytic functions, but do not require the generic conditions. This will allow $f(0) = 0$, and $B^c(.)$ and $B(.)$ have only finite number of intersections. We can still establish the some results with appropriate modifications. In section 6.2 we simply drop the generic conditions, and deal with the most general case and allow infinitely many active bidding intervals.

6.1 Analytic Functions $F(.)$

We now drop the generic assumption, but assume that $F(.)$ is analytic. This implies that $B^c(.)$ and $B(.)$ are analytic as well. There are only finitely many intersections of $B^c(.), B(.)$ for analytic functions. However at the point of intersection, we can characterize crossing properties by higher order derivative conditions. Furthermore, when $f(0) = 0$, we need to establish the inequality $B^{c\prime}(0) < B'(0)$ for the results in the first bidding interval to hold. The following lemmata will make the arguments go through. From the analytic property, there exists an n-th order derivative such that $F^{(n)}(0) > 0, F^k(0) = 0$ for all $k < n$.

The following lemma is useful in computing derivatives at 0 of the two bidding functions $B(v), B_0(v)$ under the more general conditions.

Lemma 22 *Let $F^{(n)}(0) = a > 0$, we have (i) $\lim_{v\to 0} \frac{vf(v)}{F(v)} = n$; (ii) $r'(0) = (n+1)^{-\frac{1}{n}}$; (iii) $\lim_{v\to 0} \frac{F(r(v))}{F(v)} = \frac{1}{n+1}$.*

Proof. Prove (i) by mathematical induction. If $n = 1$, it is trivial. Assume it holds for $n = k$, we want to show it true for $n = k + 1$. We have

$$\lim_{v\div 0} \frac{vf(v)}{F(v)} = 1 + \lim_{v\div 0} \frac{vf'(v)}{f(v)} = 2 + \lim_{v\div 0} \frac{vF^{(3)}(v)}{F^{(2)}(v)} = ... = k + \lim_{v\to 0} \frac{vF^{(k+1)}(0)}{F^{(k)}(0)} = k + 1.$$

Hence it holds for $n = k + 1$, the result holds for all n. To prove (ii), let $t = r'(0) = \lim_{v\to 0} \frac{r(v)}{v}$. We have

$$\lim_{v\to 0} \frac{F(r(v))}{F(v)} = t^n, \lim_{v\to 0} \frac{f(r(v))}{f(v)} = t^{n-1}.$$

We can compute $r'(0)$ as follows. From $rf(r) + F(r) = F(v)$, we get

$$t = \lim_{v\to 0} r'(v) = \lim_{v\to 0} \frac{f(v)}{2f(r) + r(v)f'(r))} = \lim_{v\to 0} \frac{1}{2\frac{f(r)}{f(v)} + \frac{r(v)}{v}\frac{vf'(r)}{f(v)}} = \frac{1}{(n+1)t^{n-1}}.$$

Hence

$$r'(0) = t = (n+1)^{-\frac{1}{n}}.$$

We immediately have (iii). ■

Lemma 23 *We have $B_0'(0) = \frac{n(N-1)}{1+n(N-1)}$.*

Proof. We have

$$B_0'(v) = \frac{(N-1)f(v)}{F(v)}[v - B_0(v)]. \tag{27}$$

Let $d = B_0'(0)$, we can determine d from the following equation

$$d = \lim_{v\to 0} \frac{(N-1)vf(v)}{F(v)}[1 - \frac{B_0(v)}{v}] = (N-1)n(1-d),$$

hence

$$B_0'(0) = \frac{n(N-1)}{1+n(N-1)}.$$

■

Lemma 24 *We must have $K(0) < 0$, and $B^{c\prime}(0) < B'(0) < B_0'(0)$.*

Proof. From (11), we have

$$B^{c\prime}(0) - B'(0) = \frac{1}{N+1}K(0)$$

By Lemma 22, we have

$$K(0) = Nt^{(n-1)(N-1)+1} - (N-1) = N(1+n)^{-\frac{(n-1)(N-1)+1}{n}} - (N-1)$$

$$= N(1+n)^{\frac{N-2}{n} - (N-1)} - (N-1).$$

We want to show $K(0) < 0$, or

$$(1+n)^{\frac{N-2}{n} - (N-1)} < \frac{N-1}{N} = 1 - \frac{1}{N}. \tag{28}$$

The left-hand side is decreasing in n. We have $n \geq 2$, hence it is sufficient to show (28) for $n = 2$, or

$$3^{-\frac{N}{2}} < 1 - \frac{1}{N}. \tag{29}$$

Since the left-hand side of (29) is decreasing in N, and the right-hand side is increasing in N, it is sufficient to show (29) for $N = 2$, or $\frac{1}{3} < \frac{1}{2}$, which is obviously true. Hence we have $B^{c\prime}(v) - B'(v) = K(v) < 0$ in a neighborhood of 0. By Lemma 20, the lemma is proved. The proof of $B'(0) - B_0'(0) < 0$ is similar to the proof of Lemma 12 with some difference in the limit. We have

$$B'(0) - B_0'(0) = \lim_{v\to 0} \frac{vf(v)}{F(v)} \left[\frac{B^c(v) - B_0(v)}{v} - N(\frac{B(v) - B_0(v)}{v}) \right]$$

$$= n \left[\lim_{v\to 0} \frac{K(v)}{1 + \frac{(N-1)f(v)v}{F(v)}} - N(B'(0) - B_0'(0)) \right]$$

Hence

$$B'(0) - B_0'(0) = \frac{nK(0)}{(1+nN)(1+n(N-1))} < 0.$$

■

Remark 4 *Lemma 22 and 5 hold for the power function $F(v) = v^s, 0 < s < 1$ as well. In this case, we have $B^{*\prime}(0) = \frac{s(N-1)}{1+s(N-1)}, r'(0) = (s+1)^{-\frac{1}{s}}, \lim_{v\to 0} \frac{F(r(v))}{F(v)} = \frac{1}{s+1}$.*

One more thing we need for the results in sections 3 and 4 to apply to all analytic functions is the definition of the set A^*. We start with the definition of A. We define $A = \{v : v > 0, B^c(v) - B(v) = 0$, and the first non-zero $n-$th derivative of $B^c - B$ at v is $< 0\}$. The set A is finite when $F(.)$ is analytic. This is equivalent to saying that A is the set of all points $v > 0$, at which $B^c(.)$ crosses $B(.)$ from above. The definition for A^* from A is the same as before. With this modification, all results now apply to the analytic case.

When $F(.)$ is analytic, to verify the crossing property, we can simply take the higher derivatives of $K(.)$, and verify that the first non-zero derivative is negative. To see this, note that at $v > 0$, we have

$$B^{c\prime}(v) - B'(v) + (B^c(v) - B(v))\frac{d}{dv}\ln F^N(v) = K(v).$$

When the first derivative is zero, $K(v) = 0$, we take the second derivative, we have

$$B^{c\prime\prime}(v) - B''(v) + (B^{c\prime}(v) - B'(v))\frac{d}{dv}\ln F^N(v) + (B^c(v) - B(v))\frac{d^2}{dv^2}\ln F^N(v) = K'(v).$$

We have $K'(v) < 0$ if and only if $B^{c\prime\prime}(v) - B''(v) < 0$. Higher order derivatives are similar.

To verify the equilibrium property for each interval, note that in Lemma 20, the crossing from above property at each v_k implies that $B(.) > B^c(.)$ over $(v_k, v_k + \varepsilon)$. Thus Lemma 20 can be proved in the same way as before. There is also no need to change the proof of Lemma 21. Lemma **??** is also proved in exactly the same way. Thus we have shown Theorem 4 for the analytic case.

6.2 The case of infinitely many Intervals

Our purpose in this section is to show that if F is not analytic, or if we don't have the property that $B(.) - B^c(.)$ crosses zero with a non-zero derivative[7], the results still hold by allowing the possibly infinite number of active bidding intervals. This possibility requires some modifications of the construction of the bidding intervals. After this is properly done, the results we established in each interval $[v_k, v_{k+1}]$ still hold.

We say that $v \in (0, \beta)$ is a merging point for $B^c(.)$ and $B(.)$ if $B^c(.) = B(.)$ for some interval $[v, v + \varepsilon]$, and $B^c(.) \neq B(.)$ over any interval $(v - \varepsilon, v)$. Consider the set A of all points $v > 0$ at which either $B^c(.)$ crosses $B(.)$ from above or v is a merging point for $B^c(.), B(.)$. The set A is a countable set and is indexed by v_k. We say $v_j, v_k \in A$ are neighboring points if there exists no $v_t \in A$ such that $v_j < v_t < v_k$. We call v_j a left-neighbor of v_k, and v_k a right neighbor of v_j. For any pair of neighboring points v_j, v_k, define $B_j^*(.)$ over $[v_j, \beta]$ by the same differential equation with the initial condition $B_j^*(v_j) = B(v_j)$. Let $v^* = \sup\{v_j : B(.)$ crosses $B_j^*(.)$ from below$\}$. Define $B_{v^*}^*(.)$ by the same differential equation with the initial condition $B_{v^*}^*(v^*) = B(v^*)$. We divide into three cases: case 1) $B(.)$ does not cross $B_{v^*}^*(.)$ from below; case 2) $B(.)$ does cross $B_{v^*}^*(.)$ from below; or case 3) $B(.)$ crosses all $B_j^*(.)$ as well as $B_{v^*}^*(.)$.

In case 2), we can show that v_* must be a left neighbor for some v_k in A, In both casek s 1) and 2), we define $A^* = \{v_k |\ v_k \in A$ has a left neighbor v_j in A, and $B(.)$ crosses $B_j^*(.)$ from below$\}$. Note that in case 1), it is possible that $v^* = \beta$, but in case 2), we must have $v^* < \beta$.

In case 3, we let $v_L = \beta$, and $A^* = A \cup \{\beta\}$. We also call the two points v^*, β neighboring points in this case.

In case 1), define $B^*(.) = B_j^*(.)$ over the interval $[v_j, v_k)$ for any neighboring pair in A^*. If $v^* = \beta$, let $B^*(\beta) = B(\beta)$. If $v^* < \beta$, define $B^*(.) = B_{v^*}^*(.)$ over $[v^*, \beta]$. This uniquely defines $B^*(.)$.

In case 2), define $B^*(.) = B_j^*(.)$ over the interval $[v_j, v_k]$ for any neighboring pair in A^* when $v_j < v^*$, and define $B^*(.) = B_{v^*}^*(.)$ over $[v^*, \beta]$. This uniquely defines $B^*(.)$.

In case 3), define $B^*(.) = B_j^*(.)$ over the interval $[v_j, v_k)$ for any neighboring pair in A^*.

In all cases, we have a right-continuous function $B^*(.)$ over $[0, \beta]$. For each neighboring pair v_j, v_k in A^*, we now show that it has the same equilibrium properties in the main text.

[7] Such transversal crossing conditions are generic, while the set of analytic functions among the infinitely differentiable functions is sparse in some sense.

At each v_j,either $B^c(.)$ crosses $B(.)$ from above, or v_j is a merging point.

Case A: v_j is a merging point. The two functions B^c, B are equal over an interval $[v_j, y]$. Make y as large as possible while preserving $B^c = B$, until y is the crossing point. There are two subcases: either B^c crosses B from below (or rises above B), or crosses from above (or sinks below B). In the first subcase, we let $z_k = y$. In the second subcase, B^c will cross B from below at exactly one point in (y, v_k). Furthermore B will also cross $B^*(.)$ at exactly one point in (y, v_k), and we let z_k be this crossing point. By our construction, we must have $B^c(.) > B(.)$ a.e. in (z_k, v_k). By the arguments in Lemma 20, we also have $B(.) > B^*(.)$ a.e. in (z_k, v_k). Furthermore, we have $B(.) \leq B^*(.)$ over (v_j, z_k). Let $a_k = B(z_k), b_k = B(v_k)$. It is now clear that speculators are inactive over $(B(v_j), a_k)$, and active over (a_k, b_k).

Case B: $B^c(.)$ crosses $B(.)$ from above at v_j. Then $B^c(.)$ will cross $B(.)$ from below exactly once, and $B^c(.) < B(.)$ over some interval $(v_j, v_j + \varepsilon)$. By Lemma 20, $B(.) < B^*(.)$ over the interval $(v_j, v_j + \varepsilon)$. Furthermore, $B(.)$ will cross $B^*(.)$ from below exactly once. Let z_k be the unique $B(.)$ crossing of $B^*(.)$ over the interval (v_j, v_k). We now have $B(.) \leq B^*(.)$ over the interval (v_j, z_k), and $B^c(.) > B(.) > B^*(.)$ a.e. over (z_k, v_k). Let $a_k = B(z_k), b_k = B(v_k)$, and we know that speculators are inactive over $(B(v_j), a_k)$, and active over (a_k, b_k).

Thus for any two neighboring pair in A^*, we have exactly the same equilibrium properties in section 3. The only difference is there can be infinitely many pairs of neighboring points in A^*.

The case when $v_j = v_L, v_k = \beta$ can be treated similarly. Since $B^c(.)$ crosses $B(.)$ from below exactly once, and $B(.)$ crosses $B^*(.)$ exactly once at z_k. We let $v_k = \beta$. Then all properties of the interval $[v_L, \beta]$ are the same as in a typical interval $[v_j, v_k]$.

From the above arguments, the maximum of the active speculator bid is determined as follows. In case 1) above, $b_s^* = B(v^*)$. In case 2) above, we have $b_s^* = B(v_k)$, where v_k is the right neighbor of v^*. In case 3), when $(v^*, \beta) \in A^*$, all players have the same maximum equilibrium bid. The result is similar to that of the main text.

7 Proofs

We will introduce notations first for our proof.

Suppose bidder i has use value x, with an upper bound v_i, while bidder k has upper bound v_k. Let $g(x, v_i, v_k)$ denote the use value of bidder k at which the equation $J(g(x, v_i, v_k), v_k) = J(x, v_i)$ is satisfied. We assume that $g(x, v_i, v_k) = v_k$ if $J(x, v_i) \geq v_k$.

When buyer i with use value v_i buys from bidder j with use value $v_j \leq v_i$ during resale and both believe that bidder $k \neq i, j$ has use value upper bound w_k. Bidder j also believes that buyer i has use value upper bound w_i, and sets the optimal reservation price $r(w_i, v_j)$ for buyer i. Let $w_{-,j}$ be the vector $(w_k)_{k\neq,j}$. Let $J(y_i, w_i) = v_i$. The payoff of buyer i in this optimal resale auction is denoted by

$$\tilde{\pi}(v_i, v_j, w_{-,j}) = \int_{r(w_i,v_j)}^{v_i} \left(\prod_{k\neq i,j} F_k(g(x, w_i, v_k)|v_i) \right) dx. \tag{30}$$

The revenue contributed by bidder i is $v - \pi(v_i, v_j, w_{-,j})$. To explicitly specify the number of buyers, we use the notation $\pi(v_i, v_j, w_{-,j}, k)$ when the number of buyers is k. If there are two types of buyers with m type-one buyer n type-two buyers and upper bound w_1, w_2 respectively, then we use the notation $\pi(v_i, v_j, w_1, w_2, m, n)$. The conditional payoff of the seller j is denoted by $\tilde{h}(v_j, w_{-j})$, while $h(v_j, w_{-j})$ denote the unconditional payoff. The unconditional payoff from resale to buyer k is given by

$$h^k(v_j, w_{-j}) = \int_{r(w_k,v_j)}^{w_k} (J(x, w_k) - v_j) \left(\prod_{j\neq i,k} F(g(x, w_j, w_k)) \right) dF(x)$$

$$= \int_{r}^{w_k} (z - v_j) \left(\prod_{j\neq i,k} F(g^j(z)) \right) dF(x(z, w_k)),$$

and $h(v_j, w_{-j}) = \sum_k h^k(v_j, w_{-j})$.

We apply the following standard result on the existence and uniqueness of the solution to a system of differential equations with the initial boundary conditions.

Picard–Lindelöf Theorem: Let b be a real variable, z be a vector, and $f(b, z)$ be vector of functions continuous in b, and Lipschitz continuos in z. Then the system of differential equations $y'(b) = f(b, y(b))$ with the initial boundary condition $y(b_0) = z_0$ has a unique solution over $[b_0, b_1]$.

Proof of Theorem 1:

When buyer i bids b, there are payoffs from (i) winning the auction itself; (ii) resale to some buyer k after winning the auction; (iii) losing the auction to some buyer $k \neq i$, then buys the object from buyer k. The first part of the payoff is

$$\pi_w(v_i, b) = H(b) \prod_{k \neq i} F(\phi_k(b))(v_i - b).$$

Let $K_1 = \{k \neq i : \phi_k(b) \geq v_i\}, K_2 = \{k \neq i : \phi_k(b) < v_i\}$. The second part of the payoff is the sum of payoffs selling to buyer in K_1 :

$$\pi_{wi}(v_i, b) = \sum_{j \in K_1} \pi_{wij}(v_i, b),$$

with

$$\pi_{wij}(v_i, b) = H(b) \int_{r(\phi_j(b), v_i)}^{\phi_j(b)} (J(x, \phi_j(b)) - v_i) \prod_{k \neq i,j} F(g(x, \phi_j(b), \phi_k(b)))dF(x).$$

We have

$$\frac{\partial^2}{\partial b \partial v_i} \pi_{wij}(v_i, b) \geq -H(b) \frac{\partial}{\partial b} \int_{r(\phi_j(b), v_i)}^{\phi_j(b)} \prod_{k \neq i,j} F(g(x, \phi_j(b), \phi_k(b)))dF(x),$$

hence

$$\frac{\partial^2}{\partial b \partial v_i} \pi_{wi}(v_i, b) \geq -H(b) \frac{\partial}{\partial b} \left(\prod_{k \in K_2} F(\phi_k(b)) \right) \left(\prod_{k \in K_1} F(\phi_k(b)) - \prod_{k \in K_1} F(r(\phi_k(b), v_i)) \right)$$

Thus

$$\frac{\partial^2}{\partial b \partial v_i} (\pi_w(v_i, b) + \pi_{wi}(v_i, b)) \geq H(b) \frac{\partial}{\partial b} \left(\prod_{k \in K_2} F(\phi_k(b)) \prod_{k \in K_1} F(r(\phi_k(b), v_i)) \right)$$

$$> H(b) \sum_{j \in K_2} \phi_j'(b) f(\phi_j(b)) \left(\prod_{k \in K_1, k \neq j} F(r(\phi_k(b), v_i)) \prod_{k \in K_2, k \neq j} F(\phi_k(b)) \right)$$

If bidder i loses the auction to bidder j, and the use value of bidder j is lower, then he may buy back from bidder j in K_2 during resale. The payoff from losing the auction is

$$\pi_{li}(v_i, b) = \sum_{j \in K_2} \pi_{lij}(v_i, b),$$

with

$$\pi_{lij}(v_i, b) = \int_{\phi_j(b)}^{y_i} \tilde{\pi}(v_i, x, \phi_i \phi_j^{-1}(x)) H(\phi_j^{-1}(x)) \prod_{k \neq i,j} F(\phi_k \phi_j^{-1}(x))dF(x),$$

where $r(\phi_i \phi_j^{-1}(y_i), y_i) = v_i$. We have

$$\frac{\partial}{\partial v_i} \pi_{lij}(v_i, b) = \int_{\phi_j(b)}^{y_i} \frac{\partial}{\partial v_i} \left(\tilde{\pi}(v_i, x, \phi_i \phi_j^{-1}(x)) \right) H(\phi_j^{-1}(x)) \prod_{k \neq i,j} F(\phi_k \phi_j^{-1}(x))dF(x)$$

$$\frac{\partial^2}{\partial b \partial v_i}\pi_{lij}(v_i,b) = \frac{\partial}{\partial b}\left(\int_{\phi_j(b)}^{v_i} \prod_{k\in K_1, k\neq, j} F(g(v_i,\phi_i\phi_j^{-1}(x),\phi_k\phi_j^{-1}(x)))H(\phi_j^{-1}(x)) \prod_{k\in K_2,k\neq,j} F(\phi_k\phi_j^{-1}(x))dF(x)\right)$$

$$= -H(b)\phi_j'(b)f(\phi_j(b))\left(\prod_{k\in K_1,k\neq,j} F(g(v_i,\phi_i(b),\phi_k(b)) \prod_{k\in K_2,k\neq,j} F(\phi_k(b))\right).$$

Hence

$$\frac{\partial^2}{\partial b \partial v_i}\pi_{li}(v_i,b) = -H(b)\sum_{j\in K_2}\phi_j'(b)f(\phi_j(b))\left(\prod_{k\in K_1,k\neq,j} F(g(v_i,\phi_i(b),\phi_k(b)) \prod_{k\in K_2,k\neq,j} F(\phi_k(b))\right).$$

Since $g(v_i,\phi_i(b),\phi_k(b)) < r(\phi_k(b),v_i)$, we have

$$\frac{\partial^2}{\partial b \partial v_i}u(v_i,b) = \frac{\partial^2}{\partial b \partial v_i}\left(\pi_w(v_i,b)+\pi_{wi}(v_i,b)+\pi_{li}(v_i,b)\right) > 0,$$

and the proof is complete.

Proof of Lemma 1:

We will establish the lemma in several steps. First we assume that there are no speculators, or equivalently, the speculators are not active in equilibrium. Lemma 25 says that the first-order condition for the case of symmetric strategies is the same as that of the model without resale. Apply this lemma to the symmetric bidding strategy $B^*(v)$ which obviously satisfies the first-order condition. With the single-crossing property, the equilibrium property of $B^*(v)$ follows immediately as a consequence.

Lemma 25 *Let $\phi(b)$ be the same inverse bidding strategy for all bidders. In the auction with resale model, the first-order condition of equilibrium holds everywhere is the same as that of the auction without resale model.*

Proof. Note that the first-order condition holds for all use value v_i, b, $\phi(b) = v_i$. To prove that the models with and without resale have the same first-order condition, let buyer i with use value $v_i = \phi(b_0)$ bids $b < b_0$. By symmetry, there is no incentive for resale after the buyer wins the auction, but there may be resale opportunity after losing the bid. If buyer $k \neq i$ with use value $v_k \geq \phi(b)$ wins the auction by bidding $y = \phi^{-1}(v_k)$, there is in fact no incentive for buyer k to resell the object as he is unaware of buyer $i's$ deviation. But if he carries out the resale anyway, he will set the reservation price at v_k. The expected payoff of of buyer i from resale is then

$$\int_{\phi(b)}^{v_i}(v_i - x)dF^{N-1}(x).$$

Hence the overall payoff to buyer i from bidding $b < b_0$ is given by

$$F^{N-1}(\phi(b))(v_i - b) + \int_{\phi(b)}^{v_i}(v_i - x)dF^{N-1}(x). \tag{31}$$

The derivative of (31) with respect to b, evaluated at the optimal bid b_0, is given by

$$\frac{d}{db}\left[F^{N-1}(\phi(b))(v_i - b)\right]|_{b=b_0}. \tag{32}$$

If the buyer i bids $b > b_0$, there is resale after winning the auction, but no resale after losing. The payoff from resale is given by

$$(N-1)\int_{r(\phi(b),v_i)}^{\phi(b)}((x-v_i)f(x) + F(x) - F(\phi(b)))F^{N-2}(x)dx,$$

Hence the payoff from bidding b is given by

$$F^{N-1}(\phi(b))(v_i - b) + (N-1)\int_{r(\phi(b),v_i)}^{\phi(b)} ((x-v_i)f(x) + F(x) - F(\phi(b)))F^{N-2}(x)dx. \quad (33)$$

The derivative of (33) with respect to b is given by

$$\frac{d}{db}\left[F^{N-1}(\phi(b))(v_i - b)\right] - (N-1)f(\phi(b))\int_{r(\phi(b),v_i)}^{\phi(b)} F^{N-2}(x)dx.$$

When evaluated at $b = b_0$, we have $\phi(b) = v_i = r(\phi(b), v_i)$, and the second term of the above derivative is zero. Hence the payoff is differentiable at b_0 with the derivative given by (32). Hence the first-order condition of the auction with resale equilibrium is given by

$$\frac{d}{db}\left[F^{N-1}(\phi(b))(v_i - b)\right] = 0,$$

which is the first-order condition satisfied by the optimal strategy of the auction without resale. ■

Now we need to show that the symmetric bidding strategy $B_0(v)$ is the unique equilibrium bidding strategy in the auction with resale when there are no speculators. Once we establish the symmetry property of the equilibrium, the uniqueness follows from the well-known result of the auction without resale model. The symmetry result is proved first for the case when all regular buyers have the same maximum bid b^*. Lemma 26 says that the first order condition at b^* is also the same as that of the auction without resale model.

Lemma 26 *In the auction with resale with symmetry buyers, assume that all buyers have the same maximum bid b^* in their support of the equilibrium bid distributions, then the first-order condition for equilibrium at b^* is*

$$\frac{d}{db}\left((\beta - b)\prod_{k\neq i} F(\phi_i(b))\right)|_{b=b^*} = 0,$$

as if there is no resale. Furthermore $\phi_i'(b^*) = \phi_j'(b^*)$ for all i, j and

Proof. Let buyer i with use value β bids $b < b^*$. There is no resale if the buyer wins the auction, but there is resale when he loses the auction. After losing the auction to buyer k, he may buy it from the winner with the following payoff

$$\int_{\phi_k(b)}^{\beta} \pi(\beta, x, \phi_i\phi_k^{-1}(x)) \prod_{s\neq i,k} F(\phi_s\phi_k^{-1}(x))dF(x). \quad (34)$$

The derivative of (34) with respect to b is

$$-\phi_k'(b)f(\phi_k(b))\pi(\beta, \phi_k(b), \phi_i(b))\prod_{s\neq i,k} F(\phi_s(b)). \quad (35)$$

At $b = b^*$, we have $\phi_k(b) = \beta = \phi_i(b) = r(\phi_k(b), \phi_i(b))), \pi(\beta, \phi_k(b), \phi_i(b)) = 0$, so that (35) is zero. Hence the first-order condition is

$$\frac{d}{db}((\beta - b)\prod_{k\neq i} F(\phi_i(b)))|_{b=b^*} = 0.$$

■

The following is an initial more restrictive version of the symmetry result without speculators. It will be used for the proof of the full version of the symmetry property. The idea of the proof will also be used in the symmetry result when there are speculators.

Lemma 27 *Assume that there are no speculators. In the auction with resale, assume that all buyers have the same maximum bid b^* in their support of the equilibrium bid distributions, then the inverse equilibrium bidding strategy ϕ_i must be symmetric, all buyers have $[0, b^*]$ as the support of their bid distribution, and $\phi_i(b) > b, \phi_i'(b) > 0$ for all $b > 0$.*

Proof. First we show that the maximum equilibrium bid b^* is strictly less β. If not, since it is a common bid by all buyers with the use value β, the equilibrium payoff of a buyer with the use value β must be 0. By bidding slightly lower, such a buyer has a positive probability of winning at a price lower than his use value, thus the payoff will be higher than the equilibrium payoff, a contradiction. Hence we must have $b^* < \beta$ in equilibrium. Now we will show symmetry. We consider a regular bidder i with use value v_i bidding b near b^*. The payoff of winning in the first stage is given by

$$(v_i - b) \prod_{k \neq i} F(\phi_k(b)). \tag{36}$$

The derivative, whenever it exists, is given by

$$\prod_{k \neq i} F(\phi_k(b)) \left((v_i - b) \sum_{k \neq i} \frac{f(\phi_k(b))}{F(\phi_k(b))} \phi_k'(b) \right) - 1. \tag{37}$$

This expression is linear in $\phi_k'(b)'s, k \neq i$. We will show that this is the case for the resale payoffs as well. Let $K_1 = \{k : \phi_k(b) > v_i\}$, and $K_2 = \{k : \phi_k(b) < v_i\}$. We ignore the bidders k with $\phi_k(b) = v_i$ because the derivative the payoff from selling to or buying from such bidders are known to be zero at the optimal bid. The payoff from reselling to other bidders with higher use value is given by ■

$$\pi_{wi}(v_i, \phi_{-i}(b))) = \sum_{k \in K_1} \pi_{wik}(v_i, \phi_{-i}(b)),$$

with $\pi_{wik}(v_i, \phi_{-i}(b))$ given by

$$\int_{r(\phi_k(b), v_i)}^{\phi_k(b)} (J(x, \phi_k(b)) - v_i) \left(\prod_{j \notin K_1, j \neq i} F(\phi_j(b)) \right) \left(\prod_{j \in K_1, j \neq k} F(g(x, \phi_k(b), \phi_j(b))) \right) dF(x).$$

It is easy to see that the derivative of $h^k(v_i, \phi_{-i}(b))$ with respect to b, whenever it exists, is linear in all $\phi_j'(b)$ if we know that $y = g(x, \phi_k(b), \phi_j(b))$ has this property. By definition y satisfies the equation:

$$y - \frac{F(\phi_j(b)) - F(y)}{f(y)} = x - \frac{F(\phi_k(b)) - F(x)}{f(x)}.$$

When $y = \phi_j(b)$, its derivative is just $\phi_j'(b)$. When $y < \phi_j(b)$, take the implicit derivative with respect to b, we have

$$y'(b)(2 + \frac{F(\phi_j(b))f'(y)}{f(y)^2}) = \frac{f(\phi_j(b))}{f(y)} \phi_j'(b) - \frac{f(\phi_k(b))}{f(x)} \phi_k'(b), \tag{38}$$

so that $y'(b)$ is fact linear in $\phi_k'(b), \phi_j'(b)$. Thus we have shown that the payoff from selling to bidders in K_1 during resale is linear in all $\phi_j(b)'s, j \neq i$. For the payoff $\pi_{lik}(b)$ of buying from bidder k in K_2 during resale, we have

$$\pi_{lik}(b) = \int_{\phi_k(b)}^{v_i} \pi(v_i, x, \phi_i \phi_k^{-1}(x)) \prod_{j \neq k, i} F(\phi_j \phi_k^{-1}(x)) dF(x).$$

The derivative with respect to b is given by

$$-\pi(v_i, x, \phi_i(b)) \left(\prod_{j \neq k, i} F(\phi_j(b) \right) \phi_k'(b)$$

which is certainly linear in $\phi'_k(b)$. Thus the first-order conditions of the equilibrium of the model can be written as

$$\sum_{k\neq i} a_{ik}(b,\vec{\phi}(b))\phi'_k(b) = c(b,\vec{\phi}(b)),$$

where v_i has been replaced by $\phi_i(b)$, $\vec{\phi}(b)$ is the vector $(\phi_1(b),\phi_2(b),...,\phi_N(b))$, $a_{ik}(b,\vec{\phi}(b))$ and $c(b,\vec{\phi}(b))$ are functions of $(b,\vec{\phi}(b))$, continuous in b and Lipschitz continuous in $\vec{\phi}(b)$. These properties are insured by the piecewise C^2 smooth assumptions of F. Thus we can write the system of linear equations in $\phi_j(b)$ as

$$A\vec{\phi}(b) = \vec{c} \tag{39}$$

When $v_i = \beta, b = b^*$, from (37), we have

$$\phi'_k(b^*) = \frac{1}{(\beta - b^*)f(\beta)} \text{ for all } i.$$

The system of equations in $\phi_j(b^*)$ can be solved, and we get

$$\phi'_i(b) = \frac{1}{(N-1)(\beta - b^*)f(\beta)} \text{ for all } i.$$

In other words, the matrix A at $b = b^*$ can be written as an identity matrix and $c_i(b^*,\vec{\phi}(b^*)) = \frac{1}{(N-1)(\beta-b^*)f(\beta)} > 0$. Hence for b near b^*, by continuity, A must be a singular matrix, and (39) can be solved uniquely for $\vec{\phi}(b) > 0$. Therefore we have a system of differential equations satisfying the conditions of the Picard–Lindelöf Theorem. The solution is unique with the initial condition $\phi_i(b^*) = \beta$ for all i. Since it is obvious that a symmetric solution satisfying the boundary conditions exists, the uniqueness of the solution implies we have the symmetry property in a neighborhood of b^*. In the next step, we want to show that the symmetry property holds for all b. Assume this is not true, then we must have some $b_0 > 0$ such that the symmetry property holds for all $b \geq b_0$, but fails in any neighbor hood to the left of b_0. We claim that $\phi_i(b_0) > b_0$, otherwise bidder i can bid slightly lower to get a positive payoff while in the solution the payoff is 0, violating the optimal property of the bid b_0. Therefore we have $\phi_i(b_0) > b_0$. Repeating the arguments above, we can extend the definition below b_0, so that the symmetry property must hold in a neighborhood of 0. This is a contradiction, and the contradiction implies that we must have the symmetry for all $b > 0$, and $\phi_i(0) = 0$ has to be true. Our proof for the Proposition is complete.

We now show that the maximum bid of all the buyers must be the same. This gives us the full symmetry result.

Lemma 28 *Assume that there are no speculators. In the auction with resale, the support of the equilibrium bid distribution of all buyers must be the same, and hence the equilibrium bidding strategies must be symmetric.*

Proof. Assume that, by relabeling, buyer one has the largest maximum equilibrium bid b^*. There is at least another buyer, say buyer two, who also has the same maximum equilibrium bid. Let there be $m \geq 2$ buyers with the maximum equilibrium bid b^*, and let buyer $m+1, m+2, .., m+k$, be the buyers with the next highest maximum equilibrium bid $b_* < b^*$. Using the same argument in Lemma 27, we can show that for each all b, $\phi_i(b), i = 1,2,..,m$ are defined over $[b_*, b^*]$ and are all equal and $\phi'_i(b) > 0$ over the interval $[b_*, b^*]$. Let $\phi(b)$ denote $\phi_i(b)$ for all i. For $b \leq b^*$, when bidder $i = 1$ with use value β bids $b \leq b^*$ Lemma 26 gives us the first-order condition for the equilibrium bid at $b = b^*$

$$(m-1)f(\beta)\phi'(b^*)(\beta - b^*) - 1 = 0.$$

For bidder $j = m+1$ with use value β bidding $b \leq b^*$, there is no resale after winning the auction. When he loses the auction, there may be resale. In the resale, the winner bidder $k, m \geq k > 1$ resells to other bidders, believing that buyers $j \leq m, j \neq k$ has use value upper bound $\phi(b)$, while buyers $j \geq m+1$ are identical with the use value upper bound β. The payoff from resale is given by

$$R(b) = \int_{r(\beta,\phi(b))}^{\beta} \pi(\beta, x, \beta, N-m) dF^{m-1}(x),$$

with

$$R'(b^*) = 0.$$

The payoff of bidding b is therefore

$$F^{m-1}(\phi(b))(\beta - b) + R(b),$$

with the first-order condition

$$(m-1) f(\beta)\phi'(b^*)(\beta - b^*) - 1 = 0. \tag{40}$$

Since the optimal bid of bidder $j = m+1$ is below b_*, the single-crossing condition implies that the derivative of the payoff (from the left) at b^* is

$$m f(\beta)\phi'(b^*)(\beta - b^*) - 1 < 0. \tag{41}$$

Obviously, (40) and (41) are contradictory. This contradiction proves the Proposition. ■

Since the differential equations determining the equilibrium strategies of the auction with resale model are the same as those without resale for symmetric strategies, and the boundary conditions are the same, we immediately have $B^*(v)$ as the unique equilibrium bidding strategy of the model.

Proof of Lemma 2:

The proof requires similar steps above. Lemma 29 gives us the first-order equilibrium condition at b^*, when all regular buyers bid the same maximum amount b^*. When the speculators are added, it does not affect the resale payoff after a bidder wins the auction. It only affects the probability of winning the auction.

Lemma 29 *In the auction with resale with speculators, assume that all buyers have the same maximum bid* b^* *in their support of the equilibrium bid distributions, then the first-order condition for equilibrium at* b^* *is*

$$\frac{d}{db}\left[H(b)(\beta - b)\prod_{k\neq i} F(\phi_i(b)) \right] |_{b=b^*} = 0,$$

as if there is no resale. Furthermore $\phi_i'(b^*) = \phi_j'(b^*)$ *for all* i, j.

Proof. Let regular buyer i with use value β bids $b < b^*$. There is no resale if the buyer wins the auction, but there is resale when he loses the auction. After losing the auction to buyer k, he may buy it from the winner with the following payoff

$$\int_{\phi_k(b)}^{\beta} \pi(\beta, x, \phi_i\phi_k^{-1}(x)) H(\phi_k^{-1}(x)) \prod_{s\neq i,k} F(\phi_s\phi_k^{-1}(x)) dF(x). \tag{42}$$

The derivative of (34) with respect to b is

$$\phi_k'(b) f(\phi_k(b))\pi(\beta, \phi_k(b), \phi_i(b)) H(b) \prod_{s\neq i,k} F(\phi_s(b)). \tag{43}$$

At $b = b^*$, we have $\phi_i(b) = \phi_k(b) = \beta = r(\phi_k(b), \phi_i(b)))$, $\pi(\beta, \phi_k(b), \phi_i(b)) = 0$, so that (35) is zero. Hence the first-order condition is

$$\frac{d}{db}((\beta - b)H(b)\prod_{k\neq i} F(\phi_i(b)))|_{b=b^*} = 0.$$

■

Now we can establish the following symmetry property when all regular bidders bid the same maximum amount.

Lemma 30 *Assume that there are speculators in the auctions with resale, and assume that all regular buyers have the same maximum bid b^* in their support of the equilibrium bid distributions. Let $\underline{b}= \inf\{b : H(b) > 0\}$. Then the inverse equilibrium bidding strategy ϕ_i must be symmetric for all $b \geq \underline{b}$, and $\phi_i(b) > b, \phi_i'(b) > 0$ for all $b > \underline{b}$.*

Proof. Note that the speculators bid would not be higher than b^*. Using the same argument for the case with no speculators, we can show that with the speculators around, we also have $b^* < \beta$ in equilibrium as there is no resale to speculators. Now we will show symmetry. For any two different regular buyers i, j, we need to show that $\phi_i(b) = \phi_j(b)$ for all $b > \underline{b}$. We will first write down the first-order condition that need to be satisfied for bidder i. Consider the bidder i with use value $v_i = \phi_i(b)$, b near β, and define K_1, K_2 as before. We will use notations similar to the no speculator case. The optimal payoff from selling to bidders in K_1 during resale is given by $H(b)\pi_{wi}(b)$. The payoff from winning the auction is

$$H(b)\left((v_i - b)\prod_{k\neq i} F(\phi_k(b)) + \pi_{wi}(b)\right). \tag{44}$$

The payoff after losing the auction is from buying either one of the speculators or one of the buyer $k \in K_2$. This is denoted by

$$\sum_{s=1}^{p} \pi_{lis}(b) + H(b)\sum_{k\in K_2} \pi_{lik}(b). \tag{45}$$

From the same arguments in Lemma 27, we know that the derivative of (44) with respect to b are linear functions of $\phi'_{-i}(b)$ and $H'(b)$. This is also true for the second term of (45). For the first term, let b_s^* be the maximum bid of the support of $H(b)$. We have

$$\pi_{lis}(b) = \int_b^{b_s^*} \pi(v_i, 0, \phi_i(y)) \prod_{j\neq i} F(\phi_j(y))dH(y),$$

who derivative with respect to b is given by

$$-\pi(v_i, 0, \phi_i(b)) \prod_{j\neq i} F(\phi_j(b))H'(b)$$

is linear in $H'(b)$, and does not have terms involving $\phi_j'(b)'s$. From the same arguments, we have a system of linear equations in $\vec{\phi}'(b)$, while combining $H'(b)$ with the right hand side of the equations. The system of linear equations in $\vec{\phi}'(b)$ have a unique solution for b near b^*. Again we apply the Picard–Lindelöf Theorem to get a unique solution of the system of differential equations with the boundary conditions $\phi_i(b^*) = \beta$. Again a symmetric solution of the system of differential equations implies that the symmetry property must hold near b^*. The extension of this property to all of the interval $b > \underline{b}$ follows exactly the same arguments. ■

Another important implication of zero profit for speculators is that the symmetry property of the regular bidders's bidding strategies can now be extended to all $b \leq b^*$. Notice that all the arguments in that Proposition can be carried out as long as $H(b) > 0$. Since $H(b) > 0$ for all $\dot{b} > 0$, we immediately have the following result.

Lemma 31 *Assume that there are speculators in the auctions with resale, and assume that all buyers have the same maximum bid b^* in their support of the equilibrium bid distributions. Then the inverse equilibrium bidding strategy ϕ_i must be symmetric for all $b \geq 0$, and $\phi_i(b) > b, \phi_i'(b) > 0$ for all $b > 0$.*

Now we are ready to show that the maximum bid may be the same among the regular bidders.

Lemma 32 *Assume that there are speculators in the auction with resale with N identical regular buyers. In equilibrium, the support of all regular buyers must be the same, and hence the equilibrium bidding strategies must be symmetric. If b^* is the maximum equilibrium bid of all regular buyers, then their bid distributions all have the support $[0, b^*]$, and we have $\phi_i(b) > b, \phi_i'(b) > 0$ for all $b > 0$.*

Proof. The idea is similar to the proof of Proposition 28. Let b_s^* be the maximum of the support of $H(b) = \prod_{s=1}^{p} H_s(b)$. Assume that, by relabeling, buyer one has the largest maximum equilibrium bid b^*. Clearly we must have $b_s^* \leq b^*$. If $b_s^* < b$, the speculators have no influence on bids above b_s^*, hence we apply the same arguments in Proposition 28, and show that all regular buyers must have the same maximum bid b^*. Then we can apply Proposition 30 and we get the symmetry property for all $\phi_i(b)$. Therefore it is only necessary to consider that case $b_s^* = b^*$. Let there be $m \geq 1$ buyers with the maximum equilibrium bid b^*, and let buyer $m+1$ be the buyer with the next (among the regular bidders) highest maximum equilibrium bid $b_* < b^*$. Using the same argument in Proposition 30, we can show that for each $b \geq b_*$, $\phi_i(b), i = 1, 2, .., m$ are defined over $[b_*, b^*]$ and are all equal and $\phi_i'(b) > 0$ over the interval $[b_*, b^*]$. Let $\phi(b)$ denote the common value. Following the proof of Proposition 28, consider the payoff from resale to the regular bidder one with use value β after bidding $b \in (b_*, b^*)$. There may be resale after bidder one loses the auction to a speculator or a regular bidder $i \leq m$. After a speculator wins the auction by bidding y, sh/e believes that buyers $i \leq m$ have upper bound $\phi(y)$, and buyers $i > m$ have upper bound β. The optimal reservation price for the two types of buyers are $r(\phi(y), 0), r(\beta, 0)$ respectively. The payoff of buying from the speculator is

$$R_s(b) = \int_b^{b^*} \pi(v_*, 0, \phi(y), \beta, m-1, N-m) F^{m-1}(\phi(y)) dH(y).$$

We have

$$R_s'(b^*) = 0.$$

Similarly, let $R_r(b)$ be the payoff of buying the object from a winning regular bidder $i \leq m$. Clearly, we also have $R_r'(b^*) = 0$. The payoff from bidding b is then given by

$$H(b)F^{m-1}(\phi(b))(\beta - b) + R_s(b) + R_r(b)$$

The first-order condition of the equilibrium bid at b^* is

$$(m-1)f(\beta)\phi'(b^*)(\beta - b^*) + \frac{H'(b^*)}{H(b^*)}(\beta - b^*) - 1 = 0. \tag{46}$$

For bidder $m+1$ with use value β bidding b, the derivative of the payoff with respect to b at b^* is

$$mf(\beta)\phi'(b^*)(\beta - b^*) + \frac{H'(b^*)}{H(b^*)}(\beta - b^*) - 1 = 0$$

Again the single crossing property, we must have

$$mf(\beta)\phi'(b^*)(\beta - b^*) + \frac{H'(b^*)}{H(b^*)}(\beta - b^*) - 1 < 0. \tag{47}$$

The two statements (46) and (47) are clearly contradictory, and this proves the Proposition. ■

7.1

Proofs for section 3

Proof of Lemma 4: Since $B^t(x,v)$ is strictly increasing in x, we have $B^t(x,v) < B^t(v)$ when $x < v$. Hence we have

$$B(v) < B^t(v)\int_{r(v)}^{v} d\tilde{F}^N(x) < B^t(v).$$

Proof of Lemma 5: Rewrite (3) as

$$B(v) = \int_{r(v)}^{v} [x - \frac{1-\tilde{F}(x)}{\tilde{f}(x)}] d\tilde{F}^N(x).$$

Since the integrand converges to 0 as $v \to 0$, we must have $B(v) \to 0$. This proves (i). To prove (ii), we use the following formula for $B(v)$

$$B(v) = \frac{N}{F^N(v)} \int_{r(v)}^{v} [xF^{N-1}(x) - \int_{r(v)}^{x} F^{N-1}(y)dy] dF(x).$$

Since $r(v)$ is optimally chosen in a maximization problem, by the envelop theorem, we have

$$B'(v) = \frac{Nf(v)}{F^N(v)}\left(vF^{N-1}(v) - \int_{r(v)}^{v} F^{N-1}(x)dx\right) - \frac{Nf(v)}{F(v)}B(v)$$

$$= \frac{Nf(v)}{F(v)}\left[v - \frac{1}{F^{N-1}(v)}\int_{r(v)}^{v} F^{N-1}(x)dx - B(v)\right] = \frac{Nf(v)}{F(v)}\left[B^t(v) - B(v)\right]. \tag{48}$$

Lemma 4 implies that $B'(v) > 0$.

Proof of Lemma 6: Let $b^0 = \inf\{b : H(b) > 0\}$ be the infimum of the support of $H(b)$. Let $\phi(b)$ be the symmetric inverse bidding strategy of all regular bidders with use value $v \geq \phi(b^0)$. Let $b(v)$ be the inverse of ϕ. We want to show that $b^0 = 0$. Suppose $b^0 > 0$, we will find a contradiction. Let $\pi_0 \geq 0$ be the profit of the speculator. For $v \geq \phi(b^0)$, we have

$$N\int_{r(v)}^{v} [xf(x) + F(x) - F(v)]F^{N-1}(x)dx = b(v)F^N(v) + \pi_0,$$

hence

$$b(v) = \frac{N\int_{r(v)}^{v} [xf(x) + F(x) - F(v)]F^{N-1}(x)dx}{F^N(v)} - \frac{\pi_0}{F^N(v)} \tag{49}$$

$$\leq \frac{\int_{r(v)}^{v} xdF^N(x)}{F^N(v)} < \frac{\int_{r(v)}^{v} dF^N(x)}{F^N(v)} = v - \frac{F^N(r(v))}{F^N(v)}.$$

By lemma 5, $b(v)$ is continuous, hence there is some v' with $v' > b^0$, and $b(v') = b^0$. We claim that b^0 is the minimum bid of the regular buyer. A regular bidder with $v \in [b^0, v']$ will not bid below b^0, as s/he will never win. We must have $b(v) = b^0$ for any $v \in [b^0, v']$ (using non-increasing property). Hence b^0 is an atom of the bid distribution of a regular buyer. For any tie-breaking rule, we can find a regular buyer with a value $v > b^0$, who can increase profit by bidding slightly higher, thus greatly increasing the probability of winning while paying only slightly more after winning. The existence of an atom for the regular buyer is a contradiction. The contradiction implies that the minimum bid must be 0. Since a speculator makes zero profit by bidding 0, the speculator profit must be zero for all equilibrium bids.

Proof of Lemma 7: The regular bidder with use value v bidding b has the following expected payoff (assuming the case one with $v \leq \phi(b)$)

$$u(v,b) = H(b)F^{N-1}(\phi(b))(v-b) + \int_b^{\tilde{y}} \int_{r(\phi(y))}^{v} F^{N-1}(x)dxdH(y), \tag{50}$$

where $\tilde{y}$ satisfies $r(\phi(\tilde{y})) = v$. Taking the derivative with respect to b, we have

$$\frac{\partial u(v,b)}{\partial b} = H^{'}(b)F^{N-1}(\phi(b))[v-b-\int_{r(\phi(b))}^{v} F^{N-1}(x|\phi(b))dx] + H(b)\frac{d\left[F^{N-1}(\phi(b))(v-b)\right]}{db}, \tag{51}$$

which is the same as (6). The formula (6) holds in the other case as well when $v \geq \phi(b)$. When $\phi(b) < v < \phi(y)$, the payoff for the bidder from the second-price auction is the same as that in case one, which is $\int_{r(\phi(y))}^{v} F^{N-1}(x)dx$. When $v \geq \phi(y) > \phi(b)$, the payoff for the buyer from the second-price auction is now equal to

$$(v-r(\phi(y)))F^{N-1}(r(\phi(y))) + \int_{r^*(\phi(y))}^{\phi^{-1}(v)} (v-x)dF^{N-1}(x) = (v-\phi^{-1}(v))F^{N-1}(\phi^{-1}(v)) + \int_{r^*(\phi(y))}^{\phi^{-1}(v)} F^{N-1}(x)dx.$$

It follows that

$$u(v,b) = H(b)F^{N-1}(\phi(b))(v-b) + \int_b^{\phi^{-1}(v)} \int_{r^*(\phi(y))}^{v} F^{N-1}(x)dxdH(y) + C(v), \tag{52}$$

where

$$C(v) = \int_{\phi^{-1}(v)}^{\tilde{y}} [(v-\phi^{-1}(v))F^{N-1}(\phi^{-1}(v)) + \int_{r^*(\phi(y))}^{\phi^{-1}(v)} F^{N-1}(x)dx]dH(y).$$

The derivative leads to the same formula (6).

Proof of Theorem 3: By assumption, $\pi_0(b) = B(\phi_0(b)) - b \leq 0$, hence $B(v) - B_0(v) \leq 0$ for all v. If a speculators is active in equilibrium, Let $\phi(.)$ be the inverse equilibrium bidding strategy of the regular bidders. By Lemma 6, there exists an interval (b_1, b_2) of active bids such that $\eta(b) = \phi(b)$ on the interval. We claim that on the interval $[\eta(b_1), \eta(b_2)] = [x_1, x_2]$, we have $B(.) = B_0(.)$. Otherwise, there exists an active $b = B(v)$, with $\pi_s(b) = B(v) - B_0(v) < 0$ contradicting Lemma 6. However, $B(.) = B_0(.)$ implies that the second term in (6) is zero, while the first term is strictly positive, violating the equilibrium property of the regular bidders. This proves that the speculator cannot be active in equilibrium. We obtain the same equilibrium $B_0(.)$ in the model without resale as shown in Theorem 2.

Proof of Lemma 8: Using L'Hopital's rule, we know that $B^c(v) \to 0$ as $v \to 0$. From the definition, by derivative rules, we have

$$B^{c\prime}(v) = 1 - N + N\frac{F^{N-1}(r(v))r'(v)}{F^{N-1}(v)} + \frac{(N-1)f(v)}{F(v)}(v - B^c(v)).$$

Let $B'(0) = c$, we have

$$c = K(0) + (N-1)(1-c),$$

and $K(0) = \frac{N}{2^N} - (N-1)$, hence we have

$$Nc = K(0) + N - 1 = N2^{-N},$$

and we get $B^{c\prime}(0) = c = 2^{-N}$. When $N > 1$, the inequality $K(0) < 0$ can be proved by a simple mathematical induction.

Proof of Lemma 9: We can write

$$B^c(v) = v - \frac{1}{F^N(v)} \int_{r(v)}^{v} \frac{F(v)}{f(x)} dF^N(x).$$

Hence

$$\begin{aligned} B^c(v) - B(v) &= v - \frac{1}{F^N(v)} \int_{r(v)}^{v} (x + \frac{F(x)}{f(x)}) dF^N(x) \\ &= v - \frac{1}{F^N(v)} \left(\int_{r(v)}^{v} x dF^N(x) + N \int_{r(v)}^{v} F^N(x) dx \right) \\ &= v - \frac{1}{F^N(v)} \left(vF^N(v) - r(v)F^N(r(v)) + (N-1) \int_{r(v)}^{v} F^N(x) dx \right) \\ &= \frac{1}{F^N(v)} \left(r(v)F^N(r(v)) - (N-1) \int_{r(v)}^{v} F^N(x) dx \right). \end{aligned}$$

This proves the first part of the equation (8). We want to prove

$$\Delta'(v) = K(v)F^N(v). \tag{53}$$

We have

$$\Delta'(v) = NF^N(r(v))r'(v) - (N-1)F^N(v) + NF^{N-1}(r(v))f(r(v))r(v)r'(v).$$

Since $r(v)f(r(v)) = F(v) - F(r(v))$, we have

$$\begin{aligned} \Delta'(v) &= NF^N(r(v))r'(v) - (N-1)F^N(v) + NF^{N-1}(r(v))(F(v) - F(r(v)))r'(v). \\ &= F^N(v) \left[Nr'(v) \frac{F^{N-1}(r(v))}{F^{N-1}(v)} - (N-1) \right] = K(v)F^N(v). \end{aligned}$$

Since $K(v)F^N(v) \to 0$, as $v \to 0$, from (53), we have

$$\Delta(v) = \int_0^v K(x)F^N(x) dx,$$

and the second part of (8) is proved. To prove (9), note that

$$\Delta'(v) = (B^{c\prime}(v) - B'(v))F^N(v) + (B^c(v) - B(v))Nf(v)F^{N-1}(v). \tag{54}$$

This, combine (54), gives us

$$B^{c\prime}(v) - B'(v) = K(v) - \frac{Nf(v)}{F(v)} (B^c(v) - B(v)),$$

and we have (9). The statement (10) is an immediate consequence of (9). For the case of $v = 0$, we have

$$B^{c\prime}(v) - B'(v) = K(v) - \frac{Nf(v)v}{F(v)} (\frac{B^c(v) - B(v)}{v}).$$

As $v \to 0$, we have

$$B^{c\prime}(0) - B'(0) = K(0) - N(B^c(0) - B(0)),$$

and we get (11).

Proof of 10: The equation (12) follows immediately from the definitions of B^c and B_0. We have

$$B'(v) - B_0'(v) = \frac{f(v)}{F(v)} \left[N(B^t(v) - B(v)) - (N-1)(v - B_0(v)) \right]$$

$$= \frac{f(v)}{F(v)}[Nv - \frac{N}{F^{N-1}(v)} \int_{r(v)}^{v} F^{N-1}(x)dx - (N-1)(v - B_0(v))]$$

$$= \frac{f(v)}{F(v)} [B^c(v) - NB(v) + (N-1)B_0(v)] = \frac{Nf(v)}{F(v)} \left[\frac{1}{N} B^c(v) + \frac{N-1}{N} B_0(v) - B(v)\right].$$

Proof of Lemma 11: The formula (14) has been proved in Lemma 9. From Lemma 10, we have

$$B'(v) - B_0'(v) = \frac{Nf(v)}{F(v)} \left[\frac{1}{N} B^c(v) + \frac{N-1}{N} B_0(v) - B(v)\right]$$

$$= \frac{f(v)}{F(v)} [(B^c(v) - B(v)) + (N-1)(B_0(v) - B(v))]$$

$$B'(v) - B_0'(v) + \frac{(N-1)f(v)}{F(v)}(B(v) - B_0(v)) = \frac{f(v)}{F(v)}(B^c(v) - B(v))$$

$$\frac{d}{dx}[F^{N-1}(B(v) - B_0'(v))] = \frac{f(v)}{F^2(v)} \Delta(x).$$

Since $\lim_{v\to 0} \frac{f(v)}{F^2(v)}\Delta(x) = 0$, for $N \geq 2$, we have

$$F^{N-1}(B(v) - B_0(v)) = \int_0^v \frac{f(v)}{F^2(v)} \Delta(x)dx = -\frac{\Delta(v)}{F(v)} + \int_0^v \frac{f(v)}{F(v)} \Delta'(x)dx$$

$$= -\frac{\Delta(v)}{F(v)} + \int_0^v K(x)F^{N-1}(x)dx,$$

and

$$B(v) - B_0(v) = -\frac{\Delta(v)}{F^N(v)} + \int_0^v K(x)\tilde{F}^{N-1}(x)dx = B(v) - B^c(v) + \int_0^v K(x)\tilde{F}^{N-1}(x)dx,$$

or

$$B^c(v) - B_0(v) = \int_0^v K(x)\tilde{F}^{N-1}(x)dx.$$

Note that (15) still holds for $N = 1$, and the difference between (14) and (15) gives us the third equation.

Proof of Lemma 12: The first part of the inequality has been shown in Lemma 9. From (13), we have

$$B'(v) - B_0'(v) = \frac{f(v)}{F(v)} [B^c(v) - B_0(v) - N(B(v) - B_0(v))].$$

Using (15), we have

$$B'(0) - B_0'(0) = \lim_{v\to 0} \frac{vf(v)}{F(v)} \left[\frac{B^c(v) - B_0(v)}{v} - N(\frac{B(v) - B_0(v)}{v})\right]$$

$$= \lim_{v\to 0} \frac{\int_0^v K(x)F(x)^{N-1}dx}{vF^{N-1}(v)} - N(B'(0) - B_0'(0))$$

and we have

$$B'(0) - B_0'(0) = \frac{1}{N+1} \lim_{v\to 0} \frac{K(v)}{1 + \frac{(N-1)f(v)v}{F(v)}} = \frac{K(0)}{N(N+1)}.$$

By Lemma 8, we have $K(0) < 0$, hence $B'(0) - B_0'(0) < 0$.

Proof of Lemma 13: To prove (i), note that $B^c(x) < B(x)$ is an immediate consequence of $K(v) < 0$. First we show that $B(.) \leq B_0(.)$. If not, then there exists some $v > v_1$ such that $B(v) > B_0(v)$, and $B'(v) > B_0'(v)$. By Lemma 10, we have $B'(v) < B_0'(v)$ which is a contradiction. Next we show that $B(.) < B_0(.)$. If $B(v) = B_0(v)$ for some $v \in (v_1, v_1 + \varepsilon)$, then Lemma 10 implies that $B'(v) - B_0'(v) < 0$, which means that $B(v + \varepsilon') > B_0(v + \varepsilon')$ for some $v + \varepsilon' \in (v_1, v_1 + \varepsilon)$, contradicting what we have just shown. The proof for (ii) is competely similar.

Proof Lemma 14: Let v_1' be the first $B^c(.)$ crossing of $B(.)$ from below. We also know that v_1 is the first crossing from above, hence there is only one unqiue crossing v_1' from below over $(0, v_1)$. We must have $B^c(.) < B(.)$ over $(0, v_1')$, and $B^c(.) > B(.)$ over (v_0', v_1). By Lemma 12, $B(.)$ is below $B_0(.)$ in the beginning of the interval, and above $B_0(.)$ at the end of the interval. Hence $B(.)$ must cross $B_0(.)$ from below. Let $z_1 \in (0, v_1)$ be the first such crossing. We claim that $v_1' \leq z_1$. To show this, it is sufficient to show $B^c(z_1) - B(z_1) \geq 0$. If not, we have $B^c(z_1) < B(z_1)$. Lemma 10 says that $B'(z_1) - B^{*\prime}(z_1) < 0$ contradicting the crossing property at z_1. Using the same arguments in Lemma 13, we must have $B(.) > B_0(.)$ over (z_1, v_1).

Proof of Lemma 15: From (21), $L(y) > 0$ is equivalent to

$$1 - (N-1)(v - B(v))\frac{f(v)}{F(v)B'(v)} > 0,$$

or

$$B'(v)\frac{F(v)}{f(v)} > (N-1)(v - B(v)), \tag{55}$$

or

$$N(B^t(v) - B(v)) > (N-1)(v - B(v)) \tag{56}$$

Since $v - B^c(v) = N(v - B^t(v))$, (56) is equivalent to

$$B^c(v) > B(v).$$

Proof of Lemma 16: Note that $B(.) \leq B_0(.)$ over $[0, z_1]$. The same arguments in the proof of Theorem 3

show that speculators cannot be active over an interval, roughly speaking, in which "revenue" is lower than "cost". Moreover, when the speculators are not active in equilibrium, the first-order condtion of the regular bidders implies that they must bid $B_0(.)$ over $[0, z_1]$ in equilibrium. Next we want to show that speculators must be active over (a_1, b_1). Suppose that there is an equilibrium $\tilde{H}(.), \tilde{B}(v)$ such that the support of $\tilde{H}$ does not contain the interval (a_1, b_1). Let (c_1, c_2) be an interval in $[a_1, b_1]$, but not in the support of $\tilde{H}(.)$, and c_1 is a boundary point of the support of $\tilde{H}(.)$ so that $\eta(c_1) = \tilde{\phi}(c_1)$, as for any b in the support of $\tilde{H}(.)$, we must have $\tilde{\phi}(b) = \eta(b)$. Since the speculator is inactive in (c_1, c_2), let $x_1 = \eta(c_1)$, and define $\tilde{B}_0(.)$ as in (17) with the initial condition $\tilde{B}_0(x_1) = B(x_1)$. As speculators are not active, we must have $\tilde{B}(.) = \tilde{B}_0(.)$ over (x_1, x_2). We can apply Lemma 10 to the triple $B^c, B, \tilde{B}_0$. Since $B^c(x_1) > B(x_1) = \tilde{B}_0(x_1)$, we must have $B'(x_1) - \tilde{B}_0'(x_1) > 0$ by Lemma 10. Hence a speculator can make a strictly positive profit by bidding slightly above c_1. This violates the equilibrium condition of $\tilde{B}$. Thus we have shown that speculators are active at each bid in (a_1, b_1), hence active over $[a_1, b_1]$. Lemma **??** then implies that regular bidders must bid $B(.)$ over $[z_1, v_1]$ in equilibrium. Since speculators make zero profit which is the highest possible, the equilibrium property is satisfied. To verify the optimality of the bidding strategy $B(.)$ of the regular bidders, by the single-crossing property, it is sufficient to verify the first order condition. The first-order condition of the regular bidder, by Lemma 7, is equivalent to

$$\frac{H'(b)}{H(b)}F^{N-1}(\eta(b))[B^t(\eta(b)) - b] = -\frac{d\left[F^{N-1}(\eta(b))(\eta(b) - b)\right]}{db}$$

$$= F^{N-1}(\eta(b)) - (N-1)F^{N-2}(\eta(b))f(\eta(b))(\eta(b) - b).$$

Hence

$$\frac{H'(b)}{H(b)} = \frac{1 - (N-1)(\eta(b) - b)\frac{f(\eta(b))}{F(\eta(b))}\eta'(b)}{B^t(\eta(b)) - b} = L(b). \tag{57}$$

which is satisfied by the function $H(.)$ given in the lemma. This proves the optimality of the bid $B(v)$ for all $v \in (z_1, v_1)$.

Proof of Lemma 21: First assume that $k < m$. The proof in Lemma 14 applies to each k, and this proves the lemma for the case $k < m$. If $k = m$, and $m < s$, there is $v_{m+1} \in A$, the same arguments also apply, and the lemma is proved too. If $k = m = s$, let z_{m+1} be the first $B(.)$ crossing of $B^*(.)$ from below. By the same arguments earlier, we must have $B^c(z_{m+1}) \geq B(z_{m+1})$, so that $B^c(.)$ must cross $B(.)$ from below before z_{m+1}. We also have $B^c(.) > B(.)$ over (z_{m+1}, β) as there is no more crossing between the two. We have the same conclusion in this case.

Proof of Theorem 4: The arguments of Lemma 16 can be directly applied here for each interval $[v_{k-1}, v_k]$.

In each interval $[v_{k-1}, v_k]$, speculators are not active in (b_{k-1}, a_k), but active in $[a_k, b_k]$. Regular bidders bid $B^*(.)$ over $[v_{k-1}, z_k]$, and $B(.)$ over $[z_k, v_k]$. The bidding functions are optimal, if the joint bid distribution of the speculators is given by $H(b) = H_0 \exp(-\int_b^{b_k} L(y)dy)$. The only thing we need to check is the uniqueness of the bid distribution $H(.)$. Note that we have shown that for any equilibrium, the support of the speculators bid distribution must have the same support as the one described in the Theorem. In particular, it means that the maximum active speculator bid b_s^* is uniquely determined. Furthermore, it satisfies the same differential equation. With the boundary condition $H(b_s^*) = 1$, we know that $H(.)$ is uniquely determined. The uniqueness of the bidding functions have been shown above.

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