Costly location in Hotelling duopoly

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Abstract

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We introduce a cost of location into Hotelling’s (1929) spatial duopoly. We derive the general conditions on the cost-of-location function under which a pure strategy price-location Nash equilibrium exists. We then illustrate this result with several specific functional forms of the cost-of-location function.

Key words: Horizontal product differentiation, spatial competition, cost of location

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1 Introduction

To paraphrase Stigler (1964), no one has the right to invite attention to another extension of Hotelling (1929) without advance indication of the justification for doing so. Our justification is the observation that in a literature where what matters is “location, location, location,” location itself has been treated as a free good. Since economics is sometimes referred to as “the science of scarcity,” this seems an odd specification for economists to make, and we introduce a rental cost of location that rises moving closer to the center of the line to Hotelling duopoly.  

d’Aspremont et al. (1979) show that existence of a pure strategy price equilibrium in Hotelling duopoly requires that firms not locate “too close” to the center of the line. Osborne and Pitchik’s (1987) numerical analysis shows that in a two-stage location choice, price-setting game, equilibrium locations are within the region where equilibrium prices follow mixed strategies.

One interpretation is that Hotelling implicitly assumed the cost of location to be independent of location, normalized it to 0 for simplicity, and that the literature has followed this approach. However, it is of interest to consider the case of location costs that differ by location. In Europe, centrality locations are typically more expensive than those on the periphery. The same was true of the United States through the mid-1950s, and the opposite tends to be true of the United States today.

We consider a two-stage model, with location choice in the first stage, followed by price setting in a standard Hotelling-line duopoly. At the pricing stage, the cost of location is a fixed cost; it does not influence the equilibrium price. When we turn our attention to the first stage, we derive the general condition on the cost of location curve that should be met for an equilibrium to exist in Hotelling’s original setting. One view of the result is that we find conditions under which location costs ensure the existence of a pure-strategy location-price equilibrium. We illustrate this result for a particular family of cost-of-location functions, and for this case, equilibrium locations are bounded away from the ends of the line and satisfy the condition identified

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1Safire (2009) attributes the phrase to late Lord Harold Samuel, a British real estate tycoon.

2Note that “cost of location” is something other than “relocation costs.” The latter starts from a particular situation, and then tells a dynamic story (although often within a static framework). Our design corresponds to the case of a firm that must incur a location-dependent rental cost to set-up cost before it can set price.
by d’Aspremont et al. (1979).

2 Literature

The literature that flows from Hotelling (1929) is vast. Extensions include\(^3\) a finite reservation price (Lerner and Singer (1937), Salop (1979), Economides (1984), Hinloopen and Van Marrewijk (1999)), a circular road (Chamberlin (1953), Vickrey (1964/1999), Samuelson (1967), Salop (1979), Economides (1986)) and graphs (Soetevent, 2010), nonlinear transportation costs (d’Aspremont et al. (1979), Capozza and Van Order (1982), Economides (1986)), more than two firms (Chamberlin (1933), Lerner and Singer (1937), Shaked (1982)), quantity competition (Hamilton et al., 1994, Gupta et al. (1997)), sequential entry with no relocation (Prescott and Visscher (1977), Eaton and Ware (1987)), and price-taking firms (Anderson and Engers (1994), Hinloopen (2002)). If nothing else, this literature establishes that the equilibrium predictions of a spatial oligopoly model are sensitive to the modeling assumptions. Introducing a cost of location is no exception to this observation.

3 Linear transportation costs

3.1 Stage 2: price setting

Following Hotelling (1929), the market consists of a line of finite length \(l\), along which consumers are uniformly distributed. There are two firms, \(A\) and \(B\), located at distances \(a\) and \(b\) respectively from the left and right ends of the line. Without loss of generality, we suppose that firm \(A\) is located to the left of firm \(B\). Both firms supply a homogeneous product that yields gross surplus \(v\). \(v\) is large enough that all consumers always buy.\(^4\) Consumers have unit demand, and a consumer located at \(x\) (measured from the left side of the market) experiences net surplus

\[
U(x; a) = v - t |x - a| - p_A, \tag{1}
\]

\(^3\)These references do not include the closely-related literature that models basing-point and other spatial pricing policies.

\(^4\)In particular, \(v > 7tl/8\), see Hinloopen and Van Marrewijk (1999).
when buying from firm $A$, where $t > 0$ is the transportation rate, and $p_i$ is the price charged by firm $i$, $i = A, B$, and

$$U(x; b) = v - t |l - b - x| - p_B,$$

if the product is bought from firm $B$. The location of the marginal consumer follows from $U(x; a) = U(x; b)$:

$$x^* = \frac{1}{2t} (p_B - p_A + t(l + a - b)).$$

Fixed and marginal cost of production are constant, and equal to zero. Let $c(y) > 0$ be the location cost function, where $y$ is the distance from the firm’s location to the nearest end of the line ($a$ for firm $A$, $b$ for firm $B$). We assume that $c$ is twice continuously differentiable. In a familiar way, now allowing for the cost of location, the objective functions of firm $A$ and $B$ (conditional on prices and locations) are

$$\pi_A(p_A, p_B, a, b) = \begin{cases} 
lp_A - c(a) & p_A < p_B - t(l - a - b) \\
\frac{(t(l + a - b) + p_B - p_A)p_A}{2t} - c(a) & |p_A - p_B| \leq t(l - a - b) \\
-c(a) & p_A > p_B - t(l - a - b)
\end{cases}$$

and

$$\pi_B(p_A, p_B, a, b) = \begin{cases} 
lp_B - c(b) & p_B < p_A - t(l - a - b) \\
\frac{(t(l - a + b) + p_A - p_B)p_B}{2t} - c(b) & |p_A - p_B| \leq t(l - a - b) \\
-c(b) & p_B > p_A - t(l - a - b)
\end{cases}$$

respectively.

The existence of location cost rules out back-to-back, zero-price equilibria (which would imply negative payoffs). Conditions for the existence of pure-strategy price equilibria with firms at different locations are due to d’Aspremont et al. (1979); we state them in the form given by Martin (2002).

**Proposition 1** (d’Aspremont et al. (1979)) If $a + b < l$, a pure-strategy price equilibrium exists if, and only if

$$\frac{a}{l} \leq 3 + \frac{b}{l} - 6\sqrt{\frac{b}{l}}$$

(6)
\( \frac{b}{l} \leq 3 + \frac{a}{l} - 6 \sqrt{\frac{a}{l}}. \)  

If (6) and (7) are satisfied, equilibrium prices and payoffs, given locations, are

\[
p_A^*(a,b) = t \left( l + \frac{a-b}{3} \right),
\]

\[
p_B^*(a,b) = t \left( l - \frac{a-b}{3} \right),
\]

\[
\pi_A^* = \frac{1}{2t} (p_A^*)^2 - c(a),
\]

\[
\pi_B^* = \frac{1}{2t} (p_B^*)^2 - c(b),
\]

where asterisks denote second-stage equilibrium values, taking locations as given.

**Proof.** d’Aspremont *et al.* (1979).  ■

As d’Aspremont *et al.* remark, for symmetric locations, (6) and (7) simplify to

\[
a \leq b \leq \frac{1}{4}.
\]

### 3.2 Stage 1: choice of location

**Proposition 2** Necessary and sufficient conditions for the existence of a pure-strategy location-price equilibrium are that (6) and (7) be satisfied for locations satisfying the location first-order conditions

\[
\frac{t}{3} \left( l + \frac{a^*-b^*}{3} \right) - c' (a^*) \equiv 0
\]

and

\[
\frac{t}{3} \left( l + \frac{b^*-a^*}{3} \right) - c' (b^*) \equiv 0,
\]

(provided the implied \( a \geq 0, b \leq l \)), the location second-order conditions

\[
\frac{t}{9} - c''(a^*) < 0
\]
and
\[ \frac{t}{9} - c''(b^*) < 0, \]  
(16)

and the participation constraints
\[
\pi_A(a^*, b^*) = \frac{1}{2}tl^2 - c(a^*) \geq 0 \\
\pi_B(a^*, b^*) = \frac{1}{2}tl^2 - c(b^*) \geq 0.
\]  
(17) (18)

**Proof.** Substitute (8) and (9) into (10) and (11), respectively, to obtain expressions for the first-stage objective functions. The first- and second-order conditions are immediate. The first-order conditions imply that equilibrium is symmetric. Then
\[ p^*_A(a^*, b^*) = tl. \]  
(19)
With (10) and the requirement that equilibrium payoffs be nonnegative, one obtains (17). Similarly for (18). □

3.3 Example I

For illustrative purposes, let \( t = l = 1 \), and let the location cost function take the form
\[ c(y) = y^\beta, \]  
(20)
for \( y = a, b \) (now omitting asterisks where possible without confusion, for notational compactness). Note that if \( \beta > 1 \), the location cost function (20) corresponds to a proper fraction raised to a power greater than 1.⁵ Larger values of \( \beta \) then imply smaller location cost (see Figure 1).

We assume equilibrium occurs where firms set pure strategy prices, and derive consistency conditions for this to be the case. For firm A (13) gives the first-order condition (likewise for firm B)
\[
\frac{\partial \pi^*_A}{\partial a} = \frac{1}{3} \left( 1 + \frac{a - b}{3} \right) - \beta a^{\beta-1} \equiv 0, \]  
(21)
from which the symmetric equilibrium follows
\[ a^* = b^* = (3\beta)^{\frac{1}{1-\beta}}. \]  
(22)

⁵As will be shown below, \( \beta > 1 \) is one of the conditions for the existence of a pure-strategy price-location equilibrium.
For (22) to be valid, the equilibrium it identifies must satisfy four conditions: (i) the second-order condition of the location stage, (ii) the d’Aspremont et al. condition (12), (iii) the condition that profits are non-negative, and (iv) the stability condition in location space.

The second-order condition is

\[
\frac{\partial^2 \pi_A^*}{\partial a^2} = \frac{1}{9} - \beta(\beta - 1)a^{\beta-2} = \frac{1}{9} - \beta(\beta - 1)(3\beta)^{\frac{\beta-2}{\beta-1}} < 0, \tag{23}
\]

which requires $\beta > 1$. Note that $\lim_{\beta \to 1} a^* = 0$. Condition (12) implies that $(3\beta)^{\frac{1}{\beta-1}} < 1/4$, which holds for $\beta \in (1, 2.4342)$. Further, equilibrium prices are

\[
p_A^*(a^*, b^*) = p_B^*(a^*, b^*) = 1. \tag{24}
\]

Then, using (10) or (17), firm A’s equilibrium payoff is

\[
\pi_A^* = \frac{1}{2} - (3\beta)^{\frac{\beta}{\beta-1}}, \tag{25}
\]

which is positive $\forall \beta > 1$. Finally, necessary conditions for stability are that the trace of the matrix of second-order partial derivatives of payoffs functions be negative, and the determinant positive, when evaluated at equilibrium.
values. Condition (23) ensures that the trace of the matrix is negative. The determinant is positive if \(|-1/9| < \left| 1/9 - \beta(\beta - 1)(3\beta)^{2-\beta} \right|\), a condition that is less binding than (23). Figure 2 shows best response curves and equilibrium locations for two values of \(\beta\), \(\beta = 2.1\) and \(\beta = 2.3\).\(^6\) Best-response lines slope downward: location choices are strategic substitutes. A larger value of \(\beta\) means smaller location cost, all else equal, and (as one expects), the equilibrium location shifts toward the center of the line as \(\beta\) increases.

In sum, the price-location pair \((p^*, a^*) = \left(1, (3\beta)^{\frac{1}{1-\beta}}\right)\) is a pure strategy subgame perfect Nash equilibrium for the Hotelling model with linear transportation cost and cost of location (22) if, and only if, \(\beta \in (1, 2.4342]\). Figure 3 shows the equilibrium locations over the admissible range of \(\beta\).

\(^6\)The best-response functions are implicitly defined in (21). The curves in Figure (2) are based on numerical approximations.
Figure 3: Equilibrium locations in the Hotelling model with cost of location \( c(x) = x^\beta \) as a function of \( \beta \) \((l = t = 1)\).

4 Quadratic transportation cost

4.1 Stage 1: price setting

Following d’Aspremont et al. (1979), assume that a consumer located at \( x \) and that buys from firm \( A \) experiences net utility of

\[
U(x; a) = v - t |x - a|^2 - p_A,
\]

and

\[
U(x; b) = v - t |l - b - x|^2 - p_B.
\]

The location of the marginal consumer is now given by

\[
x^* = a + \frac{p_B - p_A}{2t(l - a - b)} + \frac{l - a - b}{2},
\]
from which the profits of both firms conditional on price and location follow

\[
\pi_A(p_A, p_B, a, b) = \begin{cases} 
lp_A - c(a) & a + \frac{p_B-p_A}{2l(l-a-b)} + \frac{l-a-b}{2} > l; \\
\left(a + \frac{p_B-p_A}{2l(l-a-b)} + \frac{l-a-b}{2}\right)p_A - c(a) & 0 \leq a + \frac{p_B-p_A}{2l(l-a-b)} + \frac{l-a-b}{2} \leq l; \\
-c(a) & a + \frac{p_B-p_A}{2l(l-a-b)} + \frac{l-a-b}{2} < 0,
\end{cases}
\]

and

\[
\pi_B(p_A, p_B, a, b) = \begin{cases} 
lp_B - c(b) & b + \frac{p_A-p_B}{2l(l-a-b)} + \frac{l-a-b}{2} > l; \\
\left(b + \frac{p_A-p_B}{2l(l-a-b)} + \frac{l-a-b}{2}\right)p_B - c(b) & 0 \leq b + \frac{p_A-p_B}{2l(l-a-b)} + \frac{l-a-b}{2} \leq l; \\
-c(b) & b + \frac{p_A-p_B}{2l(l-a-b)} + \frac{l-a-b}{2} < 0.
\end{cases}
\]

Absent cost of location, d’Aspremont et al. (1979) show that for this situation, a unique price equilibrium exists for any locations \(a\) and \(b\), and that it is given by

\[
p^*_A(a, b) = t(l-a-b)\left(l + \frac{a-b}{3}\right), \tag{31}
\]

\[
p^*_B(a, b) = t(l-a-b)\left(l - \frac{a-b}{3}\right). \tag{32}
\]

### 4.2 Stage 2: choice of location

**Proposition 3** Necessary and sufficient conditions for the existence of a pure-strategy location-price equilibrium are the location first-order conditions

\[
-\frac{t}{18}(3l + a^* - b^*)(l + 3a^* + b^*) - c'(a^*) \equiv 0, \tag{33}
\]

and

\[
-\frac{t}{18}(3l + b^* - a^*)(l + 3b^* + a^*) - c'(b^*) \equiv 0, \tag{34}
\]

(provided the implied \(a \geq 0, b \leq l\), the location second-order conditions

\[
-\frac{t}{9}(5l + 3a^* - b^*) - c''(a^*) < 0, \tag{35}
\]
and
\[ -\frac{t}{9}(5l + 3b^* - a^*) - c''(b^*) < 0, \]  
and the participation constraints
\[ \pi_A(a^*, b^*) = \frac{tl^2}{2}(l - a^* - b^*) - c(a^*) \geq 0, \]  
and
\[ \pi_B(a^*, b^*) = \frac{tl^2}{2}(l - a^* - b^*) - c(b^*) \geq 0. \]

**Proof.** Substitute (31) and (32) into (29) and (30), respectively, to obtain expressions for the first-stage objective functions. The first- and second-order conditions are immediate. The first-order conditions imply that equilibrium is symmetric. Then
\[ x^* = \frac{l}{2}, \]
and
\[ p^*_A(a^*, b^*) = p^*_B(a^*, b^*) = tl(l - a^* - b^*), \]
from which the participation constraints (37) and (38) follow.

### 4.3 Example II

Let
\[ c(a) = \gamma (\gamma - a), \]  
with \( \gamma > 0 \). This location cost function implies lower cost towards the centre of the market. In case of symmetric locations. The first-order condition (33) boils down to:
\[ -\frac{1}{6}(1 + 4a) = c'(a), \]
which yields, using (40)
\[ a^* = b^* = \frac{1}{4}(6\gamma - 1). \]

The condition that \( 0 \leq a^* \leq 1/2 \) translates into \( 1/6 \leq \gamma \leq 1/2 \). For these values of \( \gamma \) the second-order condition (35) is satisfied
\[ -\frac{1}{9}(5 + \frac{1}{2}(6\gamma - 1)) < 0. \]
Straightforward calculations show that participation constraint (37) requires that $-\infty < \gamma \leq 1/2 \cup 3 \leq \gamma < \infty$.

That is, $\forall \gamma \in [1/6, 1/2]$ the price-location pair $(p^*, a^*) = (3(1 - 6\gamma)/2, (6\gamma - 1)/4)$ is a pure strategy subgame perfect Nash equilibrium for the Hotelling model with quadratic transportation cost and location cost function (40). In particular, for $\gamma = 1/2$ we have $(p^*, a^*) = (0, 1/2)$ and the participation constraint is just met. That is, in equilibrium firms supply a homogeneous good and charge a price equal to their marginal cost, yielding exactly zero profits. Figure ?? illustrates.

In sum, adding a cost of location to the Hotelling model with quadratic transportation costs can reverse the conclusion of d ’Aspremont et al (1979). In particular, it can yield the situation of (endogenous) minimum differentiation that does not suffer from the problems in Hotelling (1979). Note in passing that the existence of the Bertrand paradox as derived here is not based on a discrete argument (whereby undercutting yields the entire market), but through a ’smooth’ optimization excercise.

Figure 4: Second-stage profits firm $A$, as a function of $a$, given that $b = 1/2$, and $t = l = 1$. 
5 Conclusion

We have motivated our specification with the observation that rent varies with location. In European cities (as in the United States through the first half of the twentieth century), rent tends to rise moving toward metropolitan centers. We have examined the consequences of such a rent-location relationship in a model that assumes a uniform distribution of consumers, and find that pure-strategy location-price equilibrium locations are located in a band that is, in general, bounded away from both the end and the center of the line — rather than minimum or maximum differentiation, location cost induces intermediate degrees of differentiation, with less differentiation as location cost is more sensitive to distance.

We envisage extending the present work to derive the equilibrium rent-location relationship from the distribution of the population. This will permit us to examine conditions leading to the hollowing-out of center cities that is observed in the United States from the latter part of the twentieth century.

References


