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Framing Games: Evidence-Based Decision Making in an Adversarial Setting*

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January 14, 2013

Abstract

We study framing in an adversarial setting by turning the scientific process upside down. Instead of objective truth seekers who formulate hypotheses and then gather evidence to test them, we introduce a class of games between self-interested parties who frame existing evidence to influence a decision maker. The decision maker chooses between the frames based on the likelihood of each, which allows us to characterize the equilibrium decision as a statistical estimator. We find that the estimator is generally biased, and the bias favors the party with the more extreme claim. However, the bias disappears as the amount of evidence grows, and for certain classes of distributions, this kind of adversarial framing performs better than objective inquiry.

JEL classification: C11; C12; D72; D74; D83; K41

Keywords: frames, counterframes, evidence-based decision making, adversarial justice, Bayesian hypothesis testing, litigation games

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*We wish to acknowledge thoughtful discussions with Ed Cheng, Rebecca Haw, Brian McCann, Martin Schonger, Emanuele Tarantino, and conference and seminar participants at EARIE 2012, ETH Zurich, the Naval Postgraduate School, the University of Bologna, and Vanderbilt University.

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1 Introduction

Many institutions and organizations are designed to make decisions in an adversarial setting. The most obvious example is the judiciary, where litigants propose competing theories of the case, and the court chooses between them. A two-party political system shares some of the same characteristics. Less obvious examples are firms or agencies whose divisions pursue objectives that are different from those of sister divisions. In each of these settings, a final arbiter (the court, the electorate, or the manager) resolves disagreements (among litigants, political parties, or divisions of a firm), often by appealing to evidence.

The justification for this appeal is the scientific method which helps decision makers distinguish between hypotheses whose predictions are consistent with evidence and those whose predictions are not. Modern statistical decision theory makes precise this notion and validates it with theorems about consistency and efficiency (e.g. DeGroot, 1970). However, it is a stretch to think that the scientific method is a good description of behavior in adversarial settings. Instead of objective truth seekers who formulate *ex-ante* hypotheses and then gather evidence to test them, we have self-interested parties who *frame* existing evidence to influence a decision maker. The question motivating this paper is how framing affects decsion making in such a setting.

To answer it, we introduce a class of normal form games of complete information between two self-interested players who strategically frame data after they have been produced. In this context, a "frame" is a hypothesis about the data-generating process chosen from a (common knowledge) family of admissible distributions. A non-strategic decision maker[1] chooses between competing frames based on the likelihood of each, and the players receive a payoff based on the decision. These payoffs, e.g., a payment from one player to another, is a taken to be a weighted average of the claims, and because more credible claims should win proportionally more often, the expected payoff is a credibility-weighted average of the competing claims. Unlike assumed in Daughety and Reinganum (2000a), the credibility of claims is not fixed but is determined endogenously.

In choosing an optimal frame, each player faces a tradeoff between credibility (the likelihood that the data were generated by the asserted process) and value (following a favorable decision) of a frame. Moreover, each player also takes the anticipated frame by its rival into account when choosing its own frame. We solve for the Nash equilibrium (Proposition 1) and provide conditions under which it is unique (Lemma 2). Uniqueness is not a crucial result as multiple equilibria are payoff-equivalent. We then argue that the Nash-equilibrium outcome has properties like those of a statistical estimator of the mean of the data-generating process. Compared to the maximum likelihood estimate, the equilibrium outcome of the framing game is unbiased if the players' choice set is symmetric. The estimator is biased otherwise, where the bias favors the player with the more extreme frame (Propositions 2 and 3). However, the bias disappears as the amount of data grows (Proposition 4). We conclude the paper by illustrating that for certain classes of distributions, this kind of adversarial framing performs better than objective inquiry.

[1] In the litigation setting, the adversarial process puts limits on the ability of the decision maker to drawing inference from the behavior of the players. The assumption of nonstrategic behavior by the decision maker thus comports well with practice (Daughety and Reinganum, 2000a:505). For a discussion of the role of the decision maker's sophistication in the litigation game literature, see Froeb and Kobayashi (2012).

Our findings contribute to several different strands of literature, discussed below. We contribute to the economic literature on evidence production and revelation by focusing on the interpretation of evidence after it has been produced. Most of the literature focuses on the incentives of the parties to produce and reveal evidence. In contrast, we study how parties optimally frame the evidence after its production. As such, our approach can be viewed as opening up the "trial" black box in litigation games to show exactly how decision makers reach decisions. Equivalently, the chosen approach can be viewed as a subgame that is played after evidence has been produced and discovered. We show that this subgame matters, and that ignoring it may miss important dimensions of competition between adversaries.

The paper also touches upon the literature on persuasion. However, unlike the recent work by Kamenica and Gentzkow (2011) or Dziuda (2011) that considers only one persuader, we assume two persuaders with opposing interests. We also contribute to the rhetorical literature on framing by formally modeling framing in an adversarial context. The model allows us to better understand why different frames and counterframes are chosen and what they mean. For example, we identify situations where one of the parties claims that the evidence is not very informative, and so should be given little weight. This "obfuscation strategy" is used to explain away unfavorable evidence, and essentially sacrifices credibility to gain a higher payoff.

The structure of the paper is as follows. In Section 2 we discuss the related literature. We illustrate the main ideas of the paper with a simple framing game in a litigation context in Section 3. In Section 4 we present a general characterization of the game and provide proofs for the formal results. We discuss applications in Section 5 and conclude in Section 6.

2 Related Literature

Our analysis of framing touches upon a number of different strands of research. Decision making in an adversarial setting has been extensively studied in the context of the legal system. Daughety and Reinganum (2000a) model the behavior of a trial court that is constrained by the "rules of evidence, procedure, and higher court review" (p. 502). Here, courts aggregate *credible* evidence presented by the litigants, subject to these constraints. In Daughety and Reinganum (2000b) they extend their model and study the production of this evidence by the litigants in an effort to understand the "source and nature of biases that arise in an adversarial system" (p. 366). Evidence is assumed to be *credible*, and the evidence presented is strategically chosen by the litigants. We take a different approach by assuming that evidence is a given, and litigants present their theories as to how the evidence was generated. The issue in our paper is therefore not one of the credibility of the evidence, but of an *endogenization* of the credibility of the assertions the litigants make—given this evidence—in an adversarial system.

By assuming a (potentially) rule-constrained decision maker that observes the evidence and the litigants' theories of the case and then updates her "beliefs" about the evidence-generating process, our approach is similar to some recent legal literature that models judges' or juries' decision making using the metaphor of *statistical hypothesis testing* (e.g., Cheng, 2012). The positive justification for using the metaphor are the many similarities between legal decision making and hypothesis testing, e.g., updating after seeing evidence, binary decisions, and Type I and Type II errors (Saks and Neufeld, 2011). The metaphor has been criticized as leading to a false sense of precision, as it makes objective that which is inherently subjective (Tillers, 2011).

Within the literature on litigation games, our analysis also touches upon evidence (i.e., information) production (e.g., Gilligan and Krehbiel, 1997; Froeb and Kobayashi, 1996, 2001, 2012; Yilankaya, 2002) and revelation (e.g., Milgrom and Roberts, 1986; Shin, 1994; Gentzkow and Kamenica, 2012). However, while this literature studies the incentives to produce and reveal information, we look at evidence interpretation in the subgame after production and revelation.

The objective of our paper is similar to that of the persuasion literature, to understand how persuasion affects decision making. A recent example is Kamenica and Gentzkow (2011) who consider a setting in which a sender chooses a stochastic information generation process (e.g., runs or commissions a clinical trial) and reports the realized signal to persuade a rational decision maker. They find that a sender of information "can structure his arguments, selection of evidence, etc., so as to increase the probability of conviction by a rational judge" (Kamenica and Gentzkow, 2011:2590). Their approach of one sender and one receiver[2], however, does not apply to decision making or persuasion in an adversarial setting. Similar in approach but different in objective are Gilligan and Krehbiel (1989) and Krishna and Morgan (2001a) who study the submission of competing bills to a decision-making legislature by two committee members with opposing biases; Krishna and Morgan (2001b) who show that consulting two experts with opposing viewpoints can be better in deciding on the correct action than consulting only one[3]; and Glazer and Rubinstein (2001) where two debaters with opposing interests know the true state of the world and try to convince a listener.

In the literature on how best to rhetorically frame arguments to increase their persuasiveness, framing means selecting "some aspects of a perceived reality . . . [to] make them more salient . . . , in such a way as to promote a particular problem definition, causal interpretation, moral evaluation, and/or treatment recommendation . . . " (Entman, 1993:52). Recently, this line of research has noted that "virtually all public debates involve competition between contending parties to establish the meaning and interpretation of issues" (Chong and Druckman, 2007:100) which has led to the notion of *counterframes*, offered in competition to the original frame (Brewer and Gross, 2005). Several scholars have presented case studies of framing in an adversarial context (McCright and Dunlap, 2000; Roth, Dunsby, and Bero, 2003; Dugan, 2004; Squires, 2011). Others have studied how individuals react to frames and counterframes (Schniderman and Theriault, 2004; Chong and Druckman, 2007; Hansen, 2007). The work closest to ours is Wedeking (2010) who asks "how do the parties strategically choose which frames to employ." He uses a theory of rhetoric to develop a typology of issue frames and then provides empirical evidence that parties choose frames based not only on the evidence, but also on the rival's frame.

[2] See Dziuda (2011) for a related analysis.

[3] Kawamura (2011) demonstrates that as the number of agents increases, extreme (or exaggerated) messages become less informative

3 Framing in a Simple Litigation Game

Imagine that a plaintiff (P) sues a defendant (D), and the issue before the court is whether the defendant is liable for damages. Before trial, evidence $\bar{z} = (z_1, z_2, ..., z_n)$, with $z_i \in [0, 1]$, is produced and discovered, and the parties offer competing *theories of the case* to explain how the evidence is related to the liability of the defendant. We model evidence as independent draws from an unknown probability distribution. The plaintiff asserts that this distribution is $f_P(z_i) \in \mathscr{F}$ while the defendant asserts it is $f_D(z_i) \in \mathscr{F}$ where $\mathscr{F}$ is the set of *admissible* distributions. A "frame" is thus a hypothesis about the evidence-generating process.

The distinguishing feature of our modeling approach is that the parties choose how to frame the dispute and then the decision maker chooses between the frames. Formally, let γ be the prior weight the decision maker places on the plaintiff's theory of the case, and let $(1 - \gamma)$ be the prior weight on the defendant's theory. The decision maker updates its prior belief about the evidence-generating process, $f_0 = \gamma f_P + (1 - \gamma) f_D$, using *Bayes' rule*,

$$\frac{\theta}{1-\theta} = \frac{\gamma}{1-\gamma} \cdot \frac{\mathscr{L}_P}{\mathscr{L}_D} \tag{1}$$

where

$$\mathscr{L}_P := \prod_{i=1}^{n} f_P(z_i) \qquad \text{and} \qquad \mathscr{L}_D := \prod_{i=1}^{n} f_D(z_i) \tag{2}$$

are the likelihoods of the parties' respective frames. Equation (1) says that the posterior odds are the prior odds times the likelihood ratio (i.e., the odds for P's frame to D's frame given by the data, also referred to as the *Bayes factor*). After updating, the court's posterior belief about the evidence-generating process is

$$f_1 = \theta f_P + (1 - \theta) f_D,$$

where

$$\theta = \frac{\gamma \mathscr{L}_P}{\gamma \mathscr{L}_P + (1-\gamma)\, \mathscr{L}_D} \tag{3}$$

is obtained from equation (1). With damages normalized to unity, we assume the expected award to the plaintiff P (and thus the expected payment by the defendant D) is the posterior mean

$$\mu_1 = \theta \mu_P + (1-\theta)\, \mu_D = \frac{\gamma \mathscr{L}_P \mu_P + (1-\gamma)\, \mathscr{L}_D \mu_D}{\gamma \mathscr{L}_P + (1-\gamma)\, \mathscr{L}_D} \tag{4}$$

where μ_P and μ_D are the mean values of the evidence under the two framing distributions, with $\mu_j = \sum_{i=1}^{n} z_i f_j(z_i)$ for $j = P, D$. We refer to this mean values of the evidence as the parties' *claims*. The plaintiff's payoff is thus a *likelihood-weighted average of the competing claims.*

The dispute is a zero-sum game where a plaintiff P and defendant D simultaneously choose frames $f_P \in \mathscr{F}$ and $f_D \in \mathscr{F}$. The expected payoffs, as a function of the two competing frames, are $\mu_1(f_P, f_D)$ for the plaintiff and $-\mu_1(f_P, f_D)$ for the defendant. A pure-strategy Nash equilibrium of this game is a strategy profile $(f_P^*, f_D^*) \in \mathscr{F} \times \mathscr{F}$ such that

$$\begin{aligned} \mu_1(f_P^*, f_D^*) &\geq \mu_1(f_P, f_D^*) \qquad \forall f_P \in \mathscr{F}, \quad \text{and} \\ \mu_1(f_P^*, f_D^*) &\leq \mu_1(f_P^*, f_D) \qquad \forall f_D \in \mathscr{F}. \end{aligned} \tag{5}$$

We illustrate the equilibrium of this game and the main results assuming the evidence is the result of ten independent coin tosses where $z_i = 1$ if *Head* and $z_i = 0$ if *Tail*. We view these coin tosses as $n = 10$ independent draws from a binomial distribution where k is the number of *Head*s and $n - k$ is the number of *Tail*s. This defines the set of admissible frames, i.e., $\mathscr{F} = \{f(k|n = 10, p) : p \in (0, 1)\}$ with p the probability of *Head*. Because k and n are fixed, p_P and p_D are the parties' only choice variables. These probabilities can be viewed as the parties' claimed *liability assessments*, or *claims*. The set of admissible frames in this *binomial framing game* can thus be redefined as $\mathscr{P} \in [0, 1]$ so that $p_j \in \mathscr{P}$ for $j = P, D$. A pure-strategy Nash equilibrium of this game is a strategy profile $(p_P^*, p_D^*) \in \mathscr{P} \times \mathscr{P}$ such that $p_1(p_P^*, p_D^*) \geq p_1(p_P, p_D^*)$ for all $p_P \in \mathscr{P}$ and $p_1(p_P^*, p_D^*) \leq p_1(p_P^*, p_D)$ for all $p_D \in \mathscr{P}$ with

$$p_1(p_P, p_D) = \theta p_P + (1 - \theta)\, p_D. \tag{6}$$

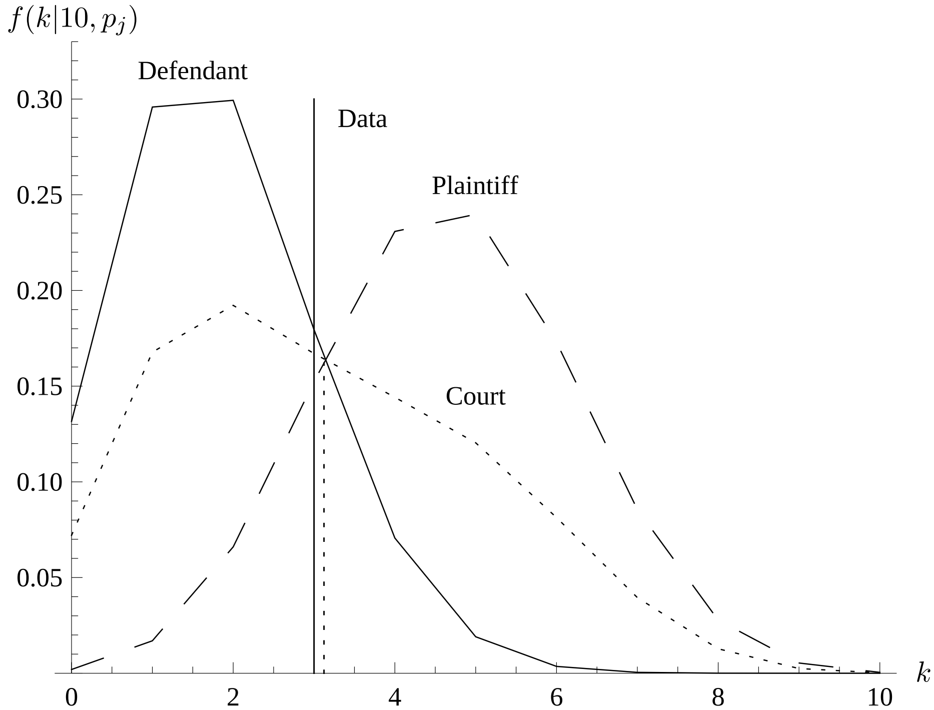
In Table A1 we present numerical results for the equilibria of the binomial framing game for values of k ranging from 0 to 10 and a prior weight $\gamma = 1/2$. For instance, let $k = 3$. In the pure-strategy Nash equilibrium the plaintiff's claim is $p_P^* = 0.464$ and the defendant's claim is $p_D^* = 0.184$. The posterior weight is $\theta = 0.459$ so that the outcome as *likelihood-weighted average of the competing claims* is $p_1(0.464, 0.184) = 0.312$. This posterior mean of the evidence is above the maximum likelihood estimator, $\hat{p}_{MLE} = 3/10$.

We graph this equilibrium outcome in Figure 1. The competing frames are represented by the probability density functions $f_P^* = f(k|10, p_P^* = 0.464)$ [dashed line] and $f_D^* = f(k|10, p_D^* = 0.184)$ [solid line]. The Court's posterior belief $f_1^* = \theta f_P^* + (1 - \theta)\, f_D^*$ is the likelihood-weighted average of the two frames, i.e., probability density function f_P^* and f_D^* [dotted line]. The posterior mean $10 \cdot p_1(p_P^*, p_D^*) = 3.1$ is plotted with a vertical dotted line, just to the right of the solid vertical line at $k = 3$ which corresponds to the maximum likelihood estimate. The decision maker places less weight, 0.459 vs. 0.541, on the more extreme claim, made by the plaintiff, because it is further from the data, $|p_P^* - \hat{p}_{MLE}| > |p_D^* - \hat{p}_{MLE}|$, and therefore less credible, $f(3|10, p_P^* = 0.464) < f(3|10, p_D^* = 0.184)$. However, the bias of the posterior mean, measured as the difference between $\hat{p}_{MLE}$ and $p_1(p_P^*, p_D^*)$, in equilibrium favors the party with the more extreme claim. For evidence in favor of the defendant (a low number of *Head*s, $k = 3$) the favored party is the plaintiff; for evidence in favor of the plaintiff (a high number of *Head*s, e.g., $k = 8$) the bias is against the plaintiff.

4 A General Framing Game

The simple model above captures a litigation setting in which a plaintiff and a defendant each propose competing *theories of the case*, and a court as decision maker delivers a verdict that

Figure 1: Binomial Framing Game with Three *Heads* ($k = 3$)



takes the liability assessments under these theories as well as their *credibility* (likelihood of the assessment given the evidence) into account. The outcome of litigation (i.e., the verdict) is a payment which is a simple transfer from the defendant to the plaintiff. Applying *Bayes' rule*, the payment is a weighted average of the credibility assessments, and because more credible theories should win proportionally more often, the expected award for the plaintiff is a *credibility weighted average of the competing assessments.*

In this section we present a more general model of the *framing game*. Given existing data, two players each present their *frames* or hypotheses of the data-generating process. A frame consists of a *claim* y (analogous to a liability assessment) that is directly payoff relevant and the *credibility* w of the claim that captures the likelihood of the claim given the data. This credibility w is analogous to the likelihood-weights in equation (4). These weights are constrained by the feasible set; and in the previous section this set corresponded to the likelihood function of the binomial distribution. For convenience, we will often refer to the *incredibility* x of the claim which is defined as the reciprocal of the credibility. The payoffs of the game are determined by a *payment* v from *player '−'* (the defendant) to *player '+'* (the plaintiff) that is a credibility-weighted average of the players' claims.

4.1 Definition and Existence of Equilibrium

We first provide a formal definition of the game. Our approach endogenizes the credibility of claims, thus extending the trial technology results by Daughety and Reinganum (2000a) to account for the adversarial nature of litigation and other situations of evidence-based decision making. We then discuss a geometrical interpretation of the game before we give conditions under which a Nash equilibrium of the framing game exists. The remainder of this section derives the results for the *(un)biasedness* and *consistency* of the estimator.

DEFINITION 1 (A Framing Game)**.** *Suppose non-empty sets* $S_+ \subseteq \mathbb{R}_+ \times \mathbb{R}$ *and* $S_- \subseteq \mathbb{R}_- \times \mathbb{R}$ *are given where* $\mathbb{R}_+$ *and* $\mathbb{R}_-$ *are the sets of strictly positive and strictly negative real numbers, respectively. A framing game* $\Gamma(S_+, S_-)$ *is a two-person zero-sum game in which players '+' and '−' simultaneously choose actions* $s_+ = (x_+, y_+) \in S_+$ *and* $s_- = (x_-, y_-) \in S_-$*. For a pair of actions* $s_+ \in S_+$ *and* $s_- \in S_-$*, the payment from player '−' to player '+' is a weighted average of values of* y_j *with weights* $w_j = 1/|x_j|$*:*

$$v(s_+, s_-) = \frac{w_+ y_+ + w_- y_-}{w_+ + w_-} = \frac{x_+ y_- - x_- y_+}{x_+ - x_-}. \tag{7}$$

For action $(x_j, y_j) \in S_j$*, call* y_j *the "claim," call* $|x_j|$ *the "incredibility" of the claim, and call the weight* $w_j = 1/|x_j|$ *the "credibility" of the claim.*

The payment rule in equation (7) is the outcome of a potentially rule-constrained decision making process, such as in liability litigation where equation (7) represents the court's liability assessment based on the presented evidence *and* the litigants' theories of the case. We therefore endogenize the credibility of claims; more specifically, a *"theory of the case"* in our analysis consists of both a claim and its credibility. This is quite contrary to the approach in Daughety and Reinganum (2000a) who assume fixed credibility of claims[4] and propose axioms of trial court behavior that are "broadly descriptive of (rule-constrained) decision making" (p. 506). Below, we reproduce these axioms for fixed levels of incredibility x_j. This allows for a straightforward assessment of where our analysis differs.

DEFINITION 2. *For notational convenience, let* $\tilde{v}(y_+, y_-) \equiv \tilde{v}(y_+, y_- | x_+, x_-)$ *be the payment rule for fixed values of* x_+ *and* x_-*.*

- *A1. Strict Monotonicity:* $\partial \tilde{v}(y_+, y_-)/\partial y_+ > 0$ *and* $\partial \tilde{v}(y_+, y_-)/\partial y_- > 0$
- *A2. Interiority:* $\max\{y_+, y_-\} \geq \tilde{v}(y_+, y_-) \geq \min\{y_+, y_-\}$
- *A3. Homogeneity:* $\tilde{v}(\lambda y_+, \lambda y_-) = \lambda \tilde{v}(y_+, y_-)$ *for all* $\lambda > 0$
- *A4. Symmetry: for all* $y_+ \neq y_-$*:* $\tilde{v}(y_+, y_-) = \tilde{v}(y_-, y_+)$
- *A5. Independence of Presentation: For all* $a_+, b_+, a_-, b_- \in \mathbb{R}$ *such that* $a_- \neq b_-$*:*

$$\tilde{v}(\tilde{v}(a_+, a_-), \tilde{v}(b_+, b_-)) = \tilde{v}(\tilde{v}(a_+, b_-), \tilde{v}(b_+, a_-)).$$

[4] "We abstract from the credibility assessment issue by assuming that both parties provide credible evidence (any noncredible evidence having been discounted or discredited)" (Daughety and Reinganum, 2000a:507). As an alternative, they propose a two-stage approach where at stage 1 the credibility of evidence is assessed "using a Bayesian approach", and at stage 2 the defendant's liability is assessed. We assume that evidence has been presented (evidence is fixed), and its credibility and the defendant's liability are assessed in a single stage.

Under *strict monotonicity* (A1) the payment $v(s_+, s_-) = \tilde{v}(y_+, y_-)$ is increasing in claim y_j; *interiority* (A2) means that the payment $\tilde{v}(y_+, y_-)$ lies within the two players' claims. It further implies that $\tilde{v}(y_+, y_-)$ is reflexive, i.e., if both players claim y, then $\tilde{v}(y, y) = y$. *Homogeneity* (A3) allows for proportional scaling of the claims. *Symmetry* (A4) means that the role of players should not matter. Finally, *Independence of Presentation* (A5) guarantees that the payment rule is "independent of such extraneous factors as the nature, style, or sequence of presentation of the cases" (Daughety and Reinganum, 2000a:510). It is the latter two axioms where our decision-making rule deviates from the trial court behavior in Daughety and Reinganum (2000a). Our payment rule $v(s_+, s_-)$ depends both on the claims and the credibility of these claims. A crucial assumption of our analysis is that a more extreme claim, i.e., one that is further away from the data, implies a lower credibility—an implication at odds with the assumption of "fixed credibility" in Daughety and Reinganum (2000a).

LEMMA 1. *Let $y_+ \neq y_-$ and $x_- > -\infty$. The payment rule $v(s_+, s_-)$ is "strictly monotonic" (A1), "interior" (A2), and "homogeneous" (A3). It is "symmetric" (A4) and "independent of presentation" (A5) if, and only if, the players' claims have equal credibility, $1/x_+ = w_+ = w_- = -1/x_-$.*

The symmetry axiom (A4) and the independence axiom (A5) hold if, and only if, the credibility of the players' claims is the same. In that case, the role of the players and the "nature of presentation" does not matter since claims come with the same relative weights. Once credibility enters the payment rule $v(s_+, s_-)$, the two axioms fail to hold. Our analysis below sheds light on how *endogenous* credibility of the theories of the case affects decision making.

The game in Definition 1 has a useful geometrical interpretation. We provide an illustration of this in Figure 2. Geometrically, S_+ and S_- are sets in the (x, y)-plane on opposite sides of the y-axis, and $v(s_+, s_-)$ is the y-intercept of the line connecting point $s_+ = (x_+, y_+) \in S_+$ with point $s_- = (x_-, y_-) \in S_-$. We can rewrite the payment rule in (7) as

$$v(s_+, s_-) = y_- - m(s_+, s_-)x_- = y_+ - m(s_+, s_-)x_+ \tag{8}$$

where

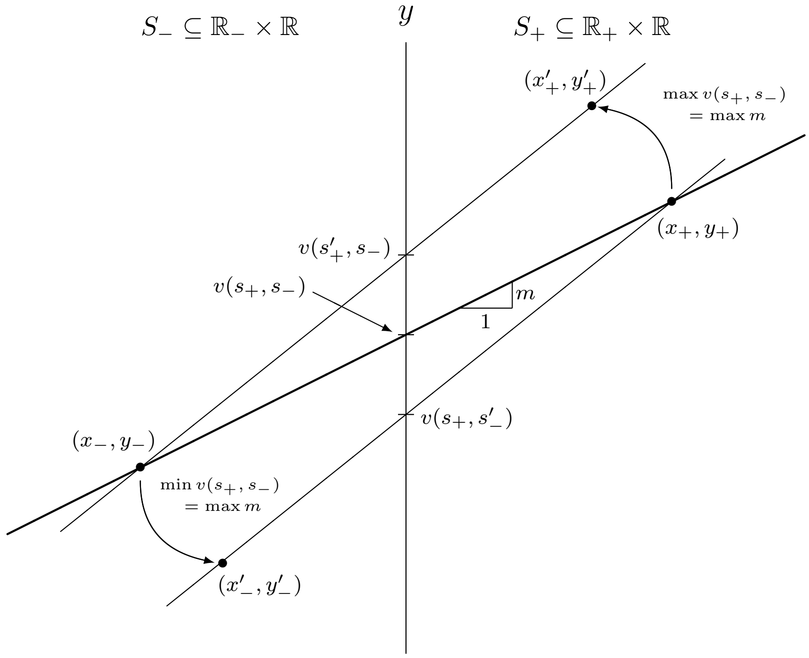
$$m(s_+, s_-) = \frac{y_+ - y_-}{x_+ - x_-} \tag{9}$$

is the slope of the line connecting points s_+ and s_-.

Player '+' seeks to maximize the payments in $v(s_+, s_-)$ whereas player '−' seeks to minimize them. Given any point $s_- = (x_-, y_-)$, to maximize $v(s_+, s_-) = y_- - m(s_+, s_-)x_-$, since $x_- < 0$, player '+' will choose an $s_+ = (x_+, y_+)$ that maximizes the slope $m(s_+, s_-)$. Likewise, given any $s_+ = (x_+, y_+)$ player '−' will choose an $s_- = (x_-, y_-)$ so as to minimize $v(s_+, s_-) = y_+ - m(s_+, s_-)x_+$ and thus will also maximize the slope $m(s_+, s_-)$.

The intuition of the players' problems is depicted in Figure 2. Given an action s_- by player '−', the best response for player '+' is some action $s'_+ = (x'_+, y'_+) \in S_+$ such that the line connecting s_- and s_+ rotates counter-clockwise in s_- and the y-intercept increases from $v(s_+, s_-)$ to $v(s'_+, s_-)$. Likewise, given an action s_+ by player '+', the best response for

Figure 2: Geometry of Framing Games



player '−' is some action $s'_- = (x'_-, y'_-) \in S_-$ such that the line rotates counter-clockwise in s_+ and the y-intercept decreases from $v(s_+, s_-)$ to $v(s_+, s'_-)$.

The following proposition provides a condition for the existence of a Nash equilibrium of the framing game. Let

$$M(S_+, S_-) \quad = \quad \left\{ \frac{y_+ - y_-}{x_+ - x_-} \; : \; (x_+, y_+) \in S_+, \; (x_-, y_-) \in S_- \right\} \tag{10}$$

be defined as the set of slopes of lines connecting a point in S_- to one in S_+.

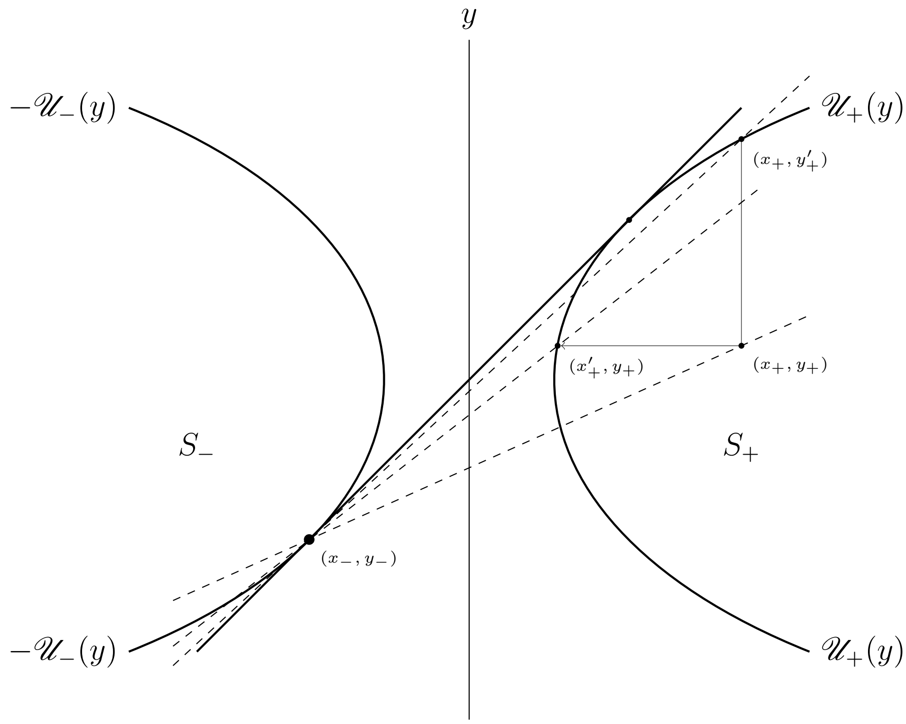
PROPOSITION 1. *A framing game $\Gamma(S_+, S_-)$ has a Nash equilibrium $\left(s^*_+, s^*_-\right)$ if and only if the set of slopes $M(S_+, S_-)$ contains a maximum value. If multiple Nash equilibria exist then the payment rule exhibits the same slope, m^*, with the same y-intercept, v^*, for all equilibria.*

The requirement of $M(S_+, S_-)$ containing a maximum value for the existence of a Nash equilibrium puts some restrictions on the choice sets S_+ and S_-. For instance, the set of slopes $M(S_+, S_-)$ does not contain a maximum if

$$S_+ = \{x \in \mathbb{R} : x \geq \underline{x} > 0\} \times \mathbb{R}$$

so that the action space is not bounded. For such a S_+ player '+' can choose any claim y_+ without compromising the credibility of that claim, i.e., it can choose the lowest feasible incredibility, $x_+ = \underline{x}$, for any y_+. The best move is then to choose $(x_+, y_+) = (\underline{x}, \infty)$, but then no maximum exists because there is always an $\epsilon > 0$ such that $m((\underline{x}, \infty + \epsilon), s_-) >$

Figure 3: Mininimum-Incredibility or Minimum-Unlikelihood $\mathscr{U}$



$m((\underline{x}, \infty), s_-)$, and $M(S_+, S_-)$ is not bounded above. Likewise, suppose

$$S_+ = \mathbb{R}_+ \times \{y \in \mathbb{R} : y \leq \overline{y} < \infty\} .$$

so that the action space is not bounded. For such a S_+ player '+' can choose any incredibility x_+ for the highest feasible claim, $y_+ = \overline{y}$. Because for any such x_+, there is an even smaller $x'_+ = x_+ - \epsilon$ with $\epsilon > 0$, i.e., a higher credibility for claim $y_+ = \overline{y}$, so that $m((x_+, \overline{y}), s_-) < m((x_+ + \epsilon, \overline{y}), s_-)$, the set of slopes $M(S_+, S_-)$ is not bounded above and no maximum exists.

The result in Proposition 1 allows us to put aside the issue of equilibrium multiplicity. Any equilibrium exhibits the same slope and, more importantly, the same payment rule. This implies that the *outcome* of the framing game, our main variable of interest, is *unique* for all equilibria. For the remainder of the paper we assume that an equilibrium exists. More specifically, S_+ and S_- are convex and compact sets. For the discussion of the equilibrium properties below we will further assume that the equilibrium is unique. Conditions for uniqueness are provided in Lemma 2 below.

Before we discuss the equilibrium properties in the next section, we reduce the action space by an immediate implication of the slope-maximization argument made above and further illustrated in Figure 3. Given $y_+ > y_-$, the action s_j that maximizes the slope $m(s_+, s_-)$ is the one with the smallest feasible value of $|x_j|$. This means, for a given claim y_j, player j chooses the most credible action s_j. Consider player '+' and a move s_- by player '$-$.' For any given feasible claim y_+, player '+' chooses the smallest feasible x_+ since, for $s_+ = (x_+, y_+) \in S_+$ and $s'_+ = \left(x'_+, y_+\right) \in S_+$, $m(s_+, s_-) > m(s'_+, s_-)$ for any $x'_+ > x_+$. We define this smallest feasible x_+ for a given y_+ as the *minimum-incredibility* and denote it by $\mathscr{U}_+(y_+)$. The optimal (in)credibility of action s_+ is thus determined by the player's choice of the claim y_+, and the action space is $\tilde{S}_+ = \{(\mathscr{U}_+(y), y) : y \in \text{dom}(\mathscr{U}_+)\}$. Likewise for player '$-$' who chooses the largest feasible x_- (or: smallest $|x_-|$). We denote this *minimum-incredibility*, given some y_-, by $\mathscr{U}_-(y_-)$. Player '$-$'s choice set is then $\tilde{S}_- = \{(\mathscr{U}_-(y), y) : y \in \text{dom}(\mathscr{U}_-)\}$.

Note that $\tilde{S}_+$ and $\tilde{S}_-$ are the graphs of the functions $\mathscr{U}_+$ and $\mathscr{U}_-$ turned 90° clockwise so that the solution of the game is more simply related to slopes. If S_+ and S_- are mirrored on the y-axis, then $\mathscr{U}_+(y) = -\mathscr{U}_-(y)$. Moreover, if S_+ and S_- are [strictly] convex sets, then the functions $\mathscr{U}_+$ and $\mathscr{U}_-$ are [strictly] concave up (or: concave right for the sideways orientation of the graph) and [strictly] concave down (or: concave left).[5] This latter property is by the fact that we defined $\mathscr{U}_+$ such that S_+ is the epigraph of $\mathscr{U}_+$, $S_+ = \text{epi}(\mathscr{U}_+)$; and $\mathscr{U}_-$ is defined such that S_- is the subgraph of $\mathscr{U}_-$, $S_- = \text{sub}(\mathscr{U}_-)$. A function is concave up [concave down] if and only if its epigraph [subgraph] is convex (Rockafellar, 1970:23).

LEMMA 2. *If the functions $\mathscr{U}_+$ and $\mathscr{U}_-$ are strictly concave up and strictly concave down, respectively, then the Nash equilibrium of $\Gamma(S_+, S_-)$ is unique.*

4.2 Equilibrium Properties

For the discussion of the properties of the equilibrium of the framing game, we assume that players '+' and '$-$' are identical but their assigned roles. More specifically, we assume their choice sets are mirrored on the y-axis so that $\mathscr{U}_+(y) = -\mathscr{U}_-(y) = \mathscr{U}(y)$ for claims $y \in Y$ where $Y \subseteq \mathbb{R}$ is a compact set. Moreover, let $\mathscr{U}(y)$ be a positive strictly concave up function with unbounded derivatives. Given these assumptions, by Lemma 2 the game has a unique Nash equilibrium in pure strategies. Write $\Gamma(\mathscr{U}) = \Gamma(S_+, S_-)$ for the corresponding game and $v(\mathscr{U}) = v(s_+, s_-)$ for the payment rule of the game.

We interpret the equilibrium outcome $v^*(\mathscr{U}) = v(s_+^*, s_-^*)$ as an estimator of the mean of the data. Similar to the binomial example in the previous section, let there be a vector of data $\bar{z} = (z_1, \ldots, z_n)$, $z_i \in Y$, of n random draws from a probability distribution function $f \in \mathscr{F}$. The true function f is unknown; instead, the players propose data-generating processes $f_+ \in \mathscr{F}$ and $f_- \in \mathscr{F}$. Any such process f_j can be characterized by two variables:

- the *mean* $\mu_j = \text{E}_j(z_i) = \sum z_i f_j(z_i)$ when $z_i \in Y$ are random draws from f_j;
- the *likelihood* $\mathscr{L}_j = \prod f_j(z_i)$ for the process f_j given data $\bar{z}$.

The likelihood-weighted average $v^*(\mathscr{U})$ of the means of the data under the competing models f_+ and f_- is an estimator of the true mean, μ_{true}, of data $\bar{z}$. Player '+' prefers high estimates of the mean of $\bar{z}$ whereas player '$-$' prefers low estimates of the mean of $\bar{z}$. In terms of the game in Definition 1 the claim y_j can be interpreted as the claimed mean μ_j, and the credibility w_j of the claim can be interpreted as the likelihood $\mathscr{L}_j$, so that $|x_j| = 1/\mathscr{L}_j$ is the unlikelihood of the data-generating process f_j. In this likelihood interpretation, we define $\mathscr{U}(\mu_j) = \min\{1/\mathscr{L}_j : \text{E}_j(z_i) = \mu_j\}$ as the *minimum-unlikelihood* of a data-generating process

[5] We use the terminology "concave up" for convex functions and "concave down" for concave functions.

f_j having a specified mean $\mu_j \in Y$. Given strict convexity of $\mathscr{U}$, the minimum-unlikelihood $\mathscr{U}$ has a unique minimum value at μ_{MLE},

$$\mu_{MLE} \equiv \arg\min_{\mu \in Y} \mathscr{U}(\mu) = \arg\max_{\mu \in Y} -\mathscr{U}(\mu). \tag{11}$$

This μ_{MLE} is the mean of a maximum-likelihood model, or the *maximum-likelihood estimator.*

LEMMA 3. *The players' Nash equilibrium claims y_+^* and y_-^* satisfy $y_-^* < \mu_{MLE} < y_+^*$ and $y_-^* < v^*(\mathscr{U}) < y_+^*$. Moreover, in Nash equilibrium, the line connecting the points $\big(\mathscr{U}(y_-^*), y_-^*\big)$ and $\big(\mathscr{U}(y_+^*), y_+^*\big)$ is tangent to $\mathscr{U}(y_+^*)$ and $-\mathscr{U}(y_-^*)$, implying*

$$\mathscr{U}'(y_+^*) = -\mathscr{U}'(y_-^*) = \frac{\mathscr{U}(y_+^*) + \mathscr{U}(y_-^*)}{y_+^* - y_-^*} = \frac{1}{m}. \tag{12}$$

Equation (12) provides a straightforward equilibrium condition that formalizes the above slope-maximization argument (illustrated in Figures 2 and 3). The line connecting the points $\big(\mathscr{U}(y_-^*), y_-^*\big)$ and $\big(\mathscr{U}(y_+^*), y_+^*\big)$ must be tangent to the minimum-unlikelihood functions $\mathscr{U}$ and $-\mathscr{U}$ evaluated at the equilibrium strategies.[6]

In equilibrium, both players will try to manipulate the outcome by shading their claims. This means, player '+' will present a claim higher than the claim with the maximum likelihood, $y_+^* > \mu_{MLE}$. This comes at the cost of lower credibility (because $\mathscr{U}(y) > \mathscr{U}(\mu_{MLE})$ for all $y \neq \mu_{MLE}$) and thus a lower weight of claim y_+^* in the payment rule $v(s_+^*, s_-^*)$. Player '$-$', on the other hand, will present a claim lower than the maximum likelihood estimator, also at the cost of its credibility. As a result, the players will in equilibrium offer different, i.e., competing claims.

4.2.1 Unbiasedness

If $v^*(\mathscr{U}) = \mu_{MLE}$, then the equilibrium outcome of the framing game is an *unbiased* estimator (relative to the maximum likelihood estimator). The equilibrium outcome is *biased* if otherwise.

PROPOSITION 2. *Suppose $\mathscr{U}(y)$ is a positive strictly concave up function. If for such a $\mathscr{U}$ there are two claims $y'_+ \in Y \subseteq \mathbb{R}$ and $y'_- \in Y \subseteq \mathbb{R}$ such that*

$$\mathscr{U}'(y'_+) = \frac{\mathscr{U}(y'_+)}{y'_+ - \mu_{MLE}} \tag{13}$$

$$-\mathscr{U}'(y'_-) = \frac{\mathscr{U}(y'_-)}{\mu_{MLE} - y'_-} \tag{14}$$

$$\frac{\mathscr{U}(y'_+)}{y'_+ - \mu_{MLE}} = \frac{\mathscr{U}(y'_-)}{\mu_{MLE} - y'_-} \tag{15}$$

then $y_+^* = y'_+$, $y_-^* = y'_-$, and the equilibrium outcome is an unbiased estimator of the mean of the data, $v^*(\mathscr{U}) = \mu_{MLE}$.

[6]Note that the expression in (12) is with the reciprocal of the slope, $1/m$. This is because the slope in equation (9) is defined in the 90° clockwise sideways view of the minimum unlikelihood functions.

Observe that the RHS of the equilibrium condition in equation (12) in Lemma 3 is less strict than the unbiased-equilibrium conditions (13)-(15) in Proposition 2. While the former characterizes *any* Nash equilibrium, the latter guarantee an *unbiased* Nash equilibrium outcome.

In order to analyze the properties of the Nash equilibrium further, let player '+'s claim be $y_+ = \mu_{MLE} + \Delta y_+$ and player '−'s claim be $y_- = \mu_{MLE} - \Delta y_-$. We can therefore characterize their claims as deviations from the maximum-likelihood estimator. We refer to such deviations as *shading*. The payment rule can be rewritten as

$$v(s_+, s_-) = \mu_{MLE} + \frac{\mathscr{U}(\mu_{MLE} - \Delta y_-)\Delta y_+ - \mathscr{U}(\mu_{MLE} + \Delta y_+)\Delta y_-}{\mathscr{U}(\mu_{MLE} + \Delta y_+) + \mathscr{U}(\mu_{MLE} - \Delta y_-)}. \tag{16}$$

The equilibrium outcome $v^*(\mathscr{U})$ is an unbiased estimator if and only if

$$\frac{\mathscr{U}(\mu_{MLE} + \Delta y_+^*)}{\mathscr{U}(\mu_{MLE} - \Delta y_-^*)} = \frac{\Delta y_+^*}{\Delta y_-^*}, \tag{17}$$

where Δy_+^* and Δy_-^* are the players' respective equilibrium deviations from μ_{MLE}. If this condition (17) is not satisfied, then we say the equilibrium outcome has an *upward bias*, $v^*(\mathscr{U}) > \mu_{MLE}$, if

$$\frac{\mathscr{U}(\mu_{MLE} + \Delta y_+^*)}{\mathscr{U}(\mu_{MLE} - \Delta y_-^*)} < \frac{\Delta y_+^*}{\Delta y_-^*}, \tag{18}$$

and a *downward bias*, $v^*(\mathscr{U}) < \mu_{MLE}$, otherwise.

The conditions in Proposition 2 and equations (16)-(18) apply to any positive concave up function $\mathscr{U}$. Consider for example a symmetric function $\mathscr{U}$ where $\mathscr{U}$ is defined to be *symmetric* if

$$\mathscr{U}(\mu_{MLE} + \Delta y) = \mathscr{U}(\mu_{MLE} - \Delta y) \quad \text{for all } \Delta y > 0. \tag{19}$$

For such a symmetric case, the Nash equilibrium satisfies the no-bias condition in equation (17). However, symmetry is not a necessary condition, and an asymmetric function $\mathscr{U}$ where

$$\mathscr{U}(\mu_{MLE} + \Delta y) \neq \mathscr{U}(\mu_{MLE} - \Delta y) \quad \text{for some } \Delta y > 0 \tag{20}$$

may yield an unbiased equilibrium outcome. In the following Proposition we summarize the estimator's bias properties for a more restrictive shape of functions $\mathscr{U}$ what we call *regular asymmetry*.

PROPOSITION 3. *Suppose $\mathscr{U}$ is a positive strictly concave up function with unbounded derivative and a minimum value at μ_{MLE}. Then*

1. *for a symmetric $\mathscr{U}$ the estimator is unbiased, $v^*(\mathscr{U}) = \mu_{MLE}$; moreover,*

2. *for a regular asymmetric $\mathscr{U}$ with $|\mathscr{U}'(\mu_{MLE}+\Delta y)| < |\mathscr{U}'(\mu_{MLE}-\Delta y)|$ for all $\Delta y > 0$, so that $\mathscr{U}(\mu_{MLE}+\Delta y) < \mathscr{U}(\mu_{MLE}-\Delta y)$, for all $\Delta y > 0$, the estimator exhibits an upward bias, $v^*(\mathscr{U}) > \mu_{MLE}$;*

3. *for a regular asymmetric $\mathscr{U}$ with $|\mathscr{U}'(\mu_{MLE}+\Delta y)| > |\mathscr{U}'(\mu_{MLE}-\Delta y)|$ for all $\Delta y > 0$, so that $\mathscr{U}(\mu_{MLE}+\Delta y) > \mathscr{U}(\mu_{MLE}-\Delta y)$, for all $\Delta y > 0$, the estimator exhibits a downward bias, $v^*(\mathscr{U}) > \mu_{MLE}$.*

Given the convexity of $\mathscr{U}$, the condition for an upward bias, $|\mathscr{U}'(\mu_{MLE}+\Delta y)| < |\mathscr{U}'(\mu_{MLE}-\Delta y)|$, can be rewritten as $\mathscr{U}(\mu_{MLE}+\Delta y) < \mathscr{U}(\mu_{MLE}-\Delta y)$. This latter condition implies that *shading y_+ greater than μ_{MLE} gives a more likely theory than shading y_- less than μ_{MLE} by the same amount.* Intuitively, player '+'s costs of shading are lower than its rival's, and as we show in the proof of Proposition 3, player '+' will in equilibrium shade more than its rival so that $\Delta y_+^*/\Delta y_-^* > 1$. However, it is not obvious what this means for the credibility of the claim. By condition (18), such a claim by player '+' could be more or less credible than its rival's since both $\mathscr{U}(\mu_{MLE}+\Delta y_+^*)/\mathscr{U}(\mu_{MLE}-\Delta y_-^*) > 1$ and $\mathscr{U}(\mu_{MLE}+\Delta y_+^*)/\mathscr{U}(\mu_{MLE}-\Delta y_-^*) < 1$ potentially satisfy the condition. In the following corollary we show that in equilibrium, more shading implies a less credible claim.

COROLLARY 1. *The player who is favored by the regular asymmetry of $\mathscr{U}$ will in equilibrium shade more and this more extreme claim is less credible.*

The player who is favored will exploit this advantage and in equilibrium offer a *more extreme* and *less credible* claim.

4.2.2 Consistency

Consider again the vector of data $\bar{z} = (z_1, z_2, \ldots, z_n)$. As n increases, the models the players propose will be more constrained by the data. This implies that the likelihood of a model f_- as a fraction of the likelihood of the maximum likelihood model should go to 0. Similarly for a model f_+. Under regularity conditions (see Greene, 2003:474), as n increases the mean of the maximum likelihood model, μ_{MLE}, converges to the true mean μ_{true}. It should follow then that players will optimally shade their claims less far away from the maximum likelihood model as data increases and the unlikelihood of exaggerated claims increases. A precise formulation requires more details of the underlying process and allowable models, but the following simplified version gives the basic idea.

PROPOSITION 4. *Suppose for positive integers n, $\mathscr{U}_n$ are positive strictly concave up functions with unbounded derivatives each with a minimum value at the same μ_{MLE}. Suppose for $y \neq \mu_{MLE}$,*

$$\lim_{n\to\infty} \frac{\mathscr{U}_n(\mu_{MLE})}{\mathscr{U}_n(y)} = 0.$$

Then $\lim_{n\to\infty} v^*(\mathscr{U}_n) = \mu_{MLE}$.

The implications of Proposition 4 for the performance of the equilibrium outcome $v^*(\mathscr{U})$ as statistical estimator are straightforward. We saw in Proposition 2 that $v^*(\mathscr{U})$ is possibly biased for small samples. However, as Proposition 4 shows, it is a consistent estimator, converging to the maximum-likelihood estimator as the number of data points increases. Intuitively, with more data, extreme claims (or any claim deviating from the maximum-likelihood estimator) become less credible and thus more "costly." In equilibrium, players will shade less as more data are available and their choices are even more constrained by credibility concerns.

5 Applications

In this section, we demonstrate the ability of our model to explain some of the features of institutions that use adversarial, evidence-based decision making. We use the model to compare these institutions to obvious alternatives, like an inquisitorial regime, e.g., as in (Froeb and Kobayashi, 2001).

5.1 The Obfuscation Strategy

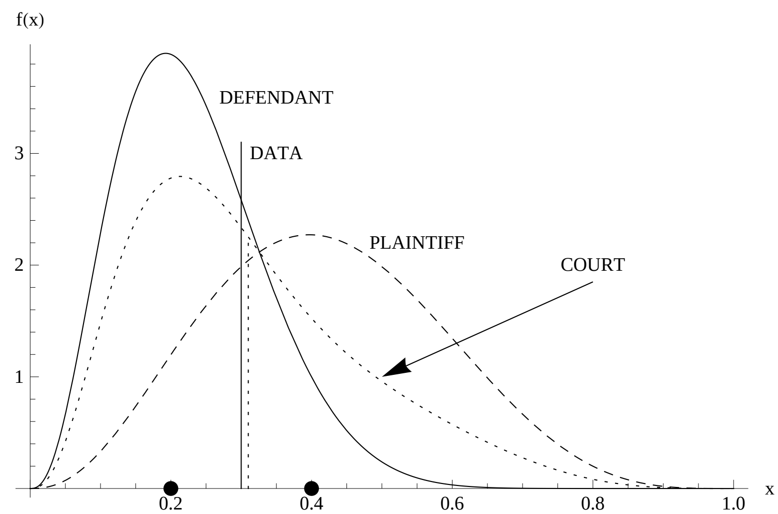
Anyone who has participated in a trial or follows political campaigns will recognize what we call the "obfuscation strategy." One party essentially claims that the evidence is not very informative, and so should be given little weight. Of course, the counter claim is that it should be given more weight. Although it is usually thought of as a rhetorical device, it arises in our model in situations where one of the parties chooses a distribution with a mean further away from the evidence. To minimize the costs of a lower likelihood, the party also claims that the distribution has a large spread or variance. This is analogous to claiming that the data are not very informative.

We illustrate the strategy with a distribution where the spread can be varied independently of the mean. Using such a class of distributions, each party's claim has two elements, the location of the distribution and the spread of the distribution. Despite the added complexity, the framework and theorems of Section 4 still apply. Suppose, for example, that the parties agree that the data are independent draws from the same Beta(α, β) distribution which we reparameterize as Beta(p, q) with $p = \mathrm{E}(z_i) = \alpha / (\alpha + \beta)$ and $q = \alpha + \beta$, so that $\mathrm{Var}(z_i) = p\,(1-p)\,/\,(q+1)$. With this parameterization, p is the location parameter, and q is inversely related to the spread.

In Figure 4, we illustrate the equilibrium outcome of the framing game where the evidence, two draws from a Beta(p, q) distribution, $\bar{z} = (0.2, 0.4)$, is plotted as dots on the horizontal axis. In equilibrium, the defendant asserts that the data are generated by a Beta$(p = 0.23, q = 15.8)$ (small spread) while the plaintiff asserts a Beta$(p = 0.42, q = 8.3)$ (large spread). We plot the defendant's frame [solid line], the plaintiff's frame [dashed line], and the court's posterior distribution [dotted line]. The mean of the posterior distribution is about 0.31, plotted with a dotted vertical line, just to the right of the mean of the data 0.3, plotted with a solid vertical line.

Note that the best response to an obfuscation strategy is what might be called an "elucidation strategy," choosing a distribution with a smaller spread and a location closer to the data. The party whose position is favored by the evidence (the defendant in Figure 4),

Figure 4: Plaintiff's Obfuscation Strategy for a Beta(α, β) Distribution



chooses a distribution with most of its probability mass closer to the data, resulting in a bigger likelihood. This is analogous to claiming that the data are informative.

5.2 The Set of Admissible Claims

Our assumption that both parties choose frames f_j from the same admissible set $\mathscr{F}$ (or probabilities p_j from the same admissible set $\mathscr{P}$) is significant. The consistency proof of the previous section, for example, follows from the collapse of the admissible set onto the maximum likelihood estimate as the number of observations increases. This requires that the parties agree on the boundaries of admissible set.

To see this why this is important, suppose that the data appeared in the order of three *Heads*, and then seven *Tails*. The order is immaterial for a sequence of independent random variables, but to a self-interested party trying to frame the data in a favorable light, this particular sequence suggests another way to frame the data. The plaintiff could claim that the data-generating process was not *i.i.d.*, but rather, after the first three flips, it became tainted, and always returned *Tails*. In other words, the plaintiff could claim that only the first three observations provide information about the unknown distribution, and that the last seven were uninformative. And of course, the defendant could counter-claim the opposite.

Our framework rules out these kinds of frames and counterframes. Because they are outside the admissible set, our decision maker cannot place a credibility weight on the claims, and so cannot evaluate them. Presented with these kind of claims, the decision maker might be forced to ignore the evidence. To avoid this outcome, one might expect institutions to evolve in ways that encourage agreement on the feasible set which would make decision making more accurate, and therefore more efficient, as in Posner (1981).

The institution of "hot tubbing" can be viewed as serving this purpose. Used in Britain, New Zealand, and Australia, the court asks the competing experts to confer with each other, by themselves in a"hot tub," and then write a report outlining areas of agreement and disagreement (Baker and Morse, 2006:10). For example, expert economists are encouraged to clearly articulate their competing theories of the case so that the court can more easily understand and test their predictions, with the goal of assessing credibility. This kind of institutional agreement can be viewed as improving the performance of the follow-on competition at trial.[7]

5.3 Comparison to Scientific Inquiry

With our formal model, we can compare adversarial evidence-based decision making to an obvious alternative, neutral objective inquiry. In this subsection, we design an experiment where we know the underlying data-generating process, and then compare the properties of decision making under the two regimes. We focus on bias but other properties, like variance, could also be considered.

Models of scientific inquiry are well known. Instead of allowing the parties to frame the data, we imagine that our decision maker appoints a neutral expert to examine and interpret the data. Like the decision maker in the framing game, the neutral expert, or "inquisitor," uses Bayesian inference. However, she updates her *own* prior belief about the unknown parameter p after seeing the evidence, i.e., without considering the parties' frames. Her prior belief is characterized by a Beta(α, β) distribution that includes the uniform ($\alpha = \beta = 1$) as a special case. We imagine that a neutral inquisitor would choose a neutral prior like the uniform, which we assume in what follows, but this is not necessary.

The Beta(α, β) distribution is the conjugate prior to a Binomial likelihood. This means that the posterior distribution is in the same family as the prior, but is updated with the evidence. Specifically, if the evidence is of $Heads = k$ and $Tails = n - k$, the inquisitor's posterior belief about the unknown data-generating process is characterized by a Beta($\alpha + Heads, \beta + Tails$) distribution. This posterior belief has a mean of

$$\hat{p}_{INQ} = \frac{\alpha + Heads}{\alpha + Heads + \beta + Tails} \tag{21}$$

which we take as the inquisitorial estimator of the unknown mean. We compare this to the adversarial estimator $\hat{p}_{ADV} = p_1(p^*_P, p^*_D)$.

As in Section 3, we assume that the evidence is generated by a binomial distribution $f(k|n = 10, p)$. For each possible realization of evidence, i.e., $k = \{0, 1, 2, ..., 10\}$, we compute the Nash equilibrium of the framing game and the posterior mean $\hat{p}_{ADV}$ in equation (6)[8] as well as the posterior mean $\hat{p}_{INQ}$ of the inquisitorial regime in equation (21). Because we know the data-generating process, for each p we can compute how likely each realization k is,

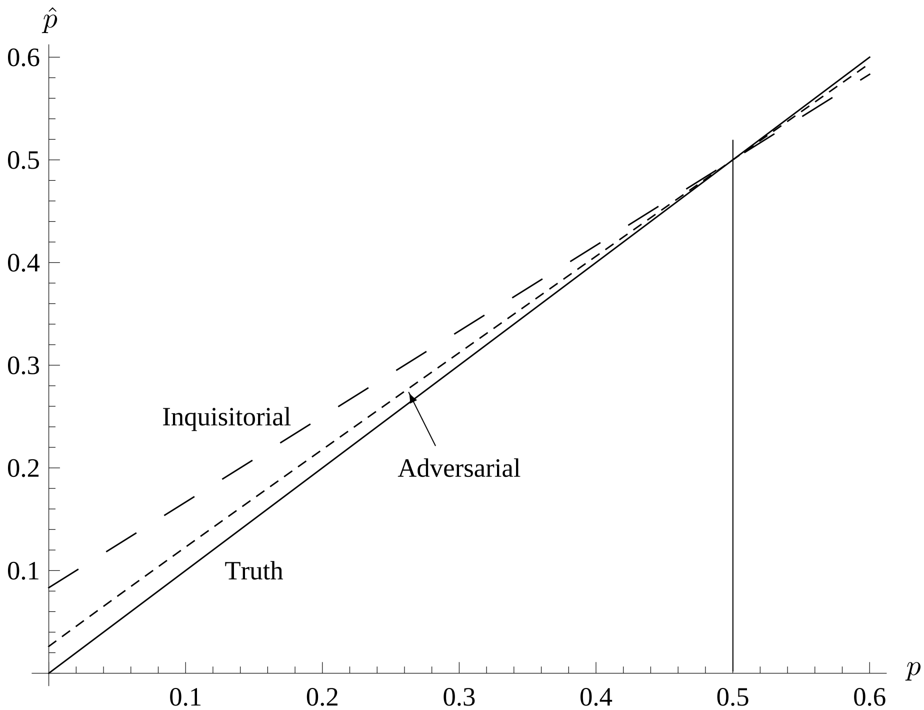
$$f(k|10, p) = \binom{10}{k} p^k (1 - p)^{10-k},$$

which allows us to calculate the expectation of the estimator for each of the two regimes.

[7] For a more detailed account of hot-tubbing see *Adam Liptak: "In U.S., Expert Witnesses Are Partisan," New York Times, August 12, 2008.*

[8] See Table A1.

Figure 5: Bias of Adversarial Framing ($\hat{p} = \hat{p}_{ADV}$) vs. Objective Inquiry ($\hat{p} = \hat{p}_{INQ}$)



In Figure 5, we plot the expected values of our estimators $\hat{p}_{ADV}$ [dotted line] and $\hat{p}_{INQ}$ [dashed line] against the "true" value of p for $p \in [0, 0.6]$. Our benchmark is the solid line which has a slope of 1, representing an unbiased estimator. The further away the estimator, $\hat{p}_{ADV}$ or $\hat{p}_{INQ}$, is from the solid line, the bigger the bias it has. We see that for $p < 0.5$, the inquisitorial regime [dashed line] is biased towards the prior mean of $\alpha/(\alpha + \beta) = 0.5$. For $p > 0.5$ the inquisitorial estimator has a downward bias.

The adversarial estimator [dotted line] does better than the inquisitorial estimator for all $p \neq 0.5$. It is closer to the true p than the inquisitorial system, except for $p = 0.5$, where the expected value of the two estimators is the same. Only in this case does the inquisitorial regime perform as well as the adversarial. We find this surprising because of the framing: two adversaries, trying to manipulate the outcome of a decision-making process, yield "better" results than a neutral third party inquisitor when $p \neq 0.5$.

While suggestive, this finding depends on the characteristics of the evidence-generating process. In this case, competition between the parties induces them to frame evidence in a way that results in better decisions than would be reached by neutral inquiry.

6 Conclusions

In this paper, we present a formal model of framing that shows how parties frame evidence. The model can be used to positively explain some of the observed features of practice, including parties following an obfuscation strategy. It also can be used to normatively compare adversarial decision making to policy relevant alternatives, like objective inquiry.

We find that the adversarial system performs remarkably well: it is consistent, and for certain processes, performs better than neutral inquiry.

We also conjecture that our model might be useful as an empirical framework for those studying framing, or as subgame of the more widely studied games of evidence production and revelation. It may be that explicitly modeling evidence production, knowing how it is going to be framed by the parties, will lead to other insights about the nature of evidence-based decision making in an adversarial setting.

A Appendix A

Table A1: Equilibria in the Binomial Framing Game for $k = 1, 2, \ldots, 10$

Heads k	D**'s claim** p_D^*	P**'s claim** p_P^*	**Posterior** θ	**Decision** $p_1(p_P^*, p_D^*)$	**MLE** $\hat{p}_{MLE}$	**Bias**
0	0.00000	0.11471	0.22822	0.02618	0.0	0.02618
1	0.04529	0.24956	0.38696	0.12433	0.1	0.02433
2	0.10989	0.36141	0.43154	0.21843	0.2	0.01843
3	0.18358	0.46399	0.45914	0.31233	0.3	0.01233
4	0.26371	0.56009	0.48069	0.40617	0.4	0.00617
5	0.34924	0.65076	0.50000	0.50000	0.5	0.00000
6	0.43992	0.73629	0.51931	0.59383	0.6	−0.00617
7	0.53601	0.81642	0.54086	0.68767	0.7	−0.01233
8	0.63859	0.89011	0.56846	0.78157	0.8	−0.01843
9	0.75044	0.95471	0.61304	0.87567	0.9	−0.02433
10	0.88529	1.00000	0.77178	0.97382	1.0	−0.02618

B Appendix B: Proofs

Proof of Lemma 1. Note that $\tilde{v}(y_+, y_-)$ is continuous and differentiable in y_j, $j = +, -$.

- *Strict Monotonicity (A1)* is given if

$$\frac{\partial \tilde{v}(y_+, y_-)}{\partial y_+} = -\frac{x_-}{x_+ - x_-} > 0 \quad \text{and} \quad \frac{\partial \tilde{v}(y_+, y_-)}{\partial y_-} = \frac{x_+}{x_+ - x_-} > 0$$

  which holds true because $x_- < 0$.

- *Interiority (A2):* Note that $\max\{y_+, y_-\} = y_+ \geq \tilde{v}(y_+, y_-) = \frac{x_+y_- - x_-y_+}{x_+ - x_-}$ if $y_+ > y_-$, and $\min\{y_+, y_-\} = y_- \leq \tilde{v}(y_+, y_-)$ if $y_+ > y_-$ because $x_- < 0$. Hence, if $y_+ > y_-$, $\max\{y_+, y_-\} \geq \tilde{v}(y_+, y_-) \geq \min\{y_+, y_-\}$. Conversely, $\min\{y_+, y_-\} = y_+ \leq \tilde{v}(y_+, y_-)$ if $y_+ < y_-$, and $\max\{y_+, y_-\} = y_- \geq \tilde{v}(y_+, y_-)$ if $y_+ < y_-$. Hence, if $y_+ < y_-$, $\max\{y_+, y_-\} \geq \tilde{v}(y_+, y_-) \geq \min\{y_+, y_-\}$.

- *Homogeneity (A3)* is given if

$$\tilde{v}(\lambda y_+, \lambda y_-) = \frac{w_+ \lambda y_+ + w_- \lambda y_-}{w_+ + w_-} = \frac{\lambda\,(w_+ y_+ + w_- y_-)}{w_+ + w_-} = \lambda \tilde{v}(y_+, y_-)$$

  which holds for all $\lambda > 0$.

- Let $y_+ \neq y_-$. *Symmetry (A4)* is given if

$$\tilde{v}(y_+, y_-) = \frac{w_+ y_+ + w_- y_-}{w_+ + w_-} = \frac{w_+ y_- + w_- y_+}{w_+ + w_-} = \tilde{v}(y_-, y_+),$$

which holds true if, and only if $w_+ = w_-$. This is because $\frac{w_+y_+ + w_-y_-}{w_+ + w_-} = \frac{w_+y_- + w_-y_+}{w_+ + w_-}$ if and only if $(w_+ - w_-)\, y_+ = (w_+ - w_-)\, y_-$ which holds (for $y_+ \neq y_-$) if and only if $w_+ - w_- = 0$.

- Let $a_- \neq b_-$, and let $x_- > -\infty$ so that $w_- > 0$. *Independence of Presentation (A5)* is given if

$$\tilde{v}(\tilde{v}(a_+, a_-), \tilde{v}(b_+, b_-)) = \tilde{v}(\tilde{v}(a_+, b_-), \tilde{v}(b_+, a_-)) \iff \frac{(a_- - b_-)\, w_-\, (w_+ - w_-)}{(w_+ + w_-)^2} = 0$$

  which holds true if and only if $w_+ = w_-$.

Q.E.D.

Proof of Proposition 1. We first prove the equilibrium existence claim for convex sets S_+ and S_- and then show that the maximum value of $M(S_+, S_-)$ is the unique Nash-equilibrium slope, m^*, and the equilibrium exhibits a unique y-intercept, i.e., payment rule v^*. We then proceed to non-convex sets.

- Let S_+ and S_- be convex sets. The set $M(S_+, S_-)$ in (10) has a maximum if, and only if, sets S_+ and S_- are also compact. If, for instance, no maximum value for y_+ exists (because the choice set for y_+ is not bounded above), or if no minimum value for y_- exists (because the choice set for y_- is not bounded below), the slope $m = \frac{y_+ - y_-}{x_+ - x_-}$ is not bounded above and $M(S_+, S_-)$ has no maximum value. The analogous argument applies to values of x_+ and x_-. Further note that $v(s_+, s_-)$ is continuous in x_j and y_j. The framing game is a two-person game with a continuous action space. Such a game has a Nash equilibrium if S_j is a convex and compact set and the payoff function is continuous (Glicksberg, 1952).

- This maximum value $\max M(S_+, S_-)$ is the Nash-equilibrium slope. Suppose there is a maximum value of $M(S_+, S_-)$ denoted by $m(\hat{s}_+, \hat{s}_-) = \max M(S_+, S_-)$, then

$$(\hat{s}_+, \hat{s}_-) \in \arg \max_{s_+ \in S_+, s_- \in S_-} m(s_+, s_-) \qquad \text{s.t.} \quad m(s_+, s_-) \in M(S_+, S_-). \tag{A1}$$

  Because '+'s best response to $\hat{s}_-$ is $s'_+ \in \arg\max_{s_+ \in S_+} m(s_+, \hat{s}_-) = \arg\max_{s_+ \in S_+} v(s_+, \hat{s}_-)$ so that $m(s'_+, \hat{s}_-) = m(\hat{s}_+, \hat{s}_-)$ and $\hat{s}_+ = s'_+$, and because '$-$'s best response to $\hat{s}_+$ is $s'_- \in \arg\max_{s_- \in S_-} m(\hat{s}_+, s_-) = \arg\max_{s_- \in S_-} v(\hat{s}_+, s_-)$ so that $m(\hat{s}_+, s'_-) = m(\hat{s}_+, \hat{s}_-)$ and $\hat{s}_- = s'_-$, $(\hat{s}_+, \hat{s}_-)$ is a Nash equilibrium of the framing game, $\left(s^*_+, s^*_-\right) = (\hat{s}_+, \hat{s}_-)$. Because there is only one maximum value of $M(S_+, S_-)$, the slope $m^* \equiv m(\hat{s}_+, \hat{s}_-) \in M(S_+, S_-)$ is unique. Moreover, the y-intercept and payment rule $v^* \equiv v(\hat{s}_+, \hat{s}_-)$ is unique. To see this, suppose two profiles $\left(s'_+, s'_-\right) \in S_+ \times S_-$ and $\left(s''_+, s''_-\right) \in S_+ \times S_-$ with slope $m^* = m(s'_+, s'_-) = m(s''_+, s''_-)$ but $v(s'_+, s'_-) > v(s''_+, s''_-)$. These two profiles are not Nash equilibrium profiles because player '$-$' would deviate from s'_- to s''_- and player '+' would deviate from s''_+ to s'_+ (per the argument in Figure 2).

  While the framing game has a unique Nash-equilibrium slope m^* and unique Nash-equilibrium payment rule v^*, the Nash equilibrium need not be unique. Any pair $\left(s'_+, s'_-\right)$ so that $m(s'_+, s'_-) = m^*$ and $v(s'_+, s'_-) = v^*$ is a pure-strategy profile in Nash equilibrium. If more than one such profile exists, then any mixture of these Nash equilibrium pure strategies will also be a Nash-equilibrium strategy on the line with slope m^* and y-intercept v^*; and any such mixture of Nash equilibrium pure strategies is itself a pure-strategy Nash equilibrium.

- Now, suppose S_+, or S_-, or both are not convex. Take S_+^h and S_-^h to be the convex hulls of S_+ and S_-. We first show that the framing game $\Gamma(S_+, S_-)$ has a solution if and only $\Gamma(S_+^h, S_-^h)$ has a solution.

  * We first establish that if $\Gamma(S_+, S_-)$ has a solution then $\Gamma(S_+^h, S_-^h)$ has a solution; more specifically, we show that if $\Gamma(S_+, S_-)$ has a solution then this solution is a solution of $\Gamma(S_+^h, S_-^h)$: Let $\left(s_+^*, s_-^*\right) \in S_+ \times S_-$ be a solution of $\Gamma(S_+, S_-)$. Suppose $\left(s_+^*, s_-^*\right)$ is not a solution of $\Gamma(S_+^h, S_-^h)$. Then there is a $\left(s_+^h, s_-^h\right) \in S_+^h \times S_-^h$ such that $m(s_+^h, s_-^h) > m(s_+^*, s_-^*)$ but $\left(s_+^h, s_-^h\right) \notin S_+ \times S_-$ (otherwise, $\left(s_+^h, s_-^h\right)$ is a solution of $\Gamma(S_+, S_-)$). This profile $\left(s_+^h, s_-^h\right)$ represents two points on a line with slope $m(s_+^h, s_-^h)$ and y-intercept $v(s_+^h, s_-^h)$. When $\left(s_+^h, s_-^h\right) \notin S_+ \times S_-$, then either $s_+^h \in S_+^h$ but $s_+^h \notin S_+$, or $s_-^h \in S_-^h$ but $s_-^h \notin S_-$, or both. Suppose $s_-^h = s_-^* \in S_-$ but $s_+^h \notin S_+$. Then, by convex hull, there must be at least two points $s'_+, s''_+ \in S_+$ that are points on the line with slope $m(s_+^h, s_-^h)$ and y-intercept $v(s_+^h, s_-^h)$. But then $m(s'_+, s_-^h) = m(s''_+, s_-^h) > m(s_+^*, s_-^h) = m(s_+^*, s_-^*)$ and $\left(s_+^*, s_-^*\right)$ is not a solution of $\Gamma(S_+, S_-)$.
  * Next we show that if $\Gamma(S_+^h, S_-^h)$ has a solution then $\Gamma(S_+, S_-)$ has a solution, although it may not be the same solution: Suppose $\left(s_+^h, s_-^h\right)$ is a solution of $\Gamma(S_+^h, S_-^h)$. If $\left(s_+^h, s_-^h\right) \in S_+ \times S_-$, then it is also a solution of $\Gamma(S_+^h, S_-^h)$. Conversely, suppose $\left(s_+^h, s_-^h\right) \notin S_+ \times S_-$ and let $s_-^h \in S_-$ but $s_+^h \notin S_+$. Then there are at least two points $s'_+, s''_+ \in S_+$ and $s'_+, s''_+ \in S_+^h$ with $m(s_+^h, s_-^h) = m(s'_+, s_-^h) = m(s''_+, s_-^h)$ and $v(s_+^h, s_-^h) = v(s'_+, s_-^h) = v(s''_+, s_-^h)$. Hence, by uniqueness of slope m^* and v^*, s'_+ and s''_+ are parts of a solution of $\Gamma(S_+, S_-)$.

  Because $\Gamma(S_+, S_-)$ has a solution if and only if $\Gamma(S_+^h, S_-^h)$ has a solution, and $\Gamma(S_+, S_-)$ (for convex S_+ and S_-) has a solution if and only if $M(S_+, S_-)$ has a maximum value, $\Gamma(S_+^h, S_-^h)$ (when S_+ or S_- are not convex) has a solution if and only if $M(S_+, S_-)$ has a maximum value. The results in the proposition thus apply to both convex and non-convex sets S_+ and S_-. Q.E.D.

Proof of Lemma 2. The action profile $(\hat{s}_+, \hat{s}_-)$ that implements the maximum value of $M(S_+, S_-)$, $m^* = m(\hat{s}_+, \hat{s}_-)$, is not unique if more than one point of S_+ and S_- lie on the line with slope m^*, connecting a point $\hat{s}_+ \in S_+$ and $\hat{s}_- \in S_-$ (Proposition 1). The same then holds for $\hat{s}_+ \in \tilde{S}_+$ and $\hat{s}_- \in \tilde{S}_-$ because a solution of $\Gamma(S_+, S_-)$ is a solution of $\Gamma(\tilde{S}_+, \tilde{S}_-)$ if $y_+ > y_-$. If $\mathscr{U}_+$ is *strictly* concave up, then there is at most one point $\hat{s}_+ \in S_+$ that lies on the line with slope m^* and on $\tilde{S}_+$. Likewise, if $\mathscr{U}_-$ is *strictly* concave down, then there is at most one point $\hat{s}_- \in S_-$ that lies on the line with slope m^* and on $\tilde{S}_-$. Thus the Nash equilibrium of $\Gamma(S_+, S_-)$ is unique. Q.E.D.

Proof of Lemma 3. 1. Recall from equation (9) that $m = (y_+ - y_-) / (x_+ - x_-)$ and therefore

$$m = \frac{y_+ - y_-}{\mathscr{U}(y_+) + \mathscr{U}(y_-)}.$$

Players maximize their payoffs by maximizing the slope m. The first derivative of this slope with respect to y_+ is:

$$\frac{dm}{dy_+} = \frac{\mathscr{U}(y_+) + \mathscr{U}(y_-) - (y_+ - y_-)\,\mathscr{U}'(y_+)}{\left(\mathscr{U}(y_+) + \mathscr{U}(y_-)\right)^2}.$$

Evaluate the slope at $y_+ = \mu_{MLE}$ so that $\mathscr{U}'(\mu_{MLE}) = 0$: Because $\mathscr{U}(y) > 0$ for all y so that $\mathscr{U}(\mu_{MLE}) > 0$ and $\mathscr{U}(\mu_{MLE}) + \mathscr{U}(y_-) > 0$, we obtain $m' > 0$. Slope m is increasing in

y_+, and player '+' can increase his payoffs by choosing $y_+ > \mu_{MLE}$. The analogous is true for player '−' where

$$\frac{dm}{dy_-} = -\frac{\mathscr{U}(y_+) + \mathscr{U}(y_-) + (y_+ - y_-)\,\mathscr{U}'(y_-)}{(\mathscr{U}(y_+) + \mathscr{U}(y_-))^2}.$$

Thus $y_-^* < \mu_{MLE} < y_+^*$ and $y_-^* < y_+^*$. Moreover, $y_-^* < v^*(\mathscr{U}) = v(s_+^*, s_-^*) < y_+^*$ by the proof of axiom A2 in Lemma 1.

2. For a given y_-, player '+'s choice of y_+^* (and $x_+^* = \mathscr{U}(y_+^*)$) solves $dm/dy_+ = 0$. Likewise for player '−'. Rearranging the expressions for dm/dy_+ and dm/dy_- we obtain

$$\mathscr{U}'(y_+^*) = \frac{x_+^* - x_-}{y_+^* - y_-} \qquad \text{and} \qquad -\mathscr{U}'(y_-^*) = \frac{x_+ - x_-^*}{y_+ - y_-^*}$$

so that the equilibrium slope is

$$\mathscr{U}'(y_+^*) = -\mathscr{U}'(y_-^*) = \frac{x_+^* - x_-^*}{y_+^* - y_-^*} = \frac{1}{m}.$$

Q.E.D.

Proof of Proposition 2. Setting $y'_+ \equiv \mu_{MLE} + \Delta y'_+$ and $y'_- \equiv \mu_{MLE} - \Delta y'_-$, we can rewrite the outcome $v(s'_+, s'_-)$ in equation (7) as:

$$v(s'_+, s'_-) = \mu_{MLE} + \Delta y'_+ - \frac{\Delta y'_+ + \Delta y'_-}{\mathscr{U}(y'_+) + \mathscr{U}(y'_-)}\mathscr{U}(y'_+).$$

The outcome is unbiased, $v(s'_+, v'_-) = \mu_{MLE}$, if:

$$\Delta y'_+ - \frac{\Delta y'_+ + \Delta y'_-}{\mathscr{U}(y'_+) + \mathscr{U}(y'_-)}\mathscr{U}(y'_+) = 0 \quad \text{or} \quad \frac{\mathscr{U}(y'_+)}{\mathscr{U}(y'_-)} = \frac{\Delta y'_+}{\Delta y'_-}. \tag{A2}$$

The expression $\mathscr{U}(y'_+)/\left(y'_+ - \mu_{MLE}\right)$ in (13) is the reciprocal of the slope of the line through points $(0, \mu_{MLE})$ and $\left(\mathscr{U}(y'_+), y'_+\right)$; the expression $\mathscr{U}(y'_-)/\left(\mu_{MLE} - y'_-\right)$ in (14) is the reciprocal of the slope of the line through points $(0, \mu_{MLE})$ and $\left(-\mathscr{U}(y'_-), y'_-\right)$. For a line going through points $\left(-\mathscr{U}(y'_-), y'_-\right)$ and $\left(\mathscr{U}(y'_+), y'_+\right)$ (so that $v(s'_+, s'_-)$ is the vertical intercept) these two slopes must be the same, as in equation (15). Rearranging (15) we obtain

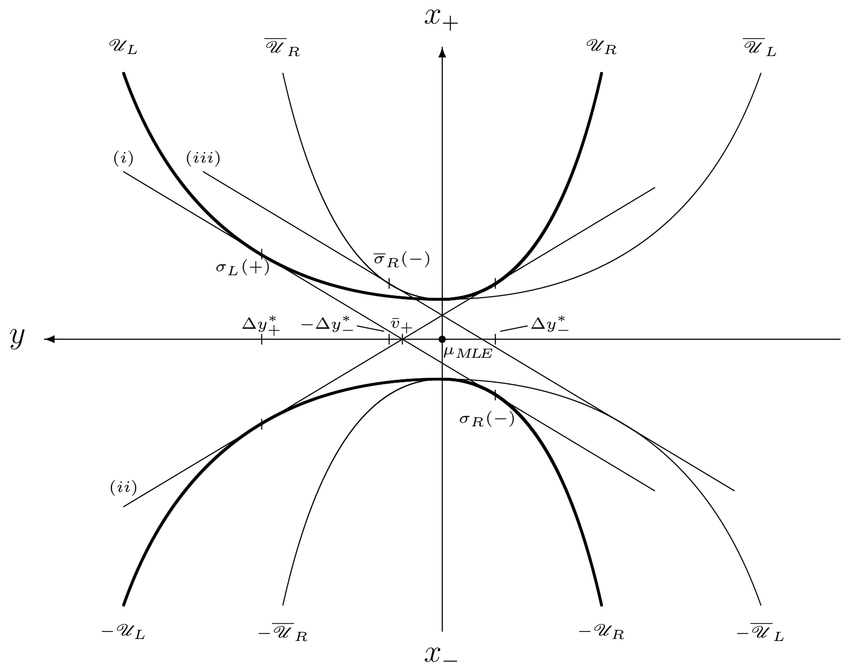
$$\frac{\mathscr{U}(y'_+)}{y'_+ - \mu_{MLE}} = \frac{\mathscr{U}(y'_-)}{\mu_{MLE} - y'_-} \quad \text{if and only if} \quad \frac{\mathscr{U}(y'_+)}{\mathscr{U}(y'_-)} = \frac{\Delta y'_+}{\Delta y'_-},$$

i.e., if and only if the condition for an unbiased estimator is satisfied. This means, the maximum-likelihood estimator $v = \mu_{MLE}$ is a point on the line connecting points $\left(-\mathscr{U}(y'_-), y'_-\right)$ and $\left(\mathscr{U}(y'_+), y'_+\right)$.

If $\mathscr{U}(y'_+)/\left(y'_+ - \mu_{MLE}\right) = \mathscr{U}(y'_-)/\left(\mu_{MLE} - y'_-\right)$, then the line connecting the points $\left(-\mathscr{U}(y'_-), y'_-\right)$ and $\left(\mathscr{U}(y'_+), y'_+\right)$ has the same slope as the two line segments (in equations (13) and (14)). This slope is equal to $\left(\mathscr{U}(y'_+) + \mathscr{U}(y'_-)\right)/\left(y'_+ - y'_-\right)$. From Lemma 3 we know that two claims y'_+ and y'_- such that

$$\mathscr{U}'(y'_+) = -\mathscr{U}'(y'_-) = \frac{\mathscr{U}(y'_+) + \mathscr{U}(y'_-)}{y'_+ - y'_-}$$

Figure A1: Biased Estimator with Regular Asymmetric $\mathscr{U}$



are the Nash equilibrium claims, $y_+^* = y_+'$ and $y_-^* = y_-'$, where $\left(\mathscr{U}(y_+^*), y_+^*\right)$ and $\left(-\mathscr{U}(y_-^*), y_-^*\right)$ are the players' respective equilibrium strategies. Hence, claims y_+' and y_-' satisfying equation (13)-(15) yield an unbiased outcome in Nash equilibrium. Q.E.D.

Proof of Proposition 3. Define $\mathscr{U}_L : (\mu_{MLE}, \max Y] \to \mathbb{R}_+$, $\mathscr{U}_L(y) = \mathscr{U}$ for $y > \mu_{MLE}$ as the LHS arm of $\mathscr{U}$; define $\mathscr{U}_R : [\min Y, \mu_{MLE}) \to \mathbb{R}_+$, $\mathscr{U}_R(y) = \mathscr{U}$ for $y < \mu_{MLE}$ as the RHS arm of $\mathscr{U}$; define $\overline{\mathscr{U}}_R : (\mu_{MLE}, \max Y] \to \mathbb{R}_+$, $\overline{\mathscr{U}}_R(y) = \mathscr{U}(2\mu_{MLE} - y)$ for $y > \mu_{MLE}$ as the RHS arm mirrored in the x axis; and define $\overline{\mathscr{U}}_L : [\min Y, \mu_{MLE}) \to \mathbb{R}_+$, $\overline{\mathscr{U}}_L(y) = \mathscr{U}(2\mu_{MLE} - y)$ for $y < \mu_{MLE}$ as the LHS arm mirrored in the x axis. See Figure A1 for an illustration of these functions and their negative values, $-\mathscr{U}_L$, $-\mathscr{U}_R$, $-\overline{\mathscr{U}}_R$, and $-\overline{\mathscr{U}}_L$.

1. We begin by showing *Claim (1)*: If $\mathscr{U}$ is symmetric then the estimator is unbiased, $v^*(\mathscr{U}) = \mu_{MLE}$. First, note that by symmetry in equation (19), $\mathscr{U}_L(y) = \overline{\mathscr{U}}_R(y)$. This is because, for $y > \mu_{MLE}$, $\mathscr{U}_L(y) = \mathscr{U}(y) = \mathscr{U}(\mu_{MLE} + \Delta y)$ and $\overline{\mathscr{U}}_R(y) = \mathscr{U}(2\mu_{MLE} - y) = \mathscr{U}(\mu_{MLE} - \Delta y) = \mathscr{U}(\mu_{MLE} + \Delta y)$ where the latter equality holds by symmetry in equation (19). Moreover, by symmetry and because $\mathscr{U}(y)$ is strictly convex so that $\mathscr{U}_L(y)$ is strictly convex for $y > \mu_{MLE}$, we have $\mathscr{U}_L'(y) = \overline{\mathscr{U}'}_R(y)$ for all $y > \mu_{MLE}$. The analogous can be shown for $\mathscr{U}_R(y) = \overline{\mathscr{U}}_L(y)$ and $\mathscr{U}_R'(y) = \overline{\mathscr{U}'}_L(y)$. By equation (12), the Nash equilibrium claims y_+^* and y_-^* are such that

$$\begin{aligned} \mathscr{U}'(y_+^*) &= -\mathscr{U}'(y_-^*) & \iff \\ \mathscr{U}'(\mu_{MLE} + \Delta y_+^*) &= -\mathscr{U}'(\mu_{MLE} - \Delta y_-^*) & \iff \\ \mathscr{U}_L'(\mu_{MLE} + \Delta y_+^*) &= -\mathscr{U}_R'(\mu_{MLE} - \Delta y_-^*) & \iff \\ \mathscr{U}_L'(\mu_{MLE} + \Delta y_+^*) &= \overline{\mathscr{U}'}_R(\mu_{MLE} + \Delta y_-^*) & \end{aligned}$$

Because $\mathscr{U}_L'(y) = \overline{\mathscr{U}'}_R(y)$ for all y and by strict convexity of $\mathscr{U}$, $\Delta y_+^* = \Delta y_-^*$ in order for the equilibrium condition to hold. Moreover, symmetry implies that for $\Delta y_+^* = \Delta y_-^*$,

$\mathscr{U}(\mu_{MLE} + \Delta y_+^*) = \mathscr{U}(\mu_{MLE} - \Delta y_-^*)$. Then, $\mathscr{U}(y_+^*)/\mathscr{U}(y_-^*) = 1 = \Delta y_+^*/\Delta y_-^*$, satisfying the no-bias condition in (A2) (or (17)).

2. In a next step we *Claim 2*: For a regular asymmetric $\mathscr{U}$ with $|\mathscr{U}'(\mu_{MLE}+\Delta y)| < |\mathscr{U}'(\mu_{MLE}-\Delta y)|$ for all Δy, the equilibrium outcome exhibits an upward bias, $v^*(\mathscr{U}) > \mu_{MLE}$. We quantify this bias as follows: $\bar{v}_+ = v^*(\mathscr{U}) - \mu_{MLE}$ so that the bias is upward if $\bar{v}_+ > 0$. For the proof, consider Figure A1. Note that it is a version of Figure 3 rotated 90° counterclockwise. The origin is at $(\mu_{MLE}, 0)$. Note that the figure is mirrored sideways, i.e., larger values of y are to the left of the origin. Let $\sigma_L(+) \equiv \big(\mu_{MLE} + \Delta y_+^*, \mathscr{U}_L(\mu_{MLE} + \Delta y_+^*)\big)$ and $\sigma_R(-) \equiv \big(\mu_{MLE} - \Delta y_-^*, -\mathscr{U}_R(\mu_{MLE} - \Delta y_-^*)\big)$. A positive bias, $\bar{v}_+ > 0$ implies that the line connecting points $\sigma_L(+)$ and $\sigma_R(-)$ has a y-intercept at $\mu_{MLE} + \bar{v}_+$ which is depicted to the left of μ_{MLE}.

   By the equilibrium condition in Lemma 3, $\mathscr{U}_L'(\mu_{MLE} + \Delta y_+^*) = -\mathscr{U}_R'(\mu_{MLE} - \Delta y_-^*)$ and therefore, by construction of $\overline{\mathscr{U}}_R$, $\mathscr{U}_L'(\mu_{MLE}+\Delta y_+^*) = \overline{\mathscr{U}'}_R(\mu_{MLE}+\Delta y_-^*)$. For $\Delta y_+^* > \Delta y_-^*$, note that by condition $|\mathscr{U}'(\mu_{MLE} + \Delta y)| < |\mathscr{U}'(\mu_{MLE} - \Delta y)|$ in the claim, $\Delta y_+^* \neq \Delta y_-^*$. Moreover, by the convexity of $\mathscr{U}_L$, $|\mathscr{U}'(\mu_{MLE} + \Delta y')| < |\mathscr{U}'(\mu_{MLE} + \Delta y'')|$ for $\Delta y' < \Delta y''$; therefore $\Delta y_+^* \not< \Delta y_-^*$. For the regular asymmetry in *Claim 2* we thus find that $\Delta y_+^* > \Delta y_-^*$. To show that $\bar{v}_+ > 0$, we show that $\bar{v}_+ \neq 0$ and $\bar{v}_+ \not< 0$.

   Suppose $\bar{v}_+ = 0$. The line connecting points (i.e., actions) $\sigma_L(+)$ and $\sigma_R(-)$ runs through the origin. Moreover, the line segments connecting points $\sigma_L(+)$ and the origin, the origin and $\sigma_R(-)$, and (equivalently) $\overline{\sigma}_R(-) \equiv \big(\mu_{MLE} + \Delta y_-^*, \overline{\mathscr{U}}_R(\mu_{MLE} + \Delta y_-^*)\big)$ and the origin must be of equal slope. For $\Delta y_+^* > \Delta y_-^*$, $\mathscr{U}_L(\mu_{MLE} + \Delta y_+^*)$ must lie in the extension of the line connecting points $\overline{\sigma}_R(-)$ and the origin. By the equilibrium condition (12), the slopes at $\overline{\mathscr{U}}_R(\mu_{MLE} + \Delta y_-^*)$ and $\mathscr{U}_L(\mu_{MLE} + \Delta y_+^*)$ must be the same. But because $\mathscr{U}_L \neq \overline{\mathscr{U}}_R$ and by strict convexity of $\mathscr{U}$, equal slopes (with both points lying on the same line) require that $\mathscr{U}_L$ and $\overline{\mathscr{U}}_R$ intersect for some $y > \mu_{MLE}$. Because $\mathscr{U}_L(\mu_{MLE}) = \overline{\mathscr{U}}_R(\mu_{MLE})$, such a crossing violates the assumption of $|\mathscr{U}'(\mu_{MLE} + \Delta y)| < |\mathscr{U}'(\mu_{MLE} - \Delta y)|$ in the claim. Hence, $\bar{v}_+ \neq 0$.

   Suppose $\bar{v}_+ < 0$. The line connecting points $\sigma_L(+)$ and $\sigma_R(-)$ runs through $\sigma(\bar{v}_+) \equiv (\mu_{MLE} + \bar{v}_+, 0)$. Moreover, the line segments connecting points $\sigma_L(+)$ and $\sigma(\bar{v}_+)$, points $\sigma(\bar{v}_+)$ and $\sigma_R(-)$, and (equivalently) the points $\overline{\sigma}_R(-)$ and $\sigma(\bar{v}_+)$ must be of equal slope. This requires that point $\overline{\sigma}_R(-)$ lies below the line connecting points $\sigma_L(-)$ and $\sigma(\bar{v}_+)$ (otherwise, the two respective line segments cross). The line segment connecting $\sigma_L(-)$ lies below $\mathscr{U}$ (and below $\mathscr{U}_L$ for $y > \mu_{MLE}$). In order for $\overline{\sigma}_R(-)$ to lie below the line it must lie below $\mathscr{U}_L$. But this violates the assumption of $|\mathscr{U}'(\mu_{MLE} + \Delta y)| < |\mathscr{U}'(\mu_{MLE} - \Delta y)|$ so that $\mathscr{U}(\mu_{MLE} + \Delta y) < \mathscr{U}(\mu_{MLE} - \Delta y)$. Hence $\bar{v}_+ \not< 0$. $\bar{v}_+ \neq 0$ and $\bar{v}_+ \not<$ imply $\bar{v}_+ > 0$ and an upward bias, $v^*(\mathscr{U}) > \mu_{MLE}$.

3. The proof for *Claim 3* is analogous to *Claim 2* and omitted. Q.E.D.

Proof of Proposition 4. Translate in the y direction to make the mimima of the $\mathscr{U}_n$ at $\mu_{MLE} = 0$. Normalize the $\mathscr{U}_n$ such that $\mathscr{U}_n(0) = 1$, say, since scaling leaves the values $v(\mathscr{U}_n)$ unchanged. Suppose there is a $v_0 > 0$, such that for infinitely many n, $v(\mathcal{U}_n) > v_0$. Then for these n, and $s_- = (x_-, y_-) = (-1, 0)$ there must be a move $s_+ = (x_+, y_+) = (\mathscr{U}_n(y_+), y_+)$ of the '+' player with $v(s_-, s_+) = y_+/\,(x_+ + 1) > v_0$. Hence $1 < \mathscr{U}_n(y_+) = x_+ < y_+/v_0 - 1$. In particular then $y_+ > v_0$, and by concavity,

$$\mathscr{U}_n(v_0) < \mathscr{U}_n(0)\,(1 - v_0/y_+) + \mathscr{U}_n(y_+)\,(v_0/y_+) < 1\,(1 - v_0/y_+) + (y_+/v_0 - 1)\,(v_0/y_+) = 2$$

contradicting the hypothesis $\lim_{n\to\infty} 1/\mathscr{U}_n(v_0) = 0$. Hence, for any $v_0 > 0$ only finitely many $v(\mathscr{U}_n) > v_0$, and similarly for any $v_0 < 0$ only finitely many $v(\mathscr{U}_n) < v_0$. Thus $\lim_{n\to\infty} v(\mathscr{U}_n) = 0 = \mu_{MLE}$. Q.E.D.

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