Revenue-Superior Variants of the Second-Price Auction

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Abstract

The equilibrium of Güth and van Damme (1986)'s "k-price auction," where the winner pays the average of his bid and the second highest bid weighted by k and $1-k$, is identical to the equilibrium of a first-price auction with transformed value distributions and converges towards truth-bidding if k vanishes. The sealed-bid and English "random k-price auctions," where the payment rule of the k-price auction applies to at most one, randomly chosen, bidder, is solvable through two rounds of elimination of weakly dominated strategies. In the first order, raising k from zero has no effect on total surplus and strictly positive effects on revenues for: the k-price auction with two bidders and "power-related" value distributions; and the random k-price auctions with an arbitrary number of bidders and value distributions ordered according to weaker concepts of stochastic dominance. The results are robust to the presence of a reserve price and can be extended to models with efficient cartels of bidders.

Keywords: revenues; efficiency; second-price auction; first-price auction; English auction; k-price auction; bidder heterogeneity. **J.E.L. Classification code:** D44.

1. Introduction

1.1 Overview of the Results

In Güth and van Damme (1988)'s k-price auction, hereafter kPA, with $0 \leq k \leq 1$, the highest bidder pays an average of his bid and the second-highest bid weighted, respectively, by k and $1 - k$. When $k = 0$ the kPA reduces to the second-price auction, or SPA, and when $k = 1$ to the first-price auction, or FPA. In our sealed-bid "random k-price auction," or RkPA, the rules of the SPA apply to all bidders except possibly one, the "k-price bidder," who is chosen at random and to whom the rules of the kPA apply. Every bidder has the same probability π, such that $0 < \pi \leq 1/n$ where n is the number of bidders, of being selected as the k-price bidder. The RkPA is dominance solvable (for a definition, see Moulin 1979) in two rounds of elimination of all weakly dominated strategies.

We obtain an explicit expression for the first-order effect of raising k from zero on the expected revenues from the RkPA and show that it is strictly positive when the bidders' values are distributed over a common interval support according to different independent and stochastically ranked probability measures. Furthermore, we show that the (negative) effect on expected total surplus is only of the second-order. Thus, for all small $k > 0$ the RkPA brings more expected revenues than the SPA while being almost as efficient.

These results hold when a "stronger" bidder's value distribution dominates a "weaker" bidder's both for the hazard-rate and the reverse-hazard-rate stochastic dominance relations. This "double hazard-rate dominance" is less restrictive than likelihood-ratio dominance and is therefore satisfied by "power-related" distributions, that is, distributions whose cumulative functions are powers of the same function.

The intuition for our results is as follows. It is weakly dominant for a bidder who has not been chosen as the k-price bidder to submit his value. Because he faces stochastically larger, in the sense of reverse-hazard-rate dominance, competing bids, a weaker bidder when chosen as the k-price bidder shades his bid below his value less than a stronger bidder would. As all bidders have the same probability of becoming the k-price bidder, the equilibrium allocation is slightly biased towards weaker bidders. Given that a stronger bidder's value distribution also hazard-rate dominates a weaker bidder's, Myerson (1981) implies higher revenues (see next subsection).

Inefficiency occurs only when a bidder different from the k-price bidder wins the auction with a smaller value. When k becomes small, the k-price bidder shades his bid slightly and the two values must differ little, which occurs with a small probability and contributes a total effect on the expected surplus only of the second (or higher) order.

In the literature (see the subsection below), an "efficient cartel" of bidders has access to its members' information and controls their bids in order to maximize the sum of their payoffs. While such a cartel would have only its highest-value member submits a "serious" bid in the FPA and SPA, it may rather have its second-highest-value member bid seriously in the RkPA when its highest-value member has been chosen as the k-price bidder. However, such a change would only imply second-order corrections on revenues and efficiency and our results are robust to the introduction of efficient cartels. They are also robust to the introduction of a reserve price[1].

If the value distributions are ordered according to the likelihood-ratio dominance, we can extend our findings to a combination of the English and RkPA auctions similar to Klemperer (1998)'s "anglo-dutch" auction.

Our results hinge on the rule of one bid per bidder. If shill bidding is allowed, the equilibria of the RkPA and the SPA become equivalent. Full value support is also necessary for our results.

We then analyze the original kPA of Güth and van Damme (1988) and, using methods from Lebrun (1999), prove that its equilibrium for given value distributions is identical to the equilibrium of the FPA where the value distributions have been raised to the exponent $1/k$–a result that Güth and van

[1] As well as of risk-averse bidders.

Damme (1988) already noticed for homogeneous bidders[2]. This link between the two auction procedures allows us to import results from the literature on the FPA with heterogeneous bidders. For example, although the kPA is not dominance solvable its equilibrium is unique. With stochastically ranked value distributions, we prove that as k tends towards zero the bidding functions tend towards the identity function and we obtain bounds on the rates of convergence.

Because of the complexity of the equilibrium of the FPA and hence of the kPA, we are able to derive further results only in the two-bidder case with power-related value distributions. In this case, we prove that, as for the RkPA, increasing k above zero has a strictly positive first-order effect on revenues and, if the reserve price is not binding, no first-order effect on efficiency. Furthermore, the first-order effect on revenues is exactly $\pi^{-1} \geq 2$ times as large as for the RkPA when no (binding) reserve price is set and at least π^{-1} as large with a reserve price. Thus, if bidders could reasonably be expected to follow the complicated equilibrium of the kPA, a revenue -maximizing seller would prefer it to the RkPA.

We examine the related literature in the next subsection and introduce the model in Section 2. Our results about the RkPA are in Section 3 and about the kPA in Section 4. Section 5 concludes. The appendices contain some proofs and a discussion of the limitations and extensions of our results. Proofs similar to proofs already included in the appendices or to proofs of existing results, as well as proofs of secondary or technical results, have been relegated to the online discussion paper Lebrun (2012).

1.2 Related Literature

In the independent private values model with risk-neutral bidders and when a single indivisible item is for sale, the Vickrey-Clarke-Groves mechanism (see, for example, Milgrom 2004 and Krishna 2009) reduces to a SPA.

[2] Güth and van Damme (1988) assume the values to be uniformly distributed.

Truthful bidding is its only Bayesian-Nash equilibrium in weakly dominant strategies and produces an efficient outcome: the item goes to the highest-value bidder. Truthful bidding above the reserve price is the unique unrestricted Bayesian-Nash equilibrium when the reserve price is binding and there are at least three bidders (see Remark 3.2, page 1488, in Maskin and Riley 1984 and Heidhues and Blume 2004). Furthermore, the rules of the SPA are simple and anonymous[3].

Increasing auction revenues is obviously a major concern to private sellers (as well as decreasing costs to private buyers, in the case of procurement auctions), but also of governments seeking to rely less on taxes[4]. Efficiency is important to governments selling public assets and, because it may increase participation, even to some private sellers[5].

Although the SPA with an optimally chosen reserve price may maximize the seller's expected revenues when values are identically distributed[6], this is no longer true when values are differently distributed. From Myerson (1981), the optimal auction must allocate the item not to the highest-value bidder but rather to the bidder with the highest "virtual value," computed from his actual value and its probability distribution. Furthermore, sale should occur only if this highest virtual value is larger than the seller's value. When a

[3]The SPA may serve as a model of auctions on Web-sites such as eBay, where bidders submit bids through "proxies." However, new strategic considerations stem from the dynamic character of online auctions (for an informal account of bidding behavior on eBay see Steiglitz 2007 and for formal theoretical models and empirical studies see Ambrus and Burns 2010 and Hendricks, Onur, and Wiseman 2012).

[4]For example, although given a low priority, it was already one of the stated goals of the first spectrum auctions. See McMillan (1994) and McAfee and McMillan (1996).

[5]For example, Cramton (1998) writes: "The conflict between revenue maximization and efficiency is further reduced when one considers the desirable effects an efficient auction has on participation. Potential bidders are attracted to the auction based on the expected gains from participation. An efficient auction maximizes the gains from trade, which is the pie to be divided between seller and buyer." As McAfee (2002, p.117) writes about the spectrum auctions, efficiency is important to bidders because it "eliminates unnecessary risk and minimizes the amount of resale that will occur after the auction."

[6]It does maximize expected revenues if the common value distribution is "regular." See Myerson (1981).

strong bidder's value distribution hazard-rate dominates (see Krishna 2009 and Section 2 below for the various concepts of stochastic dominance) a weak bidder's, his virtual value is smaller than the weak bidder's with the same actual value and the optimal auction must then be biased towards the weak bidder. Implementing the optimal auction when bidders are heterogeneous would require a very knowledgeable seller able to tailor the auction rules to each specific case[7].

Participants to many auctions should not be expected to be identical ex-ante. For example, bidders in procurement auctions may have to cope with different levels of capacity utilization or different technologies. Bidders to a construction contract differ according to their distances to the job site. Empirical estimations such as Branmann and Froeb (2000) and Krasnokutskaya (2011) are at least consistent with a ranking of the bidders from strongest to weakest[8].

Legal evidence from antitrust cases point to endemic collusion in many auctions[9]. When the individual bidders' values are independently and identically distributed according to F, an efficient cartel consisting of m bidders behaves in auctions such as the SPA or the FPA as a single bidder with value–the highest among its members'–distributed according to F^m. Larger

[7] While the random and deterministic k-price auctions of the present paper can be considered in agreement with the "Wilson doctrine."

[8] Except near the upper extremity of the cost interval, the estimated cumulative distribution functions of "regular" and "fringe" bidders in Krasnokutskaya (2011) are ranked. The confidence intervals around the point estimations are consistent with (first-order) stochastic dominance.

It is of course conceivable that neither one of two specific bidders be unambiguously stronger. Brendstrup and Paarsch (2006) present such an example. However, Hubbard, Kirkegaard, and Paarsch (2011) point out that the absence of stochastic ranking between the estimated distributions in this example may be due to "an artifact of the interaction of sampling error and the Laguerre polynomials that Brendstrup and Paarsch (2006) used to approximate the probability density functions."

[9] Quoting Froeb (1989) and the GAO report (1990), Marshall and Meurer (2001) write that 81 % of criminal cases under Sherman Section One from 1979 to 1988 were in auction markets (also quoted by Hendricks and Porter, 1989), 245 bid-rigging or price-fixing cases pertained to road constructions, and 43 cases to government procurement.

cartels are then stronger effective bidders than smaller cartels and noncollusive bidders. Pesendorfer (2000) found that bidding in the Florida and Texas school milk markets from 1980 to 1991 was consistent with the formation of efficient cartels by those firms that were later convicted of bid-rigging[10]. In their study of timber auctions by the Forest Service in the Pacific Northwest from 1975 to 1981, Baldwin, Marshall, and Richard (1997) find that a model with an efficient cartel better explains the observed bidding than models with no collusion[11].

The theoretical literature has uncovered various mechanisms that allow a cartel in a FPA or SPA to become efficient when it can control its members' bids and implement side payments[12]. See Graham and Marshall (1987), McAfee and McMillan (1992), Mailath and Zemsky (1991), and Marshall and Marx (2007)[13]. In an important recent contribution, Biran and Forges (2011) show how to construct expost budget balanced and exante individually rational efficient mechanisms given any partition of the bidders into cartels and noncollusive bidders and any auction with bidders' values arbitrarily (but independently) distributed[14]. Mailath and Zemsky (1991)'s efficient

[10]Porter and Zona (1999) also found evidence of collusion in the Ohio school milk markets.

[11]In the auctioning of highway maintenance contracts in Minnesota, North Dakota, and South Dakota from 1994 to 1998, Bajari and Ye (2003) find that collusion was unlikely to have occurred. Efficient cartels are the only cartels their model allows. Bajari and Ye (2003) argue that this assumption is reasonable for the industry they consider.

Asker (2010) estimated that the payoff of a cartel of bidders at auctions of collectible stamps in New York in 1996 and 1997 obtained only 74% of an efficient cartel's payoff. The details of the collusive agreement resulted in the cartel's bidding too aggressively. Among the cartel members, Asker (2010) was able to identify weak and strong bidders.

In their study of rice auctions in India, Banerji and Meenakshi (2008) also concluded in the presence of a cartel that was not efficient. However, contrary to the auctions we consider here, in addition to bidder/buyer another type of bidder existed in these auctions: a buyer's agent, whose payoff was increasing with the auction price (because of the commission he received).

[12]What McAfee and McMillan (1992) call a "strong cartel."

[13]Lopomo, Marshall, and Marx (2005) show that no ex-post budget balanced mechanism within a cartel of bidders in the English auction can result in full efficiency–inside and outside the cartel–without pre-auction communication.

[14]Indeed, Biran and Forges (2011)'s construction applies to any Bayesian game with

mechanism for the whole encompassing cartel in the SPA satisfies the exante participation constraints of all subgroups of bidders. Thus, no such subgroup would collectively deviate and form its own cartel. Biran and Forges (2011) establish the same result for the FPA with initially homogeneous bidders.

Froeb, Tschantz, and Crooke (2001) introduced the phrase "power related distributions" for joint distributions, such as those arising when initially homogeneous bidders form efficient cartels, where any two bidders' value cumulative distribution functions are powers of each of other. Of course, such distributions can also be used to assess the effects of mergers among bidders: see Dalkir, Logan, and Masson (2000), Tschantz, Crooke, and Froeb (2000), and Branmann and Froeb (2000). Whaerer and Perry (2003) prove that the existence of an additive scalar measure of capacity that satisfies three natural axioms also implies power-related value distributions. From Piccione and Tan (1996), pre-auction cost-reducing investments by bidders result in such joint distributions of values if the investment technology satisfies an assumption of "conditional stochastic ordering."

For stochastically ordered distributions of values over a common support, of which power-related distributions are an example, both the equilibrium allocation of the FPA and Myerson (1981)'s optimal allocation are biased towards weaker bidders: as he bids more aggressively in the FPA, a weaker bidder outbids a stronger bidder with the same value (see, for example, Maskin and Riley, 2000). It is then natural to wonder whether the FPA brings more revenues than the SPA. Lebrun (1996) does indeed prove the revenue superiority of the FPA for powers of the uniform distribution when there are two bidders. Many other existing revenue comparisons between the FPA and SPA do not apply to distributions with a common support. Among the recent results, Kirkegaard (2012) extends Maskin and Riley (2000)'s two examples–"stretches" and "shifts"–where the FPA is more advantageous to the seller. As their names indicate, these examples as well as Kirkegaard's

independent types and quasi-linear payoffs.

extensions require the (two) bidders' values to be distributed over different supports. Kaplan and Zamir (2010) prove the revenue superiority of the FPA over the SPA when the two bidders' values are uniformly distributed over different intervals. Cheng (2006) assumes two power distributions satisfying a certain constraint that implies different supports[15]. Although Swinkels and Mares (2010) consider the case of a seller who prefers dealing with one of the two bidders, translating their model into the standard framework yields a revenue comparison when one bidder's value is distributed as the sum of a constant and the other bidder's value, again ruling out common support[16].

That even power relation does not guarantee revenue superiority of the FPA follows from a particularly simple example in Maskin and Riley (1985) where two bidders' value distributions have the same two-point support and hence are power related. Maskin and Riley (1985) explicitly compute the equilibrium revenues from the FPA and show that they are strictly smaller than the revenues from the SPA when the bidders are heterogeneous[17]. From the continuity and uniqueness results in Lebrun (2002, 2006), the FPA brings strictly less revenues for some continuous power-related distributions over a common full interval support that approximate such discrete distributions.

Although the rules of the FPA are simple, if bidders are heterogeneous

[15] Kirkegaard (2012) extends Cheng (2006) and Lebrun (1996)'s results.

[16] Cantillon (2008) obtains revenue comparisons within the same auction procedure–FPA or SPA–and between the symmetric joint value distribution and asymmetric joint distributions with the same distribution of the maximum value. Here, it would translate in, for example, the comparison between the revenues from the FPA when the initially homogeneous bidders are divided into either cartels of equal sizes or cartels of different sizes. If post-auction resale is allowed, Hafalir and Krishna (2009) prove that the FPA brings higher revenues than the truth-bidding equilibrium of the SPA when there are two bidders and their value distributions are regular, that is, their virtual value functions are strictly increasing. From Cheng and Tan (2009), the SPA may bring higher revenues when this last assumption is not satisfied. In the present paper, we do not allow resale and do not make any monotonicity assumption on the virtual value functions.

[17] Maskin and Riley (2000) prove the revenue superiority of the two-bidder SPA when one value distribution is continuous and the other is obtained from the first by shifting some probability to the lower extremity of the interval support. Although their supports are identical, such value distributions are not power related.

the equilibrium strategies are solutions of a complicated system of differential equations that are singular at the lowest winning bid. Because of the lack of explicit formulas for the solutions to this differential system, the numerical estimation of the equilibrium strategies is an active area of research[18]. Only experienced bidders can reasonably be expected to discover and follow such complicated equilibria. Many experimental researchers focus on the bidding behaviors of subjects who have gone through some preliminary "practice rounds." In recovering the value distributions from the bid distributions from such experiments, Bajari and Hortaçsu (2005) have found that assuming equilibrium behavior by rational bidders[19] performed better than models of boundedly rational bidders.

2. The Model

Bidders $1, ..., n$, with $n \geq 2$, participate in the auction. Their values are randomly and independently drawn from the interval $[c, d]$, with $0 \leq c < d$, according to the probability distributions $F_1, ..., F_n$. We use the same notation for a probability distribution and its (continuous from the right) cumulative function. In most of the paper, we make the following assumption[20].

Differentiability and full-support assumption FSA1: For all $1 \leq i \leq n$, F_i is continuous over $\mathbb{R}$; continuously differentiable over $[c, d]$ with

[18] See, for example, Marshall, Meurer, Richard, and Stromquist (1994), Bajari (2001), Gayle and Richard (2008), Paarsch and Hong (2006), Li and Riley (2007), Peng and Yang (2010), Fibich and Gavish (2011 and 2012).

[19] Bidders are also assumed to be risk-averse. Numerous experiments have also shown bidder behaviors to be consistent with equilibrium behavior under risk-aversion. However, according to recent experimental studies (see Neugebauer and Perote 2008), behaviors become consistent with equilibria under risk-neutrality as bidders are more experienced and are not distracted by feedback on previous auctions.

[20] Although our results about the random k-price auctions are easiest to present and prove under FSA1 below, they also hold true under considerably weaker assumptions. However, gaps in the value support common to more than $n-2$ bidders cannot be accommodated (see Appendix 2).

a strictly positive derivative f_i over $(c,d]$; twice-continuously differentiable over $(c,d]$; and such that $\frac{d}{dv}\frac{f_i}{F_i}(v)$ is bounded from above.

As F_i is atomless, the logarithm $\ln F_i(v)$ tends towards $-\infty$ if v approaches c and hence the reverse hazard rate $\frac{d}{dv}\ln F_i(v) = \frac{f_i(v)}{F_i(v)}$ must take on unbounded positive values and its derivative $\frac{d}{dv}\frac{f_i}{F_i}(v)$ unbounded negative values. FSA1 simply requires that $\frac{d}{dv}\frac{f_i}{F_i}(v)$ never takes on unbounded positive values. Obviously, a smooth F_i that is log-concave in some interval $(c, c+\varepsilon)$, with $\varepsilon > 0$, satisfies FSA1 as then $\frac{d}{dv}\frac{f_i}{F_i}(v) \leq 0$ over this interval. As we state in the technical Lemma A1 in Appendix 1, whose proof is in Lebrun (2012), FSA1 implies $\lim_{v \to > c}\frac{f_i}{F_i}(v) = +\infty$.

Our weakest assumption of stochastic ordering requires that the value distributions be ordered both for the hazard-rate stochastic dominance and the reverse-hazard-rate stochastic dominance, what we call double-hazard-rate dominance. Before stating the assumption, we formally introduce this new concept of stochastic dominance and remind some useful older ones.

Relations of stochastic dominance:

(i) F_j hazard-rate dominates F_i, which we denote $F_j \succeq_h F_i$, if and only if $\frac{1-F_j}{1-F_i}(v)$ is nondecreasing or, equivalently:

$$\frac{f_j}{1-F_j}(v) \leq \frac{f_i}{1-F_i}(v),$$

for all v in (c,d).

(ii) F_j reverse-hazard-rate dominates F_i, which we denote $F_j \succeq_{rh} F_i$, if and only if $\frac{F_j}{F_i}(v)$ is nondecreasing or equivalently:

$$\frac{f_j}{F_j}(v) \geq \frac{f_i}{F_i}(v),$$

for all v in $(c, d]$;

(iii) F_j likelihood-ratio dominates F_i, which we denote $F_j \succeq_{lr} F_i$, if and only if $\frac{f_j}{f_i}(v)$ is nondecreasing or, equivalently:

$$\frac{f_j'}{f_j}(v) \geq \frac{f_i'}{f_i}(v)$$

over $(c, d]$;

(iv) F_j double-hazard-rate dominates F_i, which we denote $F_j \succeq_{dh} F_i$, if and only if $F_j \succeq_h F_i$ and $F_j \succeq_{rh} F_i$, that is:

$$\frac{F_i}{F_j}(v) \leq \frac{f_j}{f_i}(v) \leq \frac{1 - F_j}{1 - F_i}(v), \tag{1}$$

for all v in (c, d).

The two expressions of our assumption SOA1 below are easily seen to be equivalent: the log-supermodularity of $F_i(v)$ is equivalent to the ordering of the distributions for the reverse-hazard rate dominance and the log-supermodularity of $1 - F_i(v)$ for the hazard-rate dominance. SOA1 is satisfied if F_{i+1} likelihood-ratio dominates F_i (see Krishna 2002) and therefore also if F_{i+1} is F_i raised to a power at least equal to 1. However, SOA1 characterizes a much broader class of joint distributions of values.

Double-hazard-rate stochastic ordering assumption SOA1:

$$F_1 \preceq_{dh} F_2 \preceq_{dh} ... \preceq_{dh} F_n,$$

or, equivalently[21]:

$$F_i(v) \text{ and } 1 - F_i(v) \text{ are log-supermodular in } (i, v).$$

[21] When the functions are considered defined over the product lattice $\{1, 2, ..., n\} \times (c, d)$.

Under SOA1, in the sense that bidder $i+1$ is more likely than bidder i to draw high values, we may say that bidder $i+1$ is stronger than bidder i or that bidder i is weaker than bidder $i+1$.

The notations below for the reverse hazard rate $f_i(v)/F_i(v)$ and the virtual value $v-(1-F_i(v))/f_i(v)$ will be useful.

Notations:

$$\begin{aligned} \rho_i(v) &= \frac{f_i}{F_i}(v), \\ \omega_i(v) &= v - \frac{1-F_i(v)}{f_i(v)}; \end{aligned}$$

for all v in $(c,d]$ and all i.

An immediate consequence of the hazard-rate dominance under SOA1 of F_j over F_i, for $j \geq i$, is that, for the same actual value v, bidder j's virtual value $\omega_j(v)$ is not larger than bidder i's. Consequently, from Myerson (1981), when the ordering is strict a revenue maximizing seller would prefer to the SPA an auction mechanism whose equilibrium allocation is biased, among bidders with values close to the highest value, towards lower-indexed bidders.

3. The Random k-Price Auction

Let k be a number between zero and one. The sealed-bid random k-price auction or RkPA with reserve price r in $[c,d)$ is this auction procedure where the rules of the SPA with reserve price r apply to all bidders except possibly one, who is randomly chosen according to a public lottery[22] before bidders may submit their bids. Because the rules of Güth and van Damme (1988)'s kPA will apply to this bidder, we call him the k-price bidder. The lottery is fair in that every bidder has the same probability[23] π of becoming the k-price bidder, with $0 < \pi \leq 1/n$. With probability $1 - n\pi$, there is no k-price bidder and the auction proceeds as a SPA.

After the lottery has taken place[24], all bidders submit their bids, one per bidder. As in the SPA, the item goes unsold and no payment is made if all bids are below the reserve price. Otherwise, the highest bidder wins the auction and is awarded the item for sale. A tie among several highest bidders is broken arbitrarily. The winner of the auction pays the maximum of the reserve price and the second-highest bid, unless he is the k-price bidder, in which case he pays the weighted average of his own bid and this maximum with respective weights k and $1 - k$. No payment is made by any other bidder.

As we state in Theorem 1 below, the RkPA is dominance solvable through only two rounds of elimination. Furthermore, for small k, the surviving strategies are essentially unique, with an inconsequential indeterminacy when the bidder's value is not larger than the reserve price. For this reason, we refer to *the* equilibrium of the RkPA as any n-tuple of such strategies.

Theorem 1–Equilibrium of RkPA: *Let FSA1 be satisfied and let* r

[22] Choosing the k-price bidder in a manner that is not competely random may have its benefits (and costs), as, for example, choosing the last bidder to place his bid may encourage early bidding (on a related point, see the remark in Section 3 of the discussion paper Lebrun 2012). Also, if the auction is scheduled to occur at least n times, a fixed roster can produce the same average results as a lottery. However, the exposition of the results is simpler for a single auction and a fair lottery.

[23] Assuming π independent of the number of submitted bids simplifies the analysis of efficient cartels and shill bidding in Appendix 2. Furthermore, it allows π to become another instrument, along with k, available to the auctioneer. For example, an auctioneer could reduce the impact, in particular on efficieny, of a given k by lowering π.

[24] If the k-price bidder is given the opportunity to revise his bid, the lottery can also take place after the bidders have submitted their bids.

be in $[c, d)$.

(i) For all $0 \leq k \leq 1$*, the RkPA with reserve price* r *is a Bayesian game that is dominance solvable through two rounds of elimination of all weakly dominated strategies.*

(ii) There exists ζ *in* $(0, 1)$ *such that for all* $0 \leq k < \zeta$ *and all* $1 \leq i \leq n$*, any strategy of bidder* i *that survives two elimination rounds specifies bidding strictly below* r *for values in* $[c, r)$*; not higher than* r *at* r*; and is equal over* $(r, d]$ *to a bidding strategy* $\delta_i(.; k)$*, conditional on the outcome of the pre-bidding lottery, as follows:*

If bidder i *is not the k-price bidder:* $\delta_i(.; k)$ *is the identity function.*

If bidder i *is chosen as the k-price bidder:* $\delta_i(.; k)$ *is equal to* $\min\left(r, \gamma_i^{-1}(.; k)\right)$*, where* $\gamma_i(.; k)$ *is the strictly increasing and continuous function such that* $\gamma_i(c; k) = c$ *and, for all* b *in* $(c, d]$*:*

$$\gamma_i(b; k) = b + \frac{k}{\sum_{j \neq i} \rho_j(b)}. \tag{2}$$

Proof: See Appendix 1.

As can be seen from its proof in Appendix 1, Theorem 1 follows straightforwardly from the properties of the function γ_i listed in the technical lemma below, whose simple proof is in the online discussion paper. For example, (2) follows from the first-order condition, or FOC, of bidder i's maximization problem. Everywhere in this paper, we reserve the rounded derivative sign for the derivative with respect to the parameter k.

Lemma 1: *Let FSA1 be satisfied. Then, for all* i*, there exist* $\zeta, \mu > 0$

such that $\gamma_i(.;.)$*, defined in (2), can be continuously extended to* $[c, d+\mu) \times (-\zeta, \zeta)$ *in such a way that:*

(i) $\gamma_i(c;k) = c$*, for all* k *in* $(-\zeta, \zeta)$*;*

(ii) Over $(c, d+\mu) \times (-\zeta, \zeta)$*:* $\gamma_i(.;.)$ *is continuously differentiable; the derivative* $\gamma_i'(.;.)$ *with respect to* b *is strictly positive and bounded away from zero;* $\frac{\partial}{\partial k}\gamma_i(.;.)$ *is strictly positive and bounded from above; and* $\gamma_i(d+\mu;k) > d$*, for all* k *in* $(-\zeta, \zeta)$*.*

(iii) Over $(c, d] \times (-\zeta.\zeta)$*:* $\frac{\partial}{\partial k}\gamma_i^{-1}(.;.)$ *is strictly negative, bounded from below, and equal to* $-\left(\gamma_i'\left(\gamma_i^{-1}(v;k);k\right)\sum_{j\neq i}\rho_i\left(\gamma_i^{-1}(v;k)\right)\right)^{-1}$*.*

Proof: See Lebrun (2012).

When $k = 0$, the RkPA reduces to the SPA and the equation (2) above gives the truth-bidding strategy in the SPA. When $k > 0$, the k-price bidder shades his bid below his value[25].

Let $R_i(k)$ and $TS_i(k)$ be the expected revenues and total surplus from the RkPA conditional on bidder i's being a k-price bidder and $R(k)$ and $TS(k)$ the unconditional expected revenues and total surplus. For example: $R(k) = (1 - n\pi) R(0) + \pi \sum_{i=1}^{n} R_i(k)$. The expected revenues and total surplus from the SPA are simply $R(0) = R_i(0)$ and $TS(0) = TS_i(0)$, for all i. Using Theorem 1, we first obtain in Lemma A2 in Appendix 1 explicit expressions for the differences in revenues and total surpluses from the two auctions, from which we can then compute the first-order effects of increasing k and prove Theorem 2 below, the main result of this section.

Theorem 2–First-Order Effects on Revenues and Total Surplus: *Let FSA1 be satisfied.*

(i) For all i*, the derivative at* $k = 0$ *of the expected revenues* $R_i(k)$ *and total surplus* $TS_i(k)$ *of the RkPA with bidder* i *as a k-price bidder exist*

[25] The same functions (2) with $k = 1$ appear at the second step of Bajari (2001)'s second algorithm of numerical estimation of the equilibrium bidding functions in the FPA.

and are given by the equations below:

$$\frac{d}{dk}R_i(0) = \sum_{j\neq i}\int_r^d (\omega_j(v)-\omega_i(v))\frac{\rho_j(v)\rho_i(v)}{\sum_{t\neq i}\rho_t(v)}\prod_t F_t(v)\,dv; \quad (3)$$

$$\frac{d_r}{dk}TS_i(0) = 0. \quad (4)$$

(ii) The derivative at $k=0$ of the unconditional expected revenues $R(k)$ and total surplus $TS(k)$ of the RkPA exist and are given by the equations below:

$$\frac{d}{dk}R(0) = \frac{\pi}{2}\sum_{i,j}\int_r^d \frac{(\omega_j(v)-\omega_i(v))\left(\rho_i(v)-\rho_j(v)\right)}{\left(\sum_{t\neq j}\rho_t(v)\right)\left(\sum_{t\neq i}\rho_t(v)\right)}\rho_i(v)\rho_j(v)\prod_t F_t(v)\,dv \quad (5)$$

$$\frac{d}{dk}TS(0) = 0 \quad (6)$$

(iii) Under SOA1, we have $\frac{d}{dk}R(0)\geq 0$ and:

$$\frac{d}{dk}R(0) > 0$$

if and only if the restrictions of $F_1,...,F_n$ over $[r,d]$ are not all identical.

Proof: See Appendix 1.

Figure 1 illustrates how the equilibrium allocation in the RkPA when bidder i is the k-price bidder differs from the efficient allocation. If k increases from zero, bidder i with value v decreases his bid from $\delta_i(v;0)=v$ to $\delta_i(v;k)$, which changes the winner of the auction from bidder i to bidder j if bidder j's value lies between these two bids and all the other bidders' values are smaller. For our first-order analysis, we may as well assume that, as in Figure 1, v is smaller than $\delta_i(d;k)$. Indeed, from Lemma 1 (iii), $\frac{\partial}{\partial k}\delta_i(d;k)$ is bounded and hence the probability that the couple of bidder i and bidder j's values falls within the shaded triangle in Figure 1 is only of the second order.

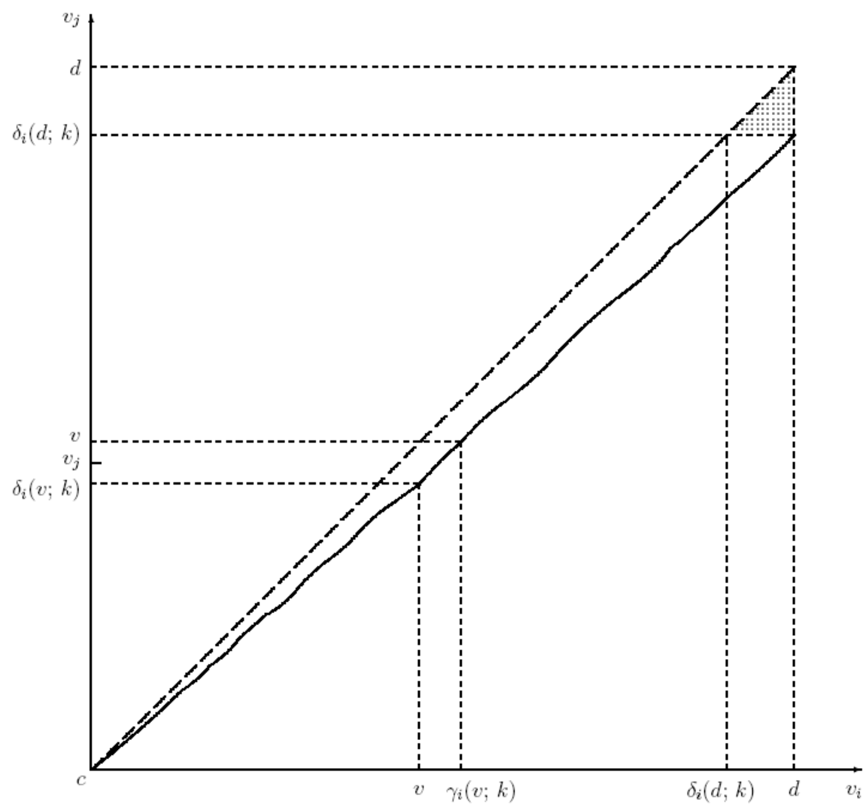


FIGURE 1: Inefficient allocation when bidder i with value v is the k-price bidder.

Under this assumption, the probability of such an inefficient allocation is approximately $f_i(v)\left(\frac{f_j(v)dk}{\sum\limits_{l\neq i}\rho_l(v)}\right)\prod\limits_{l\neq j,i}F_l(v) = \frac{\rho_j(v)\rho_i(v)}{\sum\limits_{l\neq i}\rho_l(v)}\prod\limits_{l}F_l(v)\,dk$ as, from

Lemma 1 (ii, iii), $\frac{\partial}{\partial k}\delta_i(v;k)$ is close to $-\left(\sum_{l\neq i}\rho_l(v)\right)^{-1}$. Because the allocation contributes the difference between bidders j and i's virtual values to the increase of the expected revenues, the expression (3) holds true. We do not need to consider the possibility that more than one bidder have their values between $\delta_i(v;0)$ and $\delta_i(v;k)$, for the probability of such an event is of the second order in k.

In Figure 2, we display portions of the areas where bidder i or bidder $j > i$ is the inefficient winner when one of them is the k-price bidder. Away from the extremities, these areas are strips along and, because (from Lemma 1) the derivatives of γ_i and γ_j tend towards one, approximately parallel to the 45-degree line. If F_j reverse-hazard-rate dominates F_i, which is the case under SOA1, bidder j has a higher reverse hazard rate, hence a smaller sum $\sum_{l\neq j}\rho_l$ of the other bidder's reverse hazard rates, and, from (2), shades his bid more when he is chosen as the k-price bidder. Thus, in Figure 2, the strip above the diagonal where bidder i is the inefficient winner is thicker than the strip below the diagonal when bidder j is the inefficient winner.

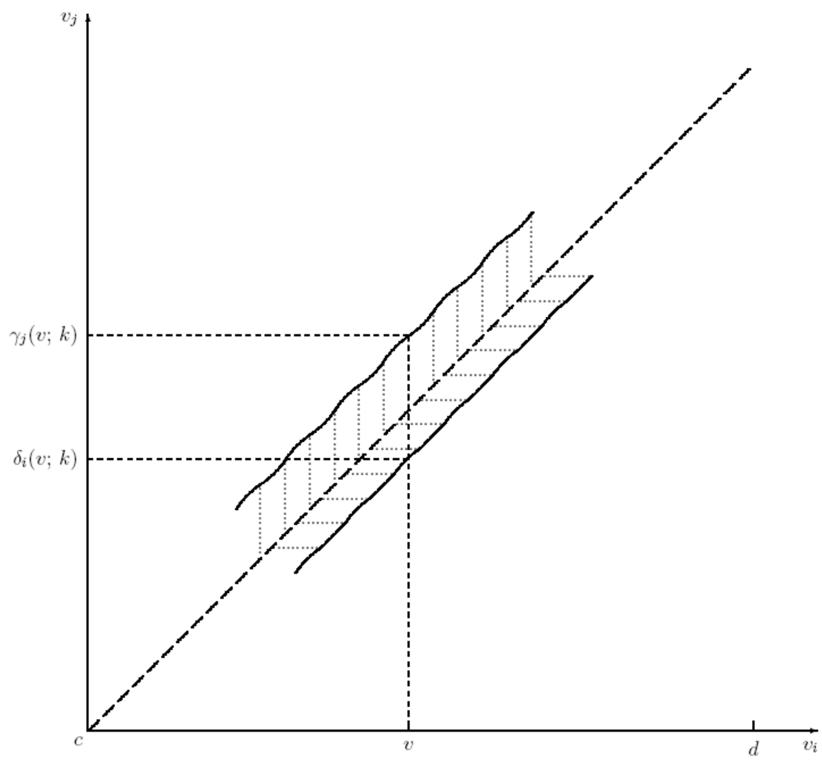


FIGURE 2: Sets of couples (v_i, v_j) resulting in an inefficient allocation when the k-price bidder is bidder i or $j > i$.

The differential bid shading is beneficial to the auctioneer when F_j hazard-rate dominates F_i, which is also the case under SOA1: as bidders i and j are as likely to be chosen as the k-price bidder, bidder j, whose virtual value is smaller, will win less often when opposing a bidder i with almost the same value. That the interplay between the two stochastic dominance relations improves expected revenues is made explicit in the equation (5)[26].

As expressed in the equations (4) and (6), because for small positive k the

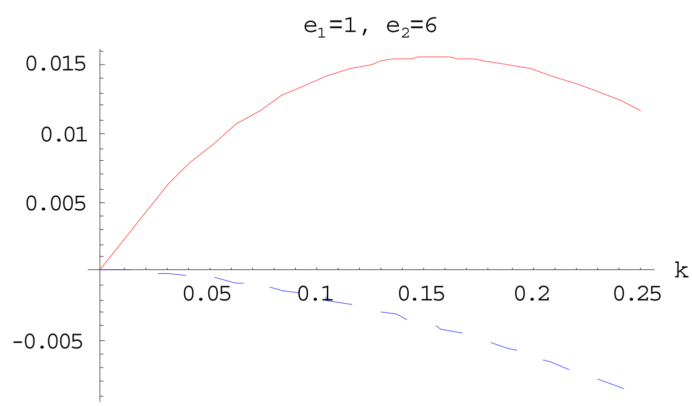
[26] Our proofs also show that: 1. a strictly negative k of small absolute value would decrease revenues; and 2. $\frac{d}{dk}R_i(0) > 0$ holds even if F_i does not dominate any given other value distribution, as long as it "hazard-rate dominates on average" the other distributions (for details, see Lebrun 2012).

winners of the SPA and the RkPA have almost the same values, increasing k above zero has only a second (or higher) order effect on efficiency.

Example 1: Consider the example with two bidders and no (binding) reserve price where the values are distributed over $[0, 1]$ according to power distributions $F_i(v) = v^{ei}$ with $e_i \geq 1$, for $i = 1, 2$. FSA1 and SOA1 are obviously satisfied. Elementary computations yield explicit expressions for $R(k)$ and $TS(k)$ (see Appendix 1 in Lebrun 2012), from which we obtain the logarithmic derivative below of the expected revenues:

$$\frac{d}{dk} \ln R(0) = \pi \frac{(e_2 - e_1)^2}{e_1 e_2 (e_1 + e_2 + 2)}.$$

In Figure 3 below, we plot the graphs of $\ln(R(k)/R(0))$ and $\ln(TS(k)/TS(0))$ for two sets of values of the exponents e_1, e_2 with $\pi = 1/2$.



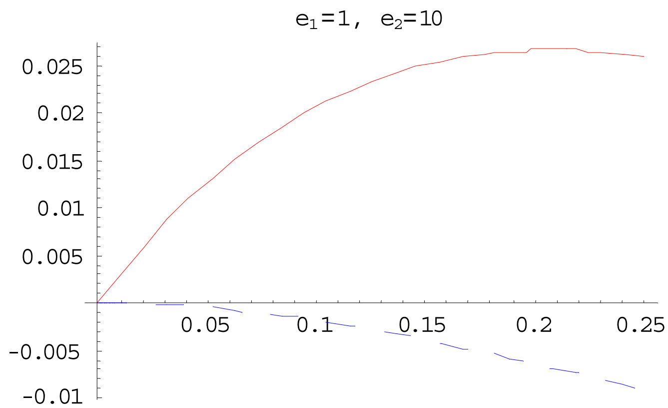


FIGURE 3: Graphs of $\ln(R(k)/R(0))$, as continuous red lines, and of $\ln(TS(k)/TS(0))$, as dashed blue lines, for two sets of values of the parameters in Example 1.

Example 2: The FPA brings more revenues than the SPA in Example 1 above[27]. As we mentioned in the introduction, Maskin and Riley (1985) show that the FPA brings strictly less revenues with two exante different bidders and two possible values, one of which strictly above the reserve price. Any two such distributions are power related and can hence be approximated (for the topology of the weak topology, the weak-* topology) by power-related differentiable distributions with the same full interval support, strictly log-concave at its lower extremity, and that satisfy FSA1. From Lebrun (2006), the equilibrium of the FPA is unique for such distributions. Lebrun (2002) then guarantees the existence of distributions such that the unique equilibrium of the FPA is sufficiently close (for the weak-* topology) to the original equilibrium in order for the SPA to bring strictly more revenues. Nevertheless, for these same distributions, Theorem 2 implies that the revenues from the RkPA exceed those from the SPA, and hence the FPA, for all small $k > 0$.

[27] See Lebrun (1996) and Kirkegaard (2012).

In Appendix 2, we extend our results to a model with efficient cartels and to a variant of the English auction, the "English random k-price auction"[28]. We also discuss the limitations of our results.

4. The Deterministic k-Price Auction

4.1 The Equilibrium and its Properties

In Güth and van Damme (1988)'s kPA, the same weights k and $1-k$ are used in the computation of the auction price whoever the highest bidder may be. While weakly dominant strategies do not exist, we construct in Theorem 3 below a standard FPA that has the same equilibrium as the kPA. Thanks to this construction, results from the literature on the FPA translate to the RkPA. In particular, the equilibrium, although complex, is unique (see Corollary 1 below).

Theorem 3-Characterization of the equilibrium of the RkPA and link with the FPA: *Let k be such that $k \in (0,1)$ and let FSA1 be satisfied.*

(i) **Equilibrium of the RkPA:** *An n-tuple of strategies $(\beta_1(.;k),...,\beta_n(.;k))$ is a Bayesian equilibrium of the kPA in weakly undominated strategies if and only if:*

1. For all i: $\beta_i(.;k)$ over $[c,r)$ specifies bidding strictly below r and $\beta_i(r;k)$ specifies bidding bids not larger than r.

2. $\beta_1(.;k),...,\beta_n(.;k)$ are bidding functions over $(r,d]$ such that their inverses $\alpha_1(.;k) = \beta_1^{-1}(.;k),...,\alpha_n(.;k) = \beta_n^{-1}(.;k)$ exist, are strictly increasing, strictly above the identity function, and form a solution

[28] As we show in Appendix 2 of the online discussion paper Lebrun (2012), our results also hold true when bidders are risk-averse.

over $(r, \eta(k)]$ *of the system of differential equations (8) below*

$$\frac{1}{k}\frac{d}{db}\ln\prod_{j\neq i}F_j\left(\alpha_j\left(b;k\right)\right)=\frac{1}{\alpha_i\left(b;k\right)-b},\ 1\leq i\leq n, \tag{7}$$

and their continuous extensions satisfy the boundary conditions:

$$\alpha_i\left(r;k\right) = r, \text{ for all except possibly one } i; \tag{8}$$

$$\alpha_1\left(\eta\left(k\right);k\right) = \ldots = \alpha_n\left(\eta\left(k\right);k\right) = d, \tag{9}$$

for a certain value in (r, d) *of the parameter* $\eta(k)$*.*

3. If $\alpha_j(r;k) > r$*, then* $\beta_j(.;k)$ *over* $(r, \alpha_j(r;k)]$ *takes the constant value* r*.*

Moreover, if $r = c$ *or if* $n = 2$*, (9) can be replaced by (11) below and 3. above never applies:*

$$\alpha_1\left(r;k\right) = \ldots = \alpha_n\left(r;k\right) = r. \tag{10}$$

(ii) **Link with the FPA:** *An n-tuple of strategies is a Bayesian equilibrium of the kPA in weakly undominated strategies if and only if it is a Bayesian equilibrium of the FPA where the bidders' values are distributed according to* $F_1^{1/k}, \ldots, F_n^{1/k}$ *.*

Proof: See Appendix 3.

The differential equations (8) in Theorem 3 are the equilibrium FOC's. In fact, the expected payoff of bidder i with value $v > r$ and bid $b > r$ is

$$\int_c^b \left(v - kb - (1-k)\max\left(w, r\right)\right) d\prod_{j\neq i}F_j\left(\alpha_j\left(w\right)\right),$$

whose derivative[29] with respect to $b > r$ is

$$(v-b)\frac{d}{db}\prod_{j\neq i}F_j(\alpha_j(b)) - k\prod_{j\neq i}F_j(\alpha_j(b)).$$

As we state in Corollary 1, the (essential) uniqueness[30] of the equilibrium holds true under FSA1 when the reserve price is binding and under the stronger assumption FSA2 below when it is not.

Stronger assumption of differentiability and full-support FSA2: FSA1 where the requirement on the reverse hazard rates is strengthened to the requirement of the existence of $\varepsilon > 0$ such that $F_1, ..., F_n$ are strictly log-concave over $(c, c+\varepsilon)$.

Thus, FSA2 requires that $\frac{d}{dv}\rho_i(v) = \frac{d}{dv}\frac{f_i}{F_i}(v)$ be strictly negative, that is, that $\rho_i(v)$ tend monotonically towards $+\infty$ as v approaches c on an interval, however small, to the right of c.

Corollary 1–Properties of the equilibrium of the k-PA: *Let FSA1 be satisfied and let k be such that $k \in (0,1)$.*

(i) **Existence:** *There exists an equilibrium in weakly undominated strategies of the kPA.*

(ii) **Expression for the bidding functions:** *If $(\beta_1(.;k), ..., \beta_n(.;k))$ is the equilibrium of kPA, we have:*

$$\beta_i(v;k) = v - \frac{\int_r^v \prod_{j\neq i}F_j\left(\varphi_{ji}(w;k)\right)^{1/k}dw}{\prod_{j\neq i}F_j\left(\varphi_{ji}(v;k)\right)^{1/k}},$$

[29] As we indicate in Appendix 3, the differentiability is proved in Lebrun (2012).

[30] Or, more precisely, uniqueness up to the inessential indeterminacy for values less than or equal to r (see (i.1) in Theorem 3).

for all v *in* $(\alpha_i(r;k), d]$*, where* $\varphi_{ji}(.;k) = \alpha_j(\beta_i(.;k);k)$ *and* $\alpha_j(.;k)$ *is the inverse of* $\beta_j(.;k)$*.*

(iii) **More aggressive bidding by weaker bidders while preserving reverse-hazard-rate bid dominance:** *Under SOA1:*

(iii.1) $\beta_i(.;k) \geq \beta_j(.;k)$*, over* $(r,d]$*, and* $\frac{d}{db}\ln F_i\alpha_i(.;k) \leq \frac{d}{db}\ln F_j\alpha_j(.;k)$*, over* $(r,\eta(k)]$*, for all* $i \leq j$*;*

(iii.2) If $\rho_i(v) < \rho_j(v)$*, for all* v *in* (r,d)*, then* $\beta_i(v;k) > \beta_j(v;k)$*, for all* v *in* (r,d)*.*

(iii.3) In the characterization (i.1, i.2, i.3) in Theorem 3, the initial condition (9) in (i.2) can be replaced by (12) below:

$$\alpha_1(r;k) = ... = \alpha_{n-1}(r;k) = r; \tag{11}$$

and, in Theorem 3 (i.3), only $j = n$ *needs be considered.*

(iv) **Uniqueness:** *If* $r > c$*, the equilibrium in weakly undominated strategies is essentially unique*[31]*. It is also essentially unique if* $r = c$ *and FSA2 is satisfied.*

Proof: See Appendix 3.

As we state in Theorem 4 (iii) below, under SOA1 the equilibrium approaches the truth-bidding equilibrium of the SPA if k tends towards zero. The proof proceeds first by showing that the bid shading in Corollary 1 (ii) by bidder 1, who is the weakest and, from Corollary 1 (iii), the most aggressive bidder, vanishes at the limit. This is a consequence of: 1. the obvious convergence towards truth-bidding of the equilibrium of the FPA with exante homogeneous bidders when the number of bidders tends towards infinity; and 2. the fact that bidder 1 bids higher than in a FPA with $(n-1)/k$ other bidders, with the same value distribution as his. Once this is established, the result for the other bidders follows. In fact, if another bidder's bid

[31] That is, unique up to the inessential indeterminacy for values not larger than r (see (i.1) in Theorem 3).

stayed away from his value, bidding closer to it would be a profitable deviation: it would increase his probability of winning by the probability with which bidder 1 bids within the gap and the other bidders below it and it would increase his payment by a negligible amount, as k tends towards zero. Theorem 4 also gives first bounds on rates of convergence. We first recall from Royden (1988) the definition of "derivates," which are generalizations of the concept of one-sided derivative, and introduce notations, one of which for the elasticity of a density with respect to the cumulative probability.

Definition: Let $h(x_1, ..., x_m)$ be a real-valued function defined over a product of intervals with nonempty interiors $P = \prod_{s=1}^{m} [c_s, d_s] \subseteq \mathbb{R}^m$. Then, for all x in P and all s the right-handed partial derivates of h at (x_{-s}, c_s) are as follows:

$$\begin{aligned} \frac{\partial^+}{\partial x_s} h(x_{-s}, c_s) &= \overline{\lim}_{\Delta x_s \to > 0} \frac{h(x_{-s}, c_s + \Delta x_s) - h(x_{-s}, c_s)}{\Delta x_s} \\ \frac{\partial_+}{\partial x_s} h(x_{-s}, c_s) &= \underline{\lim}_{\Delta x_s \to > 0} \frac{h(x_{-s}, c_s + \Delta x_s) - h(x_{-s}, c_s)}{\Delta x_s}. \end{aligned}$$

Thus, $\frac{\partial^+}{\partial x_s} h(x_{-s}, c_s)$ and $\frac{\partial_+}{\partial x_s} h(x_{-s}, c_s)$ are the supremum and infimum, respectively, of the rates of increase of h with respect to x_s to the right of c_s. The standard right-handed derivative $\frac{\partial_r}{\partial x_s} h(x_{-s}, c_s)$ exists if and only if $\frac{\partial^+}{\partial x_s} h(x_{-s}, c_s)$ and $\frac{\partial_+}{\partial x_s} h(x_{-s}, c_s)$ are equal and finite, in which case it is equal to their common value. If there is only one variable, that is, $m = 1$, the derivates and the right-handed derivative are denoted $\frac{d_+}{dx} h, \frac{d_+}{dx} h$, and $\frac{d_r}{dx} h$.

Convention and Notations:

1. Throughout the rest of the paper, we keep denoting $\beta_1(.; k), ..., \beta_n(.; k)$ and $\alpha_1(.; k), ..., \alpha_n(.; k)$ the unique direct and inverse equilibrium bidding functions of the kPA, and $\varphi_{ij}(.; k)$ the compound function $\alpha_i(\beta_j(.; k); k)$ that connect the values at which bidders i and j would tie. As a matter of convenience, we extend the inverse bidding function $\alpha_i(.; k)$ as the constant

function d above the maximum equilibrium bid $\eta(k)$, that is, we set:

$$\alpha_i(b;k) = d,$$

for all $\eta(k) \leq b \leq d$ and all i.

2. For all i, we denote $\varepsilon_i(p)$ the elasticity of the density $f_i\left(F_i^{-1}(p)\right)$ with respect to the cumulative probability p, that is:

$$\begin{aligned}\varepsilon_i(p) &= \frac{d \ln f_i\left(F_i^{-1}(p)\right)}{d \ln p} \\ &= \frac{p f_i'\left(F_i^{-1}(p)\right)}{f_i\left(F_i^{-1}(p)\right)^2}.\end{aligned}$$

Theorem 4–Convergence of the equilibrium towards truth-bidding[32]: *Let SOA1 be satisfied. Let also FSA1 be satisfied if $r > c$ and FSA2 if $r = c$. Extend the functions $\beta_i(.;k)$, $\alpha_i(.;k)$, $\varphi_{ij}(.;k)$ to $k = 0$ as follows:*

$$\beta_i(v;0) = \alpha_i(v;0) = \varphi_{ij}(v;0) = v,$$

for all v in $[r,d]$. Then:

(i) **Upper bounds on the rates of convergence of the strategies of the bidders other than the strongest bidder:** *For all $i < n$, we have, for all v* in *$[r,d]$:*

$$0 \leq -\frac{\partial_+}{\partial k}\beta_i(v;0), \frac{\partial^+}{\partial k}\alpha_i(v;0) \leq \left((n-i)\rho_i(v)\right)^{-1}.$$

[32] Our proof of Theorem 4 holds true even if, instead of SOA1, we require only that the value distributions be ordered for the reverse-hazard rate dominance. Furthermore, as our informal argument above suggests, our proof of the convergence of the equilibrium strategies towards truth-bidding (Theorem 4 (iii)) goes through even if bidders 2, ..., n value distributions are not stochastically ordered as long as they all reverse-hazard rate dominate bidder 1's (see Footnote 50 in Appendix 3). From the remark after Theorem 6 in the next subsection, some assumption of full support is necessary for the convergence towards truth-bidding.

If I is $[r,d]$ when $r>c$ and any interval $[c+\gamma,d]$, where $\gamma>0$, when $r=c$, the upper bound above on the derivates is "uniform" in v over I, that is:

$$\overline{\lim}_{k\to 0}\max_{v\in I}\left(\max\left(\frac{v-\beta_i(v;k)}{k},\frac{\alpha_i(v;k)-v}{k}\right)-((n-i)\,\rho_i(v))^{-1}\right)\leq 0.$$

Moreover, if $r=c$ and the elasticity ε_i is bounded from below, this is also true for $I=[c,d]$.

(ii) **Lower bound on the rate of convergence of the strategy of the strongest bidder:** *We have*[33]*:*

$$-\frac{\partial_+}{\partial k}\beta_n(v;0)\,,\frac{\partial^+}{\partial k}\alpha_n(v;0)\geq((n-1)\,\rho_n(v))^{-1}\,.$$

(iii) **Joint continuity of the bidders' strategies:** *For all $1\leq i,j\leq n$, the functions $\beta_i(v;k)\,,\alpha_i(v;k)\,,\varphi_{ji}(v;k)$ are continuous jointly in both variables v,k at $(v;0)$, for all v in $[r,d]$.*

Proof: See Appendix 3.

The assumption in Theorem 4 (i) that the elasticity ε_i be bounded from below is equivalent to requiring that there exists a strictly positive number T, however large, such that F_i^T is convex in an interval $(c,c+\varepsilon)$ (with $\varepsilon>0$), however small, to the right of c[34].

[33]We have abstained to state the uniformity of the lower bound in (ii) on the derivates $-\frac{\partial_+}{\partial k}\beta_n(v;0)$ and $\frac{\partial^+}{\partial k}\alpha_n(v;0)$, which can be proved similarly to the uniformity of the upper bounds in (i), because we do not need it in our later developments.

[34]This assumption is also equivalent to the requirement that the "local ρ-concavity" of F_i be bounded from above (for other applications of local ρ-concavity to auction theory, see Mares and Swinkels 2010). Similar assumptions–on the elasticity of the slope of demand functions–are commonly made in the literature on oligopoly (see, for example, Seade 1980 a and b, Suzumura and Kiyono 1987, Besley and Suzumura 1989, Suzumura 1990, Okuno-Fujiwara and Suzumura 1993)).

The analysis of the standard FPA is complicated by the singularity of the system of FOC's at the lower extremity of the winning bid interval. What makes matters even worse in the kPA with k close to zero is that, as the bidding functions tend towards the identity function, the system (8) is nearly singular everywhere over the bid interval. Partly for this reason, it is difficult to go much further than Theorem 4 while keeping the same level of generality. However, as we show in the next subsection, explicit expressions for the rate of change of expected revenues can be obtained in the two-bidder case.

4.2 The two-bidder case

In addition to assuming two bidders, we also make the stronger assumption of full support FSA2 above and strengthen the stochastic dominance relation between the two value distributions to power relation. Thus, $n = 2$ and

$$F_2 = F_1^l, \tag{12}$$

for some constant $l \geq 1$. We keep using primes and straight derivative signs for the derivatives with respect to the bid or value. The system (7) of FOC's and the terminal condition (9) in Theorem 3 reduce to the system (13) and condition (14) below:

$$\begin{aligned} \frac{d \ln F_1\left(\alpha_1\left(b;k\right)\right)}{db} &= \frac{1}{\left(\alpha_2\left(b;k\right)-b\right)/k} \\ \frac{d \ln F_2\left(\alpha_2\left(b;k\right)\right)}{db} &= \frac{1}{\left(\alpha_1\left(b;k\right)-b\right)/k} \end{aligned} \tag{13}$$

$$\alpha_1\left(\eta\left(k\right);k\right) = \alpha_2\left(\eta\left(k\right);k\right) = d, \tag{14}$$

where $\eta\left(k\right) \in \left(r, d\right)$ is the maximum equilibrium bid. Thanks to the system (13), how the derivatives of the bidding functions evolve with k informs us on the rates of convergence of the bidding functions towards the identity function.

From Corollary 1 (iii), bidder 1 bids more aggressively and, as a consequence, bidder 2 needs a higher value to tie for winner of the auction, that is, $\varphi_{21}(v;k) = \alpha_2(\beta_1(v;k);k) \geq v$. From hereon, we simplify our notations by dropping the subscripts from φ_{21}. The equilibrium allocation of the kPA is inefficient in the set of value couples (v_1, v_2) bounded from below by the 45-degree line and from above by the graph of $\varphi(.;k)$.

Although from Theorem 4 (iii), $\varphi(.;k)$ approaches the identity function, the value of its derivative does not approach one everywhere. Indeed, dividing the second equation in (13) by the first, we find:

$$\frac{d \ln F_2(\alpha_2(b;k))}{d \ln F_1(\alpha_1(b;k))} = \frac{\alpha_2(b;k) - b}{\alpha_1(b;k) - b}; \tag{15}$$

which, together with (12) and the condition (14), implies:

$$\varphi'(d;k) = \frac{1}{l}.$$

Thus, when the values are differently distributed, $\varphi'(d;k)$ stays away from one.

To acquire information about the behavior of the derivative of φ, we differentiate (15) and, after rearranging, we find:

$$\begin{aligned}
& \frac{d}{db} \ln \frac{d \ln F_2(\alpha_2(b))}{d \ln F_1(\alpha_1(b))} \\
= \; & \frac{\left(\alpha_2(b;k) + k\frac{F_2(\alpha_2(b;k))}{f_2(\alpha_2(b;k))}\right) - \left(\alpha_1(b;k) + k\frac{F_1(\alpha_1(b;k))}{f_1(\alpha_1(b;k))}\right)}{(\alpha_1(b) - b)(\alpha_2(b) - b)} \\
= \; & \frac{\gamma_1(\alpha_2(b;k);k) - \gamma_2(\alpha_1(b;k);k)}{(\alpha_1(b;k) - b)(\alpha_2(b;k) - b)},
\end{aligned} \tag{16}$$

where, remarkably, appear the inverses γ_1, γ_2 of the k-bidder's bidding functions (when above the reserve price) in the RkPA (see Theorem 1 (2), Section 3). Equally as remarkable is that the sign of the expression above does not depend directly on b: $\ln F_2(\varphi(v;k))$ is concave or convex with respect to $\ln F_1(v)$ depending on whether $\gamma_1(\varphi(v;k);k)$ is smaller or larger than $\gamma_2(v;k)$.

We are lead to consider the functions[35] $\gamma_1^{-1}(\gamma_2(.;k);k), \varphi(.;k)$, and $F_2 F_1^{-1}$ in the space of logarithms of cumulative probabilities. We denote the new functions $\Psi(.;k), \Phi(.;k)$, and Λ and define them formally below. The linearity of Λ follows immediately from the power relation (12).

Definitions: For all k in $(0,\zeta)$, where $\zeta > 0$ is as in Lemma 1 (Section 3), let γ_1 and γ_2 be as defined in Theorem 1 (2) (Section 3), that is:

$$\begin{aligned}\gamma_i(v;k) &= v + \frac{k}{\rho_j(v)} \\ &= v + k\frac{F_j(v)}{f_j(v)},\end{aligned}$$

for $i,j = 1,2,\ i \neq j$, and let $\Psi(.;k), \Phi(.;k)$, and Λ be the following functions:

(i)

$$\Psi(u;k) = \ln F_2\left(\gamma_1^{-1}\left(\gamma_2\left(F_1^{-1}(\exp u);k\right);k\right)\right),$$

for all u in $(-\infty, \ln F_1(x(k)))$, where $x(k) \in (c,d)$ is defined as follows:

$$x(k) = \gamma_1^{-1}(\gamma_2(d;k);k);$$

and

$$\Psi(u;k) = 0,$$

[35] Contrary to $\varphi(.;k)$ in the kPA, $\gamma_1^{-1}(\gamma_2(.;k);k)$ is not a "tying" function in the RkPA. Rather, it is the composition of two such functions. Bidder 2's value at which he ties with bidder 1 with value v is $\gamma_2(v;k)$ if bidder 2 is the k-price bidder and $\gamma_1^{-1}(v;k)$ if bidder 1 is the k-price bidder.

for all u in $[\ln F_1(x(k)), 0]$;

(ii)

$$\Phi(u;k) = \ln F_2\left(\varphi\left(F_1^{-1}(\exp u);k\right)\right),$$

for all u in $(\ln F_1(r), 0]$ and, if $r > c$, at $u = \ln F_1(r)$, with $\varphi(.;k) = \alpha_2(\beta_1(.;k);k)$;

(iii)

$$\Lambda(u) = lu,$$

for all u in $\mathbb{R}_-$.

As $\rho_2 = l\rho_1$, we have $\gamma_2(.;k) \geq \gamma_1(.;k)$ and $x(k)$ in (i) above is well defined. All functions above are continuous. The function $\Psi(.;k)$ is strictly increasing over $(-\infty, \ln F_1(x(k)))$ and equal to zero over $[\ln F_1(x(k)), 0]$. The function $\Phi(.;k)$ is strictly increasing over its entire definition domain, tends towards $\ln F_2(r)$ if its argument u tends towards the infimum $\ln F_1(r)$ of its domain, and vanishes at the supremum (zero).

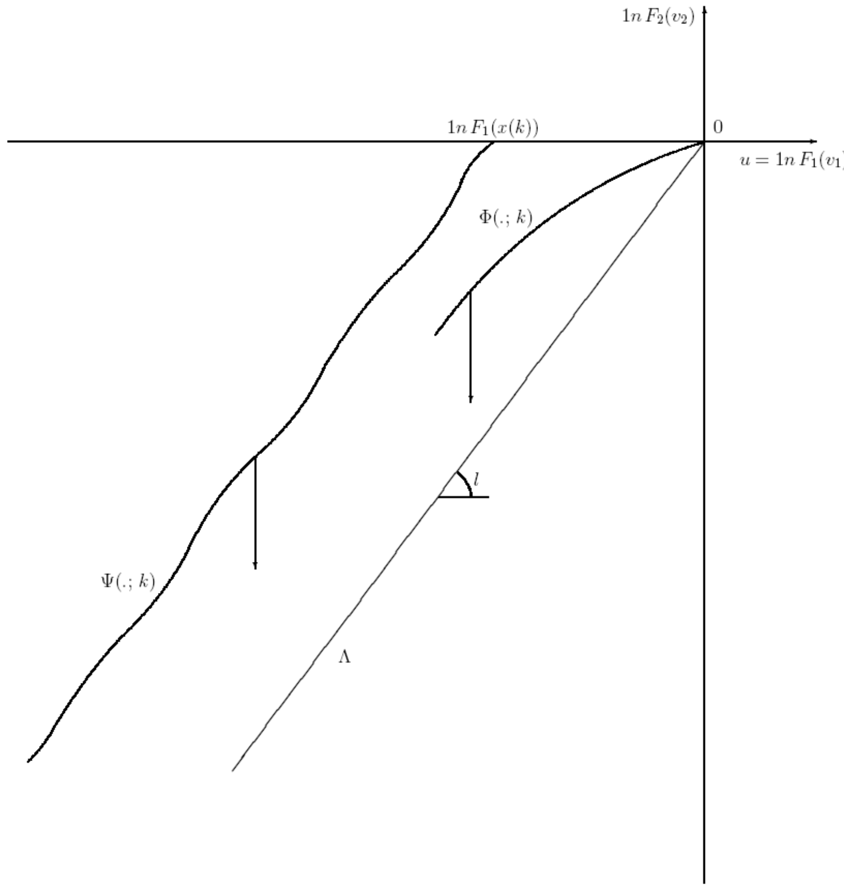


FIGURE 4: Possible configuration of the functions Ψ, Φ, and Λ.

Figure 4 displays a possible configuration of the graphs of these functions. From Corollary 1 (iii), $\Phi(.;k)$ is at least equal to Λ and strictly above it over the interior of its definition domain if $l > 1$. Because $\Psi(.;k)$ reaches zero already at $\ln F_1(x(k))$ and $\Phi(.;k)$ only at zero, $\Psi(.;k)$ is above $\Phi(.;k)$ in the neighborhood of the origin. If $l > 1$, $\gamma_2(.;k) > \gamma_1(.;k)$ and $\Psi(.;k)$ is strictly above Λ. As we already inferred from (16), the position of $\Phi(.;k)$ relative to $\Psi(.;k)$ determines the direction of its concavity. That is, Lemma 2 below holds true.

Lemma 2: *Assume $n = 2$ and $F_2 = F_1^l$, with $l > 1$. Let FSA1 be* *satisfied if* $r > c$ *and FSA2 if* $r = c$. *Then:*

(i) $\Psi(u;k) > \Lambda(u)$, *for all* u *in* $\mathbb{R}_-$;

(ii) $\Psi(u;k) > \Phi(u;k)$, *for all* u *in* $(\ln F_1(x(k)), 0)$; *and if* $r > c$: $\Psi(\ln F_1(r); k) > \Phi(\ln F_1(r); k) = \Lambda(\ln F_1(r))$.

(iii) $\Phi(u;k) > \Lambda(u)$, *for all* u *in* $(\ln F_1(r), 0)$;

(iv) $\Phi''(u;k) > (<;=) 0$ *if and only if* $\Phi(u;k) > (<;=) \Psi(u;k)$, *for all* u *in* $(\ln F_1(r), 0)$.

From the explicit expression in the definition of $\Psi(.;k)$, it is simple to prove that it and its derivative tend towards Λ and its derivative. From its definition and Theorem 4 (iii), $\Phi(.;k)$ tends towards Λ. The downward pointing arrows in Figure 4 above represent these convergences.

From Lemma 2, we can prove that the derivative of $\Phi(.;k)$ converges uniformly towards the derivative of Λ over any compact subinterval K of $(\ln F_1(r), 0)$. The main ideas of the proof are as follows. From the convergence of $\Psi(.;k)$ and its derivative, there exists k' such $\Psi'(.;k)$ is close to l over K and only possibly slightly larger than l over $[\min K, 0]$, for all $0 < k < k'$. If, at some point s in K and for small k, the derivative of $\Phi(.;k)$ was further above l, $\Phi(.;k)$ could not at the same time tend towards Λ, remain above it, and be strictly increasing. For example, if $\Phi'(.;k)$ at a point s was further above l than $\Psi'(.;k)$ is over $[\min K, 0]$ and if $\Phi(s;k)$ was not smaller than $\Psi(s;k)$, $\Phi(.;k)$ would be convex at s (from Lemma 2 (iv)), its derivative would even be higher and $\Phi(.;k)$ would hence never meet $\Psi(.;k)$ to the right of s. As depicted in Figure 5 (for the case $r = c$), $\Phi(.;k)$ would then be equal to zero to the left of $\ln F_1(x(k))$; something which is impossible as $\Phi(u;k)$ vanishes only at $u = 0$.

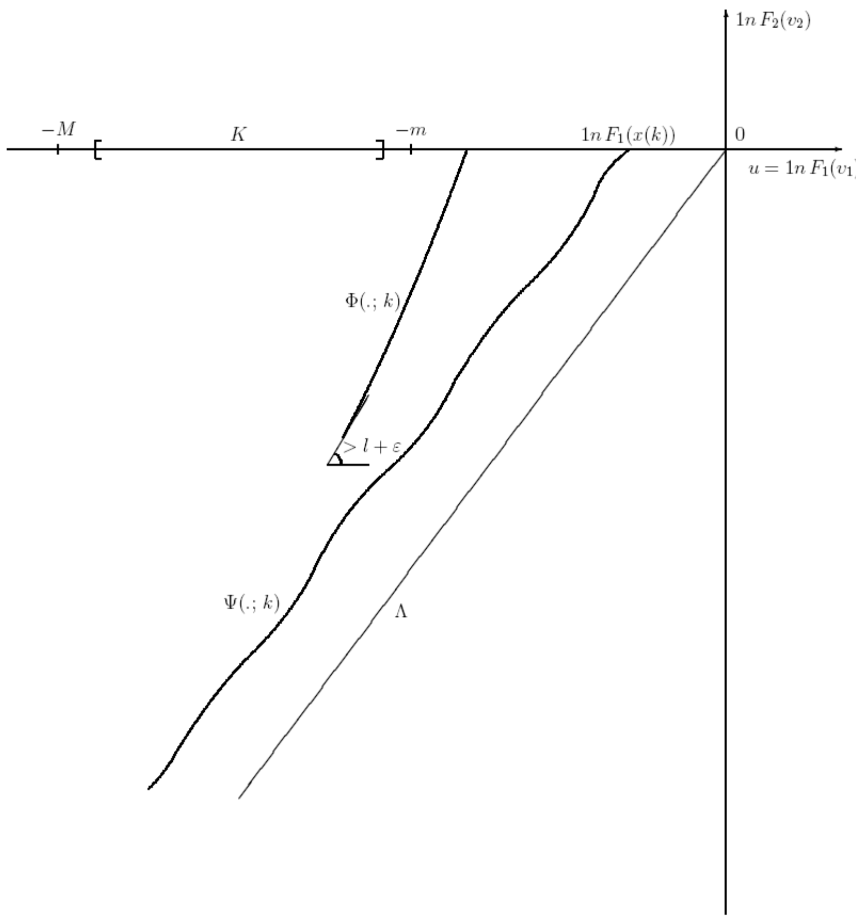


FIGURE 5: Ruling out $\Phi'(.;k)$ further above l than $\Psi'(.;k)$ is while $\Phi(.;k)$ is not smaller than $\Psi(.;k)$.

If $\Phi'(.;k)$ at a point s was again further above l than $\Psi'(.;k)$ is over K, but if $\Phi(s;k)$ was now less than or equal to $\Psi(s;k)$, $\Phi(.;k)$ would be concave at s (from Lemma 2 (iv)), its derivative would be higher and hence $\Phi(.;k)$ would never meet $\Psi(.;k)$ and would remain concave to the left of s. However, as the derivative of Λ is the constant l, a derivative of $\Phi(.;k)$ uniformly bounded away from l to the left of s is incompatible with $\Phi(.;k)$ tending everywhere towards Λ. For an illustration of this case, see Figure 6 below. Ruling out $\Phi'(.;k)$ further below l than $\Psi'(.;k)$ is over K proceeds along similar lines.

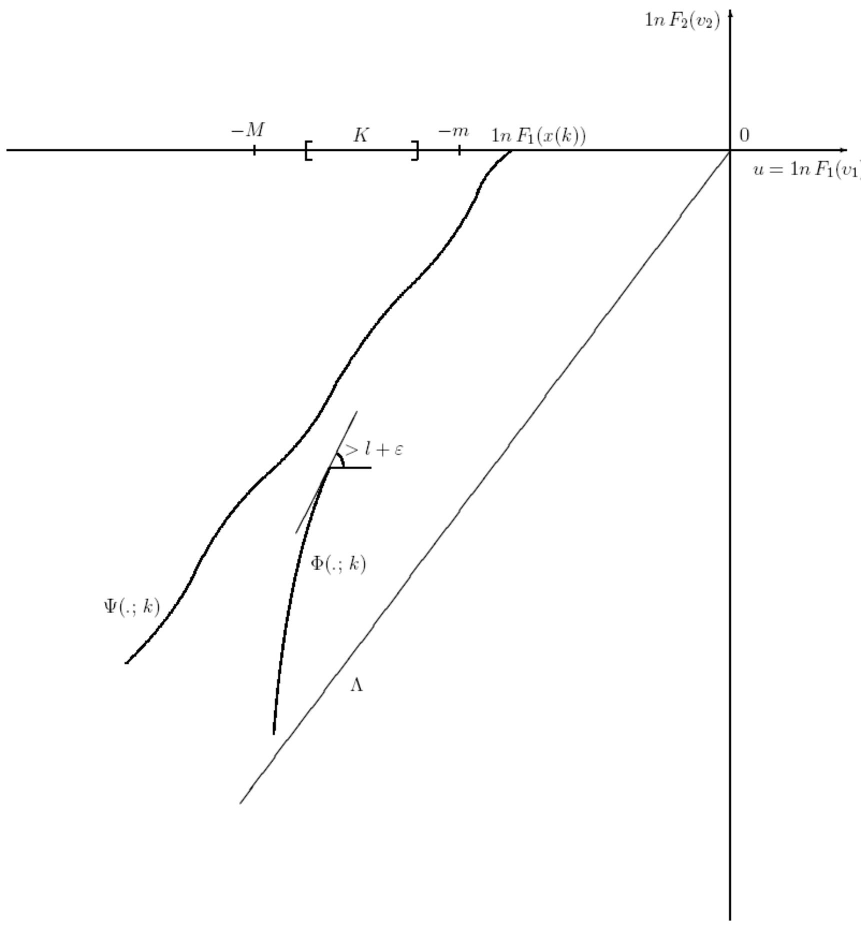


FIGURE 6: Ruling out $\Phi'(.;k)$ further above l than $\Psi'(.;k)$ is while $\Phi(.;k)$ is not larger than $\Psi(.;k)$.

We have Lemma 3 below[36], whose detailed proved is in the online discussion paper. When $r = c$, in order to bound $\Phi'(s;k)$ for s tending towards

[36] Because of the discontinuity of the derivative of $\Psi(.;k)$ at $\ln F_1(x(k))$, we express statement (ii) below in terms of the inverse function $\Psi^{-1}(.;k)$. As it can be easily

$-\infty$, that is, for bidder 1's value approaching c, we assume in (v) below that the elasticity ε_1 of the density f_1 with respect to F_1 is bounded from below.

Lemma 3: *Assume $n = 2$ and $F_2 = F_1^l$, with $l \geq 1$. Let FSA1 be satisfied if $r > c$ and FSA2 if $r = c$. Then:*

(i) $\lim_{k\to 0} x(k) = d$;

(ii) $\Psi^{-1}(.;k)$ *tends towards* Λ^{-1} *in* $C^1(K)$*, for all compact subinterval* K *of* $\mathbb{R}_-$;

(iii) $\Phi(.;k)$ *tends towards* Λ *in* $C^1(K)$*, for all compact subinterval* K *of* $(\ln F_1(r), 0)$*; if* $r > c$*,* $\Phi(.;k)$ *also tends towards* Λ *in* $C^0([\ln F_1(r), 0])$*;*

(iv) $\overline{\lim}_{(s;k)\to(0;0)} \Phi'(s;k) \leq l$;

(v) If the elasticity ε_1 *is bounded from below, then:* $\overline{\lim}_{(s;k)\to(-\infty;0)} \Psi'(s;k) \leq l$ *and, when* $r = c$*,* $\overline{\lim}_{(s;k)\to(-\infty;0)} \Phi'(s;k) \leq l$. .

Proof: See Lebrun (2012).

Thanks to Lemma 3, we can prove that, away from the extremities of the value interval, the slope of $\varphi(.;k)$ tends towards one. The set of value couples resulting in an inefficient allocation is then, similarly to the RkPA (see Figure 1), a strip along the 45-degree line and of approximately constant width. We list this and other useful consequences from Lemma 3 about the limits of derivatives in Theorem 5 below.

Theorem 5–Convergence of the derivatives with respect to value or bid: *Assume $n = 2$ and $F_2 = F_1^l$, with $l \geq 1$. Let FSA1 be satisfied if*

---

verified by appealing to (i), (ii) is equivalent to the uniform convergence over any interval $[\underline{u}, \ln F_1(x(k))]$ of $\Psi(.;k)$ and its derivative towards Λ and its derivative, that is, to the following statement: (ii)' $\max_{u\in[\underline{u},\ln F_1(x(k))]} \max(|\Psi'(u;k) - l|, |\Psi(u;k) - lu|)$ tends towards zero with k, for all finite $\underline{u} < 0$.

$r > c$ and FSA2 if $r = c$. Extend the derivatives $\beta_i'(.;k)$, $\alpha_i'(.;k), \varphi'(.;k)$ to $k = 0$ as follows:

$$\beta_i'(v;0) = \alpha_i'(v;0) = \varphi'(v;0) = 1,$$

for all v in (r,d) and all $i = 1,2$. Then:

(i)[37] $\beta_i'(v;k), \alpha_i'(v;k), \varphi'(v;k)$ *are continuous jointly in both variables at $(v;0)$, for all v in (r,d);*

(ii) $\overline{\lim}_{(v;k)\to(d;0)}\varphi'(v;k) \leq 1$;

(iii) If $r = c$ and the elasticity ε_1 is bounded from below, then $\overline{\lim}_{(v;k)\to(c;0)}\frac{d\ln F_2(\varphi(v;k))}{d\ln F_1(v)} \leq l$.

Proof: See Appendix 4.

From the initial system of differential equations (13) and the convergence of the derivatives, we obtain in Theorem 6 below the rates of convergence of the bidding functions and the function φ, which determines the equilibrium allocation, and hence of the seller's expected revenues[38]. Let $R^D(k)$ and $TS^D(k)$ be the expected revenues and total surplus from the equilibrium of the deterministic kPA. Thus, $R^D(0)$ and $TS^D(0)$ are the expected revenues and total surplus from the SPA.

Theorem 6–Rates of convergence with respect to k: *Assume*

[37] As we already observed, $\varphi'(d;k) = \frac{1}{l}$ and is strictly smaller than 1, for all k, if $l > 1$. Consequently, in this case, $\varphi'(v;k)$ is discontinuous at $(d;0)$.

[38] As for the RkPA (see Appendix 2), our results depend on full value support. In the two-bidder two-value example of Maskin and Riley (1998), we discussed in Appendix 2, a high-value bidder's equilibrium bid distribution in the kPA becomes concentrated at the high and low values and the revenues tend towards the, smaller, revenues from the FPA as k approaches zero. Again, as for the RkPA, our results imply the existence of small revenue-improving k's for any given approximation of this discrete example by power-related value distributions that satisfy FSA2. Contrary to the RkPA (see Appendix 2), it is in the interest of no efficient cartel to have a member other than it highest-value member to bid seriously and of no bidder to submit shill bids.

$n=2$ and $F_2 = F_1^l$, with $l \geq 1$. Let FSA1 be satisfied if $r > c$ and FSA2 if $r = c$. Then:

(i) For all v in (r,d) and all $i,j = 1,2$ with $i \neq j$, we have:

$$\begin{aligned}
\lim_{(b;k)\to(v;0)} \frac{\alpha_i(b;k)-b}{k} &= \lim_{(u,k)\to(v,0)} \frac{u-\beta_i(u;k)}{k} = \rho_j(v)^{-1}; \\
\lim_{(u,k)\to(v,0)} \frac{\varphi(u;k)-u}{k} &= \frac{l-1}{l}\rho_1(v)^{-1}.
\end{aligned}$$

(ii) For all $\gamma > 0$, $\overline{\lim}_{k\to 0} \max_{u\in[r+\gamma,d]} \left(\frac{\varphi(u;k)-u}{k} - \frac{l-1}{l}\frac{F_1(u)}{f_1(u)}\right) \leq 0$. When $r = c$, this inequality also holds true for $\gamma = 0$ if the elasticity ε_1 is bounded from below.

(iii) If $r = c$ and the elasticity ε_1 is bounded from below, then $\frac{d_r}{dk}R^D(0)$ and $\frac{d_r}{dk}TS^D(0)$ exist and we have:

$$\begin{aligned}
\frac{d_r}{dk}R^D(0) &= (l-1)\int_c^d (\omega_1(v)-\omega_2(v))\,F_2(v)\,dF_1(v) \geq 0, && (17)\\
\frac{d_r}{dk}TS^D(0) &= 0; && (18)
\end{aligned}$$

and $\frac{d_r}{dk}R^D(0) > 0$ if and only if F_1, F_2 are not identical.

(iv) If $r > c$, we have:

$$\frac{d_+}{dk}R^D(0) \geq (l-1)\int_r^d (\omega_1(v)-\omega_2(v))\,F_2(v)\,dF_1(v) \geq 0;$$

and if the distributions F_1, F_2 are not identical: $\frac{d_+}{dk}R^D(0) > 0$.

Proof: See Appendix 4.

Because we cannot uniformly bound the rate of increase of φ in a neighborhood of the reserve price if it is binding and because bidder 2's virtual value is bounded above bidder 1's in such a neighborhood, we cannot rule out rates of increase even higher than the expression in (iv) above.

Comparing the explicit expressions in Theorems 2 and 6, we see that the first-order effect of linking the winner's payment to his own bid is π^{-1} times, hence at least twice, as large in the kPA than in the RkPA. Intuitively, as every bidder is a k-price bidder with probability one in the kPA, the total effect is double the effect of having any given bidder's being the k-price bidder only with probability $\pi = 1/2$, as in the RkPA where a single k-price bidder is always chosen[39].

Corollary 2–Revenue comparison between the RkPA and the kPA: *Assume $n = 2$ and $F_2 = F_1^l$, with $l \geq 1$. Then:*

(i) If $r = c$, FSA2 is satisfied, and the elasticity ε_1 is bounded from below:

$$\frac{d_r}{dk} R^D(0) = \pi^{-1} \frac{d}{dk} R(0).$$

(i) If $r > c$ and FSA1 is satisfied:

$$\frac{d_+}{dk} R^D(0) \geq \pi^{-1} \frac{d}{dk} R(0).$$

Proof: See Appendix 4.

For Example 1 in Section 3 with $\pi = 1/2$, the relative rate of increase

[39] However, one should not believe that this property follows from some immediate joint differentiability with respect, for example, to bidder-specific probabilities π_1, π_2 of becoming a k-price bidder. As this subsection illustrates, the differentiability with respect to the single variable k–the uniform share of the own-bid in the payment of either winner–in the kPA is already difficult to prove.

$\frac{d_r}{dk} \ln R^D(0)$ of the revenues is then $\frac{(e_2-e_1)^2}{e_1 e_2 (e_1+e_2+2)}$[40]. While increasing k from zero to 1 in Example 2 decreases revenues, a small increase of k above zero increases revenues.

5.Conclusion

From Myerson (1981), improving on the revenues from the SPA or the English auction would require "handicapping" the stronger bidders. However, a given bidder may be stronger than some bidders and weaker than others. Even if the auctioneer knew the "strength" ranking of bidders, he could not be expected to infer the exact optimal sizes of the handicaps. Moreover, implementing such handicaps may significantly decrease efficiency. Here, we showed that increase in revenues without significant effect on efficiency follows from applying uniformly to all bidders a small link between the auction winner's payment and his bid. In fact, the equilibrium allocation will be slightly biased towards weaker bidders as stronger bidders will depress their bids more in reaction to this link.

We reviewed three possible implementations of such a link: through the sealed-bid RkPA, the hybrid open/sealed-bid English RkPA, and Güth and van Damme (1988)'s sealed-bid kPA. In the first two implementations, the link applies to no more than one bidder, who is chosen according to a fair lottery. In the third implementation, the link applies to all bidders simultaneously.

Linking the winner's payment to his bid has a strictly positive first-order effect on revenues and no first-order effect on total surplus if the bidders'

[40] This formula gives $\frac{d}{dk} \ln R^D(0) = 0.3214$ when $e_1 = 1$ and $e_2 = 4$. As the numerical estimations in Marshall, Meurer, Richard and Stromquist (1994) imply $\ln R^D(1) - \ln R^D(0) = 0.0803$, the relative rate of increase $\frac{d}{dk} \ln R^D(k)$ must obviously decrease at some k's between 0 and 1. This is also the case for $e_1 = 2$ and $e_2 = 3$, where $\ln R^D(1) - \ln R^D(0) = 0.0071$ while $\frac{d}{dk} \ln R^D(0) = 0.0238$.

values distributions are: ordered according to the double-hazard rate dominance, for the sealed-bid RkPA; ordered according to the likelihood-ratio dominance, for the English RkPA; power related and there are two bidders, for the kPA. Thus, under these respective assumptions, the ratio of the increase in expected revenues to the decrease in expected total surplus can be made arbitrarily large.

We proved that the equilibrium of the kPA is identical to the equilibrium of a FPA for some transformed value distributions. On the other hand, the equilibrium strategies of the sealed-bid and English RkPA's are simpler than those of the FPA. Indeed, the RkPA's are solved through only two rounds of elimination of weakly dominated strategies. Our results hold true even if the auctioneer sets a reserve price and can be extended to bidders who form efficient cartels.

Appendix 1

Lemma A1:

(i) Under FSA1: $\lim_{v \to > c} \frac{f_i}{F_i}(v) = +\infty$*;*

(ii) Under FSA1 and SOA1: If $j \geq i$*, either* $F_i = F_j$ *or* $F_i(v) > F_j(v)$*, for all* v *in* (c, d)*. Furthermore, in the latter case, there exists* v *in* (c, d) *where neither inequality in (1) is binding.*

(iii) Under FSA1 and SOA1, and $c \leq r < d$*: The restrictions of* F_i *and* F_j *are not identical over* $[r, d]$ *if and only if there exists* v *in* (r, d) *where neither inequality in (1) is binding.*

Proof: See Lebrun (2012).

Proof of Theorem 1:

Proof of (i): In the first round of elimination, all strategies that specify bidding at or above r for values strictly smaller than r, or bidding strictly above r when the value is r, or submitting a bid different from the value v in $(r,d]$ when the bidder is not the k-price bidder, as well as all strategies that specify, over $(r,d]$, bidding strictly above the value or strictly below r when the bidder is the k-price bidder, are eliminated. At the second round, the only weakly undominated strategies among the remaining strategies are those that make the k-price bidder follow a best reply to truth-bidding over $(r,d]$ from the other bidders. Thanks to FSA1, the expected payoff $P_i(v;b)$ below of bidder i as the k-price bidder with value $v>r$ when he submits b in the compact $[r,v]$ is continuous in b and such best replies therefore exist and furthermore belong to $[r,v)$:

$$P_i(v;b)=\int_c^b\left(v-kb-(1-k)\max\left(\widetilde{b},r\right)\right)d\prod_{j\neq i}F_j\left(\widetilde{b}\right). \qquad \text{(A1.1)}$$

Proof of (ii): There remains to prove the statement about the bidding function of the k-price bidder. Let $\zeta>0$ be as in Lemma 1 (ii).

Obviously, any optimal bid from bidder i as the k-price bidder with value $v>r$ is at least r and not larger than v. From (A1.1), we see that, for all $b>r$, $\frac{\partial}{\partial b}P_i(v;b)$ does not depend on r and:

$$\frac{\partial}{\partial b}P_i(v;b)=-k\prod_{j\neq i}F_j(b)+(v-b)\frac{d}{db}\prod_{j\neq i}F_j(b).$$

As $\frac{\partial^2}{\partial v\partial b}P_i(v;b)=\frac{d}{db}\prod_{j\neq i}F_j(b)>0$, bidder i's expected payoff has strictly increasing differences in $(v,b)\in(r,d]^2$, and δ_i is nondecreasing when strictly above r (see, for example, Theorem 2.8.4 in Topkis, 1998). Finally, $\frac{\partial}{\partial b}P_i(v;b)>(<)$ $\frac{\partial}{\partial b}P_i(\gamma_i(b;k);b)=0$, for all $r<v\leq d$ and all b in $\left(r,\gamma_i^{-1}(v;k)\right)$ $\left(\left(\gamma_i^{-1}(v;k),d\right)\right)$. Consequently, $\delta_i(.;k)$ is equal to $\gamma_i^{-1}(.;k)$, when $\gamma_i^{-1}(.;k)$ is strictly above r, and r otherwise. ||

Lemma A2: *Let FSA1 be satisfied. Let* $\Delta R_i(k)$ *and* $\Delta TS_i(k)$ *be the differences between the expected revenues and total surpluses from the RkPA with bidder* i *as the k-price bidder and the SPA, both with the reserve price* r*, that is:* $\Delta R_i(k) = R_i(k) - R_i(0)$ *and* $\Delta TS_i(k) = TS_i(k) - TS_i(0)$*. Then, the following formulas hold true for all* k such that $\delta_1(d;k), ..., \delta_n(d;k) > r$*:*

$$\begin{aligned}\Delta R_i(k) &= \sum_{j\neq i}\int_r^{\delta_i(d;k)}\int_{v_j}^{\gamma_i(v_j;k)}(\omega_j(v_j)-\omega_i(v_i))\prod_{t\neq i,j}F_t(v_i)\,dF_i(v_i)\,dF_j(v_j)\\ &\quad+\sum_{j\neq i}\int_{\delta_i(d;k)}^{d}\int_{v_j}^{d}\\ &\quad(\omega_j(v_j)-\omega_i(v_i))\prod_{t\neq i,j}F_t(v_i)\,dF_i(v_i)\,dF_j(v_j)\,; \qquad\text{(A1.2)}\end{aligned}$$

$$\begin{aligned}\Delta TS_i(k) &= \sum_{j\neq i}\int_r^{\delta_i(d;k)}\int_{v_j}^{\gamma_i(v_j;k)}(v_j-v_i)\prod_{t\neq i,j}F_t(v_i)\,dF_i(v_i)\,dF_j(v_j)\\ &\quad+\sum_{j\neq i}\int_{\delta_i(d;k)}^{d}\int_{v_j}^{d}(v_j-v_i)\prod_{t\neq i,j}F_t(v_i)\,dF_i(v_i)\,dF_j(v_j)\,. \qquad\text{(A1.3)}\end{aligned}$$

Proof: Consider the extension of $\gamma_1(.;.)$,...,$\gamma_n(.;.)$ over $[c, d+\mu) \times (-\zeta, \zeta)$ as in Lemma 1. From Lemma 1, as $\gamma_i(r;0) = r$, decreasing ζ if necessary, we may assume $\gamma_1(r;k), ..., \gamma_n(r;k) < d$ and consequently $\delta_1(d;k) = \gamma_1^{-1}(d;k), ..., \delta_n(d;k) = \gamma_n^{-1}(d;k) > r$, for all k in $(-\zeta, \zeta)$.

For all k such that $0 \leq k < \zeta$, the equilibrium allocation in the RkPA differs from the equilibrium allocation in the SPA when bidder i wins the SPA because his value v_i is the highest but loses the RkPA because v_i is smaller than $\gamma_i(v_j;k)$, where v_j is the highest value from the other bidders. As the winner of an auction contributes his virtual value to the expectation of the revenues and any bidder with value r is left with no payoff in either auction, we find (A1.2). We find (A1.3) similarly. ||

Proof of Theorem 2:

Proof of (i): As in the proof of Lemma A2, we may assume that the extension in Lemma 1 of $\gamma_1(.;.)$,...,$\gamma_n(.;.)$ over $[c, d+\mu) \times (-\zeta, \zeta)$ is such that $\delta_1(d;k) = \gamma_1^{-1}(d;k), ..., \delta_n(d;k) = \gamma_n^{-1}(d;k) > r$, for all k in $(-\zeta, \zeta)$. The derivative $I_{ji}(v_j;k)$ with respect to k of the function $\int_{v_j}^{\gamma_i(v_j;k)} (\omega_j(v_j) - \omega_i(v_i)) \prod_{t \neq i,j} F_t(v_i) \, dF_i(v_i)$ inside the first integral sign in the term corresponding to the value $j \neq i$ of the index in the first sum in the RHS of (A1.2) is:

$$
\begin{aligned}
& I_{ji}(v_j;k) \\
= \; & (\omega_j(v_j) - \omega_i(\gamma_i(v_j;k))) f_i(\gamma_i(v_j;k)) f_j(v_j) \\
& \frac{\partial}{\partial k}(\gamma_i(v_j;k)) \prod_{t \neq i,j} F_t(\gamma_i(v_j;k)) \\
= \; & \left\{ \begin{array}{c} (v_j - \gamma_i(v_j;k)) f_i(\gamma_i(v_j;k)) f_j(v_j) \\ -(1 - F_i(\gamma_i(v_j;k))) f_j(v_j) - (1 - F_j(v_j)) f_i(\gamma_i(v_j;k)) \end{array} \right\} \\
& \frac{\partial}{\partial k}(\gamma_i(v_j;k)) \prod_{t \neq i,j} F_t(\gamma_i(v_j;k)).
\end{aligned}
$$

From FSA1, f_i and f_j are continuous. From Lemma 1, $\gamma_i(.;.)$ is continuous over $[c, d+\mu) \times (-\zeta, \zeta)$ and $\frac{\partial}{\partial k}(\gamma_i(.;.))$ is bounded over $(c, d+\mu) \times (-\zeta, \zeta)$. The derivative above is then continuous and bounded over $(c, d+\mu) \times (-\zeta, \zeta)$. In particular, differentiating inside this first integral sign gives a continuous function of the bound of integration $\delta_i(d;k)$.

The derivatives with respect to the bounds of integration are also continuous. Therefore, from Lemma 1 (iii), the derivative of (A1.2) with respect to k exists. As the derivative with respect to k in the argument of the bounds of integration $\delta_i(d;k)$ in the terms corresponding to j in both sums in the RHS cancel out ($\gamma_i(\delta_i(d;k);k) = d$), the derivative of (A1.2) is equal to:

$$
\sum_{j \neq i} \int_r^{\delta_i(d;k)} I_{ji}(v_j;k) \, dv_j,
$$

for all k in $(-\zeta, \zeta)$. (3) then follows by setting $k = 0$ and using $\frac{\partial}{\partial k}(\gamma_i(v_j;k)) =$

$$\left(\sum_{t\neq i}\rho_t\left(v_j\right)\right)^{-1}.$$

A similar (and simpler) reasoning shows that (A1.3) is differentiable with respect to k. As the SPA obviously maximizes total surplus, (4) follows.

Proof of (ii): Adding up the equations (3) over all $1\leq i\leq n$, we find:

$$\begin{aligned}&\frac{d}{dk}R\left(0\right)\\ =\;&\pi\sum_i\frac{d}{dk}R_i\left(0\right)\\ =\;&\pi\sum_{i,j}\int_r^d\left(\omega_j\left(v\right)-\omega_i\left(v\right)\right)\frac{1}{\sum_{t\neq i}\rho_t\left(v\right)}\rho_j\left(v\right)\rho_i\left(v\right)\prod_t F_t\left(v\right)dv.\end{aligned}$$

We find another expression $\frac{d}{dk}R\left(0\right)$ by permuting the indices i,j in the previous expression. Averaging the two expressions, we find:

$$\begin{aligned}&\frac{d}{dk}R\left(0\right)\\ =\;&\frac{\pi}{2}\sum_{i,j}\int_r^d\left(\omega_j\left(v\right)-\omega_i\left(v\right)\right)\left(\frac{1}{\sum_{t\neq i}\rho_t\left(v\right)}-\frac{1}{\sum_{t\neq j}\rho_t\left(v\right)}\right)\rho_j\left(v\right)\rho_i\left(v\right)\prod_t F_t\left(v\right)dv,\end{aligned}$$

which is (5). (6) follows immediately from (4).

Proof of (iii): The first statement is an immediate consequence of SOA1 and (5). From Lemma A1 (iii), there exists v in (r,d) where both inequalities in (1) for some i,j are strict if and only if the restrictions of F_i and F_j over $[r,d]$ are not identical. The second statement follows[41]. ||

Appendix 2
Extensions

[41] It would be possible to have $\frac{d}{dk}R\left(0\right)=0$ even with different distributions if their derivatives f_i,f_j were discontinuous. See Footnote 44 in Lebrun (2012).

Efficient Cartels: One of our motivations to study stochastically ranked value distributions over the same interval was the possibility of efficient collusion among bidders. In auctions such as the SPA an efficient cartel has only its highest-value member submit a serious bid. However, an efficient cartel in the RkPA may want its second-highest value member to bid seriously[42]. In fact, if there is only one highest value member in a cartel and he has been chosen as the k-price bidder, the cartel must decide whether to have him submit its only serious bid under the harsher payment rule of the kPA or to have rather its second-highest value member submit a bid under the rules of the SPA. Nevertheless, this complication brings about only second-order corrections on revenues and total surplus and therefore does not affect our main conclusion: the RkPA still brings more revenues than the SPA with a negligible effect on total surplus.

The results, whose detailed statements and proofs can be found in the online discussion paper Lebrun (2012), hold true if SOA1 is replaced by the assumption SOA1' below, where $F_1^{(1)}, ..., F_m^{(1)}$ are the probability distributions of the highest values of the efficient cartels $S_1, ..., S_m$ that form a partition of the set of bidders:

SOA1':

$$F_1^{(1)} \preceq_{dh} ... \preceq_{dh} F_m^{(1)}.$$

SOA1' reduces to SOA1 in the competitive case, where no S_i counts more than one bidder. Obviously, SOA1' holds true if the individual bidders' value distributions are power related.

English Random k-Price Auction: Assume now that the auction starts as Bikhchandani and Riley (1991)'s open English "button" auction, with irrevocable exit and full information about the bidders activities, and where the price rises[43] from a reserve price r' in $[c, d)$. Assume also the

[42] The mechanism inside the cartel could be constructed as in Biran and Forges (2011). See the introduction and Footnote 14.

[43] A bidder who does not want to take part in the auction "drops out" just before the price is set to rise.

auctioneer has announced the price level r in (r', d) at which the auction rules are set to change: if the price ever hits this level, the auction switches to the sealed-bid RkPA where every still active bidder has to submit a bid at least equal to r. For our revenue-ranking result (Theorem 2 (iii)) to apply, the restrictions of the value distributions to $[r, d]$ have to be ordered for the double hazard-rate dominance. This is the case for all r if the unrestricted value distributions are ordered for the likelihood-ratio dominance, that is, if they satisfy SOA2 below, which is stronger than SOA1. Power-related distributions satisfy SOA2.

Likelihood–ratio stochastic ordering assumption, or SOA2:

$$F_1 \preceq_{lr} F_2 \preceq_{lr} ... \preceq_{lr} F_n,$$

or, equivalently[44]:

$$f_i(v) \text{ is log-supermodular in } (i, v).$$

Again, details of the statements and proofs of the results can be found in Lebrun (2012).

The auctioneer can announce instead that an RkPA will be run when the number of remaining bidders falls to a certain level (at least equal to two). The auction will then be more similar to Klemperer (1998)'s anglo-dutch auction and the price will never be determined by the rules of the English auction alone. SOA2 is then the appropriate assumption to guarantee our results as the RkPA may start at any value r of the price between r' and d.

Limitations

[44] Over the product lattice $\{1, 2, ..., n\} \times (c, d)$.

Shill Bids: That one bidder may only submit one bid is a crucial rule of the RkPA. If bidders could duplicate themselves, the RkPA and SPA would have equivalent equilibria and would then bring the same revenues. For the sake of simplicity, assume that every bidder may only send a maximum of $q \geq 2$ bidding agents to the auction (without being detected and incurring some severe punishment, for example, the exclusion from the auction) and such that $\pi \leq (qn)^{-1}$. It is in a bidder's best interest to have as many agents as possible, as it increases his probability of having one of them chosen as the k-price bidder[45], in which case he will instruct this agent to bid below (or at) the reserve price and another agent to bid his value (if above the reserve price). The game is then identical to the RkPA with efficient cartels (see above) where all members of the same cartel have the same value. As every bidder sends more than one agent, no bidder will ever have his agent chosen as the k-price bidder submit a serious bid, that is, larger than r. Only the rules of the SPA will apply and the same equilibrium outcome as in the SPA will follow.

Gaps in the Value Support: Our results do not hold without a full support assumption, such as FSA1. Consider a setting as in Maskin and Riley (1985) with two bidders whose values are differently distributed over the pair $\{r, d\}$ and where, for the sake of simplicity, $r = 0$. If $k = 0$, both bidders submit their values and the auctioneer's expected revenues is d multiplied by the probability that d is the value of both bidders. If k becomes strictly positive[46], when chosen to be the k-price bidder any bidder with value d will bid $r = 0$. In fact, the only opportunity to win the auction such a bidder forgoes–when he ties with his opponent at d–would bring him no payoff. Expected revenues therefore fall. Although no revenue-improving $k > 0$ exists here, there do exist some for any given couple of value

[45] While it does not change the probability that the k-price bidder be an agent of another bidder, which stays equal to π times the number of such agents.

[46] And assuming a tie involving the k-price bidder is broken in his favor.

distributions that approximate those discrete distributions while satisfying FSA1 (see Example 2, Section 3).

Appendix 3

Proof of Theorem 3: The proof of the characterization (i.1, i.2, i.3), which can be found in Lebrun (2012), proceeds along similar lines[47] as the proof in Lebrun (1997, 1999) of the characterization in Theorem 1 in Lebrun (1999) of the equilibria of the FPA.

From Theorem 1 in Lebrun (1999), the differential system and boundary conditions in (i) are the same as those that characterize the equilibria of the FPA where the value distributions are $F_1^{1/k}, ..., F_n^{1/k}$ and (ii) follows.

That all bidding functions must start rising from r if $r = c$ or if $n = 2$ then follows from Theorem 1 (3)[48] and Corollary 6 in Lebrun (1999). (10) therefore holds true. ||

Proof of Corollary 1:

Proof of (i): This follows from Theorem 3 (ii) above and Lebrun (1999) (or Lebrun 1997 or Maskin and Riley 2000).

Proof of (ii): The formula is a direct consequence of Theorem 3 (ii) above and the envelope theorem or Myerson (1981).

Proof of (iii): Because $F_j^{1/k}/F_i^{1/k} = (F_j/F_i)^{1/k}$, $F_i \preceq_{rh} F_j$ implies $F_i^{1/k} \preceq_{rh} F_j^{1/k}$. From Corollary 3 (ii) in Lebrun (1999), we then have $\beta_i(.; k) \geq \beta_j(.; k)$ or, equivalently, $\alpha_j(.; k) \geq \alpha_i(.; k)$. From the formula (A2.2) in Lebrun (1999) or from the difference between the equations (7) for i and j, we find:

[47] Although it is more than a mere translation of the arguments in Lebrun (1997, 1999).
[48] Also from Lemma A2-7 in Lebrun (1997).

$$
\begin{aligned}
& \frac{d}{db}\ln\frac{F_j\alpha_j}{F_i\alpha_i}(b;k) \\
= \; & \frac{d}{db}\ln F_j\alpha_j(b;k) - \frac{d}{db}\ln F_i\alpha_i(b;k) \\
= \; & \frac{k}{\alpha_i(b;k)-b} - \frac{k}{\alpha_j(b;k)-b} \\
\geq \; & 0,
\end{aligned}
$$

and (iii.1) is proved.

Assume $\rho_i(v) < \rho_j(v)$, for all v in (r,d). If there existed b in $(r,\eta(k))$ such that $\alpha_i(b;k) = \alpha_j(b;k)$, we would have from (7) (see the proof of (iii.1) above) $\alpha_j'(b;k)\rho_j(\alpha_j(b;k)) = \frac{d}{db}\ln F_j(\alpha_j(b;k)) = \frac{d}{db}\ln F_i(\alpha_i(b;k)) = \alpha_i'(b;k)\rho_i(\alpha_i(b;k))$. From $\rho_i < \rho_j$ over (r,d), we would then have $\alpha_j'(b;k) < \alpha_i'(b;k)$ and $\alpha_j(.;k)$ would be strictly larger than $\alpha_i(.;k)$ to the right of b, which would contradict (iii.1). No such b then exists and we have proved (iii.2).

Under SOA1, from (iii.1), all bidders 1 to $n-1$ must bid at least as aggressively as bidder n and only bidder n's bidding function may take the constant value r and the modification (iii.3) to the characterization in Theorem 3 follows.

Proof of (iv): Uniqueness when $r > c$ follows from Lebrun (1997, 1999) (specifically, Corollary 2 (i) in Lebrun 1999 or Corollary 2 in Lebrun 1997). Strict log-concavity of F_i and strict log-concavity of $F_i^{1/k}$ are equivalent. Uniqueness when $r = c$ then follows from Lebrun (2006). ||

Proof of Theorem 4:

Proof of (i): For all $i < n$ and $w \leq v$, from SOA1 and Corollary 1 (iii):

$$
\frac{\prod_{j>i} F_j\left(\varphi_{ji}(w;k)\right)}{\prod_{j>i} F_j\left(\varphi_{ji}(v;k)\right)} \leq \left(\frac{F_i(w)}{F_i(v)}\right)^{n-i}.
$$

Obviously, we also have:

$$\frac{\prod_{j<i} F_j\left(\varphi_{ji}(w;k)\right)}{\prod_{j<i} F_j\left(\varphi_{ji}(v;k)\right)} \leq 1.$$

Consequently, from Corollary 1 (ii) we find:

$$0 \leq \frac{v-\beta_i(v;k)}{k} \leq \frac{1}{k}\int_r^v \left(\frac{F_i(w)}{F_i(v)}\right)^{(n-i)/k} dw, \tag{A3.1}$$

from which it follows:

$$\begin{aligned} 0 &\leq \overline{\lim}_{k\to 0}\frac{v-\beta_i(v;k)}{k} \\ &\leq \lim_{k\to 0}\frac{1}{k}\int_r^v \left(\frac{F_i(w)}{F_i(v)}\right)^{(n-i)/k} dw \\ &= \frac{F_i(v)}{(n-i)\,f_i(v)}, \end{aligned}$$

where the equality is from Lemma A3 (i) below.

That the upper bound is uniform over I, that is,

$$\overline{\lim}_{k\to 0}\max_{v\in I}\left(\frac{v-\beta_i(v;k)}{k} - \frac{F_i(v)}{(n-i)\,f_i(v)}\right) \leq 0, \tag{A3.2}$$

for $r > c$ and $I = [r, d]$ is a consequence of (A3.1) and Lemma A3 (iv) and for $r = c$ and $I = [c+\gamma, d]$, where $\gamma > 0$, of (A3.1) and Lemma A3 (iii). From Lemma A3 (v), it is uniform over $[c, d]$ if $r = c$ and ε_i is bounded from below. We have proved the statements in (i) about $\beta_i(v;k)$.

Let ε and γ be arbitrary strictly positive numbers. From (A3.2), for all $I = [r, d]$ if $r > c$ or $I = [c+\gamma, d]$ with $\gamma > 0$ if $r = c$, there exists $k' > 0$

such that for all $0 < k < k'$ we have:

$$\frac{v - \beta_i(v;k)}{k} \leq \frac{F_i(v)}{(n-i) f_i(v)} + \varepsilon,$$

for all v in I. Because $b \leq \alpha_i(b;k)$, we have $\alpha_i(b;k) \in I$ and the inequality above applies to $v = \alpha_i(b;k)$, for all b in I. We find[49]:

$$\begin{aligned} 0 &\leq \frac{\alpha_i(b;k) - b}{k} \\ &\leq \frac{\alpha_i(b;k) - \beta_i(\alpha_i(b;k);k)}{k} \\ &\leq \frac{F_i(\alpha_i(b;k))}{(n-i) f_i(\alpha_i(b;k))} + \varepsilon. \end{aligned} \tag{A3.3}$$

From the continuity of $\frac{F_i}{(n-i)f_i}$, we then obtain that $\alpha_i(b;k)$ tends towards b uniformly over I when k tends towards zero and the statements in (i) about $\alpha_i(b;k)$ then follow from (A3.3). If ε_i is bounded from below, the same proof (but using the convergence of $F_i(v)/f_i(v)$ towards zero when v tends towards c) goes through with $I = [c,d]$ and establishes the remainder of (i).

Proof of (ii): The proof of similar to the proof of (i) above. For details, see Lebrun (2012).

Proof of (iii): We first prove the statement about β_i and α_i with $i < n$. If $r > c$, from (i) we have that $\beta_i(v;k)$ and $\alpha_i(v;k)$ are continuous in k uniformly for all v in $[r,d]$ and thereby jointly continuous. We may then assume $r = c$. From (i), $\beta_i(v;k)$ and $\alpha_i(v;k)$ are continuous in k locally uniformly with respect to v at $(v;0)$, for all v in $(c,d]$. The joint continuity at all such points when $r = c$ follows. As $c = \beta_i(c;k) \leq \beta_i(v;k) \leq v$, for all v and k, the joint continuity of $\beta_i(v;k)$ at $(c;0)$ if $r = c$ is obvious. Take any $w > c$. Then, for all v in $[c, \beta_i(w;k)]$ we have, from the monotonicity of α_i, $c = \alpha_i(c;0) \leq \alpha_i(v;k) \leq \alpha_i(\beta_i(w;k);k) = w$. As $\beta_i(w;k)$ tends

[49] $\beta_i(\alpha_i(b;k);k) = b$ only when $b \leq \eta(k)$. Otherwise, $\beta_i(\alpha_i(b;k);k) \leq b$.

towards c as $(w;k)$ tends towards $(c;0)$, $\alpha_i(v;k)$ is continuous with respect to both variables at $(c;0)$.

We next prove the statement about β_n and α_n. For all $v > r$, $\beta_n(v;k)$

must be optimal for bidder n with value v and in particular better than the bid v. After integration by parts, his expected payoff $(1-k)\int_c^v (v - \max(b,r))\, d\prod_{j<n} F_j(\alpha_j(b))$ if he submits v is:

$$(1-k)\int_r^v \prod_{j<n} F_j(\alpha_j(b;k))\, db$$

and if he submits $\beta_n(v;k)$:

$$\begin{aligned}
& k(v-\beta_n(v;k))\prod_{j<i} F_j(\alpha_j(\beta_n(v;k);k)) \\
& +(1-k)\int_c^{\beta_n(v;k)} (v-\max(b,r))\, d\prod_{j<n} F_j(\alpha_j(b;k)) \\
= \ & (v-\beta_n(v;k))\prod_{j<n} F_j(\alpha_j(\beta_n(v;k);k)) + (1-k)\int_r^{\beta_n(v;k)} \prod_{j<n} F_j(\alpha_j(b;k))\, db.
\end{aligned}$$

Consequently, we must have:

$$\begin{aligned}
& (v-\beta_n(v;k))\prod_{j<n} F_j(\alpha_j(\beta_n(v;k);k)) \\
= \ & \int_{\beta_n(v;k)}^{v} \prod_{j<n} F_j(\alpha_j(\beta_n(v;k);k))\, db \\
\geq \ & (1-k)\int_{\beta_n(v;k)}^{v} \prod_{j<n} F_j(\alpha_j(b;k))\, db.
\end{aligned}$$

and then:

$$
\begin{aligned}
& k\left(d-c\right) \\
\geq\ & k\int_{\beta_n(v;k)}^{v}\prod_{j<n}F_j\left(\alpha_j\left(\beta_i\left(v;k\right);k\right)\right)db \\
\geq\ & \int_{\beta_i(v;k)}^{v}\left(\prod_{j<n}F_j\left(\alpha_j\left(b;k\right)\right)-\prod_{j<n}F_j\left(\alpha_j\left(\beta_n\left(v;k\right);k\right)\right)\right)db \\
\geq\ & 0. \qquad \text{(A3.4)}
\end{aligned}
$$

Let $(v_l, k_l)_{l\geq 1}$ be a sequence tending towards $(v, 0)$ with v in $(r, d]$ and such that $k_l > 0$, for all l. Suppose $\beta_n(v_l; k_l)$ does not tend towards v. Extracting a subsequence if necessary, we may assume that it tends towards $b' \neq v$. As $\beta_n(v_l; k_l) \leq v_l$, we must have $b' < v$. From (A3.4) applied to (v_l, k_l), we have[50]:

$$
\begin{aligned}
& k_l\left(d-c\right) \\
\geq\ & \int_{\beta_n(v_l;k_l)}^{v_l}\left(\prod_{j<n}F_j\left(\alpha_j\left(b;k_l\right)\right)-\prod_{j<n}F_j\left(\alpha_j\left(\beta_n\left(v_l;k_l\right);k_l\right)\right)\right)db \\
\geq\ & 0.
\end{aligned}
$$

Making l tends towards $+\infty$ and using the local uniformity of the convergence of $\alpha_j(b; k)$ towards b, with $j < n$, we find:

$$\int_{b'}^{v}\left(\prod_{j\neq i}F_j\left(b\right)-\prod_{j\neq i}F_j\left(b'\right)\right)db=0,$$

which is impossible. We have proved that $\beta_n(v; k)$ is continuous at $(v; 0)$

[50] The integrand below is not smaller than $\left(F_1\left(\alpha_1\left(b;k_l\right)\right)-F_1\left(\alpha_1\left(\beta_n\left(v_l;k_l\right);k_l\right)\right)\right)\prod_{1<j<n}F_j\left(\alpha_j\left(\beta_n\left(v_l;k_l\right);k_l\right)\right)$, itself not smaller than $\left(F_1\left(\alpha_1\left(b;k_l\right)\right)-F_1\left(\alpha_1\left(\beta_n\left(v_l;k_l\right);k_l\right)\right)\right)\prod_{1<j<n}F_j\left(\beta_n\left(v_l;k_l\right)\right)\geq 0$. Continuing the proof with this last expression instead would follow more closely the informal argument in the main text. See Foonote 32.

jointly in $(v;k)$, for all $v > r$. Continuity at $(r;0)$ is proved as for β_i, $i < n$, that is: $r \leq \beta_n(v;k) \leq v$ implies that $\beta_n(v;k)$ tends towards r if $(v;k)$ tends towards $(r;0)$.

Proving the continuity of α_n from the result we have already established is straightforward. See Lebrun (2012) for details.

At this point, the joint continuity of φ_{ji} at all $(v;k)$ is immediate. ||

Lemma A3–Technical lemma: *Let G be a continuous function over $[c,d]$ that is continuous differentiable and strictly positive over $(c,d]$ with a strictly positive derivative g over this semi-open interval. Then:*

(i) For all v in $(c,d]$, we have:

$$\lim_{l \to +\infty} l \int_c^v \left(\frac{G(w)}{G(v)} \right)^l dw = \frac{G(v)}{g(v)}.$$

(ii) For all v in $(c,d]$, we have:

$$\lim_{l \to +\infty} \int_c^v \left(\frac{G(w)}{G(v)} \right)^l dw = 0,$$

and the convergence is uniform in v over any interval $[c+\gamma,d]$, with $\gamma > 0$.

(iii) If G is twice continuously differentiable over $(c,d]$, then the convergence in (i) is uniform in v over any interval $[c+\gamma,d]$, with $\gamma > 0$.

(iv) If G is twice continuously differentiable and strictly positive over $[c,d]$ and if $g(c) > 0$, then the convergence in (i) is uniform in v over $(c,d]$.

(v) If G is log-concave over an interval $[c,c+\eta]$, with $\eta > 0$, $G(c) = 0$, and the elasticity ε of g with respect to G is bounded from below, then the convergence in (i) is uniform in v over $(c,d]$.

Proof: See Lebrun (2012).

Appendix 4

Proof of Theorem 5:

Proof of (i):

(a) For all u in $(\ln F_1(r), 0)$, the first equality below follows from Lemma 3 (iii):

$$\begin{aligned} & l \\ = & \lim_{(s,k)\to(u,0)} \Phi'(s;k) \\ = & \lim_{(s,k)\to(u,0)} \frac{\alpha f\left(\varphi\left(F^{-1}(\exp s);k\right)\right)}{F\left(\varphi\left(F^{-1}(\exp s);k\right)\right)} \frac{\exp s}{f\left(F^{-1}(\exp s)\right)} \varphi'\left(F^{-1}(\exp s);k\right) \\ = & \; l \lim_{(s,k)\to(u,0)} \varphi'\left(F^{-1}(\exp s);k\right). \end{aligned}$$

The second equality follows from the definition of Φ and the third from the joint continuity, according to Theorem 4 (iii), of $\varphi(v;k)$. Consequently, $\lim_{(s,k)\to(u,0)} \varphi'\left(F^{-1}(\exp s);k\right)$, or equivalently $\lim_{(v,k)\to(w,0)} \varphi'(v;k)$ where $w = F^{-1}(\exp u) \in (r,d)$, exists and is equal to 1.

(b) Let v be in (c,d). From Corollary 1 (ii), we have $\beta_1(u;k) = u - \int_r^u \left(\frac{F_2(\varphi(w;k))}{F_2(\varphi(u;k))}\right)^{1/k} dw$, for all u in (r,d). Differentiating this equation, we find:

$$\begin{aligned} \beta_1'(u;k) &= \frac{1}{k}\int_r^u \left(\frac{F_2(\varphi(w;k))}{F_2(\varphi(u;k))}\right)^{1/k} dw \frac{f_2(\varphi(u;k))}{F_2(\varphi(u;k))}\varphi'(u;k) \\ &= \frac{1}{k}\int_r^e \left(\frac{F_2(\varphi(w;k))}{F_2(\varphi(u;k))}\right)^{1/k} dw \frac{f_2(\varphi(u;k))}{F_2(\varphi(u;k))}\varphi'(u;k) \\ &\quad + \int_e^u \frac{F_2(\varphi(w;k))}{\varphi'(w;k)\, f_2(\varphi(w;k))} d\left(\frac{F_2(\varphi(w;k))}{F_2(\varphi(u;k))}\right)^{1/k} \\ &\quad \frac{f_2(\varphi(u;k))}{F_2(\varphi(u;k))}\varphi'(u;k), \qquad \text{(A4.1)} \end{aligned}$$

where e is a fixed number strictly between r and v. The first term in the

RHS of the last equality above tends towards zero if $(u;k)$ tends towards $(v;0)$. In fact, we have:

$$0 \leq \frac{1}{k}\int_r^e \left(\frac{F_2(\varphi(w;k))}{F_2(\varphi(u;k))}\right)^{1/k} dw \frac{f_2(\varphi(u;k))}{F_2(\varphi(u;k))}\varphi'(u;k) \leq \frac{1}{k}\left(\frac{F_2(\varphi(e;k))}{F_2(\varphi(u;k))}\right)^{1/k} \frac{f_2(\varphi(u;k))}{F_2(\varphi(u;k))}\varphi'(u;k),$$

and, from Theorem 4 (iii) and what we have just proved in (a) above, $\varphi(u;k)$ and $\varphi'(u;k)$ are jointly continuous in $(u;k)$ at $(v;0)$.

We may consider $\left(\frac{F_2(\varphi(w;k))}{F_2(\varphi(u;k))}\right)^{1/k}$ in the second term as a probability distribution over the couples $(w;k')$ that is the product of two distributions: the distribution over $[e,u]$ whose cumulative function is $\left(\frac{F_2(\varphi(w;k))}{F_2(\varphi(u;k))}\right)^{1/k}$ and hence that has a mass point at e; and the degenerate distribution at k. As $(u;k)$ tends towards $(v;0)$, this product distribution tends weakly towards the distribution concentrated at $(v;0)$. From Theorem 4 (iii) and (a) above, the integrand is a continuous function of $(u;k)$. Consequently, we have:

$$\begin{aligned} & \lim_{(u;k)\to(v;0)} \int_e^u \frac{F_2(\varphi(w;k))}{\varphi'(w;k) f_2(\varphi(w;k))} d\left(\frac{F_2(\varphi(w;k))}{F_2(\varphi(u;k))}\right)^{1/k} \\ = & \frac{F_2(v)}{f_2(v)}. \end{aligned}$$

(A4.1) then implies $\lim_{(u;k)\to(v;0)} \beta_1'(u;k) = 1$, for all v in (r,d).

(c) Let b be in (c,d). For all k and all $\widetilde{b}$ such that $c < \widetilde{b} < \beta(d;k)$, we obviously have $\alpha_1'\left(\widetilde{b};k\right) = 1/\beta_1'\left(\alpha_1\left(\widetilde{b};k\right);k\right)$. (b) above then implies $\lim_{\left(\widetilde{b};k\right)\to(b;0)} \alpha_1'\left(\widetilde{b};k\right) = 1$. From the identity $\alpha_2\left(\widetilde{b};k\right) = \varphi\left(\alpha_1\left(\widetilde{b};k\right);k\right)$, we have $\alpha_2'\left(\widetilde{b};k\right) = \varphi'\left(\alpha_1\left(\widetilde{b};k\right);k\right)\alpha_1'\left(\widetilde{b};k\right)$ and consequently $\lim_{\left(\widetilde{b};k\right)\to(b;0)} \alpha_2'\left(\widetilde{b};k\right) = 1$. Finally, from $\beta_2'(u;k) = 1/\alpha_2'(\beta_2(u;k))$, for all u in (c,d) and k, we have $\lim_{(u;k)\to(v;0)} \beta_2'(u;k) = 1$, for all v in (c,d).

Proof of (ii): The proof of (ii) from Lemma 3 (iv) is similar to the proof in (i.a) above.

Proof of (iii): (iii) is an immediate consequence of Lemma 3 (v). ||

Proof of Theorem 6:

Proof of (i): Let v be in (r, d). From (13), we have $(\alpha_i(b;k) - b)/k = F_j(\alpha_j(b;k)) / \left(f_j(\alpha_j(b;k))\, \alpha_j'(b;k)\right)$, for all b close enough to v and all k small enough. Letting $(b;k)$ tend towards $(v;0)$ and using Theorem 5 (i) and Theorem 4 (iii), we find the statement in (i) about α_i.

We have $(u - \beta_i(u;k))/k = (\alpha_i(\beta_i(u;k);k) - \beta_i(u;k))/k$, for all u close enough to v and k small enough. The statement in (i) about β_i then follows from the statement about α_i and from Theorem 4 (iii).

From (15), we have:

$$\begin{aligned} & \frac{\varphi(u;k) - u}{k} \\ = \; & \frac{u - \beta_1(u;k)}{k} \left\{ \frac{d \ln F_2(\varphi(u;k))}{d \ln F_1(u)} - 1 \right\} && \text{(A4.2)} \\ = \; & \frac{u - \beta_1(u;k)}{k} \left\{ l \varphi'(u;k) \frac{f_1(\varphi(u;k))\, F_1(u)}{F_1(\varphi(u;k))\, f_1(u)} - 1 \right\}. && \text{(A4.3)} \end{aligned}$$

Letting $(u;k)$ tend towards $(v;0)$ and using the statement about β_1 and Theorem 5 (i) and Theorem 4 (iii), we obtain the statement in (i) about φ.

Proof of (ii): Let γ and ε be arbitrary strictly positive numbers. From (A4.3), (i) above, Theorem 4 (i), Theorem 5 (i, ii), and the compactness of $[r+\gamma, d]$, there exists $k' > 0$ such that for all $0 < k < k'$ and u in $[r+\gamma, d]$, we have:

$$\begin{aligned} & \frac{\varphi(u;k) - u}{k} \\ \leq \; & \left(\frac{F_2(u)}{f_2(u)} + \varepsilon \right) (l - 1 + \varepsilon) \\ \leq \; & \frac{F_2(u)}{f_2(u)} (l-1) + \varepsilon \max(M, l - 1 + \varepsilon), \end{aligned}$$

where M is the maximum of F_2/f_2. Consequently, $\overline{\lim}_{k \to 0} \sup_{u \in [r+\gamma, d]} \left(\frac{\varphi(u;k)-u}{k} - \frac{l-1}{l} \frac{F_1(u)}{f_1(u)} \right)$

$\leq \varepsilon \max(M, l-1+\varepsilon)$ and the first statement in (ii) follows by making ε tend towards zero.

The proof of the second statement proceeds from (A4.2) along similar lines and makes use of Theorem 4 (i) and Theorem 5 (iii).

Proof of (iii): From Corollary 1 (iii), bidder 1 bids more aggressively and, similarly to Lemma A2, the differences $\Delta R^D(k)$ and $\Delta TS^D(k)$ between the expected revenues and total surpluses from the kPA and the SPA are as follows:

$$\begin{aligned}
\Delta R^D(k) &= \int_c^d \int_{v_1}^{\varphi(v_1;k)} (\omega_1(v_1) - \omega_2(v_2))\, dF_2(v_2)\, dF_1(v_1) \\
\Delta TS^D(k) &= \int_c^d \int_{v_1}^{\varphi(v_1;k)} (v_1 - v_2)\, dF_2(v_2)\, dF_1(v_1).
\end{aligned}$$

The rate of change of the expected revenues can then be written as:

$$\frac{\Delta R^D(k)}{k} = \int_c^d \frac{1}{k} \int_{v_1}^{\varphi(v_1;k)} \Delta(v_1, v_2)\, dv_2 dv_1, \tag{A4.4}$$

where:

$$\begin{aligned}
&\Delta(v_1, v_2) \\
&= (\omega_1(v_1) - \omega_2(v_2))\, f_2(v_2)\, f_1(v_1) \\
&= (v_1 f_1(v_1) - (1 - F_1(v_1)))\, f_2(v_2) \\
&\quad - (v_2 f_2(v_2) - (1 - F_2(v_2)))\, f_1(v_1).
\end{aligned}$$

From FSA2, the densities f_1, f_2 are continuous over $[c,d]$ and hence so is the difference $\Delta(v_1, v_2)$ over $[c,d]^2$. Let K be the maximum of $|\Delta(v_1, v_2)|$ over this square. From (ii) above, there exists $k' > 0$ such that, for all $0 < k < k'$ and all u in (c,d), we have: $0 \leq (\varphi(u;k) - u)/k \leq 1 + (l-1) F(u)/f(u)$. As $F(u)/f(u)$ is continuous over $[c,d]$, it is bounded. Consequently, there exists L such that $0 \leq (\varphi(u;k) - u)/k \leq L$, for all u in $[c,d]$ and $0 < k < k'$.

For all $0 < k < k'$, we then have for all v_1 in $[c, d]$:

$$\begin{aligned} \left| \frac{1}{k} \int_{v_1}^{\varphi(v_1;k)} \Delta(v_1, v_2)\, dv_2 \right| &\leq K \left(\frac{\varphi(v_1;k) - v_1}{k} \right) \\ &\leq KL, \end{aligned}$$

and the integrand in the RHS of (A4.4) is bounded.

From the previous paragraph, if we can find the almost everywhere pointwise limit of the integrand of (A4.4) we will be entitled to apply the Lebesgue dominated convergence theorem. Let v_1 be in (c, d). The integrand of (A4.4) at v_1 can be rewritten as:

$$\begin{aligned} & \frac{1}{k} \int_{v_1}^{\varphi(v_1;k)} \Delta(v_1, v_2)\, dv_2 \\ = & \frac{\varphi(v_1;k) - v_1}{k} \Delta(v_1, v_1) \\ & + \frac{1}{k} \int_{v_1}^{\varphi(v_1;k)} (\Delta(v_1, v_2) - \Delta(v_1, v_1))\, dv_2. \end{aligned}$$

Let ε be an arbitrary strictly positive number. By continuity of $\Delta(v_1, v_2)$ over $[c, d]^2$, there exists $\xi > 0$ such that $|\Delta(v_1, v_2) - \Delta(v_1, v_1)| < \varepsilon$ for all v_2 such that $|v_1 - v_2| < \xi$. From Theorem 4 (iii), there exists k'', which we may assume smaller than k', such that, for all $0 < k < k''$, we have $\varphi(v_1; k) - v_1 < \xi$. We then obtain:

$$\begin{aligned} & \left| \frac{1}{k} \int_{v_1}^{\varphi(v_1;k)} (\Delta(v_1, v_2) - \Delta(v_1, v_1))\, dv_2 \right| \\ \leq & \varepsilon \left(\frac{\varphi(v_1;k) - v_1}{k} \right) \\ \leq & \varepsilon L, \end{aligned}$$

for all $0 < k < k''$. As ε was arbitrary, we have proved

$$\lim_{k \to +\infty} \frac{1}{k} \int_{v_1}^{\varphi(v_1;k)} \left(\Delta\left(v_1, v_2\right) - \Delta\left(v_1, v_1\right)\right) dv_2 = 0.$$

From (ii) above, we also have $\lim_{k \to +\infty} \frac{\varphi(v_1;k) - v_1}{k} = \frac{l-1}{l} \frac{F_1(v_1)}{f_1(v_1)}$ and we have proved:

$$\begin{aligned} & \lim_{k \to +\infty} \frac{1}{k} \int_{v_1}^{\varphi(v_1;k)} \Delta\left(v_1, v_2\right) dv_2 \\ = \; & \frac{l-1}{l} \frac{F_1\left(v_1\right)}{f_1\left(v_1\right)} \Delta\left(v_1, v_1\right) \\ = \; & \frac{l-1}{l} \left(\omega_1\left(v_1\right) - \omega_2\left(v_1\right)\right) F_1\left(v_1\right) f_2\left(v_1\right) \\ = \; & (l-1) \left(\omega_1\left(v_1\right) - \omega_2\left(v_1\right)\right) F_1\left(v_1\right)^l f_1\left(v_1\right) \end{aligned}$$

From the Lebesgue theorem of dominated convergence, $\frac{d_r}{dk} R^D(0) = \lim_{k \to >0} \frac{\Delta R^D(k)}{k}$ exists and (17) holds true. That $\frac{d_r}{dk} TS^D(0)$ exists and satisfies (18) can be proved similarly. If F_1, F_2 are different, we have $l > 1$ and $\omega_2 < \omega_1$ over (c, d). The strict positivity of $\frac{d_r}{dk} R^D(0)$ follows from (17).

Proof of (iv): The proof is similar to the proof of (iii) and can be found in the online discussion paper Lebrun (2012). ||

Proof of Corollary 2: From (5) in Theorem 2, we have:

$$\begin{aligned} & \frac{d}{dk} R(0) \\ = \; & \pi \int_r^d \frac{\left(\omega_1(v) - \omega_2(v)\right)\left(\rho_2(v) - \rho_1(v)\right)}{\rho_1(v)\, \rho_2(v)} \rho_1(v)\, \rho_2(v)\, F_1(v)\, F_2(v)\, dv \\ = \; & \pi \int_r^d \left(\omega_1(v) - \omega_2(v)\right)(l-1) \frac{f_1(v)}{F_1(v)} F_1(v)\, F_2(v)\, dv \\ = \; & \pi (l-1) \int_r^d \left(\omega_1(v) - \omega_2(v)\right) F_2(v)\, dF_1(v). \end{aligned}$$

The result then follows from Theorem 6 (iii, iv). ||

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