A theory of multi-dimensional auctions with non-quasi-linear scoring rules*

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Abstract

We establish a model of scoring auctions which generally accepts quasi-linear (QL) and price-quality ratio (PQR) scoring rules. Verifying the existence and uniqueness of the equilibrium in scoring auctions with PQR rules, we demonstrate that equivalence fails in both expected exercised scores and bidders' expected payoffs: The procurement buyer obtains higher utility in SS while bidders earn higher expected profits in FS auctions if scoring rules are on the basis of PQR.

Key words: scoring auctions, existence of a Bayesian Nash equilibrium, price-to-quality ratio
JEL classification: D44,

1 Introduction

Public sector spending amounts to 15 percent of GDP in OECD member countries on average (OECD (2007)). The call for efficient and effective use of public funds cannot be greater than ever. The public in those countries are not just concerned about the amount of public spending; they are also becoming more and more aware of the value for money spent in public projects. The challenge of the governments is therefore how to pay less for more value. Put differently, value for money is now a key criterion that a sensible government should consider.

In procurement, low-price auctions have been widespreadly used as a competitive, transparent, and accountable allocation mechanism. However, by the growing pressure to seek

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value for money, more and more procurement buyers introduce mechanisms with which relevant prices and qualities of proposals in the whole procurement cycle are assessed. *Scoring auctions*, or equivalently multi-parameter bidding, is actually the most prevailing mechanism that meets their need.

In a scoring auction, bidders are asked to submit a set of multiple-dimensional bids that include price, completion period, and quality. The seminal research by Che (1993) found that the multi-dimensional auction can be analyzed with the standard methodology on price-only auctions. Branco (1997) extended the results to the case in which bidder's signals are correlated. More recently, a thorough analysis by Asker and Cantillon (2008) showed that those properties are maintained even if the player's type is multi-dimensional.[1] These analyses are, however, restricted to quasi-linear scoring rules, in which the score is linear in the price bid and non-monetary attributes (the quality bid) are additively separable from the price bid.

In fact, quasi-linear scoring rules, or equivalently the so-called A+B bidding, is no more than just commonly used forms of scoring auctions, and scoring rules based on price-quality ratio are extensively used in practice. In U.S. highway construction, for instance, many state departments of transportation (DOTs) including ones in Alaska, Michigan, North Carolina, and South Dakota adopt "adjusted bid," under which the price bid is adjusted by being divided by the quality bid (Molenaar and Yakowenko (2007)). In Japan, most of the public procurement contracts are allocated to the bidders with the highest price-to-quality bid ratio. Despite such common use of price-quality ratio for scoring rules, little is known on the properties of this class of scoring auctions.[2] The methodology developed for auctions under quasi-linear scoring rules does not apply to those under price-quality ratio.[3]

In this paper, we establish a model of scoring auctions which accepts both quasi-linear (QL) and price-quality ratio (PQR) scoring rules. As in the existing literature, ex ante symmetric, risk neutral bidders have a convex cost function which is parameterized via a single-dimensional private signal.[4] Two auction formats are considered: first-score (FS) and second-score (SS) auctions. In a FS auction, the bidder with the lowest score wins and follows the contract as specified in his winning bid. In a SS auction, the bidder with the lowest score wins and is free to choose price and quality of the finalized contract as long

[1] Bajari et al. (2006) empirically examines the procurement auction with unit price bids. In this auction, bidders submit multiple prices and the winner is the bidder whose weighted sum of all multiple bids is lowest. Their theoretical model is on the basis of Che (1993) and Asker and Cantillon (2008).

[2] A theoretical research in economics on the scoring auction with PQR rules is conducted by Hanazono (2010). He established a model in which n risk-neutral bidders receive single-dimensional private signals independently from a common uniform distribution. Having derived an equilibrium bidding strategy, he verified that there exists a symmetric Bayesian Nash equilibrium in a FS auction with a PQR scoring rule. For empirical research, Nakabayashi (2009) constructed a model of scoring auctions with PQR rule in which he assumes that the bidder's cost function is an inverse L shape so that bidders' optimal quality choice is uniquely determined by the bidder's signal even under PQR scoring rule. Because of the restriction on either a specific distribution or shape of the cost function, both analyses are silent on general properties of scoring auctions examined in this paper.

[3] Taking the logarithm of price-quality ratio does not a solution, since it results in price being non-linear in the score. A necessary condition for quasi-linearity is violated.

[4] We relax this assumption to discuss the multi-dimensional signal environment.

as the score value based on the price and quality matches the minimal rival score.

We extend the approach by Che and Asker-Cantillon. For each bidder, the *pseudo value*[5] is computed by substituting the *optimal* quality level and the associated cost into the score function as a score is computed by substituting a quality level and a price bid into the score function. Here, the *optimal* quality level is the one at which the difference between the bidder's price bid and cost is maximized. The pseudo value thus summaries the bidder's signal, the optimal choice in quality, and the scoring function. In addition, without any loss, the bidder is considered to submit a scoring bid and quality instead of submitting a price bid and quality. Thus, the equilibrium analysis becomes simplified by focusing on a reduced-form auction game in which the bidder strategically chooses the optimal scoring bid taking into account his expected payoff that is a function of the pseudo value and the scoring bid.

These authors found that, if the scoring rule is QL, the pseudo value is independent of the scoring bid. More specifically, if the scoring rule is QL, the pseudo value is equivalent with the *best willing score*, *i.e.,* the lowest possible score the bidder can make with a non-negative payoff.[6] For any bidder with a convex cost function, the best willing score is unique. Hence, the standard auction theory applies to scoring auctions in the way that the pseudo value is regarded as the bidder's effective *type*. In particular, we observe that bidders truthfully report their pseudo values in SS auctions and that the bidder's optimal scoring bid equals the conditional expectation of the second lowest pseudo value in FS auctions is straightforward. Under PQR rules, however, the pseudo value hinges on the scoring bid. That is, the distribution of the competitor's pseudo value is endogenous given the equilibrium bidding strategy. The existing theory has no clear guidance to analyze such auctions that the player's valuation is strategy-dependent.

We find that, despite the endogeneity in the bidder's valuation, the relationship between the pseudo value and the optimal scoring bid observed under QL scoring rules is preserved under PQR scoring rules. This is quite easily seen in SS auctions, where bidders truthfully report their best willing scores as a weakly dominant strategy (Theorem 1). Just as in second-price auctions, the bidder's price-cost margin based on his bid has nothing to do with the payoff upon winning in SS auctions. Therefore, there is no reason for bidders to place a strictly higher scoring bid than the best willing score. Consequently, the optimal scoring bid is equal to the pseudo value in SS auctions even under PQR rules.

The procedure of characterizing a Bayesian Nash equilibrium in FS auctions with PQR rules requires two additional steps on the procedure used under QL rules: i) application of the envelop theorem and ii) a proof of the existence of an equilibrium. Just as in the analysis of first-price auctions, we first assume that the optimal scoring bid is strictly increasing

[5] In the independent private value paradigm, the cost given a quality level is the bidder's *valuation*. Since the value in question is either divided by quality under PQR rules or subtracted by quality under QL rules, we call it as a *pseudo* value. The pseudo value is a generalization of *productive potential* in Che (1993) and *pseudo type* in Asker and Cantillon (2008).

[6] The best willing score is determined by optimizing the score w.r.t. quality subject to price being equal to the cost. This is an analog of the willingness to pay in high-price auctions, which gives the break-even price.

in the pseudo value. Although the pseudo value is *per se* endogenous, the discrepancy between the scoring bid and the pseudo value is linear in the scoring bid if the quality is optimally chosen. That is, by applying the envelop theorem, we can treat the pseudo value as being independent of the scoring bid. As a consequence, the first-order condition with respect to the optimal scoring bid is a tractable differential equation, and the solution is easily obtained (Theorem 2). The obtained solution suggests that the optimal score value equals the conditional expectation of the second-lowest pseudo value (Lemma 1). However, the solution includes the distribution of the pseudo value, which is not explicitly described. We thus apply the contraction mapping theorem to verify that the distribution in equilibrium is consistent with the optimal choice of the scoring bid as the conditional second-lowest statistic of the equilibrium pseudo value. The solution of the differential equation is indeed (and only) the equilibrium bid function (Theorem 3).

The characterization of equilibrium bidding strategy clarifies the following three remarkable features of scoring auctions with PQR scoring rules. FS auctions lead to, i) a higher expected exercised score,[7] ii) a greater quality level, iii) a lower expected payoff for bidders than SS auctions, where the exercised score is the lowest scoring bid in FS and the second-lowest scoring bid in SS auctions, respectively. The intuition for i) comes from the difference in equilibrium score bids and the fact that the expected exercised score in a FS is in fact the expectation of the second-order statistic of the pseudo value. The equilibrium score bid in a SS is the best willing score, which is by definition better than that in a FS auction. ii) is due to the fact that the profit maximizing quality is an increasing and concave function of the exercised score.[8] Since the second lowest pseudo value has a higher expected value and a lower variance than its conditional expectation, the offered quality level is likely to be higher in FS auctions. Since both the expected exercised score and quality are greater, bidders earn higher payoffs in FS auctions (result iii).

The remaining part of this paper is organized as follows. Section 2 describes the model on scoring auctions and analyzes equilibrium properties in SS auctions. Section 3 analyzes a symmetric Bayesian Nash equilibria in FS auctions and compares expected winning scores and bidders' expected payoffs in FS and SS auctions. Section 4 delivers an extension of the analysis to a multi-dimensional signal environment. The final section is the conclusion. Proofs are given in the Appendix.

2 The model

Consider a procurement buyer who auctions off a project contract to n risk neutral bidders. All bidders are *ex ante* symmetric. At the bid preparation stage, each bidder obtains a signal $\theta \in [\underline{\theta}, \bar{\theta}]$ distributed following the publicly known cumulative distribution $F(\theta)$. The bidder's cost schedule is represented by an increasing and strictly convex function $C(q|\theta) > 0$, where $q \in \mathbb{R}_+$ denotes the non-monetary attribute (quality) with which the

[7]The exercised score is the lowest score in FS and the second-lowest score in SS auctions.

[8]It is because of the convexity of the cost function and the first-order condition of the profit maximizing quality satisfying, $s - C_q(q) = 0$.

bidder performs the contract.[9]

In the scoring auction, the bidder submits a price-bid $p \in \mathbb{R}_+$ and quality q. The publicly known scoring rule $S(p,q) : \mathbb{R}^2_+ \to \mathbb{R}_+$ maps these multi-dimensional bids into the score, $s \in \mathbb{R}_+$. The following two classes of scoring rules are analyzed in this research:

1. $S(p,q) = p/q$: Price-quality-ratio (PQR)
2. $S(p,q) = p - q$: Quasi-linear (QL).

As usual, a scoring rule is invariant to positive monotonic transforms. In addition, the ascending order of scores is equivalent to the descending order of the inverse scores. Furthermore, q is invariant to positive monotonic transforms. Hence, many practically-used scoring rules are included in these classes.[10]

The lowest-scored bidder wins the contract in both first-score (FS) and second-score (SS) auctions. The winner receives the payment p and performs the contract providing the quality level q in a FS auction. In a SS auction, the winning bidder is free to choose p and q finalized in the contract as long as $S(p,q)$ is equal to the minimum rival scoring bid. In both auctions, the bidder's p, q, and the resulting score value s satisfy $s = S(p,q)$. In other words, the winning bidder's payment p is a function of q and s. Therefore, it is no loss of generality to consider that each bidder chooses a scoring bid s and a quality bid q. Further, let s^e denote the exercised score, which is the winning bidder's scoring bid $s_{(1)}$ in FS auctions and is the second-lowest scoring bid $s_{(2)}$ in SS auctions. The bidder's payoff upon winning is given by

$$q^{\rho} \cdot [s^e - S(C(q|\theta), q)],$$

where ρ is an indicator such that $\rho = 0$ under a QL rule while $\rho = 1$ under a PQR rule.

Throughout, the following regularity conditions are imposed on the cost function.

Assumption 1. *1. For any θ, $S(C(q|\theta), q)$ is strictly convex and U-shaped in q.*

2. $\Theta = [\theta^-, \theta^+] \subset \mathbb{R}$,

3. For all θ, $C(q|\theta)/q$ is strictly convex, and smooth,

4. $\min_q C(q,\theta)/q$ is increasing in θ. $\max_\theta \min_q C(q,\theta)/q$ is well-defined. [11]

Assumption 1 ensures that, for each type θ, there exists a unique first-best quality level q^{FB} such that

$$q^{FB}(\theta) \equiv \arg\min_q S(C(q|\theta), q). \tag{1}$$

[9] A single dimensional quality is easily extended to multi-dimensional quality. See Appendix A for more details.

[10] The scoring rule does not have to be linear in price. Consider, for instance, the non-linear score rule $S(p,q) = \log(p) - q$. A monotonic transformation based on the exponential function yields $\exp(S(p,q)) = \exp(\log(p) - \log(\exp(q))) = p/\exp(q)$. Redefining $\exp(q)$ as q' suggests that the scoring rule is PQR.

[11] This assumption excludes the cost functions with "Inada conditions," for instance $C = \theta q^2$, for which $k^-(\theta) = 0$ for all θ. We still have abundant cost functions to consider such as $C = \theta q^2 + F$, $F > 0$.

Based on the first-best quality level, the best willing score s^{BW} is defined as

$$s^{BW}(\theta) = S(C(q^{FB}(\theta)|\theta), q). \tag{2}$$

In words, the best willing score is the lowest possible score the bidder with type θ can make with a non-negative profit. Assumption 1 ensures that such a break even score is unique for any θ.

In scoring auctions, the bidder's optimal quality choice is, in fact, endogenous given a score value. For notational convenience, define $k(q, \theta) = S(C(q|\theta), q)$. Then, for any score value $\{s, s^e\} \in S$, *i.e.,* a submitted or exercised score, the *optimal* quality choice is given as

$$q(S, \theta) = \arg\max_q q^{\rho} \cdot [S - k(q, \theta)]. \tag{3}$$

Two points are worth mentioning regarding $q(S, \theta)$. First, the optimal quality choice is the analogy of the profit maximizing quantity supplied by a producer. In scoring auctions, the price of quality is S under PQR rules while it is exogenously given by the score function $S(p, q)$ under QL rules, which is equal to one.[12] The optimal quality is the one at which the marginal cost of providing an additional quality unit equals the price of quality.

Second, under QL scoring rules, the optimal quality choice satisfies $q(S, \theta) = q^{FB}(\theta)$, being independent of S.[13] In other words, bidders are price takers under QL rules. Furthermore, choosing the profit maximizing quality and exploring the quality that achieves the break even score fulfill the duality relationship under QL rules.[14] Therefore, the optimal quality choice results in the best willing score in the scoring auction with QL scoring rules.

In summary, (3) suggests that S is a sufficient statistic to examine the bidder's problem in scoring auctions. Henceforth, we basically consider the reduced form auction game in which bidders endogenizing the quality selection choose their optimal scoring bids.

To explore the optimal scoring bid s, it is convenient to define the bidder's pseudo value denoted by $k(q(s, \theta), \theta)$ by substituting the optimal quality bid $q(s, \theta)$ given his scoring bid s and the associated cost $C(q(s, \theta)|\theta)$ into the scoring function.

Definition 1. *The bidder's pseudo value*

$$k(q(s, \theta), \theta) = S(C(q(s, \theta)|\theta), q(s, \theta)). \tag{4}$$

The pseudo value is a generalization of *productive potential* in Che (1993) or *pseudo-type* in Asker and Cantillon (2008) for the analysis to generally accept both QL and PQR scoring rules.

[12]In SS auctions, the quality price is proposed by each bidder as a scoring bid s and the market price is equalized to be the second-lowest score.

[13]With (3), the optimal quality choice is given by

$$q(S, \theta) = \arg\max_q S - C(q, \theta) + q,$$

which is independent of S.

[14]This is because $\arg\max_q S - C(q, \theta) + q = \arg\min_q C(q, \theta) - q$ holds under QL rules.

Literature has discovered that the pseudo value serves as the bidder's effective type under QL rules; bidders truthfully report their pseudo value in SS auctions, and a Bayesian Nash equilibrium exists in FS auctions in which bidders submit the conditional expectation of the second lowest pseudo value. These results are not only elegant but also quite reasonable if one realizes that, under QL scoring rules, the pseudo value coincides with the best willing score and that the pseudo value is unique for any signal θ.

Figure 1 illustrates the pseudo value under QL scoring rules. The price-cost margin is measured by the vertical distance between lines S and K. For any s, the bidder's optimal quality choice equals the first-best level.

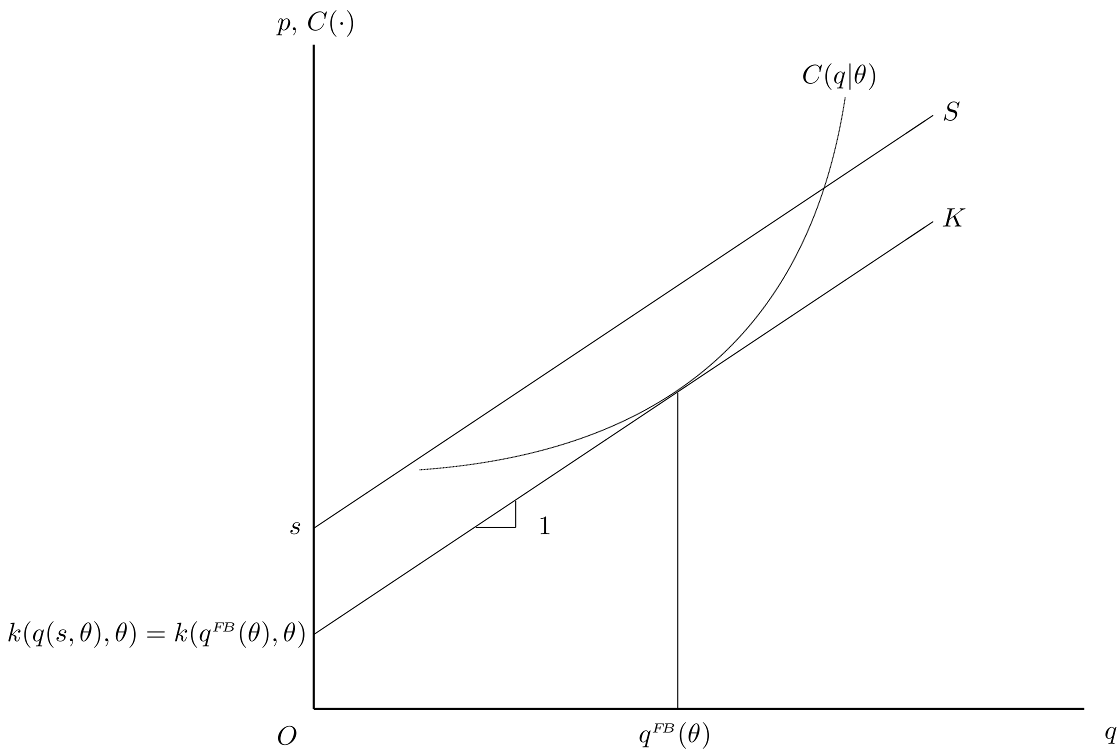

Figure 1: The pseudo value under QL scoring rules

When the scoring rule is PQR, however, the pseudo value hinges on the scoring bid s.[15] In other words, the pseudo value is endogenous in the equilibrium strategy.

The pseudo value under a PQR scoring rule is illustrated in Figure 2. Given a scoring bid s, the resulting pseudo value is given by the slope of line K. Again, the vertical distance between Line S and K represents the price-cost margin. The payoff is maximized

[15]Under the PQR scoring rules, the optimal quality bid depends on s as

$$q(s,\theta) = \{q|C_q(q|\theta) = s\}.$$

at $q(s^{FS}(\theta),\theta)$ where the marginal cost $C_q(q|\theta)$ equals s.

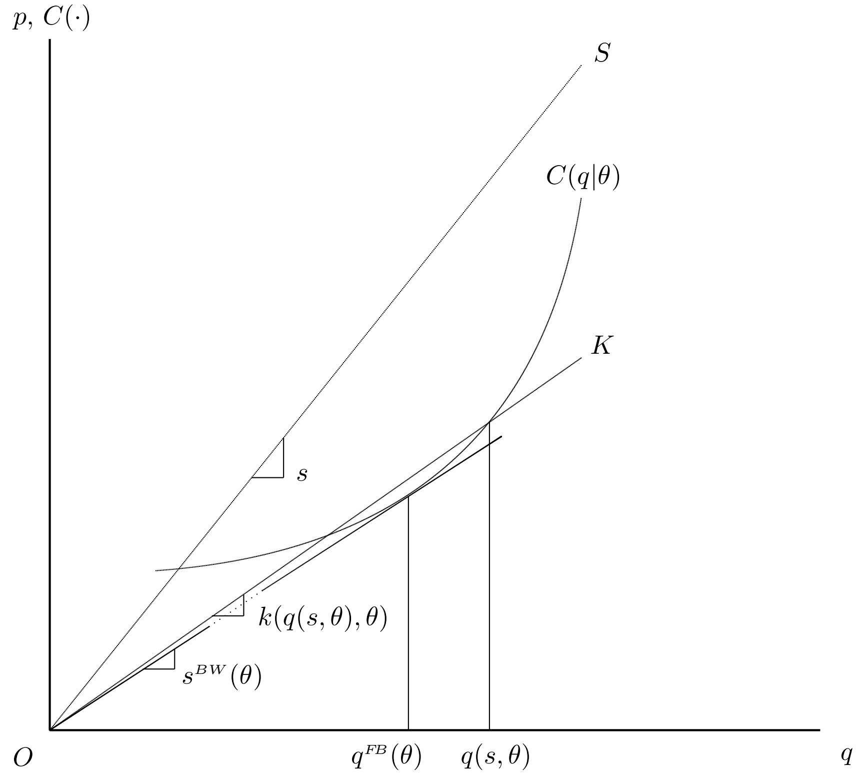

Figure 2: The pseudo value under PQR scoring rules

Despite the endogeneity of the pseudo value, the equilibrium analysis for SS auctions is, in fact, quite simple; bidders truthfully report their best willing score as a dominant strategy. In general, the bidder's payoff upon winning is given by

$$(q(s^e,\theta))^\rho \cdot [s^e - k(q(s^e,\theta),\theta)].$$

Since $s^e = s_{(2)}$ in SS auction, the bidder's payoff upon winning is independent of his own scoring bid. Furthermore, the bidder submitting the lowest scoring bid wins. Therefore, as in second-price auctions, where bidders submit their willingness to pay, bidding the break even score is a dominant strategy in SS auctions. The following theorem summarizes this point.

Theorem 1. *In a SS auction, there exists a dominant strategy equilibrium in which bidders report their best willing scores as their optimal scoring bids, i.e.,*

$$s^{SS}(\theta) = s^{BW}(\theta).$$

As a consequence, the pseudo value is equal to the equilibrium scoring bid in SS auctions under PQR rules.[16] Thus, the property that the equilibrium scoring bid and the bidder's pseudo value are identical in SS auctions is preserved under PQR scoring rules.

In SS auctions, the winning bidder's payoff is a function of the exercised score, which is given for the winning bidder. Thus, the quality price is still exogenous for the winning bidder as under QL scoring rules. In this sense, the winning bidder is a price taker, determining the profit maximizing quality in the finalized contract given the exercised score. In FS auctions, where the winning bidder's scoring bid becomes the exercised score, each bidder has a chance to determine the quality price under PQR scoring rules. Therefore, bidders are price makers. In exchange of reducing the winning probability, they can charge a higher price of quality than the break even one. Such market power results in the pseudo value being greater than the best willing score in FS auctions under PQR rules. In the next section, a symmetric Bayesian Nash equilibrium of FS auctions is examined. We demonstrate that, even under PQR scoring rules, the optimal scoring bid is equal to the conditional expectation of the second lowest bidder's pseudo value in equilibrium.

3 A Bayesian Nash equilibrium of FS auctions

Suppose that there exists a Bayesian Nash equilibrium $s^{FS}(\theta) : [\underline{\theta}, \bar{\theta}] \to \mathbb{R}_+$. Then, let $\bar{k}$ and $\underline{k}$ be the upper and lower bound of $k(q(s^{FS}(\theta), \theta), \theta)$ such that $\underline{k} \equiv \inf k(q(s^{FS}(\theta), \theta), \theta)$ and $\bar{k} \equiv \sup k(q(s^{FS}(\theta), \theta), \theta)$.[17] Given the equilibrium, $k(q(s^{FS}(\theta), \theta), \theta)$ is a random variable measured with the following cumulative distribution function:

$$F_k(k|s^{FS}(\cdot)) = \int_{\{\theta | k(q(s^{FS}(\theta),\theta),\theta) \leq k\}} f(\hat{\theta}) d\hat{\theta}.$$

Suppose also that there exists a strictly increasing and differentiable function $\sigma(\cdot) : \mathbb{R} \to \mathbb{R} : k(q, \theta) \to s$ such that $s^{FS}(\theta) \equiv \sigma(k(q(s^{FS}(\theta), \theta), \theta))$ for any $k(q(s^{FS}(\theta), \theta), \theta) \in [\underline{k}, \bar{k}]$, $\sigma(k) = \sigma(\underline{k})$ for any $k < \underline{k}$, and $\sigma(k) = k$ for any $k > \bar{k}$.[18] Then, the bidder's expected payoff in a FS auction is given by:

$$\pi^{FS}(s|\theta) = (q(s, \theta))^{\rho} \cdot [s - k(q(s, \theta), \theta)][1 - F_k(\sigma^{-1}(s))]^{n-1}.$$

For $s^{FS}(\theta)$ to be a Bayesian Nash equilibrium, it necessarily satisfies $s^{FS}(\theta) = \arg\max_s \pi^{FS}(s|\theta)$.

By the definition of the optimal quality choice given by (3), the derivative of the bidder's price-cost margin $q^{\rho} \cdot [s - k(q, \theta)]$ with respect to q equals zero at $q = q(s, \theta)$. Thus, although the pseudo value $k(q(s, \theta), \theta)$ remains endogenous given s, the first-order condition of this maximization problem is obtained by the envelop theorem. A symmetric Bayesian Nash equilibrium in FS auctions is thus characterized as follows.

[16]Under PQR scoring rules, the equilibrium scoring bid $s^{SS}(\theta)$ satisfies $s^{SS}(\theta) = k(q(s^{SS}(\theta), \theta), \theta) = k(q^{FB}(\theta), \theta)$. In other words, given the break even score, the bidder's optimal quality is the first-best level.

[17]Under a QL scoring rule, $q(s^{FS}(\theta), \theta) = q^{FB}(\theta)$.

[18]We will verify that this assumption is legitimate later on.

Theorem 2. *Suppose that there exists a symmetric Bayesian Nash equilibrium $s^{FS}(\theta)$ in FS auctions with a PQR scoring rule. Then, it is given by*

$$s^{FS}(\theta) = \sigma(k(q(s^{FS}(\theta),\theta),\theta)), \tag{5}$$
$$\textit{subject to } \sigma(k) = k + \frac{\int_k^{\bar{k}} \left[1 - F_k(\hat{k})\right]^{n-1} d\hat{k}}{[1 - F_k(k)]^{n-1}}.$$

Proof. The envelope theorem suggests that $d\pi^{FS}(s|\cdot)/ds = \partial\pi^{FS}(s|\cdot)/\partial s$. Suppose that $F_k(k|s^{FS}(\cdot))$ is differentiable for all k. Let $f_k(k|s^{FS}(\cdot))$ be the derivative. Then, the first-order condition gives

$$\begin{aligned}&[1 - F_k(k(q(s,\theta),\theta))]^{n-1}\,\sigma'(k(q(s,\theta),\theta)) \\ &\quad - [s - k(q(s,\theta),\theta)]\,(n-1)\,[1 - F_k(k(q(s,\theta),\theta))]^{n-2}\,f_k(k(q(s,\theta),\theta)) = 0.\end{aligned}$$

Replacing $k(q(s(\theta),\theta),\theta)$ with k and s with $\sigma(k)$ and taking the integral from k to $\bar{k}$ yield $\sigma(k)$ in (5). Since the second term of the constraint in (5) is non-negative and strictly decreasing in k, σ is indeed increasing. The sufficiency is guaranteed by a standard argument, just as in the case of a first-price sealed-bid auction, which is, for instance, shown in Krishna (2009). □

Note that the strategy (5) coincides with Equation (1) in Che (1993) if the scoring rule is QL.[19] Since σ is explicitly obtained, the existence of the equilibrium is guaranteed. If the scoring rule is PQR, however, σ includes F_k, which is not an explicit form, due to the endogeneity of the pseudo value. To guarantee the existence and uniqueness of the equilibrium, we need to show that $s^{FS}(\theta)$ indeed exists uniquely. We apply the Contraction Mapping Theorem and verify that the distribution of the pseudo value in equilibrium is consistent with the optimal scoring bid described in (5).

Theorem 3. *The strategy $s^{FS}(\theta)$ characterized in (5) is indeed and only a symmetric Bayesian Nash equilibrium.*

Proof. See Appendix B. □

A non-trivial observation regarding the function σ is that the buyer's information cost is measured by the order statistics of the pseudo value $k(q(s,\theta),\theta)$. In particular, the expected exercised score is equal to the conditional expectation of the the second lowest pseudo value. The following lemma summarizes this point. For notational convenience, we introduce $k(s^{FS}(\theta),\theta) \equiv k(q(s^{FS}(\theta),\theta),\theta)$ to denote the bidder's pseudo value.

[19] (5) becomes

$$p^{FS}(\theta) = c(q(s^{FS}(\theta),\theta),\theta) + \frac{\int_\theta^{\bar{\theta}} \left[1 - F_\theta(\hat{\theta})\right]^{n-1} \frac{dk(q(s^{FS}(\hat{\theta}),\hat{\theta}),\hat{\theta})}{d\hat{\theta}} d\hat{\theta}}{[1 - F_\theta(\theta)]^{n-1}},$$

where changes of variables are applied to the second term in the right-hand side. By the envelop theorem, $dk(q(s(\theta),\theta),\theta)/d\theta = \partial k(q(s(\theta),\theta),\theta)/\partial\theta$ for any s. Furthermore, $\partial k(q,\theta)/\partial\theta = \partial c(q|\theta)/\partial\theta$ under QL scoring rules. Thus, the desired result is obtained.

Lemma 1. *Let $\theta_{(j)}$, $j = 1, \ldots, n$, be the type of the bidder whose best willing score is jth lowest. Then, in FS auctions, σ satisfies*

$$\sigma(k(s^{FS}(\theta_{(1)}), \theta_{(1)})) = E[k(s^{FS}(\theta_{(2)}), \theta_{(2)})|\theta_{(1)}]. \tag{6}$$

Proof. Applying integration by parts to (5) yields

$$\sigma(k(s^{FS}(\theta), \theta)) = \int_{k(s^{FS}(\theta),\theta)}^{\bar{k}} k f_{k_{(2)|(1)}}(k)dk,$$

where $f_{k_{(2)|(1)}}(k) = (n-1)f_k(k)\,[1 - F_k(k)]^{n-2}/[1 - F_k(k(s^{FS}(\theta), \theta))]^{n-1}$. Thus,

$$\int_{k(s^{FS}(\theta_{(1)}),\theta_{(1)})}^{\bar{k}} k f_{k_{(2)|(1)}}(k)dk = E[k(s^{FS}(\theta_{(2)}), \theta_{(2)})|\theta_{(1)}]$$

□

Consequently, the relationship between the pseudo value and the optimal scoring bid that we have seen under QL rules is well-maintained under PQR rules. However, this does not imply that the expected exercised score is equivalent between FS and SS auctions under PQR rules. To see this, one should first notice that the winning bidder provides an excessive level of quality under PQR rules regardless of the auction format. The following lemma summarizes this point.

Lemma 2. *If the scoring rule is based on PQR, the optimal quality level given by the exercised score is strictly greater than the first-best level $q^{FB}(\theta)$ and, as the number of bidders rises to infinity, converges to $q^{FB}(\theta)$.*

Proof. From (3), the quality level under PQR scoring rules satisfies

$$s^e - k(q(s^e, \theta), \theta) = q(s^e, \theta)k_q(q(s^e, \theta), \theta), \tag{7}$$

where $s^e = s_{(1)}$ in FS auctions and $s^e = s_{(2)}$ in SS auctions. In both cases, the left-hand side in (7) is positive while $k_q(q^{FB}(\theta), \theta) = 0$. Since, by Assumption 1, $k_{qq}(\cdot) > 0$, it must be that $q(s^e, \theta) > q^{FB}(\theta)$. Finally, we show that $q(s(\theta), \theta)$ converges to $q^{FB}(\theta)$. In FS auctions, Equation (5) suggests that $s^{FS}(\theta) - k(s^{FS}(\theta), \theta)$ monotonically decreases to the limit zero as n rises. In SS auctions, $E(k(q^{FB}(\theta_{(2)}), \theta_{(2)})) - E(k(q^{FB}(\theta_{(1)}), \theta_{(1)}))$ monotonically decreases to the limit zero as n rises. Hence, $k_q(\cdot) > 0$ monotonically decreases to the limit zero as n rises. Together with the fact that $k_{qq} > 0$, $q(s^e, \theta) - q^{FB}(\theta)$ monotonically decreases to the limit zero as n rises. □

Excessive quality provision in the final contract is inherent in the PQR scoring rule, under which the first-best quality is equivalent to the efficient scale. Because of the winning bidder's informational rents, the price of quality is above the minimum average cost under PQR rule. Specifically, price making bidders exercise a market power in FS auctions, charing a strictly higher price of quality (the scoring bid) than the best willing score (the minimum average cost). Although bidders are price takers in SS auctions, the price of quality (the second-lowest scoring bid) is greater than the winning bidder's minimum average cost. Thus, the quality level chosen by the winning bidder is above the efficient scale. Under QL scoring rules, the quality price is exogenously given and constant. Any quality at which the price equals marginal cost is, by definition, the first-best. Thus, private and social incentives coincide toward quality provision if the scoring rule is QL.

Even though the excessive quality provision is observed in both FS and SS auctions with PQR rules, losing bidders always choose the first-best quality in SS auctions, whereas they bid excessive quality in FS auctions. Consequently, the losing bidder's pseudo values are smaller in SS than FS. The exercised score non-equivalence thus results between FS and SS auctions under PQR rules.

Theorem 4. *The expected exercised score is strictly greater in FS than in SS auctions if the scoring rule is PQR.*

Proof. Lemma 2, *i.e.,* $q(s^{FS}(\theta),\theta) > q^{FB}(\theta)$, implies that $k(q(s^{FS}(\theta_{(2)}),\theta_{(2)}),\theta_{(2)}) > k(q^{FB}(\theta_{(2)}),\theta_{(2)})$ for all $\theta_{(2)}$. By Lemma 1, $E[\sigma(k(q(s^{FS}(\theta_{(1)}),\theta_{(1)}),\theta_{(1)})] = E[k(q(s^{FS}(\theta_{(2)}),\theta_{(2)}),\theta_{(2)})]$. Thus, $E[\sigma(k(q(s^{FS}(\theta_{(1)}),\theta_{(1)}),\theta_{(1)}))] > E[k(q^{FB}(\theta_{(2)}),\theta_{(2)})]$. □

The revenue equivalence theorem (Myerson (1981); Riley and Samuelson (1981)) has first been extended to multi-dimensional auction games by Che (1993) who shows that if a QL scoring rule is used, the expected exercised scores in both FS and SS auctions are equivalent. Theorem 4 shows that the equivalence holds only under QL rules.

Under PQR scoring rules, informational rents allow every bidder to set the price of quality such that the pseudo value is greater than the best willing scores. In other words, losing bidders also exercises the market power in FS auctions, which differentiates the situation of SS auctions in which losing bidders has no chance to exercise market power. In FS auctions the winning bidder takes advantage of the market power exercised by losing bidders. The buyer thus has to incur greater informational costs to the winning bidder, which raises the expected exercised score in FS auctions.

In QL scoring rules, both the buyer and sellers are a monetary payoff maximizer. Thus, the buyer's lower welfare is equivalent to the sellers' greater payoffs if both are risk neutral. The buyer is, however, a utility maximizer under PQR rules. Therefore, the equivalence is not clear to be held under PQR rules. To conclude this section, we explore the equivalence in the bidder's expected payoff in FS and SS auctions under PQR rules. To simplify the analysis, we impose an additional assumption on the bidder's cost function.[20]

Assumption 2. *The cost function satisfies* $C_{qqq} \geq 0$.

Then, the following lemma claims that average quality bids are greater in FS than SS auctions.

[20]The intuition of Assumption 2 is that the increasing speed of the marginal cost never falls. A convex cost function which violates this is the one whose marginal cost converges to some value $s < \bar{k}$. In this case, the bidder's optimal quality choice as well as the winning payoff are infinite if the (expected) second-lowest k is greater than s in a FS (SS) auction. If such situations are not common practices, Assumption 2 is not considered to be far from being unrealistic.

Lemma 3. *If the scoring rule is PQR, the optimal level of quality in the FS auction is greater than the mean of the quality level finalized in the SS auction.*

Proof. Given $\theta_{(1)}$, a conditional random variable $k(q^{FB}(\theta_{(2)}), \theta_{(2)})$ has a mean strictly less than $s^{FS}(\theta_{(1)})$. In addition, $q(s, \theta)$ is a concave function of s.[21] Hence, given θ_1, $q(s^{FS}(\theta_{(1)}), \theta_{(1)}) > E_{\theta_{(2)}|\theta_{(1)}}[q(k(q^{FB}(\theta_{(2)}), \theta_{(2)}), \theta_{(1)})]$ holds by Jensen's inequality. □

With this result, the payoffs between FS and SS auctions are compared.

Theorem 5. *The bidder's expected payoff is greater in FS than in SS auctions in any PQR scoring rule if $C_{qqq} \geq 0$.*

Proof. By Theorem 4, $k(q^{FB}(\theta_{(2)}), \theta_{(1)})$ is, on average, lower than $s^{FS}(\theta_{(1)})$ for any $\theta_{(1)}$. By Lemma 3, the equilibrium quality level is, on average, lower in the SS auction. □

Assumption 2 is sufficient to obtain the results. Furthermore, Assumption 2 is implied by Assumption 1 when $k_{qqq} \geq 0$.[22] Thus, payoff non-equivalence likely occurs if scoring rules are based on PQR.

4 Application to multi-dimensional signals

We so far assume a single dimensional signal. The assumption is crucial to ensure the existence of a Bayesian Nash equilibrium in our proof. As pointed out by Asker and Cantillon (2008), a limitation of the single dimensional signal lies in that equilibrium price-quality pairs are described as a locus in the price quality space, which is obviously inconsistent to the real-world scoring auctions. In this section, we explore the possibility that the assumption is relaxed.

We consider the following two-dimensional signal $\theta = \{\theta^1, \theta^2\}$. Then, the cost function is assumed to be homothetic as below.

Assumption 3. *(Homothetic cost functions) Let $\eta(k, \theta)$ be the inverse of function $k(q, \theta)$ with respect to q such that for any q and $k = k(q, \theta)$, $q = \eta(k, \theta)$. Then, for any pair of $\theta \neq \widehat{\theta}$, we have either*

$$\begin{aligned} i)\ & C_q(\eta(k, \theta)|\theta) > C_q(\eta(k, \widehat{\theta})|\widehat{\theta}), \\ ii)\ & C_q(\eta(k, \theta)|\theta) = C_q(\eta(k, \widehat{\theta})|\widehat{\theta}), \text{ or} \\ iii)\ & C_q(\eta(k, \theta)|\theta) < C_q(\eta(k, \widehat{\theta})|\widehat{\theta}). \end{aligned}$$

for all $k \geq \max\{k(q^{FB}(\theta), \theta), k(q^{FB}(\widehat{\theta}), \widehat{\theta})\}$.

[21] If $q = q(s, \theta)$, then $C_q(q|\theta) = s$ holds. On the other hand, $C_{qqq} \geq 0$ implies that $C_q(q|\theta)$ is convex in q. Therefore, $q(C_q(q|\theta), \theta) \equiv q(s, \theta)$ is a concave function in s.

[22] Note that $C_{qqq}(q|\theta) = qk_{qqq}(k, \theta) + 3k_{qq}(q, \theta)$. Thus, if $k_{qqq} = 0$, then Assumption 1 and 2 are equivalent. Furthermore, if $k_{qqq} \leq 0$, then Assumption 2 implies Assumption 1.

An example of the class of cost functions is given as follows: With $\theta = \mathbb{R}^2_+$, $d(\theta^1) > 0$ and $d'^1) < 0$,

$$C(q|\theta) = \begin{cases} \theta^2 \left[(\frac{q}{\theta^2} - \theta^1)^2 + d(\theta^1) \right] & \text{if } q \geq \theta^1 \cdot \theta^2 \\ d(\theta^1) & \text{otherwise} \end{cases}$$

Without any loss, the first dimension of private information, θ^1, represents the efficiency parameter; the lower is θ^1, the lower is the bidder's cost given all other things are constant. Specifically, $C(q|\theta)$ is monotonically increasing in θ^1, and, for any k, $C_q(\eta(k,\theta)|\theta)$ is decreasing in θ^1. The second dimensional signal θ^2 describes the scale parameter of the bidder; the higher is θ^2, the greater is the bidder's quality provision level at the best willing score even if the value of the best willing score is constant. Given θ^1, cost functions are homothetic for any θ^2.

Since a class of homothetic cost functions can be grouped, a single dimension, θ^1 for instance, ends up governing the differentiation of the bidder's productivity. Regarding the signal of that dimension as the single dimensional signal θ in our model discussed before, our model can be effectively extended to the multi-dimensional signal environment. As a result, the equilibrium exercised score and quality can be scattered in the price quality space.

The extension of our theoretical analysis to the multi-dimensional signals may be quite attractive to structural estimation of scoring auctions. It would be quite interesting to empirically compare the performance between QL and PQR scoring rules, or FS and SS auctions with a given scoring rules. Due to the homothetic cost function assumption, the winning bidder becomes always the same in FS and SS auctions even under PQR scoring rules. Yet, the restriction is not unreasonable to estimate the latent variables from the scoring auction data.

5 Conclusion

In this paper, we analyze scoring auctions, relaxing the assumption that scoring rules be QL. Establishing a model of scoring auctions that also accept PQR scoring rules, we demonstrate the existence and uniqueness of equilibriums and characterize the equilibrium bidding strategies in FS and SS auctions. We found that equivalence fails between FS and SS mechanisms under PQR scoring rules. The procurement buyer's expected utility is higher in SS than in FS auctions, and the bidder's expected payoff is generally greater in FS than SS auctions. These results contrast to those observed under QL scoring rules.

A complexity associated with PQR rules is that the bidder's effective type *i.e.,* the pseudo value, may not be deterministic. Nonetheless, in SS auctions, the bidder's payoff upon winning is independent of the scoring bid. Therefore, as in the case of second-price auctions, bidders truthfully report their best-willing score. The best-willing score is, thus, considered as the pseudo value. In FS auctions, however, the bidder's pseudo value is positively related with the strategy, *i.e.,* the scoring bid. Therefore, the solution of the differential equation stemming from the bidder's first-order condition is not a closed form. The equilibrium characterization, thus, requires the demonstration of the existence of a Bayesian Nash equilibrium.

The buyer's preference may be either PQR or QL depending on the item to be procured. Consider, for instance, a bridge construction project. Suppose that the quality is measured by the lifetime of the bridge. If a longer lifetime helps the procurement buyer save the replacement cost, PQR (the total cost of maintaining the bridge divided by the quality (lifetime)) is more plausible to be considered as the procurement buyer's preference. On the other hand, suppose that a procurement buyer needs to compensate local residents because the construction work involves some environmental destruction. A better technology quality) reduces the environmental destruction and, as a result, the amount of compensation to local residents. If the procurement buyer's preference over the contract is the benefit from the project minus the total cost of the procurement which is the sum of the construction cost plus compensation, then his preference is QL. Procurement buyers should properly design the scoring rule and choose the auction format depending on his preference.

The issue of the awarding procedure between PQR and QL is also discussed in the context of cost-benefit analysis. It is concluded that the advantage of either way is thoroughly depending on the buyer's preference. Our results, however, suggest that when the scoring rule is PQR, more rents can be left over to bidders in the way that the buyer pays a higher unit price on quality if a FS auction is used. Thus, if a buyer is risk-neutral and has the preference based on PQR, the buyer should use a SS auction. Unlike the case of second-price auctions, losing bidders only exhibit their best willing scores in SS auctions. Therefore, the privacy is better maintained in SS auctions.

An extension from these theoretical results can be the structural analysis on scoring auctions. The equilibrium bid functions characterized here help identifying the bidder's pseudo value and costs with the existing technique on structural auction analysis. Furthermore, the extension of our theoretical analysis to a multi-dimensional signal environment makes it possible to handle real-world scoring auction data for structural analyses. Identifying the form of the cost function from the auction data is challenging but quite useful to design a proper scoring auction mechanism. Further econometric studies are called on for structurally analyzing scoring bid data.

Appendix A

Consider the cost function $C(q|\theta)$ where the quality bid is multidimensional such that $q \in \mathbf{R}_+^L$ with $L > 1$. Let $\phi(q)$ be a quality index, with $\phi_q(q) > 0$, and $\phi_{qq}(q)$ is negative semi-definite. Suppose that the scoring rule $S(p,q)$ is given by $S(p,q) = p/\phi(q)$ for PQR and $S(p,q) = p - \phi(q)$ for QL scoring rules. Then, without loss of generality, ϕ can be substituted with $q \in \mathbf{R}_+^L$ as follows: Define

$$C(\phi, \theta) = \min C(q|\theta) \text{ s.t. } \phi(q) = \phi$$

for $\phi \in$Range of $\phi(q)$. If FOC is necessary for the minimization problem, we have

$$\begin{aligned}
&C(\phi,\theta) = C\left(q(\phi|\theta),\theta\right) \\
&\text{where } (q(\phi|\theta),\lambda(\phi|\theta)) \text{ is a solution to} \\
&\begin{cases} C_q(q|\theta) - \lambda\phi_q(q) = 0 \in R^M \\ \phi(q) - \phi = 0 \in R \end{cases}
\end{aligned}$$

Suppose C is strictly convex in q and $\phi(q)$ is weakly concave. Then

$$\begin{aligned}
C_\phi &= \lambda(\phi|\theta) > 0 \\
C_{\phi\phi} &= \lambda_\phi(\phi|\theta) > 0.
\end{aligned}$$

To see the first condition, note that from the second equation for optimality

$$\phi_q\left(q(\phi|\theta)\right) q_\phi(\phi|\theta) = 1$$

which implies, by the first equation,

$$C_q\left(q(\phi|\theta),\theta\right) q_\phi(\phi|\theta) = \lambda.$$

This shows $C_\phi = \lambda(\phi|\theta)$. For the second condition, note that from the implicit function theorem

$$\begin{pmatrix} C_{qq} - \lambda\phi_{qq} & -\phi_q^T \\ \phi_q & 0 \end{pmatrix} \begin{pmatrix} q_\phi \\ \lambda_\phi \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

This implies

$$\begin{aligned}
&\phi_q^T\lambda_\phi = \left(C_{qq} - \lambda\phi_{qq}\right) q_\phi \\
&\Rightarrow \underbrace{q_\phi^T\phi_q^T}_{1}\lambda_\phi = q_\phi^T\underbrace{(C_{qq} - \lambda\phi_{qq})}_{\text{positive definite}}q_\phi \\
&\Rightarrow \lambda_\phi > 0
\end{aligned}$$

This shows that there is no loss of generality to restrict attention to one dimensional quality, as long as ϕ and C satisfy a certain regularity conditions.

Appendix B

For an $s \in S$, define a transformation $T : S \to S$ by[23]

$$T(s(\cdot)) = \int_{\theta}^{\theta^+} \frac{(n-1)[1-F(\hat{\theta})]^{n-2} f(\hat{\theta})}{[1-F(\theta)]^{n-1}} k(s(\hat{\theta}), \hat{\theta}) d\hat{\theta}.$$

This transformation is a contraction:

$$T(s(\cdot)) - T(\tilde{s}(\cdot)) = \int_{\theta}^{\theta^+} \frac{(n-1)[1-F(\hat{\theta})]^{n-2} f(\hat{\theta})}{[1-F(\theta)]^{n-1}} [k(s(\hat{\theta}), \hat{\theta}) - k(\tilde{s}(\hat{\theta}), \hat{\theta})] d\hat{\theta}$$

and we will show that

$$|k(s(\theta), \theta) - k(\tilde{s}(\theta), \theta)| \leq \frac{1}{2} |s(\theta) - \tilde{s}(\theta)|$$

[23]This transformation comes from a change of variables in the expression

$$\sigma(k|s(\cdot)) = k + \frac{\int_k^{k^+} [1 - F_k(\hat{k}|s(\cdot))]^{n-1} d\hat{k}}{[1 - F_k(k|s(\cdot))]^{n-1}}.$$

To see this, suppose $k(\theta)$ is increasing and let $\theta(k) = k^{-1}(k)$. Then, $F_k(k(s(\theta), \theta)|s(\cdot)) = F(\theta)$, and

$$\begin{aligned} \sigma(k(s(\theta), \theta)|s(\cdot)) &= k(s(\theta), \theta) + \int_{k(s(\theta),\theta)}^{k^+} \left[\frac{1 - F_k(k(s(\hat{\theta}), \hat{\theta})|s(\cdot))}{1 - F_k(k(s(\theta), \theta)|s(\cdot))} \right]^{n-1} \frac{dk(s(\hat{\theta}), \hat{\theta})}{d\theta} d\hat{\theta} \\ &= \int_{\theta}^{\theta^+} \frac{(n-1)[1-F(\hat{\theta})]^{n-2} f(\hat{\theta})}{[1-F(\theta)]^{n-1}} k(s(\hat{\theta}), \hat{\theta}) d\hat{\theta}. \end{aligned}$$

We ignore the monotonicity of $k(\theta)$ in verifying the transformation being a contraction, but check after deriving the unique equilibrium solution.

so that

$$\begin{aligned}
\rho(T(s),T(\tilde{s})) &= \sup_\theta |T(s)(\theta)-T(\tilde{s})(\theta)| \\
&= \sup_\theta \left| \int_\theta^{\theta^+} \frac{(n-1)[1-F(\hat{\theta})]^{n-2}f(\hat{\theta})}{[1-F(\theta)]^{n-1}} (k(s(\hat{\theta}),\hat{\theta}) - k(\tilde{s}(\hat{\theta}),\hat{\theta}))d\hat{\theta} \right| \\
&\le \sup_\theta \int_\theta^{\theta^+} \frac{(n-1)[1-F(\hat{\theta})]^{n-2}f(\hat{\theta})}{[1-F(\theta)]^{n-1}} \left|(k(s(\hat{\theta}),\hat{\theta}) - k(\tilde{s}(\hat{\theta}),\hat{\theta}))\right| d\hat{\theta} \\
&\le \sup_\theta \int_\theta^{\theta^+} \frac{(n-1)[1-F(\hat{\theta})]^{n-2}f(\hat{\theta})}{[1-F(\theta)]^{n-1}} \frac{1}{2} \left|s(\hat{\theta})-\tilde{s}(\hat{\theta})\right| d\hat{\theta} \\
&\le \sup_\theta \int_\theta^{\theta^+} \frac{(n-1)[1-F(\hat{\theta})]^{n-2}f(\hat{\theta})}{[1-F(\theta)]^{n-1}} \frac{1}{2} \sup_{\theta'} \left|s(\theta')-\tilde{s}(\theta')\right| d\hat{\theta} \\
&= \sup_\theta \sup_{\theta'} \frac{1}{2} \left|s(\theta')-\tilde{s}(\theta')\right| \\
&= \sup_\theta \frac{1}{2} |s(\theta)-\tilde{s}(\theta)| \\
&= \frac{1}{2}\rho(s,\tilde{s})
\end{aligned}$$

Lemma 4. $|k(\theta)-\tilde{k}(\theta)| \le \frac{1}{2}|s(\theta)-\tilde{s}(\theta)|$.

Proof. Since $qk_q(q,\theta) = C_q(q|\theta) - k(q,\theta)$, taking the derivative on both sides with respect to q gives

$$qk_{qq}(q,\theta) = C_{qq}(q|\theta) - 2k_q(q,\theta) > 0, \tag{8}$$

where the inequality holds due to the convexity of k.

Suppose $s(\theta) \ge \tilde{s}(\theta)$. Since $C(q|\theta)$ is convex in q, $q(s(\theta),\theta) \ge q(\tilde{s}(\theta),\theta)$. Since $\tilde{s}(\theta) \ge \min_q C(q,\theta)/q$ and $C(q,\theta)/q$ is strictly convex in q, $k(\theta) \ge \tilde{k}(\theta)$. For notational simplicity, we henceforth suppress θ and denote $q = q(s(\theta),\theta)$ and $\tilde{q} = q(\tilde{s}(\theta),\theta)$.

Taking the integral of (8) over $[\tilde{q}, q]$ yields

$$\begin{aligned}
& C_q(q,\theta) - 2\frac{C(q,\theta)}{q} - \left\{ C_q(\tilde{q},\theta) - 2\frac{C(\tilde{q},\theta)}{\tilde{q}} \right\} \ge 0 \\
& \Leftrightarrow C_q(q,\theta) - C_q(\tilde{q},\theta) \ge 2\left\{ \frac{C(q,\theta)}{q} - \frac{C(\tilde{q},\theta)}{\tilde{q}} \right\} \\
& \Rightarrow s(\theta) - \tilde{s}(\theta) \ge 2(k(s(\theta),\theta) - k(\tilde{s}(\theta),\theta)) \quad (\ge 0) \\
& \Rightarrow \frac{1}{2}|s(\theta)-\tilde{s}(\theta)| \ge |k(s(\theta),\theta) - k(\tilde{s}(\theta),\theta)|,
\end{aligned}$$

(equality holds only if $s = \tilde{s}$). This is the desired result. □

So far we have verified that there is a unique solution in S for

$$s(\theta) = \int_\theta^{\theta^+} \frac{(n-1)[1-F(\tilde{\theta})]^{n-2}f(\tilde{\theta})}{[1-F(\theta)]^{n-1}} k(\tilde{\theta})d\tilde{\theta}$$

where $k(\theta) = C(q(\theta, s(\theta)), \theta)/q(\theta, s(\theta))$. We need to verify that $s(\theta)$ and $k(\theta)$ are both increasing. For $\theta < \theta^+$,

$$\begin{aligned} s'(\theta) = & \frac{(n-1)f(\theta)}{[1-F(\theta)]} \int_{\theta}^{\theta+} \frac{(n-1)[1-F(\tilde{\theta})]^{n-2} f(\tilde{\theta})}{[1-F(\theta)]^{n-1}} k(\tilde{\theta}) d\tilde{\theta} \\ & - \frac{(n-1)[1-F(\theta)]^{n-2} f(\theta)}{[1-F(\theta)]^{n-1}} k(\theta) \\ = & \frac{(n-1)f(\theta)}{[1-F(\theta)]} [s(\theta) - k(\theta)] \geq 0 \end{aligned}$$

If $s(\theta) = k(\theta)$, $s(\theta) = k^-(\theta) = \min_q C(q, \theta)/q$. However, $s'(\theta) = 0$, $s(\theta) = k^-(\theta)$, and $k^{-\prime}(\theta) > 0$ imply $s(\tilde{\theta}) < k^-(\tilde{\theta})$ for some $\tilde{\theta} > \theta$, a contradiction. Therefore $s'(\theta) > 0$.

$$\begin{aligned} k'(\theta) &= \frac{1}{q^2} \left\{ \left(C_q[q_\theta + q_s s'] + C_\theta \right) q - [q_\theta + q_s s'] C \right\} \\ &= \frac{1}{q} \left\{ \left((C_q - \frac{C}{q})[q_\theta + q_s s'] + C_\theta \right) \right\} \end{aligned}$$

Note that $q_\theta(\theta, s(\theta)) = -C_{q\theta}/C_{qq}$, which is typically negative. Since $q_s = 1/C_{qq} > 0$, a sufficient condition for $k' > 0$ is[24]

$$\frac{d}{d\theta} \left(\frac{C(q(\theta, s), \theta)}{q(\theta, s)} \right) = \frac{1}{q} \left\{ (C_q - \frac{C}{q}) q_\theta + C_\theta \right\} > 0.$$

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[24]For example, $C = \theta q^2 + F$ satisfies this condition;

$$\left(C_q - \frac{C}{q} \right) \left(-\frac{C_{q\theta}}{C_{qq}} \right) + C_\theta = \left(2\theta q - \theta q - \frac{F}{q} \right) \left(-\frac{2q}{2\theta} \right) + q^2 = -q^2 + \frac{F}{\theta} + q^2 = \frac{F}{\theta} > 0.$$

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