Multi-dimensional Product Differentiation*

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Abstract

We employ a multi-dimensional product differentiation model with general consumer distribution to analyze the impacts of product differentiation. The degree of product differentiation is measured by unit transport costs as well as distance between firms. Holding firms' locations fixed, we find that a change in unit transport cost has two impacts on equilibrium prices: a *shifting effect* and a *rotating effect*. Depending on which effect dominates, equilibrium prices and profits can increase or decrease with unit transport cost. We then fix unit transport costs and endogenize firms' location choices in a two-stage location-then-price game. Existing studies have found that when consumer distribution is uniform on all dimensions, there always exist an equilibrium where firms maximize differentiation on one dimension but minimize differentiation on other dimension(s). We show that this is not true under general consumer distribution and there may also exist equilibria where firms choose intermediate differentiation on a dimension.

Keywords: Multi-dimensional Product differentiation; Hotelling model; Location choice

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1 Introduction

Economists have been interested in how firms optimally differentiate their products at least since Hotelling (1929). In a pioneering study, Hotelling (1929) finds that firms have an incentive to move to the middle of the (Hotelling) line when transport cost (disutility resulting from buying imperfectly matched product) is linear in the distance traveled. He argued that firms choose minimum differentiation, *Principle of Minimum Differentiation*.[1] D'Aspremont, Gabszewicz and Thisse (1979) show that this argument is invalid and find that firms actually have an incentive to maximize product differentiation when transport cost is quadratic in the distance traveled. Since then, various studies have extended their analysis in several aspects of the model including transport cost (Economides (1986a)), consumer distribution (Anderson, Goeree and Ramer (1997)), probabilistic purchase (De Palma et. al. (1985)), the consideration of mixed strategy in prices (Osborne and Pitchik (1987)) as well as multi-dimensional product characteristics space (Irmen and Thisse (1998)).

In consumers' perception, competing products become more differentiated when firms are located further away from each other or when unit transport cost increases. In contrast to the abundance of studies examining product differentiation captured by firms' locations, little has been said on unit transport cost – the other measure of the degree of product differentiation. This is not surprising given that in most existing studies, product differentiation is modeled on a single dimension (whether in Hotelling's linear city or Salop's circular city models), and in such models it is intuitive that equilibrium prices/profits increase (often linearly) with unit transport cost.

In this paper, we consider a setting where products have multiple attributes and firms differentiate from each other in a multi-dimensional characteristics space. We are interested in both measures of product differentiation: unit transport cost and firm location. We investigate two intertwined questions related to product differentiation: (1) How do unit transport costs affect equilibrium profits? and (2) How do firms choose their locations? With multi-dimensional product differentiation and general distribution, several new results emerge.

First, we fix firms' locations and investigate how equilibrium changes with unit transport costs. We identify two effects that an increase in unit transport cost has on equilibrium prices/profits. The *shifting effect* (when present) always works to raise equilibrium prices while the direction of the *rotating effect* is ambiguous. Overall, an increase in unit transport cost on either dimension can have ambiguous impact on the equilibrium prices and profits. This result has practical implications. For example, suppose that firms can invest (e.g., through advertising) to increase the perceived product differentiation measured by unit transport costs.[2] Then our results imply that firms may or may

[1] Eaton and Lipsey (1975) relax several assumptions in the original Hotelling model and check the robustness of the Principle of Minimum Differentiation.

[2] One interpretation of a change in unit transport cost is the following. Suppose that the two firms' products differ

not have an incentive for such an investment. They may even have an incentive to invest and lower the perceived product differentiation measured by unit transport costs.

Second, we fix unit transport costs and endogenize firms' locations by analyzing a two-stage location-then-price game. Existing studies (e.g., Tabuchi (1994) and Irmen and Thisse (1998)) have shown that when consumer distribution is uniform on all dimensions, it is an equilibrium where firms maximize product differentiation on the dominant dimension and minimize product differentiation on other dimensions (*Max-Min*). We allow consumer distribution to be non-uniform and we are interested in (1) whether there is always an equilibrium where firms maximize (minimize) differentiation on one (the other) dimension (i.e., *Max-Min* or *Min-Max*) and, (2) whether intermediate level of product differentiation can be an equilibrium feature. The answers are yes to (1) and no to (2) in the case of uniform distribution. But we show that the answers can be reversed once we allow non-uniform distribution. In particular, when consumer distributions on the first dimension (the dominant dimension) and the second dimension are truncated normal and uniform respectively (truncated normal-uniform), under certain conditions, neither *Max-Min* nor *Min-Max* is an equilibrium and there exists an *Intermediate-Min* equilibrium.

We identify two effects a unilateral change in a firm's location has on its profit. The first effect is the *competition effect*, which captures rival's response in price to the location change. The second effect is the *market share effect* in the sense that a move toward the middle of the line (1/2) will attract more customers holding prices fixed. Market share effect always provides incentive for firms to move to the middle. In the case of uniform distribution, under *Max-Min*, the competition effect always dominates the market share effect when a firm deviates on the first dimension while the result is exactly the opposite if a firm deviates on the second dimension. Under truncated normal-uniform distribution, however, market share effect can dominate competition effect when a firm deviates on the first dimension. In this case, firms have an incentive to deviate and move away from the middle, destructing the *Max-Min* equilibrium and creates an *Intermediate-Min* equilibrium.

1.1 Review of related literature

One strand of literature analyzes firms' location choice in Hotelling models (Hotelling (1929)).[3] D'Aspremont, Gabszewicz and Thisse (1979) is among the earliest studies formally modeling firms'

in colors (red vs. black) or flavors (chocolate vs. vanilla). Firms can advertise to change consumers' perception of how these two colors or flavors differ from each other in terms of their valuation (t) but advertising will not change the colors/flavors of the products (firms' locations). Note that change in unit transport cost is different from change in firms' locations. Unit transport cost (t) is common to both firms so a change in t affects both firms symmetrically. In contrast, a change in one firm's location affects the two firms asymmetrically. We analyze firms' location choice in Section 4.

[3]Another strand of literature extends the Hotelling line to the Salop (1979) circle, and analyzes optimal product differentiation in circular models, in particular, whether firms will be symmetrically located on the circle. See Gong, Liu and Zhang (2011) of a survey of this literature.

location choice. Later studies extend their work in several aspects of the model. For example, Economides (1986a) considers transport cost function in the form of d^α with $\alpha \in [1, 2]$, i.e., between linear and quadratic transport cost. He finds that firms maximize product differentiation when α is sufficiently large ($\alpha \geq \bar{\alpha} \approx 1.26$) but there is no equilibrium when $\alpha < \bar{\alpha}$. Osborne and Pitchik (1987) revisit the initial Hotelling setup but allow mixed strategy equilibrium since pure strategy equilibrium in prices exists only for a proper subset of all location pairs. They characterize subgame perfect Nash equilibria where one firm locates around 0.27 and the other symmetrically around 0.73. While most existing studies assume that consumers are uniformly distributed on the Hotelling line, there are some exceptions. Tabuchi and Thisse (1995) study the impact of consumer concentration in the market center on the equilibrium location choices. Using a triangular distribution, they find that there is no symmetric pure strategy equilibrium but asymmetric pure strategy equilibria exist. Anderson, Goeree and Ramer (1997) consider general log-concave consumer densities. They show that a unique subgame perfect Nash equilibrium in pure strategies exists if the density is not "too asymmetric" and not "too concave." The main differences between our paper and this strand of literature are that we consider a multi-dimensional product characteristics space with general consumer distribution and we also analyze the impact of unit transport cost on the equilibrium.

Our paper is most closely related to the literature which analyzes optimal location choices in a multi-dimensional product differentiation setting.[4] Irmen and Thisse (1998) consider a general n-dimensional differentiation model with uniform distribution on all dimensions.[5] They characterize equilibria where firms maximize differentiation on the dominant dimension and minimize differentiation on all other dimensions. Our paper differs from Irmen and Thisse in two important ways. First, we analyze the impacts of unit transport costs on the equilibrium. Second, we consider general distribution in contrast to uniform distribution in Irmen and Thisse.

Besides Irmen and Thisse, there are several other studies examining product differentiation in a multi-dimensional space. Economides (1986b) considers a 2-dimensional differentiation model with a class of transport cost functions. He finds that Nash equilibrium in prices exists for all symmetric varieties, a result in sharp contrast to the case of one-dimensional model. Tabuchi (1994) analyzes a two-stage location-then-pricing game in a two-dimensional Hotelling model. He considers only uniform consumer distribution but allows firms to choose locations sequentially.[6]

[4]There are existing studies consider multi-dimensions with uniform consumer distribution or one-dimension with general distribution. To our knowledge, this paper is the first study to consider both multi-dimensions and general distribution.

[5]Analogous to Irmen and Thisse who extend the one-dimensional Hotelling horizontal differentiation model to multi-dimensions, there are studies (e.g., Vandenbosch and Weinberg (1995) and Lauga and Ofek (2011)) extending the one-dimensional Shaked and Sutton (1982) vertical differentiation model to multi-dimensions. Desai (2001) and Jentzsch, Sapi and Suleymanova (2010) introduce consumer heterogeneity in both brand preference and unit transport cost. Some studies (e.g. Gilbert and Matutes (1993), Johnson and Myatt (2003) and Schmidt-Mohr and Villas-Boas (2008)) analyze product line and quality competition.

[6]Ansari, Economides and Ghosh (1994) employ a disutility function different from the standard forms in Hotelling

Ansari, Economides and Steckel (1998) analyze both two- and three-dimensional differentiation under uniform distribution. Their two-dimensional results are similar to ours if we assume uniform distribution.

Chen and Riordan (2010) also investigate a multi-dimensional product differentiation setting. Our paper and theirs share a feature in allowing more general consumer distribution than what is typical in the literature. In particular, we consider non-uniform distribution with independence across dimensions, while Chen and Riordan consider uniform but correlated distributions on the two dimensions, a unique feature as most existing studies assume that consumer distributions on different dimensions are independent of each other. They use copulas to disentangle the preference dependence and characterize interesting conditions such as when symmetric single-product oligopoly prices are above or below the single-product monopoly price.[7] The main difference between our paper and theirs is the underline issues of interest. We consider how unit transport costs affect equilibrium prices while they are concerned with how equilibrium prices vary with market structure. In addition, they assume exogenous firm locations while we consider the issue of location choice.

Another related paper is Alexandrov (2008). In his model, firms can offer "fat products" so their choices include both prices as well as how "fat" they make their products (the degree of product customization). An increase in unit transport cost raises equilibrium price, but it also increases firms' investment in product customization (higher cost). When the latter dominates the former, firms' profits decrease with unit transport cost. This differs from our paper in several aspects. First, we consider duopoly with multi-dimensional product differentiation and general distribution while he considers $n \geq 2$ firms with one-dimensional differentiation and uniform distribution. Second, firms choose only prices in our unit transport cost analysis while firms also make product customization decisions in his model. If product customization is fixed, then profit would always increase with unit transport cost.[8] Third, equilibrium prices increase linearly with unit transport cost in his model even when profits decrease. In our model, however, equilibrium prices can increase or decrease with unit transport cost.

The rest of the paper is organized as follows. The model is presented in Section 2. We investigate the impact of unit transport costs in Section 3 and analyze the two-stage location-then-price game in Section 4. In Section 5 we conclude. Proofs of lemmas and propositions can be found in Appendix

models, which allows them to convert the two-dimensional space into a one-dimensional space. Similar to ours, they find that non-uniform distributions can generate equilibria substantially different from those under uniform distributions. Other related studies include Braid (1991), Nilssen and Sorgard (1998) and Dearmon and Kosmopoulou.

[7] In addition to product differentiation, this method has been used in other settings such as behavioral price discrimination (e.g., Chen and Pearcy (2010).

[8] There exist other studies with results of this flavor. For example, if one fixes firms' promotion intensity, then an increase in coupon distribution cost (Bester and Petrakis (1996)) or an increase in consumer arbitrage (Kosmopoulou, Liu and Shuai (2011)) would lower firms' profits. Once their impact on firms' promotion intensity is taken into account, the results are exactly the opposite, i.e., an increase in coupon distribution cost or in consumer arbitrage improves firms' profits.

$A.1$ and we extend the benchmark 2-dimensional model to n-dimensions in Appendix $A.2$.

2 The model

We consider a duopoly horizontal differentiation model where products have multiple attributes and production differentiation occurs in a multi-dimensional space. In the main model, we consider the case of 2-dimensional product differentiation.[9] Consumers are distributed on the square with length of each side being L. For simplicity, we normalize $L = 1$ so the support of consumer distribution becomes the unit square $[0, 1] \times [0, 1]$. Consumers are distributed on the two dimensions according to distribution function $F_1(x)$ and $F_2(y)$ respectively, where the subscript '1' and '2' refers to the first and second dimension (or the x and y dimension) respectively. The corresponding density functions are $f_1(x)$ and $f_2(y)$. Consumer distributions on different dimensions are assumed to be independent. Both $f_1(x)$ and $f_2(y)$ are continuously differentiable on their respective supports and are symmetric about $\frac{1}{2}$, i.e., $f_i(z) = f_i(1 - z)$, $\forall z \in [0, 1]$, $i = 1, 2$.

Let $[L_{1A}, L_{2A}]$ and $[L_{1B}, L_{2B}]$ denote firm A and B's locations respectively. When a consumer buys from a firm at a different location, she incurs a disutility which is quadratic in the distance traveled on each dimension.

For example, if the consumer located at (x, y) buys from firm A, she would enjoy an indirect utility of

$$u_A = V - p_A - t_1(x - L_{1A})^2 - t_2(y - L_{2A})^2,$$

where p_A is the price firm A charges and t_i is the unit transport cost on dimension $i = 1, 2$. Without loss of generality, we assume that $t_1 \geq t_2$. If the consumer buys from firm B instead, her utility will be

$$u_B = V - p_B - t_1(L_{1B} - x)^2 - t_2(L_{2B} - y)^2.$$

Each consumer buys at most one unit from the firm which maximizes her utility. We assume that V is sufficiently large so all consumers buy one unit in the equilibrium (covered market). Both firms have constant marginal cost which we normalize to zero. Firms choose prices simultaneously.

Next, we conduct two types of analysis, both concerning the degree of product differentiation. In spatial differentiation models, the degree of production differentiation is represented by firms' locations as well as the unit transportation costs. Most existing studies deal with the former but not the latter. In contrast, in this paper, we consider both measures of product differentiation. In Section 3, we fix firms locations and analyze how the unit transports costs (t_1 and t_2) affect the

[9] We generalize the analysis to n-dimensional product differentiation in Appendix $A.2$ and show that our main results continue to hold. In particular, equilibrium prices/profits may increase or decrease with unit transport cost and *Max-Min-$\cdots$-Min* may not be an equilibrium under non-uniform distributions.

equilibrium. Then, in Section 4, we fix unit transport costs and endogenize firms' locations in a two-stage location-then-price game. Before we proceed to analysis, we need to ensure that there always exists a pure strategy Nash equilibrium (PSNE) (unique or not). A sufficient condition for the existence of PSNE is that the consumer distribution is ρ-concave. Following Caplin and Nalebuff (1991) (Definition on p. 29), ρ-concavity is defined as follows.

Definition 1 *Consider $\rho \in [-\infty, +\infty]$. For $\rho > 0$, a nonnegative function, f, with convex support $B \subset R^n$ is called ρ-concave if $\forall \alpha_0, \alpha_1 \in B$,*

$$f(\alpha_\lambda) \geq [(1-\lambda) f(\alpha_0)^\rho + \lambda f(\alpha_1)^\rho]^{1/\rho}, \quad 0 \leq \lambda \leq 1,$$

with $\alpha_\lambda = (1-\lambda)\alpha_0 + \lambda\alpha_1$. For $\rho < 0$, the condition is exactly as above except when $f(\alpha_0) f(\alpha_1) = 0$, in which case there is no restriction other than $f(\alpha_\lambda) \geq 0$. Finally, the definition is extended to include $\rho = -\infty$, 0, ∞ through continuity arguments.

Assumption 1 *The probability density function $f_1(x) \cdot f_2(y)$ is ρ-concave on its support $[0,1]^2$ with $\rho = -\frac{1}{3}$.*

Our utility function satisfies Assumption $A1$ in Caplin and Nalebuff (1991). Our Assumption 1 is a special case ($n = 2$ dimensions) of Assumption $A2$ in Caplin and Nalebuff. Therefore, under Assumption 1, there always exists a pure strategy equilibrium in prices for any (t_1, t_2) and firm locations (Caplin and Nalebuff, Theorem 2, p. 39).[10]

3 Unit transport costs

We start by analyzing how unit transport costs affect the equilibrium with firms' locations held fixed. In particular, firms are located at the end points of the square, with firm A at (0,0) and firm B at (1,1) respectively. Consider a consumer located at (x, y). Her indirect utility from buying from either of the two firms becomes

$$u_A = V - p_A - t_1 x^2 - t_2 y^2, \quad u_B = V - p_B - t_1(1-x)^2 - t_2(1-y)^2.$$

The marginal consumers (x, y) defined by

$$u_A = u_B \Rightarrow y = \frac{(p_B - p_A) + (t_1 + t_2) - 2t_1 x}{2t_2}.$$

[10] If $f_i(\cdot)$ is twice continuously differentiable on its support, then demand functions are twice-continuously differentiable. In addition, if the ρ-concavity of consumer distribution is strengthened to log-concavity, then the pure strategy equilibrium is unique (Caplin and Nalebuff, Proposition 6, p. 42).

We focus on symmetric equilibrium. Given the assumption $t_1 \geq t_2$, the marginal consumers line will touch the horizontal lines (see Figure 1). We are particularly interested in the two marginal consumers $(x_1, 1)$ and $(x_2, 0)$, where

$$x_1 = \frac{(p_B - p_A) + (t_1 - t_2)}{2t_1}, \quad x_2 = \frac{(p_B - p_A) + (t_1 + t_2)}{2t_1}.$$

Any consumer to the left of the marginal consumers line buys from firm A whose demand is

$$q_A = F_1(x_1) + \int_{x_1}^{x_2} F_2(y) f_1(x) dx.$$

All other consumers buy from firm B and $q_B = 1 - q_A$.

Firm $j = A, B$'s problem is

$$\max_{p_j} \quad \pi_j = p_j q_j, \quad j = A, B.$$

We look for a symmetric pure strategy Nash equilibrium (PSNE) in prices and the results are presented in the next Proposition.

Proposition 1 *When firms are located at the end points of the square, in the unique symmetric pure strategy equilibrium candidate,*

(i) Both firms choose a price of

$$p^* = \frac{t_2}{\int_{x_1^*}^{x_2^*} f_2(y) f_1(x) dx}, \tag{1}$$

where

$$x_1^* = \frac{t_1 - t_2}{2t_1}, \quad x_2^* = \frac{t_1 + t_2}{2t_1}, \quad y = \frac{-2t_1 x + t_1 + t_2}{2t_2}.$$

(ii) Firms split the market and each earns a profit of $\pi^* = \frac{p^*}{2}$.

Proof. See Appendix A.1. ■

$(p_A, p_B) = (p^*, p^*)$, as a solution to firms' first-order conditions, constitutes an equilibrium candidate. Whether it is indeed an equilibrium depends on whether firms have an incentive to deviate. If the assumption on consumer distribution is strengthened to log-concave, then the PSNE is unique (Caplin and Nalebuff). Since asymmetric equilibria always come in pairs, the equilibrium must be symmetric and there is thus no need to check firms' incentive to deviate. Next, suppose that $p_A = p_B = p^*$ is an equilibrium. We want to check how p^* changes with t_1 and t_2.

3.1 Comparative statics

In the case of single-dimensional product differentiation. Let t denote the unit transport cost and let $f(\cdot)$ denote the density function on $[0,1]$ which is symmetric about $1/2$. It can be easily shown that the solution to first-order conditions leads to $p^* = \frac{t}{f(\frac{1}{2})}$. Therefore, firms' prices and profits increase linearly with the unit transport cost t. An implication of this standard result is that, if firms have means to increase t without incurring too much cost, they would have an incentive to do so.[11]

In the case of multi-dimensional product differentiation, the impact of unit transport costs on prices and profits is less clear. In the case of two dimensions, the equilibrium price p^* in general depends on both t_1 and t_2. Since we have closed-form solution for the equilibrium price p^*, we can calculate its derivatives with respect to t_1 and t_2 respectively, and see how unit transport costs affect the equilibrium price p^*. The results are summarized in the next Proposition.

Proposition 2 *Under multi-dimensional product differentiation, equilibrium price (thus profit) may increase with, decrease with or be independent of unit transport cost on either dimension. The results depend on the signs of the following partial derivatives:*

$$\frac{\partial p^*}{\partial t_1} = \frac{\frac{t_2^2}{t_1^2} f_2(0) f_1(x_2^*) + \frac{1}{2}\int_{x_1^*}^{x_2^*} f_2'(y)(2x-1) f_1(x)dx}{\left(\int_{x_1^*}^{x_2^*} f_2(y) f_1(x)dx\right)^2}, \tag{2}$$

$$\frac{\partial p^*}{\partial t_2} = \frac{\int_{x_1^*}^{x_2^*} f_2(y) f_1(x)dx - \frac{t_1}{t_2}\left[\frac{t_2^2}{t_1^2} f_2(0) f_1(x_2^*) + \frac{1}{2}\int_{x_1^*}^{x_2^*} f_2'(y)(2x-1) f_1(x)dx\right]}{\left(\int_{x_1^*}^{x_2^*} f_2(y) f_1(x)dx\right)^2}, \tag{3}$$

where

$$x_1^* = \frac{t_1 - t_2}{2t_1}, \quad x_2^* = \frac{t_1 + t_2}{2t_1}, \quad y = \frac{-2t_1 x + t_1 + t_2}{2t_2}, \forall x \in [x_1, x_2].$$

Proof. See Appendix $A.1$. ■

The intuition of why equilibrium price can decrease with uniform transport cost is as follows. When t_1 or t_2 changes, it changes a firm, say firm A's incentive to deviate measured by $\frac{\partial q_A}{\partial p_A}$. We identify two effects a change in t_1 or t_2 has on $\frac{\partial q_A}{\partial p_A}$. The first effect is the *shifting effect*. When a firm lowers its price slightly, the marginal consumers line shifts parallelly (see Figure 2). The length of parallel shift ($\frac{\partial x_1}{\partial p_A} = \frac{\partial x_2}{\partial p_A} = -\frac{1}{2t_1}$) decreases with t_1 but is independent of t_2. That is, when t_1 increases, the same price cut can steal fewer customers, so firms have fewer incentive to undercut

[11] Of course this is constrained by consumers' reservation value v. As t and thus p^* increase, the assumption that v is sufficiently large so that the market is covered becomes increasingly restrictive.

each other's price, supporting higher equilibrium price. The second effect is the *rotating effect*. Note that the marginal consumer line has a slope of $\frac{dy}{dx} = -\frac{t_1}{t_2}$ (see Figure 3) so it rotates clockwise when t_1 increases or t_2 decreases. The consequent impact of such a rotation on $\frac{\partial q_A}{\partial p_A}$ depends on consumer distribution. Let us see why. In Figure 3, there are two parallel solid lines (under the initial (t_1, t_2)) and two parallel dashed lines (under the new (t_1', t_2')).[12] Let us start with the solid lines, i.e., under the initial (t_1, t_2). Start with $p_A = p_B = p^*$. If firm A lowers its price by dp_A, the marginal consumer line will shift parallelly and the extra sales is given by the measure of consumers in the area $ABCD$. Next, consider the dashed lines. Start with $p_A = p_B = p^*$. If firm A lowers its price by dp_A, the marginal consumer line will shift parallelly and we restrict the length of parallel shift to be the same as that in $ABCD$.[13] The extra sales under (t_1', t_2') is captured by the measure of consumers in the area $EFGH$. It is in general ambiguous how the measure of consumers in the area $ABCD$ and that in the area $EFGH$ compare.

We have explained that the shifting effect is zero when t_2 changes. Thus $\frac{\partial p^*}{\partial t_2}$ captures only the rotating effect, while $\frac{\partial p^*}{\partial t_1}$ captures a mixture of both the shifting effect and the rotating effect.[14] Since the shifting effect appears in $\frac{\partial p^*}{\partial t_1}$ but not in $\frac{\partial p^*}{\partial t_2}$, it must be proportional to $\int_{x_1^*}^{x_2^*} f_2(y) f_1(x) dx$ so as to offset the corresponding term in the $\frac{\partial p^*}{\partial t_2}$ expression.

3.1.1 Some specific examples

Next, we utilize the results in Proposition 2 and consider several cases where $\frac{\partial p^*}{\partial t_i}$ $(i = 1, 2)$ takes different signs.

Case 1(i): p^* increases with t_1

Suppose that $f_2'(y) \geq 0$, $\forall y \in [0, 1/2]$.[15] By symmetry, $f_2'(y) \leq 0$, $\forall y \in [1/2, 1]$. When $x \in [x_1^*, 1/2]$, we have $2x - 1 \leq 0$ and $y \in [1/2, 1]$, implying $f_2'(y) \leq 0$. Therefore, $f_2'(y)(2x-1) \geq 0$ when $x \in [x_1^*, 1/2]$. Similarly, it can be shown that $f_2'(y)(2x-1) \geq 0$ when $x \in [1/2, x_2^*]$. Combined,

$$\frac{1}{2} \int_{x_1^*}^{x_2^*} f_2'(y)(2x-1) f_1(x) dx \geq 0 \Longrightarrow \frac{\partial p^*}{\partial t_1} > 0.$$

Case 1(ii): p^* decreases with t_1

[12] One of the unit transport cost is assumed to remain unchanged, i.e., either $t_1 = t_1'$ or $t_2 = t_2'$.

[13] The impact of unit transport cost on the length of parallel shift is already taken into account in the shifting effect.

[14] Note that the rotating effect of a change in t_1 has opposite sign of the rotating effect of a change in t_2. For example, x_1^* increases with t_1 but decreases with t_2.

[15] A special case is $f_2'(y) = 0$, i.e., uniform distribution on the second dimension.

Suppose that $f_2'(y) \le 0$, $\forall y \in [0, 1/2]$. Follow similar logic as above, we have

$$\frac{1}{2}\int_{x_1^*}^{x_2^*} f_2'(y)(2x-1)f_1(x)dx \le 0.$$

If it dominates $\frac{t_2^2}{t_1^2} f_2(0) f_1(x_2^*)$ (e.g., when $f_1(x_2^*)$ is sufficiently small), then we have $\frac{\partial p^*}{\partial t_1} < 0$.[16]

<u>Case 2(i): p^* increases with t_2</u>

If $f_2'(y) \le 0$, $\forall y \in [0, 1/2]$, then $\frac{1}{2}\int_{x_1^*}^{x_2^*} f_2'(y)(2x-1)f_1(x)dx \le 0$ as above. If it dominates $\frac{t_2^2}{t_1^2} f_2(0) f_1(x_2^*)$, then $\frac{\partial p^*}{\partial t_2} > 0$.

<u>Case 2(ii): p^* decreases with t_2</u>

Suppose that $f_2(y) = 1$, $\forall y \in [0, 1]$. Then $\frac{1}{2}\int_{x_1^*}^{x_2^*} f_2'(y)(2x-1)f_1(x)dx = 0$. Further assume that $f_1'(x) \ge 0$, $\forall x \in [1/2, x_2^*]$. Then

$$\int_{x_1^*}^{x_2^*} f_2(y) f_1(x)dx = \int_{x_1^*}^{x_2^*} f_1(x)dx \le \int_{x_1^*}^{x_2^*} f_1(x_2^*)dx = f_1(x_2^*)(x_2^* - x_1^*) = \frac{t_2}{t_1} f_1(x_2^*).$$

This implies that $\frac{\partial p^*}{\partial t_2} \le 0$ and the inequality is strict if $f_1'(x) < 0$ for some $x \in [1/2, x_2^*]$.

We have assumed that $p_A = p_B = p^*$ is an equilibrium. Since we are imposing few restrictions on the parameters and distribution functions, in general one should be able to find examples where log-concavity of distribution (or second-order conditions) is satisfied to ensure that $p_A = p_B = p^*$ is indeed an equilibrium. Also, we have only considered cases where p^* either increases/decreases with t_1 or t_2. By considering intermediate cases, one should be able to find examples where $\frac{\partial p^*}{\partial t_i} = 0$, $i = 1, 2$. A special example is the case where consumer distributions on both dimensions are uniform ($f_1(x) = f_2(y) = 1$, $x, y \in [0, 1]$). In this case, it is easy to see that the rotating effect is zero and there is only the shifting effect which occurs when t_1 changes, leading to

$$\frac{\partial p^*}{\partial t_1} = 1, \quad \frac{\partial p^*}{\partial t_2} = 0.$$

One can also check how p^* changes with t_1 and t_2 simultaneously and see whether it may increase or decrease with both t_i, or increase with one but decrease with the other. We leave this exercise for interested readers. Note that the numerator in expression (2) also appears in the numerator in expression (3) (inside the square bracket). Another direction for extension is to endow firms with the ability to affect t_1 and t_2 with certain costs and endogenize unit transport costs.

[16] When x_1^* follows truncated normal distribution and the distribution parameter σ is sufficiently small, $f_1(x_2^*)$ is sufficiently small.

4 Location choice

In the previous section, we fix firms' locations at the opposite ends of the unit square and focus on the impacts of unit transport costs on the equilibrium price. In this section, we will hold unit transport costs fixed and instead endogenize firms' locations by analyzing a two-stage location-then-price game. Firms first choose locations simultaneously. After observing each other's location choice, they make pricing decisions. We look for a subgame perfect Nash equilibrium ($SPNE$) characterizing firms' location choices and the corresponding pricing strategies.

4.1 Exogenous symmetric location

Before we look at firms' actual location choices, we want to consider an intermediate situation where firms' locations are restricted to be symmetric. That is, if firm A is located at (L_1, L_2), then firm B is located at $(1 - L_1, 1 - L_2)$. Without loss of generality, we assume that $L_i \in [0, 1/2]$, $i = 1, 2$. We then investigate how equilibrium price varies with L_1 and L_2. This intermediate situation is not the same as location choice where a firm can change its location unilaterally. However, it offers us insights on the competition effect involved in actual location choices. The idea is the following. Suppose that a firm, say firm A, unilaterally changes its location on the first dimension L_{1A}, this affects its profit through two channels. First, it moves closer to firm B and firm B will change its price in response, the *competition effect*. Second, fixing firms' prices, a move closer to the center of the market will allow firm A to sell to more consumers, the *market share effect*.

There are two cases depending on whether the marginal consumers line touches the horizontal or vertical lines of the unit square. Under certain condition (exact condition is presented in the next Lemma), the marginal consumers line touches the horizontal lines. Otherwise, it touches the vertical lines. In the former case, we can define two of the marginal consumers located at $(x_1, 0)$ and $(x_2, 1)$ respectively, with $x_1, x_2 \in [0, 1]$, similar to the demand structure in the previous section. In the latter case, the two marginal consumers are denoted by $(0, y_1)$ and $(1, y_2)$ respectively, with $y_1, y_2 \in [0, 1]$. The candidate equilibria for the two cases are presented in the next Lemma.

Lemma 1 *Firm A and B are located at (L_1, L_2) and $(1 - L_1, 1 - L_2)$ respectively with $L_1, L_2 \in [0, 1/2]$.*

(i) Case 1: $t_1(1 - 2L_1) \geq t_2(1 - 2L_2)$. The unique candidate for symmetric pure strategy equilibrium is characterized by both firms choosing a price of

$$p^*(L_1, L_2) = \frac{t_2(1 - 2L_2)}{\int_{x_1}^{x_2} f_2(y) f_1(x) dx},$$

where

$$x_1 = \frac{t_1(1-2L_1) - t_2(1-2L_2)}{2t_1(1-2L_1)}, \quad x_2 = \frac{t_1(1-2L_1) + t_2(1-2L_2)}{2t_1(1-2L_1)},$$

$$y = \frac{t_1(1-2L_1) + t_2(1-2L_2) - 2t_1(1-2L_1)x}{2t_2(1-2L_2)}, \quad \forall x \in [x_1, x_2].$$

(ii) Case 2: $t_1(1-2L_1) < t_2(1-2L_2)$*. The unique candidate for symmetric pure strategy equilibrium is characterized by both firms choosing a price of*

$$p^*(L_1, L_2) = \frac{t_1(1-2L_1)}{\int_{y_2}^{y_1} f_1(x) f_2(y) dy},$$

where

$$y_1 = \frac{t_2(1-2L_2) + t_1(1-2L_1)}{2t_2(1-2L_2)}, \quad y_2 = \frac{t_2(1-2L_2) - t_1(1-2L_1)}{2t_2(1-2L_2)},$$

$$x = \frac{t_1(1-2L_1) + t_2(1-2L_2) - 2t_2(1-2L_2)y}{2t_1(1-2L_1)}, \quad \forall y \in [y_2, y_1].$$

(iii) In both cases, firms split the market equally and each earns a profit of $\pi^*(L_1, L_2) = \frac{p^*(L_1, L_2)}{2}$.

Proof. See Appendix A.1. ■

The threshold of $t_1(1-2L_1) = t_2(1-2L_2)$ is quite intuitive. At equal prices ($p_A = p_B$), we want to see from which firm the consumer located at $(0, 1)$ buys from. $t_1(1-2L_1)$ represents firm A's advantage due to transport cost savings on the first dimension, and $t_2(1-2L_2)$ captures its disadvantage on the second dimension. If $t_1(1-2L_1) > t_2(1-2L_2)$, then firm A is advantageous at equal prices and will sell to this consumer, implying that the marginal consumers line touches the horizontal lines.

Next, we use the results in Lemma 1 to pin down the competition effect for two specific distributions which we will use later in Section 4.2. In the first distribution, consumers are uniformly distributed on both dimensions. If $t_1(1-2L_1) \geq t_2(1-2L_2)$, then $p^* = t_1(1-2L_1)$ which implies

$$\frac{\partial \ln p^*}{\partial L_1} = \frac{2}{2L_1 - 1}, \quad \frac{\partial \ln p^*}{\partial L_2} = 0.$$

The derivative of $\ln p^*$ gives us the competition effect in percentage, that is, by what percentage will prices change in response to a location change. Similarly, if $t_1(1-2L_1) < t_2(1-2L_2)$, then $p^* = t_2(1-2L_2)$ which implies

$$\frac{\partial \ln p^*}{\partial L_1} = 0, \quad \frac{\partial \ln p^*}{\partial L_2} = \frac{2}{2L_2 - 1}.$$

In the second distribution, consumers are still uniformly distributed on the second dimension. On the first dimension, however, consumers follow a truncated normal distribution with the following density function

$$f_1(x) = \begin{cases} \frac{e^{-\frac{\left(x-\frac{1}{2}\right)^2}{2\sigma^2}}}{2\sqrt{2}\sigma\int_0^{\frac{\sqrt{2}}{4\sigma}} e^{-s^2}ds}, & \text{if } x \in [0,1], \\ 0, & \text{if } x \notin [0,1]. \end{cases} \tag{4}$$

For competition effect, we will focus on the case where $L_2 = \frac{1}{2}$, i.e., there is minimum differentiation on the second dimension. Substituting $f_1(x)$ above, $f_2(x) = 1$ and $L_2 = 1/2$ into the p^* equation in Lemma 1, we can obtain

$$p^* = -t_1\sigma\sqrt{2\pi}(2L_1 - 1)erf\left(\frac{\sqrt{2}}{4\sigma}\right),$$

where $erf(\cdot)$ is the error function. Then

$$\frac{\partial \ln p^*}{\partial L_1} = \frac{2}{2L_1 - 1}.$$

4.2 Location choice

In section 4.1 we have analyzed the situation where firms' locations are exogenously restricted to be symmetric. Next, we relax this assumption and endogenize location choice. We will restrict our attention to symmetric equilibria, that is, we can use (L_1, L_2) and $(1 - L_1, 1 - L_2)$ to denote the two firms' equilibrium locations. Let $degree_1 - degree_2$ denote the type of equilibrium location choices, where $degree_i \in \{Max, Min, Intermediate\}$ represents the level of product differentiation on dimension $i = 1, 2$. Due to symmetry, we can obtain closed-form solution for the equilibrium price for general consumer distributions. However, once a firm unilaterally deviates, the resulting firm locations will be asymmetric and we cannot obtain explicit solutions for general distributions. To get around this problem, in the rest of this section, we will consider specific distributions which allow us to obtain explicit solutions for equilibrium prices.[17] In particular, we will consider two cases. In the first case (*uniform-uniform*), consumer distribution is uniform on both dimensions. In the second case (*truncated normal-uniform*), consumer distribution is still uniform on the second dimension but is truncated normal on the first dimension. In both cases, we check whether certain product differentiation (location) patterns can be supported as subgame perfect Nash equilibria.[18]

[17]Specific distribution also allows us to ensure the existence and uniqueness of price equilibrium for each location pair, which is needed for the derivation of subgame perfect Nash equilibrium.

[18]Our adoption of quadratic transport cost satisfies Assumption A1 in Caplin and Nalebuff (1991). It can be easily verified that both uniform-uniform and truncated normal-uniform density functions are log-concave. Moreover, demand functions are twice continuously differentiable. Altogether they guarantee the existence and uniqueness of price equilibrium for any location pair (Caplin and Nalebuff).

We are particularly interested in whether the results in the second case can differ qualitatively from those in the first case which has been extensively studied in the literature. We will start with the *uniform-uniform* distribution case.

4.2.1 Uniform - Uniform Distribution

When consumer distributions on both dimensions are uniform, we can solve for equilibrium prices for any location combination. We can then check whether firms have an incentive to deviate from any location combination. We check several types of candidate equilibria, including *Max-Min*, *Min-Max* and those involve intermediate degree of product differentiation. The results are presented in the next Lemma.[19]

Lemma 2 *Suppose that consumers are uniformly distributed on both dimensions.*

(i) Max-Min is always an equilibrium. The corresponding equilibrium prices and profits are:

$$p_A = p_B = t_1, \quad \pi_A = \pi_B = \frac{t_1}{2}.$$

(ii) Min-Max is also an equilibrium, provided that t_2 *is not too small relative to* t_1 *(around* $\frac{t_1}{t_2} = 2.46$*). The corresponding equilibrium prices and profits are:*

$$p_A = p_B = t_2, \quad \pi_A = \pi_B = \frac{t_2}{2}.$$

(iii) There is no other pure strategy subgame perfect Nash equilibrium.

Proof. See Appendix A.1. ■

Next, we explain the intuitions as why *Max-Min* and *Min-Max* can be supported as equilibrium, starting with *Max-Min*.[20] It is intuitive that when transport cost is quadratic in the distance traveled, firms choose to maximize product differentiation on the dominant dimension (the standard Hotelling result in the case of single dimension). However, it is interesting as to why firms choose to

[19] When we consider uniform-uniform distribution, our setup becomes a special case of the Irmen and Thisse (1998) setup ($n = 2$) where *Max-Min* is shown to be a global equilibrium under certain condition. Results similar to those in Lemma 2 have also been found in other studies. For example, Ansari, Economides and Steckel show that both *Max-Min* and *Min-Max* can be supported as equilibria. Tabuchi (1994) takes one step further and shows that there is no other equilibrium, using a slightly different but seemingly equivalent setup.

[20] Irmen and Thisse provide an example where *Newsweek* and *Time* are close in content and layout, but differentiate on the cover story (possibly the dominant characteristics). We define dominant/dominated dimension by comparing the unit transport costs. That is, with $t_1 \geq t_2$, the first dimension is the dominant dimension. In contrast, Irmen and Thisse define dominant/dominated dimensions ex-post, after firms choose locations. In particular, if firms choose to maximize differentiation on any dimension (say dimension i) while minimize differentiation on all other dimensions, then dimension i would be the dominant dimension and all other dimensions are dominated.

minimize differentiation on the other dimension. When firm A changes its location unilaterally on the second dimension, it has two effects. The first effect is the *competition effect*, which is captured by $\frac{\partial \ln p^*}{\partial L_2}$.[21] *Max-Min* ($L_1 = 0$ and $L_2 = 1/2$) fits the case $t_1(1-2L_1) \geq t_2(1-2L_2)$ and we have shown that $\frac{\partial \ln p^*}{\partial L_2} = 0$, i.e., competition effect is absent. The second effect is the *market share effect*. If firm A lowers L_2, it moves away from market center and loses market share, holding the intensify of competition (prices) fixed. The market share effect works to reduce firm A's profit when firm A moves away from $L_2 = 1/2$. Combining the two effects, firm A has no incentive to deviate on the second dimension.

Min-Max ($L_1 = 1/2$ and $L_2 = 0$) fits the case $t_1(1-2L_1) > t_2(1-2L_2)$. We have shown that $\frac{\partial \ln p^*}{\partial L_1} = 0$ and $\frac{\partial \ln p^*}{\partial L_2} = \frac{2}{2L_2-1}$. If firm A deviates on the first dimension (lowers L_1), the competition effect is absent but the market share effect works to reduce firm A's profit. If firm A deviates on the second dimension (increases L_2), the competition effect reduces firm A's profit but the market share effect improves its profit. Our result suggests that the competition effect dominates the market share effect when t_2 is not too small relative to t_1. Combined, firm A has no incentive to deviate on either dimension.

In the case of uniform-uniform distribution, there is always an equilibrium where firms maximize differentiation on one dimension and minimize differentiation on the other dimension. Moreover, there exists no other types of pure strategy equilibrium. Next, we show that once we move away from *uniform-uniform* distribution, both results may not hold.

4.2.2 Truncated Normal - Uniform Distribution

We assume that consumer distribution on the second dimension is still uniform. On the first dimension, however, consumers follow a truncated normal distribution with density function as defined in equation (4). Our results show that under truncated normal - uniform distribution, it is possible that neither *Max-Min* nor *Min-Max* can be supported as an equilibrium. Moreover, there exist other types of pure strategy equilibrium such as *Intermediate-Min*. The results are summarized in the next Proposition.

Proposition 3 *For the two-stage location-then-price game,*

(i) Max-Min is an equilibrium when σ is not too small (e.g., $\sigma > \bar{\sigma} \approx 0.177$ when $t_2 = 0.8t_1$).[22]

[21] A discussion of a somewhat similar decomposition into competition and market share effects is offered in Irmen and Thisse. Here, we use a hypothetical case where locations are exogenously restricted to be symmetric to pin down the competition effect.

[22] When a firm deviates and results in asymmetric location, we have to rely on numerical methods to calculate equilibrium prices and profits. For the various parameters we tried, our results suggest that there is a threshold $\bar{\sigma}$ such that *Max-Min* is an equilibrium when $\sigma \geq \bar{\sigma}$. However, due to the numerical nature, it is impossible to verify that this indeed holds for all $\sigma > \bar{\sigma}$.

When σ is sufficiently small, firms have an incentive to deviate on both dimensions.

(ii) Min-Max is an equilibrium if t_2 is not too small relative to t_1 ($t_2 \geq 0.408t_1$), or t_2 is small but σ is also small. When t_2 is small and σ is sufficiently large (e.g., $t_2 = 0.33t_1$ and $\sigma \geq 0.49$), firms have an incentive to deviate on both dimensions.

(iii) There exists parameter values (e.g, $t_2 = 0.1t_1$ and $\sigma = 0.17$) such that

(iii-a) Neither Max-Min nor Min-Max can be supported as an equilibrium.

(iii-b) There exists other types of equilibrium such as Intermediate-Min.

Proof. See Appendix A.1. ■

Truncated normal-uniform vs. uniform-uniform

The results in Proposition 3 differ from those in Lemma 2 qualitatively in two aspects. First, *Max-Min* is not always an equilibrium. Second, there exists another type of equilibrium: *Intermediate-Min*. Let us see what contributed to these differences. We will focus on the first dimension since all these equilibrium candidates have minimum differentiation on the second dimension. Let L_{1A} and L_{1B} denote firms' locations on the first dimension with $L_{1A} \leq 1/2$ and $L_{1B} = 1 - L_{1A}$. Now fix L_{1B} and let firm A change L_{1A} slightly. We calculate how much percentage its market share will increase by, holding firms' prices fixed ($p_B = p_A$). With minimum differentiation on the second dimension, the marginal consumer is defined by

$$V - p_A - t_1(L_{1A} - x)^2 = V - p_B - t_1(L_{1B} - x)^2$$
$$\Rightarrow \bar{x} = \frac{L_{1A} + L_{1B}}{2}, \quad \frac{\partial \bar{x}}{\partial L_{1A}} = \frac{1}{2}.$$

Firm A's market share is $F(\bar{x})$ and

$$\frac{\partial \ln F(\bar{x})}{\partial L_{1A}} = \frac{f(\bar{x})}{2F(\bar{x})}.$$

Evaluated at $L_{1B} = 1 - L_{1A}$ (symmetry), we have $\bar{x} = \frac{1}{2}$. Then $F(\bar{x}) = \frac{1}{2}$ and

$$\frac{\partial \ln F(\bar{x})}{\partial L_{1A}} = f(x = 1/2).$$

Under uniform-uniform distribution, competition effect is captured by $\frac{\partial \ln p^*}{\partial L_1} = \frac{2}{2L_1 - 1}$ and the market share effect is $f(x = 1/2) = 1$. Since the competition effect dominates the market share effect, *Max-Min* is always an equilibrium and *Intermediate-Min* cannot be an equilibrium. Under truncated normal-uniform distribution, competition effect is still captured by $\frac{2}{L_{1A}-1}$ but market share effect becomes $\frac{1}{\sqrt{2\pi}\sigma erf\left(\frac{\sqrt{2}}{4\sigma}\right)}$. Note that the competition effect increases with L_{1A} but is independent of σ, while the market share effect is independent of L_{1A} but decreases with σ. It can be easily verified that when σ is small (e.g., $\sigma = 0.17$), at $L_{1A} = 0$ the market share effect dominates the competition effect and *Max-Min* is not an equilibrium. As L_{1A} increases (firms move closer), the competition effect is strengthened and will offset the fixed market share effect at some $L_{1A} \in (0, 1/2)$.

5 Conclusion

This paper employs a multi-dimensional product differentiation (Hotelling) model with general consumer distribution. The degree of product differentiation is measured by both unit transport costs and firms' locations. Correspondingly, we analyze the impact of both measures on firms' profits, one measure at a time. We first fix firms' locations and investigate how unit transport costs affect equilibrium prices and profits. We find that equilibrium prices and profits may increase or decrease with unit transport cost on either dimension. This is in sharp contrast to the results under either (1) single-dimension nonuniform distribution or (2) multi-dimension with uniform distribution (both are special cases of our model). We identify two potential effects of a change in unit transport cost on equilibrium prices. First, an increase in unit transport cost leads to a smaller shift of the marginal consumer line (hyperplane in general n-dimensions) when a firm lowers its price, the *shifting effect*. In the mean time, a change in unit transport cost also rotates the marginal consumer line, the *rotating effect*. The shifting effect always serves to increase the equilibrium prices while the direction of rotating effect is ambiguous. In particular, if the rotating effect is negative and dominates the shifting effect, then equilibrium prices and profits decrease with unit transport cost.

We then fix unit transport costs and analyze firms' location choice in a two-stage location-then-price game. We find that it is not always an equilibrium where firms maximize differentiation one dimension and minimize differentiation on the other dimension (*Max-Min* or *Min-Max*) and there exists other types of equilibrium such as *Intermediate-Min*.[23] We decompose the effect of a unilateral location change on profit into two effects: the *market share* effect and the *competition effect*. The market share effect always pushes firms to the middle, while the competition effect does the opposite. The strength of the two effects depend on consumer distribution and it is this dependence which leads to the qualitatively different results that *Max-Min* is not always an equilibrium and there exists other types of equilibria such as *Intermediate-Min*.

While we endogenize firms' locations in the second analysis, unit transport cost remains exogenous throughout the paper. It would be interesting to analyze how equilibrium unit transport costs emerge endogenously and we reserve this for future research.

[23]Under uniform distribution, it is always an equilibrium when there are two dimensions, and remains an equilibrium when there are general n-dimensions so long as the dominant dimension is sufficiently dominant (Irmen Thisse (1998)).

A Appendix

A.1 Proof or Lemmas and Propositions

Proof of Proposition 1

Recall that the two specific marginal consumers are $(x_1, 1)$ and $(x_2, 0)$ with

$$x_1 = \frac{(p_B - p_A) + (t_1 - t_2)}{2t_1}, \quad x_2 = \frac{(p_B - p_A) + (t_1 + t_2)}{2t_1},$$

from which we can calculate

$$\frac{\partial x_1}{\partial p_A} = \frac{\partial x_2}{\partial p_A} = -\frac{1}{2t_1}.$$

In general, marginal consumer (x, y) is defined by

$$y = \frac{(p_B - p_A) - 2t_1 x + t_1 + t_2}{2t_2}, \forall x \in [x_1, x_2],$$

which leads to

$$\frac{\partial x}{\partial p_A} = 0, \quad \frac{\partial y}{\partial p_A} = -\frac{1}{2t_2}.$$

Firm A's profit is

$$\pi_A = p_A q_A = p_A \left[F_1(x_1) + \int_{x_1}^{x_2} F_2(y) f_1(x) dx \right].$$

Taking derivative of π_A with respect to p_A, we can obtain

$$\begin{aligned} \frac{\partial \pi_A}{\partial p_A} &= \left[F_1(x_1) + \int_{x_1}^{x_2} F_2(y) f_1(x) dx \right] \\ &+ p_A \left[f_1(x_1) \frac{\partial x_1}{\partial p_A} + F_2(y(x_2)) f_1(x_2) \frac{\partial x_2}{\partial p_A} - F_2(y(x_1)) f_1(x_1) \frac{\partial x_1}{\partial p_A} + \int_{x_1}^{x_2} \frac{\partial F_2(y)}{\partial p_A} f_1(x) dx \right] \\ &= F_1(x_1) + \int_{x_1}^{x_2} F_2(y) f_1(x) dx - p_A \int_{x_1}^{x_2} \frac{1}{2t_2} f_2(y) f_1(x) dx. \end{aligned} \tag{A.1}$$

Note that $F_2(y(x_2)) = 0$ and $F_2(y(x_1)) = 1$ since $y(x_2) = 0$ and $y(x_1) = 1$. Setting $\frac{\partial \pi_A}{\partial p_A} = 0$ and solving for p_A, we can obtain

$$p_A = \frac{F_1(x_1) + \int_{x_1}^{x_2} F_2(y) f_1(x) dx}{\int_{x_1}^{x_2} \frac{1}{2t_2} f_2(y) f_1(x) dx}.$$

We look for a symmetric solution ($p_A = p_B$). The numerator in the p_A expression above is simply firm A's demand, which must be 1/2 when $p_A = p_B$. Therefore,

$$p^* = p_A = \frac{t_2}{\int_{x_1^*}^{x_2^*} f_2(y) f_1(x) dx},$$

where $x_1^* = \frac{t_1 - t_2}{2t_1}$, $x_2^* = \frac{t_1+t_2}{2t_1}$ and $y = \frac{-2t_1 x + t_1 + t_2}{2t_2}$. ■

Proof of Proposition 2

Recall that $x_1^* = \frac{t_1 - t_2}{2t_1}$, $x_2^* = \frac{t_1+t_2}{2t_1}$ and $p^* = \frac{t_2}{\int_{x_1^*}^{x_2^*} f_2(y) f_1(x) dx}$. Let $A = \int_{x_1^*}^{x_2^*} f_2(y) f_1(x) dx$, the denominator in the p^* expression. We first calculate $\frac{\partial A}{\partial t_2}$ and $\frac{\partial A}{\partial t_1}$.

$$\begin{aligned}
\frac{\partial A}{\partial t_2} &= \left(f_2(y(x_2^*)) f_1(x_2^*) \frac{\partial x_2^*}{\partial t_2} - f_2(y(x_1^*)) f_1(x_1^*) \frac{\partial x_1^*}{\partial t_2} \right) + \int_{x_1^*}^{x_2^*} f_2'(y) \frac{\partial y}{\partial t_2} f_1(x) dx \\
&= \left(f_2(0) f_1(x_2^*) \frac{1}{2t_1} + f_2(1) f_1(x_1^*) \frac{1}{2t_1} \right) + \int_{x_1^*}^{x_2^*} f_2'(y) \frac{t_1(2x-1)}{2t_2^2} f_1(x) dx \\
&= \frac{1}{t_1} f_2(0) f_1(x_2^*) + \frac{t_1}{2t_2^2} \int_{x_1^*}^{x_2^*} f_2'(y)(2x-1) f_1(x) dx
\end{aligned}$$

$$\begin{aligned}
\frac{\partial A}{\partial t_1} &= \left(f_2(y(x_2^*)) f_1(x_2^*) \frac{\partial x_2^*}{\partial t_1} - f_2(y(x_1^*)) f_1(x_1^*) \frac{\partial x_1^*}{\partial t_1} \right) + \int_{x_1^*}^{x_2^*} f_2'(y) \frac{\partial y}{\partial t_1} f_1(x) dx \\
&= \left(-f_2(0) f_1(x_2^*) \frac{t_2}{2t_1^2} - f_2(1) f_1(x_1^*) \frac{t_2}{2t_1^2} \right) - \int_{x_1^*}^{x_2^*} f_2'(y) \frac{2x-1}{2t_2} f_1(x) dx \\
&= -\frac{t_2}{t_1^2} f_2(0) f_1(x_2^*) - \frac{1}{2t_2} \int_{x_1^*}^{x_2^*} f_2'(y)(2x-1) f_1(x) dx
\end{aligned}$$

Since $p^* = \frac{t_2}{A}$, we have

$$\frac{\partial p^*}{\partial t_2} = \frac{\int_{x_1^*}^{x_2^*} f_2(y) f_1(x) dx - \left[\frac{t_2}{t_1} f_2(0) f_1(x_2^*) + \frac{t_1}{2t_2} \int_{x_1^*}^{x_2^*} f_2'(y)(2x-1) f_1(x) dx \right]}{\left(\int_{x_1^*}^{x_2^*} f_2(y) f_1(x) dx \right)^2},$$

$$\frac{\partial p^*}{\partial t_1} = \frac{\frac{t_2^2}{t_1^2} f_2(0) f_1(x_2^*) + \frac{1}{2} \int_{x_1^*}^{x_2^*} f_2'(y)(2x-1) f_1(x) dx}{\left(\int_{x_1^*}^{x_2^*} f_2(y) f_1(x) dx \right)^2}.$$

■

Proof of Lemma 1

Case 1: $t_1(1-2L_1) \geq t_2(1-2L_2)$

In this case, the demand structure is such that the line representing marginal consumers crosses the two horizontal axes. Let $(x_1, 1)$ and $(x_2, 0)$ denote these two marginal consumers, with $0 \leq x_1 \leq x_2 \leq 1$.[24] It can be shown that

$$x_1 = \frac{(p_B - p_A) + t_1(1-2L_1) - t_2(1-2L_2)}{2t_1(1-2L_1)}, \quad x_2 = \frac{(p_B - p_A) + t_1(1-2L_1) + t_2(1-2L_2)}{2t_1(1-2L_1)}.$$

The line representing all marginal consumers (x, y) with $x \in [x_1, x_2]$ must satisfy the following equation,

$$y = \frac{(p_B - p_A) + t_1(1-2L_1) + t_2(1-2L_2) - 2t_1(1-2L_1)x}{2t_2(1-2L_2)},$$

which leads to

$$\frac{\partial y}{\partial p_A} = -\frac{1}{2t_2(1-2L_2)}.$$

Firm A's profit is

$$\pi_A = p_A \left[F_1(x_1) + \int_{x_1}^{x_2} F_2(y) f_1(x) dx \right].$$

$$\begin{aligned}
\frac{\partial \pi_A}{\partial p_A} &= \left[F_1(x_1) + \int_{x_1}^{x_2} F_2(y) f_1(x) dx \right] \\
&+ p_A \left[f_1(x_1) \frac{\partial x_1}{\partial p_A} + F_2(y(x_2)) f_1(x_2) \frac{\partial x_2}{\partial p_A} - F_2(y(x_1)) f_1(x_1) \frac{\partial x_1}{\partial p_A} + \int_{x_1}^{x_2} f_2(y) \frac{\partial y}{\partial p_A} f_1(x) dx \right] \\
&= [F_1(x_1) + \int_{x_1}^{x_2} F_2(y) f_1(x) dx] + p_A \left[- \int_{x_1}^{x_2} \frac{1}{2t_2(1-2L_2)} f_2(y) f_1(x) dx \right].
\end{aligned}$$

Note that $F_2(y(x_2)) = 0$ and $F_2(y(x_1)) = 1$. Setting $\frac{\partial \pi_A}{\partial p_A} = 0$ and solving for p_A, we can obtain

$$p_A = \frac{F_1(x_1) + \int_{x_1}^{x_2} F_2(y) f_1(x) dx}{\int_{x_1}^{x_2} \frac{1}{2t_2(1-2L_2)} f_2(y) f_1(x) dx}.$$

Imposing symmetry ($p_A = p_B$), the numerator in the above expression above is simply firm A's demand in equilibrium, which must be 1/2. Therefore,

$$p^*(L_1, L_2) = p_A = \frac{t_2(1-2L_2)}{\int_{x_1}^{x_2} f_2(y) f_1(x) dx},$$

[24] To ease on notation, here we will use x_1 for both $x_1(p_A, p_B)$ and the equilibrium x_1 when $p_A = p_B = p^*(L_1, L_2)$. Similarly for x_2.

where

$$x_1 = \frac{t_1(1-2L_1) - t_2(1-2L_2)}{2t_1(1-2L_1)}, \quad x_2 = \frac{t_1(1-2L_1) + t_2(1-2L_2)}{2t_1(1-2L_1)},$$

$$y = \frac{t_1(1-2L_1) + t_2(1-2L_2) - 2t_1(1-2L_1)x}{2t_2(1-2L_2)}.$$

The proof of Case 2: $t_1(1-2L_1) < t_2(1-2L_2)$ is symmetric to that of case 1 and is skipped.[25]

■

Proof of Lemma 2

We divide the proof into three parts. Part 1, 2 and 3 correspond to *Max-Min*, *Min-Max* and other differentiation types respectively in Lemma 2.

Part 1. Show that *Max-Min* is always an equilibrium

Due to symmetry, we will only show that firm A has no incentive to deviate. Fix firm B's location at $(1, 1/2)$. Let (L_{1A}, L_{2A}) denote firm A's location. With uniform-uniform distribution, there is a unique price equilibrium for any (L_{1A}, L_{2A}), and we can also write down firm A's deviation profit. We verify that firm A has no incentive to deviate under either dimension alone or deviate under both dimensions simultaneously. We skip the details as they are similar to the proof in Tabuchi (1994) and in Ansari, Economides and Steckel (1998).

Part 2. Show that *Min-Max* is also an equilibrium when t_2 is not too small relative to t_1

Similar to Part 1, we verify that firm A never has incentive to deviate under either dimension alone or deviate under both dimensions simultaneously.

Part 3. Show that there is no other pure strategy equilibrium

Besides *Max-Min* and *Min-Max*, the other types of candidate equilibria include: (1) *Max-Max*; (2) *Min-Min*, *Min-Intermediate* and *Intermediate-Min*; (3) *Max-Intermediate* and *Intermediate-Max*; (4) *Intermediate-Intermediate*. Next, we explain that none of them can be supported as an equilibrium. Due to symmetry, we only check firm A's incentive to deviate.

(1) *Max-Max* ($L_1 = L_2 = 0$): Firm A has incentive to increase L_2.

(2) *Min-Min*, *Min-Intermediate* and *Intermediate-Min*

When firms minimizes differentiation on dimension i ($L_{iA} = L_{iB} = 1/2$), the two-dimensional problem is reduced to the standard one-dimensional Hotelling model on dimension $j \neq i$. It is

[25] Details are available upon request.

commonly understood that firms want to maximize differentiation on that then, i.e., no equilibrium of the type *Min-Min*, *Min-Intermediate* or *Intermediate-Min*.

(3) *Max-Intermediate* and *Intermediate-Max*: Firm A has incentive to lower product differentiation on the dimension where there is intermediate product differentiation.

(4) *Intermediate-Intermediate*: There are two cases when $L_1, L_2 \in (0, 1/2)$. If $t_1(1-2L_1) \geq t_2(1-2L_2)$, then firm A has incentive to increase L_2 (There is no competition effect and market share effect improves its profit). Similarly, if $t_1(1-2L_1) < t_2(1-2L_2)$, then firm A has incentive to increase L_1.■

Proof of Proposition 3

We divide the proof into four parts. Part 1, 2, 3 and 4 correspond to (i) *Max-Min*, (ii) *Min-Max*, (iii-a) neither and (iii-b) *Intermediate-Min* in Proposition 3. Now that the distribution is nonuniform, in general when firms are asymmetric in location, we need to solve for the equilibrium prices numerically. Without loss of generality, we normalize $t_1 = 1$. We then consider various combinations of (t_2, σ) and check whether *Max-Min*, *Min-Max* and *Intermediate-Min* can be supported as an equilibrium, i.e., whether firms have incentives to deviate in location. Due to symmetry, we only analyze firm A's incentive to deviate.

Part 1. Show that *Max-Min* is an equilibrium when σ is not too small

In this case, firm A and B are located at $(0, 1/2)$ and $(1, 1/2)$ respectively (*Max-Min*). We first calculate the corresponding price and profit (π_A^{nodev}). We then allow firm A to change its location on either or both dimensions. After firm A's location change is observed (firm B's location is still fixed at $(1, 1/2)$), both firms choose prices accordingly. We can then calculate firm A's deviation profit π_A^{dev} and compare it with π_A^{nodev}.

Under *Max-Min*, the marginal consumer line crosses the unit square at $(x_1, 1)$ and $(x_2, 0)$ (i.e., crosses the two horizontal lines). A change in firm A's location will shift the marginal consumer line and may lead to structural changes. In particular, the unit square has 4 sides (2 horizontal and 2 vertical). The structure depends on which two of the four sides the marginal consumer line crosses, which determine the deviation type. In all deviations, $x_1, x_2, y_1, y_2 \in [0, 1]$.

- Type 1 deviation: The marginal consumer line still cross the unit square at $(x_1, 1)$ and $(x_2, 0)$ (same structure)
- Type 2 deviation: The marginal consumer line crosses the unit square at $(0, y_1)$ and $(x_2, 0)$
- Type 3 deviation: The marginal consumer line crosses the unit square at $(x_1, 1)$ and $(1, y_2)$

- Type 4 deviation: The marginal consumer line crosses the unit square at $(0, y_1)$ and $(1, y_2)$

Type 1 deviation

In this type of deviation, our results suggest that if firm A has incentive to deviate, it will deviate on both dimensions and choose a location of $(1/2, 0)$. We then find that at this location, $\pi_A^{dev} \leq \pi_A^{nodev}$ unless σ is sufficiently small (e.g., $\sigma < 0.177$ approximately for $t_2 = 0.8$).

Type 2 deviation

We find that this type of deviation can never be optimal. In particular, the "optimal" deviation always leads to $y_1 > 1$, $x_2 < 0$ or both.[26]

Type 3 and Type 4 deviation

Similar to Type 1 deviation, in Type 3 and Type 4 deviations, we also find that firm A has no incentive to deviate unless σ is sufficiently small. The threshold σ's are slightly different that in Type 1 deviation. For example, when $t_2 = 0.8$, the threshold σ is about 0.171 for Type 3 deviation and about 0.162 for Type 4 deviation.

Combined we conclude that *Max-Min* is an equilibrium when σ is not too small (e.g., $\sigma > 0.177$ when $t_2 = 0.8$).

Part 2: Show that *Min-Max* is an equilibrium under certain conditions

Now the two firms are located at $(1/2, 0)$ and $(1/2, 1)$ respectively. The proof of this part is quite similar to that of Part 1. In particular, when firm A deviates, there may be structural change in terms of where the marginal consumer line crosses the unit square. For example, when $t_2 = 0.33$ and $\sigma \geq 0.49$, firm A will deviate on both dimensions to a new location around $(0.01, 0.45)$.

Part 3: Neither *Max-Min* nor *Min-Max* is an equilibrium under certain conditions

In Part 1 and 2, we have derived the set of parameter values (t_2 and σ) such that *Max-Min* or *Min-Max* is not an equilibrium respectively. Let S_1 and S_2 denote these two sets. Then whenever $(t_2, \sigma) \in S_1 \cap S_2$, neither *Max-Min* nor *Min-Max* is an equilibrium. It can be easily verified that $S_1 \cap S_2$ is nonempty. In particular, $(t_2, \sigma) \in S_1 \cap S_2$ when t_2 is sufficiently small and σ is within certain range (e.g., $t_2 = 0.1t_1$ and σ is around 0.17).

Part 4: Show that *Intermediate-Min* is an equilibrium under certain conditions

Using the intuition right after Proposition 3 on page 17 as a guidance, we look for a symmetric *Intermediate-Min* equilibrium, that is $L_{1B} = 1 - L_{1A}$ and $L_{2A} = L_{2B} = 1/2$. We assign $t_2 = 0.1t_1$

[26] Once these violations are corrected, Type 2 deviations become special cases of Type 1 deviations. For example, if $y_1 > 1$ is replaced by $y_1 = 1$, the marginal consumer $(0, y_1)$ becomes $(0, 1)$, which is a special case of $(x_1, 1)$ in Type 1 deviations with $x_1 = 0$.

and $\sigma = 0.17$, and the unique *Intermediate-Min* candidate we find is $L_{1A} = 0.1815$. We then verify that firm A has no incentive to deviate on any single dimension or on both dimensions simultaneously.■

A.2 Extension to $n \geq 2$ dimensions

We now extend our benchmark 2-dimensional model to general n-dimensions.

A.2.1 The model

Suppose that consumers are distributed on an n-dimensional hypercube $[0,1]^n$. Let $F_i(x_i)$ and $f_i(x_i)$ denote the distribution function and density function respectively on dimension i, $i = 1, \ldots, n$. We assume that $f_i(x_i)$ is continuously differentiable on its support $[0,1]$ and distributions on different dimensions are independent. Firm A and B are distributed at the two end points of the hypercube with firm A at $[0, \ldots, 0]$ and firm B at $[1, \ldots, 1]$. Consider a consumer located at $(x_1, \ldots, x_n)$. Her indirect utility from buying from the two firms become

$$u_A = V - p_A - \sum_{i=1}^{n} t_i x_i^2, \quad u_B = V - p_B - \sum_{i=1}^{n} t_i (1 - x_i)^2,$$

where t_i is the unit transport cost on dimension i. Without loss of generality, we assume that $t_1 \geq t_2 \geq ... \geq t_n$.

Marginal consumers are defined by

$$u_A = u_B \Rightarrow p_A + \sum_{i=1}^{n} t_i x_i^2 = p_B + \sum_{i=1}^{n} t_i (1 - x_i)^2,$$

which implies

$$p_B - p_A = \sum_{i=1}^{n} t_i (2x_i - 1).$$

We can see that the set of all marginal consumers form a hyperplane.

A.2.2 Unit transport costs

Before we get into the details, let us first illustrate how our analysis in the two-dimensional model can be extended to general n-dimensions. The idea is the following. Let $p_A = p_B = p^*$ denote the equilibrium price and suppose that we want to calculate $\frac{\partial p^*}{\partial t_i}$. What we do is to aggregate all the $n-1$ dimensions other than i into one dimension $-i$. We then end up with dimension i and this aggregative dimension $-i$. Our previous analysis of the 2-dimensional model by and large carries through, once some technical issues are taken care of which we explain in detail next.[27]

Let $z_i \equiv t_i(2x_i - 1)$ and $z \equiv \sum_{i=1}^{n} z_i$. Let $l_i(z_i)$ denote the density function of z_i and let $G(z)$ and $g(z)$ denote the distribution function and density function of z respectively.

Claim 1: $l_i(z_i)$ is CD1 and symmetric on its support $[-t_i, t_i]$.

Claim 2: $g(z)$ is continuous and symmetric on its support, and is differentiable except at no more than $2^n - 2$ interior points on its domain $[-\sum_{i=1}^{n} t_i, \sum_{i=1}^{n} t_i]$.

Claim 1 is straightforward and so is the "continuous and symmetric" part of Claim 2. Next, we show that $g(z)$ is differentiable except at finitely many points on its support. We will illustrate the idea with two variables, say z_i and z_j.

Let $l_i(z_i)$ and $l_j(z_j)$ denote their density functions which are continuously differentiable on their support $[\underline{z_i}, \bar{z}_i]$ and $[\underline{z_j}, \bar{z}_j]$ respectively. The density function of their sum is

$$g(z_i + z_j = y) = \int_{-t_j}^{t_j} l_i(y - z_j) l_j(z_j) dz_j,$$

and its derivative (when defined) is

$$g'(z_i + z_j = y) = \int_{-t_j}^{t_j} l_i'(y - z_j) l_j(z_j) dz_j,$$

which is defined on its support except when $l_i'(y - z_j)$ is undefined. When can that happen? Note that the density function of each variable has structural change at either boundary of its support. For example, $l_i(z_i)$ takes different functional forms when (i) $z_i < \underline{z_i}$, (ii) $z_i \in [\underline{z_i}, \bar{z}_i]$ and (iii) $z_i > \bar{z}_i$. This structural change does not affect the continuity of $g(z_i + z_j)$ but can render it indifferentiable. Both z_i and z_j have two boundaries. There are at most 2^2 combinations of boundaries. These combinations are $\underline{z_i} + \underline{z_j}$, $\underline{z_i} + \bar{z}_j$, $\bar{z}_i + \underline{z_j}$, $\bar{z}_i + \bar{z}_j$, two of which (the smallest and the largest one) are the boundaries of the support of $z_i + z_j$. $g(z_i + z_j)$ will be differentiable on its support except at these finitely many combinations. Following similar logic, it can be seen that $g(\sum_{i=1}^{n} z_i)$ is differentiable except at most 2^n points on its support $\Omega \equiv [-\sum_{i=1}^{n} t_i, \sum_{i=1}^{n} t_i]$.[28]

Note that all consumers with $z \leq p_B - p_A$ will buy from firm A, i.e.,

$$q_A = \int_{-\infty}^{p_B - p_A} g(z) dz = \int_{-\sum_{i=1}^{n} t_i}^{p_B - p_A} g(z) dz. \tag{A.2}$$

[27] There are two new issues in an n-dimension model relative to the benchmark 2-dimensional model. First, the density function of the aggregate dimension in general will be indifferentiable at finitely many points on its support. Second, it is unclear out of dimension i and the aggregate dimension $-i$, which dimension is dominant. In contrast, in the benchmark 2-dimensional model, density function on each dimension is assumed to be continuous differentiable on its support, and dimension 1 dominates dimension 2 ($t_1 \geq t_2$).

[28] When the number of dimensions is sufficiently large, by central limit theorem the distribution of their sum z will be close to normal distribution.

Firm A's profit is

$$\pi_A = p_A q_A.$$

Taking derivative with respect to p_A, we can obtain

$$\frac{\partial \pi_A}{\partial p_A} = q_A + p_A(-1)g(z = p_B - p_A).$$

<u>Solving first-order conditions</u>

Setting $\frac{\partial \pi_A}{\partial p_A} = 0$ and imposing symmetry ($p_A = p_B$), we have $q_A = \frac{1}{2}$ and[29]

$$\frac{1}{2} - p_A \cdot g(z = 0) = 0 \Rightarrow p^* = p_A = \frac{1}{2g(z = 0)}. \tag{A.3}$$

Let $z_{-i} = \sum_{j \neq i} t_j(2x_j - 1)$. let $H(z_{-i})$ and $h(z_{-i})$ denote the corresponding distribution function and density function respectively. Its support is $[-\sum_{j \neq i} t_j, \sum_{j \neq i} t_j]$. Note that $h(\cdot)$ is symmetric about $z_{-i} = 0$.[30] Since $z_i = t_i(2x_i - 1)$, we have

$$l(z_i) = \frac{1}{2t_i} f_i(x_i).$$

In the equilibrium, $p_A = p_B$ and marginal consumers are defined by

$$z_i + z_{-i} = t_i(2x_i - 1) + z_{-i} = 0 \Rightarrow x_i = \frac{t_i - z_{-i}}{2t_i} \Rightarrow \frac{\partial x_i}{\partial t_i} = \frac{z_{-i}}{2t_i^2}.$$

Marginal consumer can be written as

$$\begin{aligned} g(z = 0) &= \int_{-\sum_{j \neq i} t_j}^{\sum_{j \neq i} t_j} l(z_i(z_{-i}))h(z_{-i})dz_{-i} \\ &= \int_{-\sum_{j \neq i} t_j}^{\sum_{j \neq i} t_j} h(z_{-i})\frac{1}{2t_i} f_i(x_i(z_{-i}))dz_{-i} \end{aligned} \tag{A.4}$$

Substituting (A.4) in to (A.3), we have

$$p^* = \frac{t_i}{\int_{-\sum_{j \neq i} t_j}^{\sum_{j \neq i} t_j} h(z_{-i}) f_i(x_i(z_{-i}))dz_{-i}}.$$

<u>Comparative statics</u>

[29] Note that this is the unique candidate symmetric pure strategy equilibrium and it is an equilibrium if the second-order conditions are satisfied.

[30] From Claim 2, $h(z_{-i})$ is continuous, symmetric on its support and differentiable except at finitely many points on its support.

Next, we analyze how p^* varies with t_i. Depending on how t_i compares with $\sum_{j\neq i} t_j$, $x_i = \frac{t_i - z_{-i}}{2t_i}$ may be outside the support of x_i which is $[0,1]$. To take this into account, we will consider two scenarios. $\frac{t_i - z_{-i}}{2t_i} \in [0,1]$ always holds in Scenario 1 where it is assumed that $t_i \geq \sum_{i\neq j} t_j$ but can be violated in Scenario 2 where it is assumed $t_i < \sum_{i\neq j} t_j$.

Scenario 1: $t_i \geq \sum_{i\neq j} t_j$

We will call this the *strong dominance* scenario. Correspondingly, Scenario 2 will be called the *weak dominance* scenario. Given our assumption that $t_1 \geq t_2 \geq \cdots \geq t_n$, this can only happen when $i = 1$ and $t_1 \geq \sum_{j\neq 1} t_j$. In this scenario, $x_i = \frac{t_i - z_{-i}}{2t_i}$ is always in $[0,1]$.

Recall that

$$p^* = \frac{t_i}{\int_{-\sum_{j\neq i} t_j}^{\sum_{j\neq i} t_j} h(z_{-i}) f_i(x_i(z_{-i})) dz_{-i}}$$

Note that the lower and upper bound of the integral in the p^* expression are independent of t_i. Next, we calculate $\frac{\partial p^*}{\partial t_i}$. Let A denote the denominator in the p^* expression, i.e.,

$$A = \int_{-\sum_{j\neq i} t_j}^{\sum_{j\neq i} t_j} h(z_{-i}) f_i(x_i(z_{-i})) dz_{-i}.$$

Then

$$\begin{aligned} \frac{\partial A}{\partial t_i} &= \int_{-\sum_{j\neq i} t_j}^{\sum_{j\neq i} t_j} h(z_{-i}) f_i'(x_i(z_{-i})) \left(\frac{\partial x_i}{\partial t_i}\right) dz_{-i} \\ &= \int_{-\sum_{j\neq i} t_j}^{\sum_{j\neq i} t_j} h(z_{-i}) f_i'(x_i(z_{-i})) \frac{z_{-i}}{2t_i^2} dz_{-i}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial p^*}{\partial t_i} &= \frac{A - t_i \frac{\partial A}{\partial t_i}}{A^2} \\ &= \frac{\int_{-\sum_{j\neq i} t_j}^{\sum_{j\neq i} t_j} h(z_{-i}) f_i(x_i(z_{-i})) dz_{-i} - \int_{-\sum_{j\neq i} t_j}^{\sum_{j\neq i} t_j} h(z_{-i}) f_i'(x_i(z_{-i})) \frac{z_{-i}}{2t_i} dz_{-i}}{A^2}. \end{aligned}$$

We can see that the first integral term in the numerator is always positive, but the sign of second integral term is ambiguous. Overall, the impact of t_i on p^* is ambiguous. The intuition is similar to the intuition in our 2-dimension benchmark model, once we aggregate $n-1$ dimensions to a single "dimension" with density $h(z_{-i})$. That is, in the strong dominance case, a change in t_i has both shifting effect and rotating effect on p^*, while in the weak dominance case, a change in t_i has only rotating effect.

We have explained that p^* may increase or decrease with t_i. Next, we provide some examples.[31]

Case 1: $\frac{\partial p^*}{\partial t_i} > 0$

Suppose that $f_i'(x_i) \leq 0, \forall x_i \in [0, 1/2]$.[32] Due to symmetry, $f_i'(x_i) \geq 0, \forall x_i \in [1/2, 1]$. When $z_{-i} \geq 0$, $x_i \in [0, 1/2]$ which implies $f_i'(x_i) \leq 0$. Then $f_i'(x_i(z_{-i}))z_{-i} \leq 0$ when $z_{-i} \in [0, \sum_{j \neq i} t_j]$. Similarly, $f_i'(x_i(z_{-i}))z_{-i} \leq 0$ when $z_{-i} \in [-\sum_{j \neq i} t_j, 0]$. Therefore,

$$\int_{-\sum_{j \neq i} t_j}^{\sum_{j \neq i} t_j} h(z_{-i}) f_i'(x_i(z_{-i})) \frac{z_{-i}}{2t_i} dz_{-i} \leq 0,$$

which further implies that $\frac{\partial p^*}{\partial t_i} > 0$.

Case 2: $\frac{\partial p^*}{\partial t_i} < 0$

Suppose that $f_i'(x_i) \geq 0, \forall x_i \in [0, 1/2]$. Following similar logic as above, we have

$$\int_{-\sum_{j \neq i} t_j}^{\sum_{j \neq i} t_j} h(z_{-i}) f_i'(x_i(z_{-i})) \frac{z_{-i}}{2t_i} dz_{-i} \geq 0.$$

If it dominates $\int_{-\sum_{j \neq i} t_j}^{\sum_{j \neq i} t_j} h(z_{-i}) f_i(x_i(z_{-i})) dz_{-i}$, then $\frac{\partial p^*}{\partial t_i} < 0$.

Scenario 2: $t_i < \sum_{i \neq j} t_j$

This scenario applies to $\forall i$ when $t_1 < \sum_{j \neq 1} t_j$ and to $\forall i \neq 1$ when $t_1 \geq \sum_{j \neq 1} t_j$. When $t_i < \sum_{i \neq j} t_j$, using $x_i = \frac{t_i - z_{-i}}{2t_i}$ can lead to x_i outside $[0, 1]$. We will restrict the range of z_{-i} to ensure that $x_i \in [0, 1]$. It can be shown that when $z_{-i} \in [-t_i, t_i]$, $x_i \in [0, 1]$ always holds. In particular, $x_i(z_{-i} = -t_i) = 1$ and $x_i(z_{-i} = t_i) = 0$.

Since $f_i(x_i(z_{-i})) = 0$ whenever $z_{-i} \neq [-t_i, t_i]$, we have

$$\begin{aligned} p^* &= \frac{t_i}{\int_{-\sum_{j \neq i} t_j}^{\sum_{j \neq i} t_j} h(z_{-i}) f_i(x_i(z_{-i})) dz_{-i}} \\ &= \frac{t_i}{\int_{-t_i}^{t_i} h(z_{-i}) f_i(x_i(z_{-i})) dz_{-i}}. \end{aligned}$$

[31] Instead of integrating over z_{-i} as above, one can integrate over x_i to obtain p^* and then calculate $\frac{\partial p^*}{\partial t_i}$. In this case, if we consider only two dimension and set $i = 1$, the corresponding results are equivalent to the $\frac{\partial p^*}{\partial t_1}$ results in our benchmark 2-dimension model (see Section 3.1).

[32] A special case of this is $f'(x_i) = 0$, i.e., consumer distribution on dimension i is uniform. In this case, it can be shown that $p^* = t_i$ regardless of $h(z_{-i})$.

Then

$$\frac{\partial p^*}{\partial t_i} = \frac{\int_{-t_i}^{t_i} h(z_{-i}) f_i(x_i(z_{-i})) dz_{-i} - \left(2t_i h(z_{-i} = t_i) f_i(x_i = 0) + \int_{-t_i}^{t_i} h(z_{-i}) f_i'(x_i(z_{-i})) \frac{z_{-i}}{2t_i} dz_{-i}\right)}{\left(\int_{-t_i}^{t_i} h(z_{-i}) f_i(x_i(z_{-i})) dz_{-i}\right)^2}.$$

Next, we consider two cases.

Case 1: $\frac{\partial p^*}{\partial t_i} > 0$.

Suppose that $f_i'(x_i) \leq 0, \forall x_i \in [0, 1/2]$. When $z_{-i} \leq 0$, $x_i \geq 1/2$, so $f'(x_i) \geq 0$ and $f_i'(x_i) z_{-i} \leq 0$. Similarly, it can be shown that $f_i'(x_i) z_{-i} \leq 0$ when $z_{-i} \geq 0$. Then $\int_{-t_i}^{t_i} h(z_{-i}) f_i'(x_i(z_{-i})) \frac{z_{-i}}{2t_i} dz_{-i} < 0$. If $h(z_{-i} = t_i)$ is sufficiently small, then $\frac{\partial p^*}{\partial t_i} > 0$.

Case 2: $\frac{\partial p^*}{\partial t_i} < 0$.

Suppose that $f_i'(x_i) \geq 0, \forall x_i \in [0, 1/2]$. Follow similar logic as above, it can be shown that $\int_{-t_i}^{t_i} h(z_{-i}) f_i'(x_i(z_{-i})) \frac{z_{-i}}{2t_i} dz_{-i} \geq 0$. Then

$$2t_i h(z_{-i} = t_i) f_i(x_i = 0) + \int_{-t_i}^{t_i} h(z_{-i}) f_i'(x_i(z_{-i})) \frac{z_{-i}}{2t_i} dz_{-i} \geq 0.$$

If it dominates $\int_{-t_i}^{t_i} h(z_{-i}) f_i(x_i(z_{-i})) dz_{-i}$, we have $\frac{\partial p^*}{\partial t_i} < 0$. A special example is when $f_i(x_i) = 1, \forall x_i \in [0, 1]$ and $h(z_{-i} = t_i) \geq h(z_{-i}), \forall z_{-i} \in [0, t_i]$. In this case, it can be shown that

$$\int_{-t_i}^{t_i} h(z_{-i}) f_i(x_i(z_{-i})) dz_{-i} \leq 2t_i h(z_{-i} = t_i) f_i(x_i = 0).$$

A.2.3 Location choice

A.2.3.1 Hypothetical symmetric location

Let L_{Ai} and L_{Bi} denote firm A and B's location on dimension i respectively, $i = 1, \ldots, n$. We will impose symmetry, i.e., $L_{Bi} = 1 - L_{Ai}$ with $L_{Ai} \in [0, 1/2]$. Next, we will calculate the equilibrium price p^* and $\frac{\partial p^*}{\partial L_{Ai}}$.

Recall that marginal consumers are defined by

$$\begin{aligned} p_A + \sum_i t_i (L_{Ai} - x_i)^2 &= p_B + \sum_{i=1}^n t_i (1 - L_{Ai} - x_i)^2 \\ \Rightarrow p_B - p_A &= \sum_{i=1}^n (1 - 2L_{Ai}) t_i (2x_i - 1). \end{aligned}$$

Let $\tilde{z} \equiv \sum_{i=1}^{n}(1-2L_{Ai})t_i(2x_i-1)$ and let $g(\tilde{z})$ denote its density function. Following similar analysis as in the previous section, it can be shown that

$$p^* = \frac{1}{2g(\tilde{z}=0)}.$$

Let $\tilde{z}_{-i} \equiv \sum_{j\neq i}(1-2L_{Aj})t_j(2x_j-1)$ and let $h(\tilde{z}_{-i})$ denote its density function on its support $[-\sum_{j\neq i} t_j(1-2L_{Aj}), \sum_{j\neq i} t_j(1-2L_{Aj})]$. Similarly, let $\tilde{z}_i = (1-2L_{Ai})t_i(2x_i-1)$ and let $l_i(\tilde{z}_i)$ denote its density function. Then

$$l_i(\tilde{z}_i) = \frac{1}{2t_i(1-2L_{Ai})} f_i(x_i).$$

Note that if we set $L_{Ai} = 0$, $\forall i$, then the definitions of $(\tilde{z},\, \tilde{z}_i,\, \tilde{z}_{-i})$ will be the same as (z, z_i, z_{-i}) in the previous section. Similarly, we will consider two scenarios: *strong dominance* and *weak dominance* respectively.

Scenario 1: $t_i(1-2L_{Ai}) \geq \sum_{j\neq i} t_j(1-2L_{Aj})$

Under this scenario (strong dominance), it can be shown that

$$\begin{aligned} g(\tilde{z}=0) &= \int_{\underline{z}^s}^{\bar{z}^s} h(\tilde{z}_{-i}) l_i(\tilde{z}_i) d\tilde{z}_{-i} \\ &= \frac{\int_{\underline{z}^s}^{\bar{z}^s} h(\tilde{z}_{-i}) f_i(x_i(\tilde{z}_i)) d\tilde{z}_{-i}}{2t_i(1-2L_{Ai})}, \end{aligned}$$

where $\bar{z}^s = \sum_{j\neq i} t_j(1-2L_{Aj})$, $\underline{z}^s = \sum_{j\neq i} -t_j(1-2L_{Aj})$.

Substituting this in to p^* and taking derivative with respect to L_{Ai}, we have

$$\frac{\partial p^*}{\partial L_{Ai}} = t_i \times \frac{(-2)\int_{\underline{z}^s}^{\bar{z}^s} h(\tilde{z}_{-i}) f_i(x_i(\tilde{z}_i)) d\tilde{z}_{-i} - (1-2L_{Ai}) \int_{\underline{z}^s}^{\bar{z}^s} h(\tilde{z}_{-i}) f_i'(x_i(\tilde{z}_i)) \left(\frac{\partial x_i}{\partial L_{Ai}}\right) d\tilde{z}_{-i}}{\left(\int_{\underline{z}^s}^{\bar{z}^s} h(\tilde{z}_{-i}) f_i(x_i(\tilde{z}_i)) d\tilde{z}_{-i}\right)^2}.$$

It is easy to see that the sign of $\frac{\partial p^*}{\partial L_{Ai}}$ is ambiguous under general distribution.

Scenario 2: $t_i(1-2L_{Ai}) < \sum_{j\neq i} t_j(1-2L_{Aj})$

Under this scenario (weak dominance), we need to restrict $\tilde{z}_{-i} \in [\underline{z}^w, \bar{z}^w]$ where $\underline{z}^w = -t_i(1-2L_{Ai})$ and $\bar{z}^w = t_i(1-2L_{Ai})$. Then it can be shown that

$$g(\tilde{z}=0) = \frac{\int_{\underline{z}^w}^{\bar{z}^w} h(\tilde{z}_{-i}) f_i(x_i(\tilde{z}_i)) d\tilde{z}_{-i}}{2t_i(1-2L_{Ai})},$$

and

$$\frac{\partial p^*}{\partial L_{Ai}} = \frac{t_i}{\left(\int_{\underline{z}^w}^{\bar{z}^w} h(\tilde{z}_{-i}) f_i(x_i(\tilde{z}_i)) d\tilde{z}_{-i}\right)^2} \times \Bigg[(-2)\int_{\underline{z}^w}^{\bar{z}^w} h(\tilde{z}_{-i}) f_i(x_i(\tilde{z}_i)) d\tilde{z}_{-i}$$

$$- (1-2L_{Ai})\Big(-4t_i h(\bar{z}^w) f_i(0) + \int_{\underline{z}^w}^{\bar{z}^w} h(\tilde{z}_{-i}) f_i'(x_i(\tilde{z}_i)) \left(\frac{\partial x_i}{\partial L_{Ai}}\right) d\tilde{z}_{-i}\Big)\Bigg],$$

where

$$\frac{\partial x_i}{\partial L_{Ai}} = -\frac{1}{2}\frac{\tilde{z}_{-i}}{t_i(L_{Ai}-1)^2}.$$

Similar to Scenario 1, the sign of $\frac{\partial p^*}{\partial L_{Ai}}$ here is also ambiguous under general distribution.

A.2.3.2 Equilibrium location choice

In our 2-dimensional benchmark model, we have explained that *Max-Min* is not always an equilibrium once we consider nonuniform distributions. Here, we prove its counterpart in n-dimension model, i.e., show that *Max-Min* - $\cdots$ - *Min* is not always an equilibrium. The idea is the following. Let t_i and $f_i(x_i)$, $i = 1, 2$ be the same as in our benchmark model. For any dimension $i > 2$, we will choose minimum differentiation. With minimum differentiation, t_j and $f_j(x_j)$ are redundant for any $j > 2$. That is, they do not enter into the marginal consumer expression or firms' demand functions. In our 2-dimensional model, we have identified conditions such that firm A has incentive to deviate on dimension 1 and 2. Intuitively the same condition can be applied to the first two dimensions here. That is, under *Max-Min* - $\cdots$ - *Min*, firm A has incentive to deviate on dimension 1 and 2.[33] The same logic can be applied to show, for example, that *Min-Max-Min* $\cdots$-*Min* is not an equilibrium in the n-dimensional model if *Min-Max* is not an equilibrium in the corresponding 2-dimensional model.

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[33] The same intuition also applies here. It can be shown that at *Max-Min* - $\cdots$ - *Min*, under uniform distribution, competition effect is zero when firm A deviates slightly on the second dimension. Market share effect then prevents from A from moving toward the middle. Under nonuniform distribution, however, the market share effect can be negative and dominate the market share effect. In this case, firm A would have an incentive to move toward the middle on the second dimension.

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