Unit price auction procedures

Yumiko Baba*
4-4-25
Shibuya Shibuya-ku
Tokyo 150-8366 JAPAN

Abstract

This paper proposes four unit price auction procedures: the pay your bid auction, the lowest winner's bid auction, the highest loser's auction, and the pay the next highest bid to yours auction. Our model is the same as the one analyzed by Varian (2007) and Edelman, Ostrovsky, and Schwarz (2007) and is a special case of Baba (1997) and Baba (1998) which assumes that the value of the item is supermodular with respect to a bidder's type and a public signal and multiplication is a special example of supermodularity. All four unit price auction procedures yield the same expected revenue to the seller and implement the optimal auction under the assumptions of unit demand, indivisible items, no collusive behavior, and risk-neutrality of bidders and the seller. Further, the lowest winner's auction and the highest loser's auction satisfy a fair criterion in the sense that each winner pays the same unit price regardless of the item he wins. A fair criterion is important in procurement auctions by the governments and used widely in road repair contract in Europe.

JEL Classifications: C72, D44, M37

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1. Introduction

Auctioning sponsored links is almost the only source of Google's revenue. Not only Google, but other internet search engines such as Microsoft and Yahoo! also sell sponsored links by auction. Sponsored links are the advertisement of private companies which appear on the top and on the right of the search results when a consumer types keywords to acquire relevant information before making his/her consumption decision. When a consumer clicks one of these sponsored-links, s/he jumps to the advertiser's web page. A search engine usually charges per click fee on the advertiser. All big three search engines use a similar auction procedure to sell

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sponsored links, which is called the generalized second price (hereafter, the GPS) auctions by Edelman, Ostrovsky, and Schwarz (2007) (hereafter EOS). A search engine auctions off several advertisement positions where the winner of the kth position pays $(k+1)$th highest bid. An advertiser's (a bidder's) value to win the kth position is assumed to be his[1] per click profit multiplied by the (expected) number of clicks he enjoys when he wins the kth position. It is assumed that a bidder's profit per click is the same regardless of his position and the (expected) number of clicks for the kth position is exogenously given and commonly known by bidders and the seller. This structure is a special case of Baba (1997) and Baba (1998) where she considers the optimal privatization scheme. In her model, the government auctions off multiple heterogeneous items and a bidder's value of the kth item is a supermodular function of a bidder's type and a public signal. She analyzes sequential first and second price auctions and show that they implement the optimal auction mechanism. Her model includes a multiplicative function as a special case and sponsored link auctions are a special example when we interpret a bidder's type as a bidder's profit per click and a public signal as the number of clicks a bidder enjoys when he wins the certain position. Varian (2007) call the same problem as position auctions and independently obtains similar results to EOS. Both Varian and EOS characterize the Nash equilibrium under perfect information and show that the difference and the equivalence between the GSP auction and the VCG mechanism. It is shown that truth-telling is not a dominant strategy in the GSP auction while it is in the VCG mechanism, but the outcomes of the GSP auction and the VCG mechanism are the same. EOS also examine the generalized English auction which corresponds to the GSP auction under incomplete information and show the difference and the equivalence between the GSP auction and the VCG mechanism under perfect information also hold under incomplete information.

EOS and Varian focus on the special feature of sponsored link auctions: bidders initially have private information about their types, can gradually learn the values of their competitors, can respond over time by updating information. Therefore, they modelize sponsored link auctions as infinitely repeated games although they do not explicitly analyze the equilibrium of infinitely repeated games. We consider situations simpler and treat them as one shot static games. Therefore, we formalize the problem as sealed-bid auctions. Although the view of EOS and of Varian might be suitable to position auctions, our model might be appropriate to procurement auctions such as road

[1] Female pronouns are used for the seller and male pronouns are used for bidders without any intension of sexual discrimination.

repair service, garbage collection service, milk procurement by school district, and so on. As Milgrom (1987) pointed out, the governments use sealed-bid auctions because English auctions are vulnerable to collusion. Rene (2011) reports that the governments use unit price auction procedures to auction off road repair contract of the next year when no one know how many holes to be repaired in the following year. Other procurement auctions such as garbage collection service and milk procurement by school district are essentially per household and per capita service and they can fit to our model. All of four unit price auction procedures we propose in this paper are robust to collusion because they are sealed-bid auctions. Further, two of four auction procedures satisfy a fair criterion in the sense that each winner pays the same per-unit price regardless of the item he wins. In position auctions, this means that each winner pays the same per-click price to the search engine regardless of the position he wins. Although we do not need to worry about a fair criterion in sponsored link auctions because they are private auctions, it is an important criterion when we consider public procurement auctions. Athey and Ellison (2011) and Chen and He (2011) introduce consumer search into the EOS model and endogenize the value per click to a bidder when he wins the kth position and characterize the Bayesian Nash equilibrium under incomplete information. Ostorovsky and Schwarz (2009) conduct a field experiment and showed that introducing reserve prices increase the seller's revenue.

2. The model.

The basic environment we analyze is the same as the one proposed by Baba (1997), Baba (1998), EOS (2007), and Varian (2007). A special application of our model is sponsored link auctions; however, we explain the model in general terms because there are other applications which fit to our model such as road repair contract, garbage collection service, milk procurement by school district, and so on. The risk-neutral seller auctions off m heterogeneous items to $n > m$ risk-neutral bidders. We denote the set of the auctioned items by M and the set of bidders by N. The valuation of the item $k \in M$ for bidder i is denoted by x_i^k and is expressed as $x_i^k = t_k x_i$. In sponsored link auctions, x_i is bidder i's per click profit and t_k is the (expected) number of clicks he enjoys when he wins the kth position. We assume $x_i \sim F(\cdot)$, $\forall i \in N$ and that the probability distribution function $F(\cdot)$ is differentiable and its density is denoted by $f(\cdot)$. We also assume that the support of $F(\cdot)$ is $[0,1]$ without loss of generality. Further, we impose the unit demand assumption and the items are indivisible. The timing of the game is described in figure1.

Figure1.

t=0 — The seller announces the auction procedure and she commits to it.

t=1 — Bidder i observes the realized value of x_i and submits a single bid, b_i, $^\forall i \in N$.

t=2 — The allocation and the payment are carried out by following the auction procedure announced by the seller in period 0.

We consider four unit price auction procedures which are simpler than the mechanisms proposed by Baba (1997), Baba (1998), EOS (2007), and Varian (2007) in the sense that they are static auctions and a bidder submits only one bid instead of m bids, one for each different item. The unit price auction procedures work well when a bidder's private signal is one dimensional. Now, we explain four unit price auction procedures in detail and characterize their symmetric Bayesian equilibrium bidding functions.
The first one is the pay your bid auction. The second one is the lowest winner's bid auction. The third one is the highest loser's bid auction. The fourth one is the pay the next highest bid to yours auction. Common to four auction procedures, we focus on the case of m=2 for simplicity, but all the arguments are straightforwardly applicable to general case of m items. First, we assume that a bidder's utility function is quasi linear with respect to money. Therefore, bidder $i's$, $^\forall i \in N$ utility when his type is x_i, he wins the item k $(k = 1{,}2)$, and pays z^{ik} is expressed as $u(x_i) = t_k x_i - z^{ik}$ and his utility is zero when he loses the auction and acquires no item. Each bidder submits only 1 bid and the final allocation and the payments are determined by n bids, one from each bidder.
Suppose bidder i submits a bid of b_1^i in the pay your bid auction. He acquires the first item if his bid of b_1^i is the highest among n bids, one from each bidder, that is, if $b_1^i > b_1^j$, $^\forall j \neq i$. Then, his payment to the seller is determined by $t_1 b_1^i$. He wins the second item if his bid of b_1^i is the second highest among n bids, one from each bidder, that is, if $b_1^k > b_1^i > b_1^j$, $^\exists k \neq i$, $^\forall j \neq i, k$. Then, bidder k wins the first item and pays $t_1 b_1^k$ and bidder i wins the second item and pays $t_2 b_1^i$. The second one is the lowest winner's bid auction. Suppose bidder i submits a bid of b_2^i in the lowest winner's bid auction. Then the allocation rule is the same as in the pay your bid auction and bidder i wins the first item if b_2^i is the highest among n bids, one from each bidder and wins the second item if b_2^i is the second highest among n bids, one from each bidder. The only difference is the payment function. Now, bidder i pays $t_1 b_2^h$ if he

wins the first item when bidder h submits the second highest bid of b_2^h among n bids, one from each bidder. Bidder i pays $t_2\, b_2^i$ if he wins the second item. The highest loser's bid auction works in the following way. Suppose bidder i submits a bid of b_3^i. Again, the allocation rule is the same as in the pay your bid auction and the lowest winner's bid auction, but the payment rule is as follows. Now, bidder i pays $t_1\, b_3^l$ if he wins the first item and pays $t_2\, b_3^l$ if he wins the second item, where b_3^l is the third highest bid among n bids, one from each bidder. Lastly, consider the pay the next highest bid to yours auction and suppose bidder i submits a bid of b_4^i. The allocation rule is the same as the other three auction procedures, but the payment rule is as follows. When bidder i wins the first item, he pays $t_1\, b_4^h$ where b_4^h is the second highest bid among n bids, one from each bidder. When bidder i wins the second item, he pays $t_2 b_4^l$. where b_4^l is the third highest bid among n bids, one from each bidder. The payment rule of the pay the next highest bid to yours auction is similar to that of the GSP auction proposed by EOS. Since we formalize it as a sealed-bid auction instead of an ascending auction, we can characterize the symmetric Bayesian equilibrium bidding function easily. As we mentioned before, our set up is more appropriate to procurement auctions such as road repair contract, garbage collection service, milk procurement by school district, and so on, but can include sponsored link auctions as an example. We characterize the Bayesian equilibrium of each auction procedure in the following section.

3. Analysis

This section consists of four subsections. 3-1 characterizes the symmetric Bayesian equilibrium bidding function of the pay your bid auction, 3-2 characterizes the symmetric Bayesian equilibrium bidding function of the lowest winner's bid auction, 3-3 characterizes the symmetric Bayesian equilibrium bidding function of the highest loser's auction, and 3-4 characterizes the symmetric Bayesian equilibrium bidding function of the pay the next highest bit to yours auction.

3-1. The pay your bid auction.

This subsection characterizes the symmetric Bayesian equilibrium bidding function of the pay your bid auction. As being explained in the previous section, bidder i wins the first item if his bid, b_1^i, is the highest among n bids and pays $t_1 b_1^i$. He wins the second item if his bid is the second highest among n bids and pays $t_2 b_1^i$. He wins nothing if his bid is lower than the second highest bid among n bids. We omit subscript i from now on if there is no risk of confusion especially because we focus on

the symmetric equilibrium bidding function.
We need to consider two cases separately.

Case A. $\tilde{x} > x$

Since we assume a quasi linear utility function as explained in section2, bidder i's expected payoff when his type is x and he submits a bid of $b_i(\tilde{x})$ is expressed as follows as long as there exists a symmetric equilibrium bidding function w.r.t. a bidder's type.

$EU_1(\tilde{x};x) = F(\tilde{x})^n t_1\big(x - b_1(\tilde{x})\big) + n\big(1 - F(\tilde{x})\big)F(\tilde{x})^{n-1}t_2\big(x - b_1(\tilde{x})\big)$

We know that there exists a symmetric equilibrium bidding function w.r.t. a bidder's type due to supermodularity of the objective function w.r.t. a bidder's type and his bid.
Bidder i solves the following problem in the pay your bid auction.

$$Max_{\tilde{x}}\ EU_1(\tilde{x};x) = F(\tilde{x})^n t_1\big(x - b_1(\tilde{x})\big) + n\big(1 - F(\tilde{x})\big)F(\tilde{x})^{n-1}t_2\big(x - b_1(\tilde{x})\big) \quad \ldots(3\text{-}1\text{-}1)$$

F.O.C. of (3-1-1) w.r.t. $\tilde{x}$ is expressed as follows.

$$nF(\tilde{x})^{n-1}f(\tilde{x})t_1\big(x - b_1(\tilde{x})\big) - F(\tilde{x})^n t_1 b_1'(\tilde{x}) + n(n-1)\big(1 - F(\tilde{x})\big)F(\tilde{x})^{n-2}f(\tilde{x})t_2\big(x - b_1(\tilde{x})\big)$$
$$-nF(\tilde{x})^{n-1}f(\tilde{x})t_2\big(x - b_1(\tilde{x})\big) - n\big(1 - F(\tilde{x})\big)F(\tilde{x})^{n-2}b_1'(\tilde{x})=0 \quad \ldots(3\text{-}1\text{-}2)$$

Evaluate (2) at $\tilde{x} = x$ becomes

$$nF(x)^{n-1}t_1\big(x - b_1(x)\big) - F(\tilde{x})^n t_1 b_1'(x) + n(n-1)\big(1 - F(x)\big)F(x)^{n-2}f(x)t_2\big(x - b_1(x)\big)$$
$$-nF(x)^{n-1}f(x)t_2\big(x - b_1(x)\big) - n\big(1 - F(x)\big)F(x)^{n-2}b_1'(x)=0 \quad \ldots(3\text{-}1\text{-}3)$$

Case B. $\tilde{x} < x$

It is easily shown that the F.O.C. of this case evaluated at $\tilde{x} = x$ is exactly the same as (3-1-3) in case A. Therefore, it suffices to solve (3-1-3) for $b_1(x)$.

To do so, we rewrite (3-1-3) as follows.

$$b_1'(x) + \frac{\big(nF(x)^{n-1}t_1 + n(n-1)\big(1 - F(x)\big)F(x)^{n-2}f(x)t_2 - nF(x)^{n-1}f(x)t_2\big)}{(F(x)^n t_1 + n(1 - F(x))F(x)^{n-2})}b_1(x)$$

$$= \frac{\big(nF(x)^{n-1}t_1 + n(n-1)\big(1 - F(x)\big)F(x)^{n-2}f(x)t_2 - nF(x)^{n-1}f(x)t_2\big)}{(F(x)^n t_1 + n(1 - F(x))F(x)^{n-2})}x \quad \ldots(3\text{-}1\text{-}4)$$

Let us define

$$A(x) = F(x)^n t_1 + n(1 - F(x))F(x)^{n-2}t_2 \quad \ldots(3\text{-}1\text{-}5)$$

By using (3-1-5), we can rewrite (3-1-4) as follows.

$$b_1'(x) + \frac{A_1'(x)}{A_1(x)}b_1(x) = \frac{A_1'(x)}{A_1(x)}x \quad \ldots(3\text{-}1\text{-}6)$$

(3-1-6) is a first order linear differential equation w.r.t. $b_1(x)$ and the solution takes the form of

$$b_1(x) = e^{\int_0^x \left(-\frac{A_1'(x)}{A_1(x)}\right)dx}\left[\int_0^x \left\{\left(\frac{A_1'(x)}{A_1(x)}x\right) e^{\int_0^x \left(-\frac{A_1'(x)}{A_1(x)}\right)dx}\right\} dx + C\right],$$

where C is a constant of integration of integration.

Since our equilibrium bidding function satisfies the initial condition of b(0) = 0, we obtain $C = 0$.

Next, we need to show the global optimality condition holds. To do so, it suffices to show the following formula holds.

$$sgn\left(-\frac{\partial U_1(\hat{x};x)}{\partial \hat{x}}\Big|_{\hat{x}=x} + \frac{\partial U_1(\hat{x};x)}{\partial \hat{x}}\Big|_{\hat{x}=\tilde{x}}\right) = sgn(x - \tilde{x}) \qquad \text{...(3-1-7)}$$

We can rewrite l.h.s. of (3-1-7) as follows.

$$sgn\left(-\frac{\partial U_1(\hat{x};x)}{\partial \hat{x}}\Big|_{\hat{x}=x} + \frac{\partial U_1(\hat{x};x)}{\partial \hat{x}}\Big|_{\hat{x}=\tilde{x}}\right)$$

$$=sgn\left(\begin{array}{c}\left(\begin{array}{c} nF(\tilde{x})^{n-1}t_1\big(x - b_1(\tilde{x})\big) - F(\tilde{x})^n t_1 b_1'(\tilde{x}) \\ +n(n-1)\big(1 - F(\tilde{x})\big)F(\tilde{x})^{n-2}f(\tilde{x})t_2\big(x - b_1(\tilde{x})\big) \\ -nF(\tilde{x})^{n-1}f(\tilde{x})t_2\big(x - b_1(\tilde{x})\big) - n\big(1 - F(\tilde{x})\big)F(\tilde{x})^{n-2}b_1'(\tilde{x}) \end{array}\right) \\ -\left(\begin{array}{c} nF(\tilde{x})^{n-1}f(\tilde{x})t_1\big(\tilde{x} - b_1(\tilde{x})\big) - F(\tilde{x})^n t_1 b_1'(\tilde{x}) \\ +n(n-1)\big(1 - F(\tilde{x})\big)F(\tilde{x})^{n-2}f(\tilde{x})t_2\big(x - b_1(\tilde{x})\big) \\ -nF(\tilde{x})^{n-1}f(\tilde{x})t_2\big(\tilde{x} - b_1(\tilde{x})\big) - n\big(1 - F(\tilde{x})\big)F(\tilde{x})^{n-2}b_1'(\tilde{x}) \end{array}\right)\end{array}\right)$$

$$=sgn\left(\begin{array}{c} nF(\tilde{x})^{n-1}f(\tilde{x})t_1 \\ +n(n-1)\big(1 - F(\tilde{x})\big)F(\tilde{x})^{n-2}f(\tilde{x})t_2 \\ -nF(\tilde{x})^{n-1}f(\tilde{x})t_2 \end{array}\right)(x - \tilde{x})$$

$$=sgn\left(\begin{array}{c} nF(\tilde{x})^{n-1}f(\tilde{x})(t_1 - t_2) \\ +n(n-1)\big(1 - F(\tilde{x})\big)F(\tilde{x})^{n-2}f(\tilde{x})t_2 \end{array}\right)(x - \tilde{x}) \qquad \text{...(3-1-8)}$$

$$=sgn(x - \tilde{x}).$$

We apply (3-1-2) to each term of $-\frac{\partial EU_1(\hat{x};x)}{\partial \hat{x}}\Big|_{\hat{x}=x} + \frac{\partial EU_1(\hat{x};x)}{\partial \hat{x}}\Big|_{\hat{x}=\tilde{x}}$ and use the fact that $-\frac{\partial u(\hat{x};x)}{\partial \hat{x}}\Big|_{\hat{x}=x} = -\frac{\partial u(\hat{x};\tilde{x})}{\partial \hat{x}}\Big|_{\hat{x}=\tilde{x}}$=0 to obtain the first equality. The last equality is obtained due to our assumption of $t_1 \geq t_2$.

(3-1-8) implies that the global maximization condition is satisfied.

The result is summarized in the following proposition.

Proposition1. (The symmetric Bayesian equilibrium bidding function of the pay your bid auction)

The symmetric Bayesian equilibrium bidding function of the pay your bid auction is characterized as follows.

$b_1(x) = \int_0^x \frac{A_1{}'(x)x}{A_1(x)}dx$, where $A_1(x) = t_1F(x)^n + n(1-F(x))F(x)^{n-1}t_2$

3-2. The lowest winner's bid auction

This subsection characterizes the symmetric Bayesian equilibrium of the lowest winner's bid auction. Now bidder i pays $t_1b_2(y_1)$ when he wins the first item, pays $t_2b_2(x_i)$ when he wins the second item, and pays nothing if he does not acquire either item, where we denote the highest bid among $b_2(x_j)s,\ j \neq i$ by $b_2(y_1)$. As in the previous subsection, we need to consider two cases separately.

Case A. $\tilde{x} > x$.

In this case, bidder $i's$ expected utility when his type is x and he submits a bid of $b_2(\tilde{x})$ is expressed as follows as long as there exists a symmetric equilibrium bidding function which is increasing for a bidder's type. We know that there exists a symmetric equilibrium bidding function which is increasing for a bidder's type due to supermodularity of the objective function w.r.t. a bidder's type and his bid.

$$EU_2(\tilde{x};x) = t_1\int_0^{\tilde{x}}(x - b_2\ (y_1)nF(y_1)^{n-1}f(y_1)dy_1$$

$$+t_2\int_0^{\tilde{x}}\int_{\tilde{x}}^1(x - b_2\ (\tilde{x}))\,n(n-1)F(y_2)^{n-2}f(y_1)f(y_2)dy_1dy_2,$$

Therefore, bidder i solves the following problem in the lowest winner's bid auction.

$$Max_{\tilde{x}}\ \ t_1\int_0^{\tilde{x}}(x - b_2\ (y_1)nF(y_1)^{n-1}f(y_1)dy_1$$

$$+t_2\int_0^{\tilde{x}}\int_{\tilde{x}}^1(x - b(\tilde{x}))\,n(n-1)F(y_2)^{n-2}f(y_1)f(y_2)dy_1dy_2,$$

where we denote the second highest bid among $b_2(x_j),\ j \neq i$ by $b_2(y_2)$ …(3-2-1)

Note that the following formula holds.

$$\int_0^{\tilde{x}}\int_{\tilde{x}}^1 n(n-1)F(y_2)^{n-2}f(y_1)f(y_2)dy_1dy_2$$

$$= \int_{\tilde{x}}^1\int_0^{\tilde{x}} n(n-1)F(y_2)^{n-2}f(y_2)dy_2f(y_1)dy_1$$

$$=n(1-F(\tilde{x}))\int_0^{\tilde{x}}(n-1)F(y_2)^{n-2}\,f(y_2)dy_2$$

$$=n(1-F(\tilde{x}))F(\tilde{x})^{n-1} \quad \text{…(3-2-2)}$$

We can rewrite (3-2-1) by using (3-2-2).

$$Max_{\tilde{x}} \quad t_1 \int_0^{\tilde{x}} (x - b_2 \ (y_1) n F(y_1)^{n-1} f(y_1) dy_1$$

$$+t_2 n(1 - F(\tilde{x}))F(\tilde{x})^{n-1}(x - b_2(\tilde{x})) \qquad \text{...(3-2-3)}$$

F.O.C. of (3-2-3) w.r.t. $\tilde{x}$ yields

$$t_1\big(x - b_2(\tilde{x})\big) nF(\tilde{x})^{n-1} f(\tilde{x})$$
$$+t_2 n(n-1)(1 - F(\tilde{x}))F(\tilde{x})^{n-2} f(\tilde{x})\big(x - b_2(\tilde{x})\big)$$
$$-t_2 nF(\tilde{x})^{n-1} f(\tilde{x})\big(x - b_2(\tilde{x})\big)$$
$$-t_2 n(1 - F(\tilde{x}))F(\tilde{x})^{n-1} b_2'(\tilde{x}) = 0 \qquad \text{...(3-2-4)}$$

Evaluate (3-2-4) at $\tilde{x} = x$ becomes

$$t_1\big(x - b_2(x)\big) nF(x)^{n-1} f(x)$$
$$+t_2 n(n-1)(1 - F(x))F(x)^{n-2} f(x)\big(x - b_2(x)\big)$$
$$-t_2 nF(x)^{n-1} f(x)(x - b_2(x)$$
$$-t_2 n(1 - F(\tilde{x}))F(\tilde{x})^{n-1} b_2'(\tilde{x}) = 0 \qquad \text{...(3-2-5)}$$

Simplifying (3-2-5) yields

$$b_2'(x) t_2 n(1 - F(x))F(x)^{n-1}$$
$$+nF(x)^{n-2} f(x) \Big((t_1 - t_2)F(x) + t_2(n-1)\big(1 - F(x)\big) \Big) b_2(x)$$
$$= nF(x)^{n-2} f(x)\big(t_1 F(x) + t_2(n-1)\big(1 - F(x)\big) - F(x)\big)x \qquad \text{...(3-2-6)}$$

Dividing both sides of (3-2-6) by $t_2 n(1 - F(x))F(x)^{n-1}$ yields

$$b_2'(x) + \left(\frac{nF(x)^{n-2} f(x) \Big(t_1 F(x) + t_2\big((n-1) - nF(x)\big)\Big)}{t_2 n(1 - F(x))F(x)^{n-1}} \right) b_2(x)$$

$$= \left(\frac{nF(x)^{n-2} f(x)\big(t_1 F(x) + t_2((n-1) - nF(x))\big)}{t_2 n(1 - F(x))F(x)^{n-1}} \right) x \qquad \text{...(3-2-7)}$$

We can further rewrite (3-2-7) as

$$b_2'(x) + \frac{A_2(x)}{t_2\big(1 - F(x)\big)F(x)} b_2(x) = \frac{A_2(x)}{t_2\big(1 - F(x)\big)F(x)} x,$$

where $A_2(x) = f(x) \Big(t_1 F(x) + t_2\big((n-1) - nF(x)\big)\Big)$ ...(3-2-8)

Case B. $\tilde{x} < x$

It is easily shown that the F.O.C. of this case evaluated at $\tilde{x} = x$ is exactly the same as (3-2-5) in case A. Therefore, it suffices to solve (3-2-8) for $b_2(x)$

(3-2-8) is a linear differential equation for $b_2(x)$and the solution takes the following form.

$$b_2(x) = e^{\int_0^x -\frac{A_2(x)x}{(1-F(x))F(x)}dx}\left[\int_0^x \left\{\frac{A_2(x)x}{(1-F(x))F(x)} e^{\int_0^x \frac{A_2(x)x}{(1-F(x))F(x)}dx} dx\right\} dx + C\right] \quad \ldots(3\text{-}2\text{-}9)$$

C is a constant of integration and is equal to 0 by the initial condition of $b_2(0) = 0$.
Next, we check the global optimality condition holds. To do so, we need to show the following equation.

$$sgn\left(-\frac{\partial U_2(\hat{x};x)}{\partial \hat{x}}\bigg|_{\hat{x}=x} + \frac{\partial U_2(\hat{x};x)}{\partial \hat{x}}\bigg|_{\hat{x}=\tilde{x}}\right) = sgn(x - \tilde{x}) \quad \ldots(3\text{-}2\text{-}10)$$

The l.h.s. of (3-2-10) takes the following form.

$$sgn\left(-\frac{\partial EU_2(\hat{x};x)}{\partial \hat{x}}\bigg|_{\hat{x}=x} + \frac{\partial EU_2(\hat{x};x)}{\partial \hat{x}}\bigg|_{\hat{x}=\tilde{x}}\right)$$

$$= sgn\left(\begin{matrix} -\left(\begin{matrix} -b_2'(\tilde{x})t_2 n(1-F(\tilde{x}))F(\tilde{x})^{n-1} \\ -nF(\tilde{x})^{n-2}f(\tilde{x})\Big((t_1-t_2)F(\tilde{x}) + t_2(n-1)\big(1-F(\tilde{x})\big)\Big)b_2(\tilde{x}) \\ +nF(x)^{n-2}f(\tilde{x})\Big((t_1-t_2)F(\tilde{x}) + t_2(n-1)\big(1-F(\tilde{x})\big)\Big)\tilde{x} \end{matrix}\right) \\ \left(\begin{matrix} -b_2'(\tilde{x})t_2 n(1-F(\tilde{x}))F(\tilde{x})^{n-1} \\ -nF(\tilde{x})^{n-2}f(\tilde{x})\Big((t_1-t_2)F(\tilde{x}) + t_2(n-1)\big(1-F(\tilde{x})\big)\Big)b_2(\tilde{x}) \\ +nF(\tilde{x})^{n-2}f(\tilde{x})\Big((t_1-t_2)F(\tilde{x}) + t_2(n-1)\big(1-F(\tilde{x})\big)\Big)x \end{matrix}\right) \end{matrix}\right)$$

$$= sgn\left(nF(\tilde{x})^{n-2}f(\tilde{x})\Big((t_1-t_2)F(\tilde{x}) + (n-1)\big(1-F(\tilde{x})\big)\Big)\right)(x-\tilde{x})$$

$$= sgn(x - \tilde{x}) \quad \ldots(3\text{-}2\text{-}11)$$

To obtain the first equality of (3-2-11), we apply (3-2-4) to each term of $-\frac{\partial U_2(\hat{x};x)}{\partial \hat{x}}\Big|_{\hat{x}=x} + \frac{\partial U_2(\hat{x};x)}{\partial \hat{x}}\Big|_{\hat{x}=\tilde{x}}$ and use the fact that $-\frac{\partial U_2(\hat{x};x)}{\partial \hat{x}}\Big|_{\hat{x}=x} = 0 = -\frac{\partial U_2(\hat{x};\tilde{x})}{\partial \hat{x}}\Big|_{\hat{x}=\tilde{x}}$. The last equality follows from our assumption of $t_1 \geq t_2$.
The result is summarized in the following proposition.

Proposition2. (The symmetric Bayesian equilibrium bidding function of the highest winner's bid auction)
The symmetric Bayesian equilibrium of the lowest winner's bid auction, $b_2(x)$, is characterized as follows.

$$b_2(x) = e^{\int_0^x -\frac{A_2(x)x}{(1-F(x))F(x)}dx}\left[\int_0^x \left\{\frac{A_2(x)x}{(1-F(x))F(x)} e^{\int_0^x \frac{A_2(x)x}{(1-F(x))F(x)}dx} dx\right\} dx\right],$$

where $A_2(x) = f(x)\big(t_1F(x) + t_2(n-1)\big(1-F(x)\big) - F(x)\big)$.

3-3. The highest loser's bid auction

This subsection characterizes the symmetric Bayesian equilibrium bidding function of the highest loser's bid auction. In the highest loser's bid auction, as in the your bid auction and the lowest winner's bid auction, bidder i wins the first item if his bid, b_1^i, is the highest among n bids, one from each bidder and wins the second item if his bid, b_1^i, is the highest among n bids, one from each bidder. Bidder i pays $t_1 b_3(y_2)$ when he wins the first item and pays $t_2 b_3(y_2)$ if he wins the second item, where we denote the second highest bid among $b_3(x_j)$, $j \neq i$ by $b_3(y_2)$. He pays nothing if he does not acquire any item at all. As in the previous two subsections, we need to consider two cases separately.

Case A. $\tilde{x} > x$.

In this case, bidder $i's$ expected utility when his type is x and he submits a bid of $b_3(\tilde{x})$ is expressed as follows as long as there exists a symmetric equilibrium bidding function which is increasing for a bidder's type. We know that there exists a symmetric equilibrium bidding function which is increasing for a bidder's type due to supermodularity of the objective function w.r.t. a bidder's type and his bid.

$$EU_3(\tilde{x};x) = \int_0^{\tilde{x}} \int_{y_2}^{\tilde{x}} t_1(x - b_3\,(y_2)n(n-1)F(y_2)^{n-2}f(y_2)f(y_1)dy_1dy_2$$
$$+\int_0^{\tilde{x}} \int_{\tilde{x}}^{1} t_2\,(x - b_3(y_2)n(n-1)F(y_2)^{n-2}f(y_2)f(y_1)dy_1dy_2$$

Therefore, bidder i solves the following problem in the highest loser's bid auction.

$$Max_{\tilde{x}} \int_0^{\tilde{x}} \int_{y_2}^{\tilde{x}} t_1(x - b_3\,(y_2)n(n-1)F(y_2)^{n-2}f(y_2)f(y_1)dy_1dy_2$$
$$+\int_0^{\tilde{x}} \int_{\tilde{x}}^{1} t_2\,(x - b_3(y_2)n(n-1)F(y_2)^{n-2}f(y_2)f(y_1)dy_1dy_2$$

This is equivalent to

$$Max_{\tilde{x}}\ t_1 \int_0^{\tilde{x}} (x - b_3(y_2))n(n-1)(F(\tilde{x}) - F(y_2))F(y_2)^{n-2}f(y_2)dy_2$$
$$+t_2 \int_0^{\tilde{x}} (x - b_3(y_2))n(n-1)(1 - F(\tilde{x}))F(y_2)^{n-2}f(y_2)dy_2 \quad \ldots(3\text{-}3\text{-}1)$$

F.O.C. of (3-3-1) w.r.t. $\tilde{x}$ yields

$$t_1\big(x - b_3(\tilde{x})\big)n(n-1)\big(F(\tilde{x}) - F(\tilde{x})\big)F(\tilde{x})^{n-2}f(\tilde{x})$$
$$+t_1 \int_0^{\tilde{x}} (x - b_3(y_2))n(n-1)f(\tilde{x})F(y_2)^{n-2}f(y_2)dy_2$$
$$+t_2\big(x - b_3(\tilde{x})\big)n(n-1)\big(1 - F(\tilde{x})\big)F(\tilde{x})^{n-2}f(\tilde{x})$$
$$-t_2 \int_0^{\tilde{x}} (x - b_3(y_2))n(n-1)f(\tilde{x})F(y_2)^{n-2}f(y_2)dy_2 \quad (3\text{-}3\text{-}2)$$

Evaluate (3-3-2) at $\tilde{x} = x$ becomes

$$t_1\big(x - b_3(x)\big)n(n-1)(F(x) - F(x))F(x)^{n-2}f(x)$$
$$+t_1 \int_0^{x} (x - b_3(y_2))n(n-1)f(x)F(y_2)^{n-2}f(y_2)dy_2$$

$+t_2\big(x-b_3(x)\big)n(n-1)\big(1-F(x)\big)F(x)^{n-2}f(x)$

$-t_2\int_0^x (x-b_3(y_2))n(n-1)f(x)F(y_2)^{n-2}f(y_2)dy_2$ …(3-3-3)

Rearrange (3-3-3) yields

$(t_1-t_2)\int_0^x (x-b_3(y_2))F(y_2)^{n-2}f(y_2)dy_2 + t_2\big(x-b_3(x)\big)\big(1-F(x)\big)F(x)^{n-2}=0$ …(3-3-4)

Since (3-3-4) holds for $^\forall x\in[0,1]$, we can differentiate (3-3-4) w.r.t. x and obtain the following expression.

$(t_1-t_2)\big(x-b_3(x)\big)F(x)^{n-2}f(x)$

$+(t_1-t_2)\int_0^x 1\cdot F(x)^{n-2}f(y_2)dy_2$

$+t_2\big(x-b_3(x)\big)\Big(\big(1-F(x)\big)(n-2)F(x)^{n-3}f(x)-F(x)^{n-2}f(x)\Big)=0$ …(3-3-5)

Rearranging (3-3-5) yields

$b_3'(x)t_2(1-F(x))F(x)^{n-2}$

$+b_3(x)((t_1-t_2)F(x)^{n-2}f(x)+t_2(F(x)^{n-3}f(x)((n-2)-(n-1)F(x)))$

$=((t_1-t_2)F(x)^{n-2}f(x)+t_2(F(x)^{n-3}f(x)((n-2)-(n-1)F(x)))x$

$+(t_1-t_2)\frac{1}{n-1}F(x)^{n-1}+t_2(1-F(x))F(x)^{n-2}$ …(3-3-6)

Now, we consider the other case.

Case B. $\tilde{x}<x$

It is easily shown that the F.O.C. of this case evaluated at $\tilde{x}=x$ takes exactly the same form as (3-3-3) in case A. Therefore, it suffices to slove (3-3-6) for $b_3(x)$.

Dividing both sides of (3-3-6) by $t_2(1-F(x))F(x)^{n-2}$ yields

$b_3'(x)+A_3(x)b_3(x)=B_3(x)$,

where $A_3(x)=\frac{(t_1-t_2)F(x)^{n-2}f(x)+t_2(F(x)^{n-3}f(x)((n-2)-(n-1)F(x))}{t_2(1-F(x))F(x)^{n-2}}$

and $B_3(x)=A_3(x)x+\frac{(t_1-t_2)\frac{1}{n-1}F(x)^{n-1}+t_2(1-F(x))F(x)^{n-2}}{t_2\big(1-F(x)\big)F(x)^{n-2}}$ …(3-3-7)

(3-3-7) is a first order linear differential equation w.r.t. $b_3(x)$ and the solution takes the form of

$$b_3(x)=e^{\int_0^x -A_3(x)dx}\left\{\left(\int_0^x B_3(x)e^{\int_0^x A_3(x)dx}dx\right)+C\right\},$$

where C is a constant of integration and $C=0$ because of the initial condition of $b_3(0)=0$.

Next, we check the global optimality condition holds.

$$sgn\left(-\frac{\partial EU_3(\hat{x};x)}{\partial \hat{x}}\bigg|_{\hat{x}=x}+\frac{\partial EU_3(\hat{x};x)}{\partial \hat{x}}\bigg|_{\hat{x}=\tilde{x}}\right)=sgn(x-\tilde{x}) \quad (3\text{-}3\text{-}8)$$

The l.h.s. of (3-3-8) is expressed as follows.

$$sgn\left(-\frac{\partial EU_3(\hat{x};x)}{\partial \hat{x}}\bigg|_{\hat{x}=x}+\frac{\partial EU_3(\hat{x};x)}{\partial \hat{x}}\bigg|_{\hat{x}=\tilde{x}}\right)$$

$$=sgn\begin{pmatrix}\begin{pmatrix} t_1\int_0^{\tilde{x}}(x-b_3(y_2))n(n-1)f(\tilde{x})F(y_2)^{n-2}f(y_2)dy_2 \\ +t_2\big(x-b_3(\tilde{x})\big)n(n-1)\big(1-F(\tilde{x})\big)F(\tilde{x})^{n-2}f(\tilde{x}) \\ -t_2\int_0^{\tilde{x}}(x-b_3(y_2))n(n-1)f(\tilde{x})F(y_2)^{n-2}f(y_2)dy_2\end{pmatrix} \\ -\begin{pmatrix} t_1\int_0^{\tilde{x}}(\tilde{x}-b_3(y_2))n(n-1)f(\tilde{x})F(y_2)^{n-2}f(y_2)dy_2 \\ +t_2\big(\tilde{x}-b_3(\tilde{x})\big)n(n-1)\big(1-F(\tilde{x})\big)F(\tilde{x})^{n-2}f(\tilde{x}) \\ -t_2\int_0^{\tilde{x}}(\tilde{x}-b_3(y_2))n(n-1)f(\tilde{x})F(y_2)^{n-2}f(y_2)dy_2\end{pmatrix}\end{pmatrix}$$

$$=sgn\begin{pmatrix} t_1\int_0^{\tilde{x}}n(n-1)f(\tilde{x})F(y_2)^{n-2}f(y_2)dy_2 \\ +t_2n(n-1)\big(1-F(\tilde{x})\big)F(\tilde{x})^{n-2}f(\tilde{x}) \\ -t_2\int_0^{\tilde{x}}n(n-1)f(\tilde{x})F(y_2)^{n-2}f(y_2)dy_2\end{pmatrix}(x-\tilde{x})$$

$$=sgn\big(t_1nF(\tilde{x})^{n-1}+t_2n(n-1)\big(1-F(\tilde{x})\big)F(\tilde{x})^{n-2}f(\tilde{x})-t_2nF(\tilde{x})^{n-1}\big)(x-\tilde{x})$$

$$=sgn\Big((t_1-t_2)nF(\tilde{x})^{n-1}+t_2n(n-1)\big(1-F(\tilde{x})\big)F(\tilde{x})^{n-2}f(\tilde{x})\Big)(x-\tilde{x})$$

$$=\text{sgn}(x-\tilde{x}) \quad \ldots(3\text{-}1\text{-}9)$$

To obtain the first equality of (3-1-9), we apply (3-2-2) to each term of $-\frac{\partial EU_3(\hat{x};x)}{\partial \hat{x}}\Big|_{\hat{x}=x}+\frac{\partial EU_3(\hat{x};x)}{\partial \hat{x}}\Big|_{\hat{x}=\tilde{x}}$ and use the fact that $-\frac{\partial EU_3(\hat{x};x)}{\partial \hat{x}}\Big|_{\hat{x}=x}=0=-\frac{\partial EU_3(\hat{x};\tilde{x})}{\partial \hat{x}}\Big|_{\hat{x}=\tilde{x}}$. The last equality holds because of our assumption of $t_1\geq t_2$

The result is summarized in the following proposition.

Proposition3. (The symmetric Bayesian equilibrium bidding function of the highest loser's bid auction)

The symmetric Bayesian equilibrium of the highest loser's bid auction, $b_3(x)$, is characterized as follows.

$$b_3(x)=e^{\int_0^x -A_3(x)dx}\left\{\left(\int_0^x B_3(x)e^{\int_0^x A_3(x)dx}dx\right)\right\},$$

where $A_3(x)=\dfrac{(t_1-t_2)F(x)^{n-2}f(x)+t_2(F(x)^{n-3}f(x)((n-2)-(n-1)F(x))}{t_2(1-F(x))F(x)^{n-2}}$

and $B_3(x)=A_3(x)x+\dfrac{(t_1-t_2)\frac{1}{n-1}F(x)^{n-1}+t_2(1-F(x))F(x)^{n-2}}{t_2\big(1-F(x)\big)F(x)^{n-2}}$

3-4. The pay the next highest bid to yours auction

Lastly, this subsection characterizes the symmetric Bayesian equilibrium bidding function of the pay the next highest bid to yours auction. As in the previous three subsections, bidder i wins the first item if his bid, b_1^i, is the highest among n bids, one from each bidder and wins the second item if his bid, b_1^i, is the second highest among n bids, one from each bidder. Now bidder i pays $t_1 b_4(y_1)$ if he wins the first item and pays $t_2 b_4(y_2)$ if he wins the second item, where we denote the highest bid among $b(x_j)$ $^\forall j \neq i$ by $b_4(y_1)$, and the second highest among $b(x_j)$ $^\forall j \neq i$, by $b_4(y_2)$. Bidder i pays nothing if he does not acquire either item.

As in the previous three subsections, we need to consider two cases separately.

Case A. $\tilde{x} > x$.

In this case, bidder $i's$ expected utility when his value is x and he submits a bid, $b_4(\tilde{x})$ is expressed as follows as long as there exists a symmetric equilibrium bidding function which is increasing w.r.t. a bidder's type. We know that such an equilibrium bidding function exists due to supermodularity of the objective function w.r.t. a bidder's type and his bid.

$$EU_4(\tilde{x}; x) = t_1 \int_0^{\tilde{x}} (x - b_4(y_1) n F(y_1)^{n-1} f(y_1) dy_1$$

$$+t_2 \int_0^{\tilde{x}} \int_{\tilde{x}}^1 (x - b_4(y_2)) n(n-1) F(y_2)^{n-2} f(y_1) f(y_2) dy_1 dy_2$$

Therefore, bidder i solves the following problem.

$$Max_{\tilde{x}} \; t_1 \int_0^{\tilde{x}} (x - b_4(y_1)) n F(y_1)^{n-1} f(y_1) dy_1$$

$$+t_2 \int_0^{\tilde{x}} \int_{\tilde{x}}^1 (x - b_4(y_2)) n(n-1) F(y_2)^{n-2} f(y_1) f(y_2) dy_1 dy_2 \quad \ldots(3\text{-}4\text{-}1)$$

F.O.C. of (3-4-1) w.r.t. $\tilde{x}$ yields

$$t_1 \big(x - b_4(\tilde{x})\big) n F(\tilde{x})^{n-1} f(\tilde{x})$$

$$+t_2 \int_{\tilde{x}}^1 \big(x - b_4(\tilde{x})\big) n(n-1) F(\tilde{x})^{n-2} f(y_1) f(\tilde{x}) dy_1$$

$$-t_2 \int_0^{\tilde{x}} (x - b_4(y_2)) n(n-1) F(y_2)^{n-2} f(\tilde{x}) f(y_2) dy_1 \quad \ldots(3\text{-}4\text{-}2)$$

Evaluate (3-4-2) at $\tilde{x} = x$ becomes

$$t_1 \big(x - b(x)\big) n F(x)^{n-1}$$

$$+t_2 \int_x^1 \big(x - b_4(x)\big) n(n-1) F(x)^{n-2} f(y_1) dy_1$$

$$-t_2 \int_0^x \big(x - b_4(x)\big) n(n-1) F(y_2)^{n-2} f(y_2) dy_1 \quad \ldots(3\text{-}4\text{-}3)$$

Now, we consider the other case.

Case B. $\tilde{x} < x$

It is easily shown that the F.O.C. of this case evaluated at $\tilde{x} = x$ is exactly the same as

(3-4-3) in case A. Therefore, it suffices to solve (3-4-3) w.r.t. $b_4(x)$.

Re arranging (3-4-3) becomes

$t_1\big(x-b_4(x)\big)F(x)^{n-1}+t_2(n-1)F(x)^{n-2}(1-F(x))-t_2xF(x)^{n-1}$

$+\int_0^x b_4(y_2)(n-1)F(y_2)^{n-2}f(y_2)dy_2=0$ ...(3-4-4)

Since (3-4-4) holds for $^\forall x\in[0,1]$, we can differentiate it w.r.t. x and obtain the following result.

$t_1\big(1-b_4'(x)\big)F(x)^2+t_1\big(x-b_4(x)\big)(n-1)F(x)f(x)$

$+t_2(n-1)(n-2)f(x)(1-F(x))-t_2x(n-1)F(x)f(x)-t_2F(x)^2$

$+t_2b_4(x)(n-1)F(x)f(x)=0,$

where we use the initial condition of ...(3-4-5)

Rewrite (3-4-5) becomes

$b_4'(x)t_1F(x)^2+b(x)(t_1-t_2)(n-1)F(x)f(x)$

$=(t_1-t_2)(n-1)F(x)f(x)+(t_1-t_2)F(x)^2$ ...(3-4-6)

Dividing both sides of (3-4-6) by $t_1F(x)^2$becomes

$b_4'(x)+A_4(x)b_4(x)=B_4(x),$

where, $A_4(x)=\frac{(t_1-t_2)(n-1)f(x)}{t_1F(x)}$ and $B_4(x)=A_4(x)+\frac{(t_1-t_2)}{t_1}$ ...(3-4-7)

(3-4-7) is a first order linear differential differential equation w.r.t. $b_4(x)$ and the formula gives us the following solution.

$$b_4(x)=e^{\int_0^x -A_4(x)dx}\left\{\left(\int_0^x B_4(x)e^{\int_0^x A_4(x)dx}dx\right)+C\right\},$$

where C is a constant of integration and C =0 because of the initial condition of $b_4(0)=0$.

Next, we check the global optimality condition holds. To do so, we need to show the following formula holds.

$$sgn\left(-\frac{\partial EU_4(\hat{x};x)}{\partial\hat{x}}\Big|_{\hat{x}=x}+\frac{\partial EU_4(\hat{x};x)}{\partial\hat{x}}\Big|_{\hat{x}=\tilde{x}}\right)=sgn(x-\tilde{x}) \quad ...(3\text{-}4\text{-}8)$$

The l.h.s. of (3-4-8) takes the following form.

$$=sgn\left(\begin{pmatrix} t_1\big(x-b_4(\tilde{x})\big)nF(\tilde{x})^{n-1}f(\tilde{x}) \\ +t_2\int_{\tilde{x}}^1\big(x-b_4(\tilde{x})\big)n(n-1)F(\tilde{x})^{n-2}f(y_1)f(\tilde{x})dy_1 \\ -t_2\int_0^{\tilde{x}}\big(x-b_4(y_2)\big)n(n-1)F(y_2)^{n-2}f(\tilde{x})f(y_2)dy_1 \end{pmatrix} - \begin{pmatrix} t_1\big(\tilde{x}-b_4(\tilde{x})\big)nF(\tilde{x})^{n-1}f(\tilde{x}) \\ +t_2\int_{\tilde{x}}^1\big(\tilde{x}-b_4(\tilde{x})\big)n(n-1)F(\tilde{x})^{n-2}f(y_1)f(\tilde{x})dy_1 \\ -t_2\int_0^{\tilde{x}}\big(\tilde{x}-b_4(y_2)\big)n(n-1)F(y_2)^{n-2}f(\tilde{x})f(y_2)dy_1 \end{pmatrix}\right)$$

$$=sgn\big((t_1-t_2)nF(\tilde{x})^{n-1}f(\tilde{x})+t_2n(n-1)F(\tilde{x})^{n-2}(1-F(\tilde{x})f(\tilde{x})\big)(x-\tilde{x}) \quad ...(3\text{-}4\text{-}9)$$

$=sgn(x-\tilde{x})$

To obtain the first equality of (3-4-9), we apply (3-4-2) for each term of $-\frac{\partial EU_4(\hat{x};x)}{\partial \hat{x}}\Big|_{\hat{x}=x}+\frac{\partial EU_4(\hat{x};x)}{\partial \hat{x}}\Big|_{\hat{x}=\tilde{x}}$ and use the fact that $-\frac{\partial U_4(\hat{x};x)}{\partial \hat{x}}\Big|_{\hat{x}=x}=0=-\frac{\partial U_4(\hat{x};\tilde{x})}{\partial \hat{x}}\Big|_{\hat{x}=\tilde{x}}$. The last equality holds because of our assumption of $t_1 \geq t_2$.

The result is summarized in the following proposition.

Proposition4. (The symmetric Bayesian equilibrium bidding function of the pay the next highest bid to yours auction)

The symmetric Bayesian equilibrium of the pay the next highest bid to yours auction, $b_4(x)$, is characterized as follows.

$$b_4(x)=e^{\int_0^x -A_4(x)dx}\left\{\left(\int_0^x B_4(x)e^{\int_0^x A_4(x)dx}dx\right)\right\},$$

where $A_4(\mathrm{x})=\frac{(t_1-t_2)(n-1)f(x)}{t_1F(x)}$ and $B_4(x)=A_4(x)+\frac{(t_1-t_2)}{t_1}$.

4. Comparison of the four auction procedures

Based on the results of the previous section, this section compares the seller's expected revenue of the pay your bid auction, the lowest loser's auction, the highest loser's auction, and the pay the next highest bid to yours auction. We can apply Myerson's (1981) arguments and obtain the following proposition.

Proposition5. (The revenue equivalence theorem)

The pay your bid auction, the lowest winner's bid auction, the highest loser's bid auction, and the pay the next highest bid to yours auction yield the same expected revenue to the seller and they implement the optimal auction mechanism when the seller sets the reserve price, $t_k x^r \ \ ^\forall k \in M$, to satisfy $x^r-\frac{1-F(x^r)}{f(x^r)}=0$.

Proof of proposition5.

This is a simple application of the arguments to obtain corollary1 (the revenue equivalence theorem) in Myerson (1981) because the assumptions of (1) PIV , (2) risk neutral bidders and the risk neutral seller, (3) unit demand, (4) indivisible items, and (5) quasi linear utility function hold in our model. Further, the allocation rules of the four unit price auction procedures are the same and the bidder with the highest type wins the first item and the bidder with the second highest type wins the second item. Therefore, the revenue equivalence theorem holds[2].

5. Conclusion

This paper proposes four unit price auction procedures and characterizes their symmetric Bayesian equilibrium bidding functions. Further, it is shown that the revenue equivalence theorem holds if the seller sets appropriate reserve prices. Among four unit price auctions of the pay your bid auction, the lowest winner's bid auction, the highest loser's bid auction, and the pay the next highest bid to yours auction, the lowest winner's bid auction and the highest loser's bid auction satisfy a fair criterion in the sense that each winner pays the same unit price regardless of the item he wins. A fair criterion is important when the governments design auctions because they are public auctions. Since our model includes various procurement auctions such as road repair service contract in the next year as analyzed in Rene (2011), garbage collection service, milk procurement auctions by school district, and so on. We can interpret these auctions as auctioning off per capita service. For example, when milk supply service in a school district is auctioned off, it can be considered per capita milk supply service multiplied by the number of students in a corresponding school district. Since everyone knows the (expected) number of students in the relevant school district, it is reasonable to formalize the situation as a unit price auction. It is also important that four unit auction procedures are sealed bid auction procedures and more robust to collusions compared with the ascending price auction procedure analyzed by EOS. Although our model also includes sponsored link auctions as a special example, they are private auction and a fair criterion might not be an issue there. Nevertheless, unit price auction procedures are simple way to implement the optimal auction and we believe they are very effective especially when the governments auctioning off per capita service contract where a fair criterion is important and collusion is a serious problem.

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