Patent licensing with Bertrand competitors

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Abstract

We study optimal licensing contracts in a differentiated Bertrand duopoly, and show that per-unit contracts are preferred to ad valorem contracts by the patentee, while welfare is higher under the ad valorem contract. The difference between Cournot and Bertrand case is explained in terms of quantity effect and profits effect..

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1. Introduction

Patent licensing is a fundamental channel for the dissemination of innovations and technological improvements across firms. Theoretical investigation of patent licensing of cost reducing innovations can be broadly divided into two classes of models: models where the innovator is an outsider and models where the innovator is an insider. Within the first class of papers (outsider innovator), a well established result is that the patentee

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prefers a fixed fee licensing scheme to a per-unit royalty (Kamien and Tauman, 1986, Katz and Shapiro, 1986, Kamien et al., 1992), even if there are situations under which per-unit royalty may be preferred to the fixed fee. These are: incomplete information (Gallini and Wright, 1990), Bertrand competition in differentiated markets (Muto, 1993), strategic delegation (Saracho, 2002), and limitations in the number of licenses (Sen, 2005). Within the second class of papers (insider innovator), per-unity royalty is usually preferred to fixed fee by the patentee. This happens in a homogenous Cournot market (Wang, 1998), in a differentiated Bertrand market (Wang and Yang, 1999), in a spatial model à la Hotelling (Poddar and Sinha, 2004), and in a Stackelberg model (Filippini, 2005). Other authors have investigated optimal two-part licensing contracts, where both a per-unit royalty and a fixed fee are imposed to the licensee, both when the innovator is an outsider and when it is an insider. For example, Kamien and Tauman (1984) and Erutku and Richelle (2000) consider the case of an external patentee in a Cournot oligopoly, while Faullì-Oller and Sandonìs (2002) analyse the case of an internal patentee (both in a Bertrand and in a Cournot framework) and show that in some situations licensing can decrease welfare. Sen and Tauman (2007) provide some generalizations to the optimal licensing scheme for the case of a Cournot oligopoly, both when the innovator is outsider and when it is insider.[1]

All the mentioned literature assumes that the royalties imposed to the licensee are "per-unit", that is, the licensee pays a certain royalty per any unit it sells. However, other types of royalty payment are possible. In particular, "ad valorem" royalties play a relevant role in licensing schemes. Ad valorem royalties imply that the patentee extracts a quota of the licensee's profits. Bousquet et al. (1998), by studying data of French

[1] Other contributions focus on patent auctioning. Among these, see for example Das Varma (2003), Giobe and Wolfstetter (2008), Stamatopulos and Tauman (2009), Tauman and Weng (2012) and Fan et al. (2011). In this article, patent auctioning is left aside.

firms, find that 78% of contracts include royalties (alone or together with a fixed fee) and, more importantly, amongst these, only 4% are per-unit royalties while 96% are ad valorem royalties. Another study by Lim and Veugelers (2002) on technology licensing contracts in US shows that at least 20% of licensing agreements include some forms of ad valorem royalty in the payment scheme. Given the importance and the diffusion of ad valorem royalties licensing schemes, it is quite surprising that scholars have devoted scarce attention to them until now. Bousquet et al. (1998) consider an outsider innovator and one potential licensee, and show that ad valorem royalties are superior to per-unit royalties for the patentee when there is uncertainty on the demand side. On the other hand, when the innovator is an insider, in a recent article by San Martín and Saracho (2010), it is shown in a homogeneous Cournot example that it is always better to adopt ad valorem royalties instead of per-unit royalties when licensing a cost reducing innovation.[2] San Martín and Saracho (2010) argue that this depends on the fact that an ad valorem royalties licensing contract allows the patentee to strategically commit to be less aggressive. Moreover, San Martín and Saracho (2010) show that welfare is higher under a per-unit royalty licensing contract than under an ad valorem licensing scheme.

We extend the analysis of the ad valorem royalties licensing scheme to the case of a differentiated Bertrand duopoly. In particular, we study optimal two-part license contracts, and we show that per-unit contracts are always preferred to ad valorem contracts by the patentee. In the choice between per-unit licensing or ad valorem licensing, the patentee faces the following trade-off. By adopting a per-unit royalty the patentee obtains additional revenues from the quantity produced by the licensee ("quantity effect"); by adopting an ad valorem royalty, there are additional profits for

[2] Boccard (2010) shows that the result by San Martín and Saracho (2010) holds also in the case of an outsider patentee, even in the absence of demand uncertainty, when more than one potential licensees exist.

the licensee due to the fact that its marginal costs are lower: these profits partially accrue to the patentee via the ad valorem royalty ("profits effect"). As long as the revenues from the per-unit royalty are higher than the cost savings and the consequent higher profits of the patentee via the ad valorem royalty, per-unit royalty licensing is preferred to ad valorem licensing. The revenues from per-unit royalty increase with the equilibrium quantity of the licensee, while the costs savings absorbed by the patentee increase with the equilibrium profits of the licensee. In Bertrand, the quantity effect dominates the profits effect for any degree of differentiation, thus making per-unit royalty more profitable than ad valorem royalty, while the reverse holds in Cournot. This general trade-off between the quantity effect and the profits effect can be applied also to other cases (for example, more firms, differentiated Cournot competitors, and Stackelberg competition). Also, we find that it may be preferable for the patentee not to license the innovation when it is constrained to adopt ad valorem royalties. This happens if the size of innovation is high or if the degree of substitutability between the products is high. Interestingly, this may happen even if innovation is non-drastic.[3] Finally, we show that, in contrast with previous results, the welfare is higher under the ad valorem royalties licensing scheme than in case of per-unit licensing or no-licensing. Therefore, the "welfare-reducing licensing" result by Faullì-Oller and Sandonìs (2002) does not apply in case of ad valorem royalty.

The remaining of this article proceeds as follows. In Section 2 we introduce the model. In Section 3 we characterize the optimal two-part ad valorem contract and we compare it with the optimal two-part per-unit contract and with the no-licensing case. A welfare analysis is also provided. In Section 4 we discuss the Stackelberg case. Section

[3] A cost reducing innovation is said to be drastic when the innovating firm becomes a monopolist if licensing does not occur. This happens when the monopoly price with the cost reducing innovation is equal or less than the unit production costs without the cost reducing innovation.

5 discusses the results of the theoretical model and summarizes. A technical appendix concludes the article.

2. The model

Consider an industry composed by two firms, Firm 1 and Firm 2. Firm 1 (the patentee) owns a patent for a cost-reducing innovation. Firm 2 is the potential licensee. Production occurs with a constant marginal cost equal to $c \in [0,1]$ in the absence of the cost-reducing innovation, while it occurs with a constant marginal cost equal to zero if the cost-reducing innovation is adopted. There is no fixed cost, with the exception of the license fee (if adopted by the patentee and accepted by the licensee).

The demand side of the market is described by the following utility function, separable in income, for the representative consumer:

$$U(q_i, q_{-i}) = q_i + q_{-i} - \frac{q_i^2}{2} - \frac{q_{-i}^2}{2} - \theta q_i q_{-i} + y$$

where q_i is the quantity of Firm $i = 1, 2$, and y is the income of the representative consumer. Parameter $\theta \in [0,1)$ represents the degree of product substitutability between the goods. When $\theta = 0$, the goods produced by the two firms are unrelated, while the higher is θ, the more the two goods are substitutable.[4] From the maximization of the utility function of the representative consumer, the direct demand function follows (see for example Singh and Vives, 1984):

[4] When $\theta = 1$, the demand functions cannot be defined (Singh and Vives, 1984). Therefore, the case of homogeneous good is excluded from the analysis.

$$q_i = \frac{1}{1+\theta} - \frac{1}{1-\theta^2} p_i + \frac{\theta}{1-\theta^2} p_{-i}$$

where p_i is the price adopted by Firm $i = 1, 2$.

The timing of the game is as follows:

- In the first stage, Firm 1 offers (and commits itself to) a two-part per-unit contract or a two-part ad valorem contract (take-it-or-leave-it), in order to license the cost-reducing innovation to Firm 2. Firm 1 may also decide not to offer any licensing contract to Firm 2.
- In the second stage, Firm 2 accepts or refuses the offer.
- In the third stage, Firm 1 and Firm 2 set simultaneously and non-cooperatively the prices, the profits are realized and the payments to the patentee are done according to the type of contract (if any) offered by Firm 1 in the first stage and eventually accepted by Firm 2 in the second stage.

The remaining notation shall be introduced throughout the following sections.

3. Optimal two-part tariffs

We solve the model by backward induction. We shall consider first the case where Firm 2 has not accepted the offer by Firm 1 in the second stage of the game (or Firm 1 has done no offer in the first stage of the game): this is the no-licensing case. Then, we shall consider the case where Firm 1 has set a two-part per-unit contract in the first stage of the game, and, finally, we shall consider the case where Firm 1 has set a two-part ad valorem contract in the first stage of the game.

3.1. *No license*

Consider the case where Firm 2 has not accepted the offer by Firm 1 in the second stage of the game, or Firm 1 has done no offer in the first stage of the game. In this case, the profits of Firm 1 are: $\Pi_1^N = p_1 q_1(p_1, p_2)$, while the profits of Firm 2 are: $\Pi_2^N = (p_2 - c)q_2(p_1, p_2)$, where the superscript *N* refers to the "no-licensing" case. Note that the marginal production costs of the two firms are different, being zero for Firm 1 and *c* for Firm 2. Depending on the value of *c*, three cases are possible. For future convenience, we shall refer to the three cases as follows:

1) *non-drastic innovation*: $c \leq (2-\theta-\theta^2)/(2-\theta^2)$. This case yields positive output levels for both firms in equilibrium.

2) *quasi-drastic innovation*: $(2-\theta-\theta^2)/(2-\theta^2) \leq c \leq (2-\theta)/2$. This case yields a corner solution in the Bertrand game: in equilibrium, only the firm with lower marginal production costs (Firm 1) produces a positive output level. However, the cost difference between the two firms is too low to allow Firm 1 to set the monopoly price.

3) *drastic innovation*: $c \geq (2-\theta)/2$. This case yields the monopoly price for Firm 1, while Firm 2 sells zero.

- *Non-drastic innovation*. Within the parameter region $c \in [0, \dfrac{2-\theta-\theta^2}{2-\theta^2}]$, straightforward calculations yield the following equilibrium prices for the two firms: $p_1^N = \dfrac{2-\theta-\theta^2+c\theta}{4-\theta^2}$ and $p_2^N = \dfrac{2-\theta-\theta^2+2c}{4-\theta^2}$. Therefore, the no-licensing equilibrium profits are:

$$\Pi_1^N * = \frac{(2-\theta-\theta^2+c\theta)^2}{(4-\theta^2)^2(1-\theta^2)} \tag{1}$$

$$\Pi_2^N * = \frac{(2-\theta-\theta^2-2c+c\theta^2)^2}{(4-\theta^2)^2(1-\theta^2)} \tag{2}$$

- *Quasi-drastic innovation*. Consider the parameter region $c \in [\frac{2-\theta-\theta^2}{2-\theta^2}, \frac{2-\theta}{2}]$. In equilibrium, Firm 2 sells zero, but Firm 1 cannot set the monopolistic price, because the marginal production costs of Firm 2 are still too low. The equilibrium price of Firm 1 is: $p_1^N = \frac{\theta+c-1}{\theta}$, while Firm 2 sets: $p_2^N = c$. Firm 2 gets zero profits, while Firm 1 obtains the following profits:

$$\Pi_1^N * = \frac{(1-c)(\theta+c-1)}{\theta^2} \tag{3}$$

- *Drastic innovation*. Finally, consider the parameter region $c \in [\frac{2-\theta}{2}, 1]$. In equilibrium, the monopolistic price of Firm 1 is lower than the marginal cost of Firm 2. Therefore, the equilibrium price of Firm 1 is: $p_1^N = 1/2$, while Firm 2 sets: $p_2^N = c$. Firm 2 gets zero profits, while Firm 1 gets:

$$\Pi_1^N * = 1/4 \tag{4}$$

3.2. *Two-part per-unit contract*

In this section we consider the case where the patentee has offered a two-part per-unit contract to the licensee. Therefore, let us suppose that Firm 1 sets in period 1 a two-part per-unit contract (F^U, h), where F^U is the fixed fee, while h is the per-unit royalty. The superscript U refers to the "per-unit licensing" case. If Firm 2 accepts the offer by Firm 1, the profits of Firm 1 are: $\Pi_1^U = p_1 q_1(p_1, p_2) + hq_2(p_1, p_2) + F^U$, while the profits of Firm 2 are: $\Pi_2^U = (p_2 - h)q_2(p_1, p_2) - F^U$. By maximizing the profits with respect to the prices, and then solving the system of the best-reply functions, we get: $p_1^U* = \dfrac{2-\theta-\theta^2+3h\theta}{4-\theta^2}$ and $p_2^U* = \dfrac{2(1+h)-\theta^2(1-h)-\theta}{4-\theta^2}$. The optimal two-part per-unit contract $(F^U*, h*)$ maximizes $\Pi_1^U(p_1*, p_2*)$ under the following constraints: $\Pi_2^U(p_1^U*, p_2^U*) \geq \Pi_2^N*$ and $h \in [0, c]$. The first constraint guarantees the incentive-compatibility of Firm 2's decision: obviously, Firm 2 prefers not to adopt the cost-reducing innovation if the overall profits (including the payment to Firm 1 under the contract (F^U, h)) in case of acceptance of Firm 1's contract are lower than the profits without the cost-reducing innovation. The second constraint imposes that the royalty cannot be higher than the reduction of the marginal costs due to innovation.[5]

The following proposition (Faullì-Oller and Sandonìs, 2002) establishes the optimal two-part per-unit contract:

Proposition 1. *Define* $\bar{h} \equiv \dfrac{\theta(2+\theta)^2}{8+10\theta^2}$. *The optimal two-part per-unit contract is given by:* $h* = \min\{\bar{h}, c\}$ *and* $F^U* = \Pi_2^U(h*) - \Pi_2^N$.

[5] This assumption is widely adopted in patent licensing literature. One exception is Filippini (2005).

Proof. See Faullì-Oller and Sandonis (2002, p.197). ■

We are in the position to calculate the equilibrium profits of Firm 1 under the optimal two-part per-unit contract. Inserting $(F^U*, h*)$ into Π_1^U, the following cases emerge:[6]

- *Non-drastic innovation*. If $\bar{h} \leq c$, we have:

$$\Pi_1^U* = \frac{\begin{bmatrix}(2+\theta)^2(16-32\theta+40\theta^2-36\theta^3+17\theta^4-4\theta^5-\theta^6)+8c(16-\theta+ \\ +4\theta^2-6\theta^3-16\theta^4+5\theta^5+5\theta^6)-4c^2(2-\theta^2)^2(4+5\theta^2)\end{bmatrix}}{4(4-\theta^2)^2(4+\theta^2-5\theta^4)} \quad (5)$$

If $\bar{h} \geq c$, we have:

$$\Pi_1^U* = \frac{(2-\theta-\theta^2)^2+8c(8-4\theta^2-\theta^3-2\theta^4-\theta^5)-c^2(8-3\theta^2-3\theta^4)}{(4-\theta^2)^2(1-\theta^2)} \quad (6)$$

- *Quasi-drastic and drastic innovation*. If $\bar{h} \leq c$, we have:

$$\Pi_1^U* = \frac{8+9\theta^2+\theta^3}{4(1+\theta)(4+5\theta^2)} \quad (7)$$

If $\bar{h} \geq c$, we have:

[6] Note that the equilibrium profits vary with the parameter region of *c* (Section 3.1), as, depending on the value of *c*, the incentive-compatibility constraint of Firm 2 changes.

$$\Pi_1^U * = \frac{2(1-\theta)(2+\theta)^2 + 2c\theta(1+\theta)(2+\theta)^2 - c^2(4+4\theta+5\theta^2+5\theta^3)}{(1+\theta)(4-\theta^2)^2} \quad (8)$$

3.3. *Two-part ad valorem contract*

In this section we consider the case where the patentee has offered a two-part ad valorem contract to the licensee. Therefore, let us suppose that Firm 1 sets in period 1 a two-part ad valorem contract (F^V, d), where F^V is the fixed fee and $d \in [0,1]$ is the ad valorem royalty. The superscript *V* refers to the "ad valorem licensing" case. In this case, the profits of the two firms become: $\Pi_1^V = p_1 q_1(p_1, p_2) + dp_2 q_2(p_1, p_2) + F^V$ and $\Pi_2^V = (1-d)p_2 q_2(p_1, p_2) - F^V$.

By maximizing the profits functions and then solving the system of the best-reply functions, the following equilibrium prices emerge in the third stage of the game:

$$p_1^V * = \frac{(1-\theta)(2+\theta+d\theta)}{4-(1+d)\theta^2} \quad (9)$$

$$p_2^V * = \frac{2-\theta-\theta^2}{4-(1+d)\theta^2} \quad (10)$$

After inserting $p_1^V *$ and $p_2^V *$ into the profits function of Firm 1, the maximization problem of Firm 1 can be written as follows:

$$\max_{F^V, d} \Pi_1^V (p_1^V *, p_2^V *)$$

subject to: $\Pi_2^V(p_1^V*, p_2^V*) \geq \Pi_2^N*$

where the inequality represents the incentive-compatibility constraint of the licensee. The solution of the maximization problem above yields the optimal ad valorem contract $(F^V*, d*)$, as indicated in the next proposition:

Proposition 2. *Define* $\bar{d} \equiv \frac{(2+\theta)^2(2-3\theta+\theta^2)R-K}{2\theta^4(2-\theta-\theta^2-2c+2\theta^2)^2}$.[7] *The optimal two-part ad valorem contract is given by:* $F^V* = 0$ *and* $d* \equiv \min\{\bar{d}, 1\}$.

Proof. It is immediate to see that the constraint $\Pi_2^V(p_1^V*, p_2^V*) \geq \Pi_2^N*$ must be binding in equilibrium. This yields two possible situations. First, suppose that: $\tilde{F}^V* = (1-d)p_2^V*q_2^V* - \Pi_2^N* > 0$. By setting $\tilde{F}^V*$ into $\Pi_1^V(p_1^V*, p_2^V*)$, we have: $\frac{\partial \Pi_1^V(p_1^V*, p_2^V*)}{\partial d} = \frac{\theta^2(1-\theta)(2+\theta)[(2+\theta)^2 - d(4+2\theta-\theta^2)]}{(1+\theta)[4-(1+d)\theta^2]^3}$, which is strictly positive. Therefore, it must be: $d* = 1$. However, in this case we observe that $F^V* = -\Pi_2^N* < 0$. Now, suppose that $F^V* = 0$. By setting $F^V* = 0$ into $\Pi_2^V(p_1^V*, p_2^V*)$ and then solving $\Pi_2^V(p_1^V*, p_2^V*) = \Pi_2^N*$ with respect to *d*, we get: $\bar{d} = \frac{(2+\theta)^2(2-3\theta+\theta^2)R-K}{2\theta^4(2-\theta-\theta^2-2c+2\theta^2)^2}$. The constraint $d \leq 1$ is not binding at $\bar{d}$ when $c \leq (2-\theta-\theta^2)/(2-\theta^2)$, while it is binding when $c \geq (2-\theta-\theta^2)/(2-\theta^2)$. ■

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[7] Note that:

$K = 64(1-\theta) - \theta^2(112+64c-32c^2) + 32\theta^3(3-c) + \theta^4(76-80c+40c^2) - \theta^5(44-24c) - \theta^6(25-32c+16c^2) + \theta^7(6-4c) + \theta^8(3-4c+2c^2)$

$R = \sqrt{64(1-\theta) - 16\theta^2(9-8c-4c^2) + 64\theta^3(2-c) + \theta^4(31-48c+24c^2) - \theta^5(84-64c) - \theta^6(51-96c+48c^2) + \theta^7(9-8c) + \theta^8(9-16c+8c^2)}$

Note that the optimal two-part ad valorem contract reduces to a pure ad valorem royalty. To explain this result, let us compare two extreme situations: a pure fixed fee contract and a pure ad valorem royalty contract. In the first case, at the third stage of the game the patentee maximizes: $\Pi_1 = p_1 q_1(p_1, p_2) + F$. In the second case, at the third stage of the game the patentee maximizes: $\Pi_1 = p_1 q_1(p_1, p_2) + d p_2 q_2(p_1, p_2)$. The pure ad valorem royalty is preferred by the patentee, as, through the royalty, the patentee is able to take into account the reaction of the licensee when choosing the price at the final stage of the game. In other words, as prices are strategic complements, by choosing the ad valorem royalty instead of the fixed fee, the patentee is able to limit price competition at the final stage of the game.

Having obtained the optimal ad valorem contract, we insert d^* and $F^V{}^*$ into $\Pi_1^V(p_1^V{}^*, p_2^V{}^*)$ in order to get the equilibrium profits of the patentee. Depending on the innovation size, the following cases may arise:

- *Non-drastic innovation*. From Proposition 2, we have that in this case the optimal ad valorem royalty is: $d^* = \overline{d}$. By using $\overline{d}$, the equilibrium profits of Firm 1 are:

$$\Pi_1^V{}^* = \frac{(1-\theta)[4(2+\theta)^2\theta^2 {d_d}^4 - 2d_n\theta^2(4+4\theta-\theta^2-\theta^3){d_d}^2 - (1+\theta){d_n}^2}{\theta[(1+\theta)(2+\theta)^4(2-3\theta+\theta^2)^2(8-4\theta-6\theta^2+\theta^3+\theta^4-R)^2]} \tag{11}$$

where d_n and d_d are the numerator and the denominator of $\overline{d}$, respectively.

- *Quasi-drastic and drastic innovation*. From Proposition 2 we observe that in this case the patentee expropriates the entire profits of the licensee and the optimal ad valorem royalty is: $d^* = 1$. The equilibrium profits of Firm 1 are therefore:

$$\Pi_1^V{}^* = \frac{8 - 9\theta^2 - \theta^3 + 2\theta^4}{4(1+\theta)(2-\theta^2)^2} \tag{12}$$

3.4. *Per-unit licensing, ad valorem licensing, or no-licensing?*

We are now in the position to compare the equilibrium profits of the patentee when it sets the optimal two-part per-unit contract, when it sets the optimal (pure) ad valorem contract, and when it does not license its innovation. By comparing $\Pi_1^U{}^*$, $\Pi_1^V{}^*$ and $\Pi_1^N{}^*$, we can state the following proposition:

Proposition 3.

i) The first-best solution for the innovator consists in adopting the optimal two-part per unit contract.

ii) The second-best solution for the innovator consists in adopting the optimal pure ad valorem contract (resp. no-licensing) if the degree of substitutability and/or the size of innovation is low (resp. high) enough.

Proof. See the Technical Appendix. ■

Proposition 3, point *i)*, provides the central result of this article. It says that in a duopolistic industry that produces a differentiated good and when firms compete with prices, an internal patentee always prefers licensing the innovation by means of a two-part per-unit contract. This holds for any innovation size, that is, it holds when the innovation is drastic, quasi-drastic or non-drastic.

Note that Proposition 3, point *i)*, differs from San Martín and Saracho (2010, p.286), where it is shown that in a Cournot context ad valorem royalties are always preferred to per-unit royalties. In what follows we provide the intuition for Proposition 3, point *i)*.[8] First, consider the difference between per-unit royalties and fixed fee or ad valorem royalties. When adopting a per-unit royalty, the patentee increases its revenues through the quantity sold by the licensee in equilibrium ("quantity effect"). However, the "industry" revenue is lower than under fixed fee or ad valorem royalties. In fact, in the latter case, the marginal production costs of the licensee are at the same level of the patentee: the efficiency gains and the higher profits of the licensee are then absorbed by the patentee through the fixed fee or the ad valorem royalty ("profits effect"). The per-unit royalty is preferred to the fixed fee and to the ad valorem royalty as long as the associated revenues are higher than the industry cost savings and the consequent higher profits of the patentee via the fixed fee or the ad valorem royalty, that is, as long as the quantity effect dominates over the profits effect. We have already noted (see the discussion below Proposition 2) that the ad valorem royalty is preferred to the fixed fee for an insider innovator, as it allows the patentee to limit price competition at the final

[8] Our explanation differs from the argument put forward by San Martín and Saracho (2010). In particular, the authors claim that the ad valorem royalty is preferred because "*this type of royalty allows the patentee to strategically commit to be less aggressive*" (San Martín and Saracho, 2010, p.286). However, in our opinion, this explanation has two limitations. The first is that it does not take into account the fact that also a per-unit royalty commits the patentee to be less aggressive: as the profits of the patentee increase with the quantity produced by the licensee, it is in the interest of the patentee to produce less due to strategic substitutability of quantities (Bulow at al., 1985). The second is that, based on the explanation provided for the homogeneous Cournot model, one should expect the same result for the Bertrand model: as under an ad valorem licensing scheme the patentee increases its own profits along with the licensee's profits both in Cournot and in Bertrand, the ad valorem licensing scheme allows the patentee to be less aggressive (in quantities or in prices).

stage of the game. Therefore, the comparison is between per-unit licensing and ad valorem licensing. Note that the revenues from per-unit licensing are higher the higher is the quantity produced in equilibrium by the licensee. On the other hand, in case of an ad valorem royalty, the costs savings absorbed by the patentee are higher the higher are the equilibrium profits of the licensee. When comparing a homogenous Cournot model with a Bertrand set-up, we have that the equilibrium quantities are lower in Cournot than in Bertrand, while the opposite holds with regard to the equilibrium profits. Therefore, the revenues from the per-unit royalty are lower in Cournot than in Bertrand, while the cost savings absorbed by the patentee are higher in Cournot than in Bertrand. It follows that in Cournot the ad valorem mechanism is preferred to the per-unit mechanism (San Martín and Saracho, 2010) while in Bertrand per-unit licensing is preferred to ad valorem licensing (Proposition 3 above).[9]

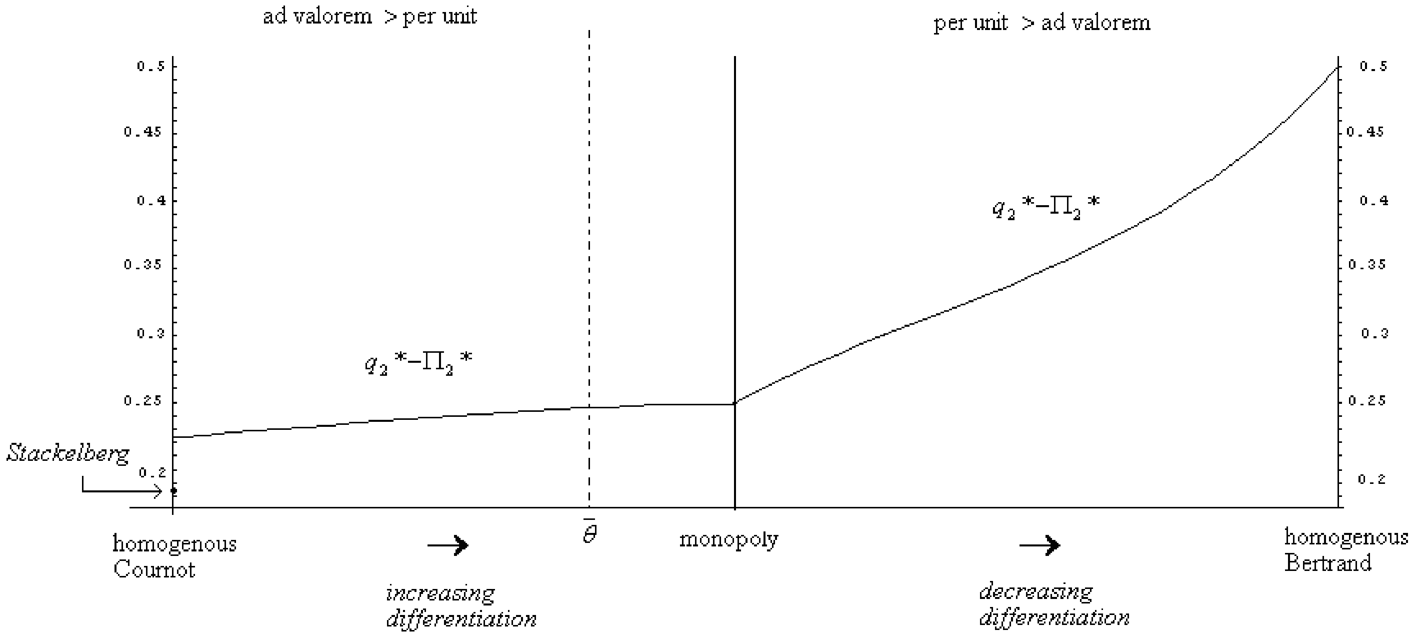
In Figure 1, we plot the difference between the licensee's equilibrium quantity and the licensee's equilibrium profits for different models before the imposition of the royalty.[10] This difference can be interpreted as a measure of the relative profitability of per-unit licensing with respect to ad valorem licensing: the higher is the difference, the higher are the revenues from the per-unit royalty with respect to the cost savings absorbed through the ad valorem royalty, that is, the stronger is the quantity effect with respect to the profits effect. As ad valorem licensing is preferred to per-unit licensing in a homogenous Cournot model, while per-unit licensing is preferred to ad valorem

[9] Rephrasing this concept using the category of anti-competitive behaviour, notice that both per-unit and ad valorem licensing have anti-competitive properties (see footnote 8). Therefore, the problem of the patentee consists in choosing the licensing mechanism that allows *better* committing to be less aggressive. In the Cournot context, the profits effect prevails over the quantity effect, thus making the ad valorem mechanism a more efficient way to commit to be less aggressive; on the contrary, in the Bertrand context, the quantity effect prevails over the profits effect, thus making the per-unit mechanism a more efficient way for the patentee to commit itself to set a higher price.

[10] Figure 1 has been obtained under the assumption of linear demand functions as those adopted in this article.

licensing in a Bertrand model for any degree of differentiation, we argue that a differentiation level must exist such that, in Cournot, the preference of the patentee shifts from the ad valorem mechanism to the per-unit mechanism. In Figure 1 we denote this differentiation level with $\bar{\theta}$.[11]

Figure 1: the profitability of ad valorem and per-unit licensing contracts



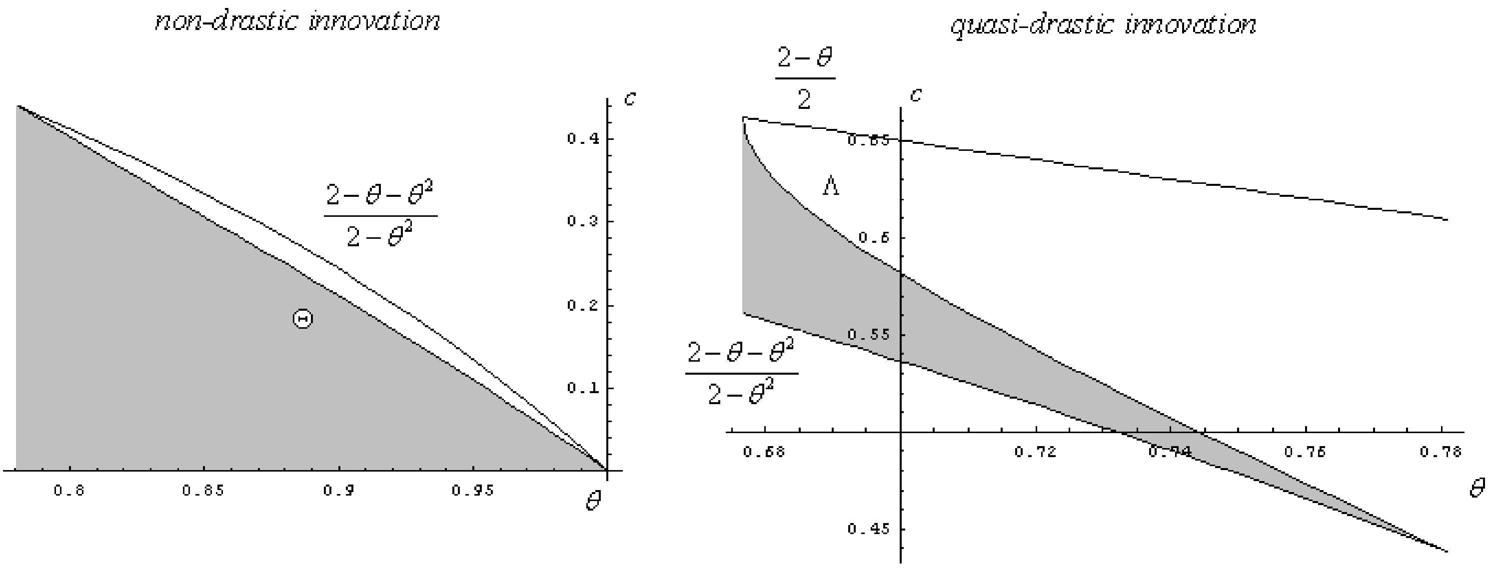
This explanation is consistent also with other results. For example, San Martín and Saracho (2010, p.286) claim that in the case of an oligopolistic industry with three firms, numerical computations show that per-unit licensing is preferred to ad valorem licensing. We argue that this depends on the fact that when passing from a duopoly to a

[11] Interestingly, this seems to be consistent with San Martín and Saracho (2011), that compare ad valorem and per-unit licensing in a differentiated Cournot context and show that when goods are sufficiently differentiated, per-unit royalties may be preferred to ad valorem royalties. Note that when the degree of differentiation increases, both the quantity and the profits increase. However, it can be easily shown that the quantity increases at a faster rate than the profits, thus making the per-unit licensing scheme relatively more and more profitable than the ad valorem licensing scheme: at $\bar{\theta}$ the ranking between the two licensing mechanism changes.

triopoly, the total equilibrium quantity of the non-patentee firms increases more than the equilibrium profits of the non-patentee firms: therefore, per-unit licensing becomes more profitable relatively to ad valorem licensing.[12] In the next section, we will test our intuition in a Stackelberg game.

Consider now point *ii*) of Proposition 3. It shows that, as a second best option, it may be preferable for the patentee not to license the innovation instead of adopting the optimal ad valorem contract. In particular, we have to distinguish between the following cases:

Figure 2: licensing through an ad valorem royalty *vs* no-licensing



[12] In an oligopoly with three firms, the total amount of quantity of the potential licensees is 1/2 while the total profits are 1/8. With two firms, the equilibrium quantity of the potential licensee is 1/3 while the profits are 1/9.

- *Non-drastic innovation*. The innovator adopts the optimal ad valorem contract instead of keeping the innovation when $\theta \leq 0.78$, or when $\theta \geq 0.78$ and the size of innovation is in the grey area indicated in Figure 2 (left side).[13]

- *Quasi-drastic innovation*. The innovator adopts the optimal ad valorem contract instead of keeping the innovation when $\theta \leq 0.67$, or when $\theta \in [0.67, 0.78]$ and the size of innovation is in the grey area indicated in Figure 2 (right side).[14]

- *Drastic innovation*. The innovator adopts the optimal ad valorem contract instead of keeping the innovation when $\theta \leq 0.67$.

Point *ii*) of Proposition 3 shows that, in contrast with the case of a two-part per-unit contract (see for example Faullì-Oller and Sandonìs, 2002, p.198), if the only option for the patentee is the ad valorem contract, it is not always true that under Bertrand competition the innovation is licensed. Interestingly, innovation may not be licensed even if it is non-drastic, provided that the two goods have a high degree of substitutability or the size of innovation is sufficiently large (however, it should also be noted that the parameter space supporting the case of no-licensing as a second-best option for the innovator is larger when the innovation is drastic). The intuition is the following. When adopting an ad valorem royalty mechanism, the patentee increases the price in order to increase the profits of the licensee and exploit part of them by means of the royalty. The incentive for the patentee to set a high price is greater the lower are the profits of the licensee. All else being equal, the profits of the licensee are lower the higher is θ (i.e. the less the firms are differentiated). Therefore, when θ is high enough,

[13] $\Theta(\theta)$ is the solution of $\Pi_1^V * = \Pi_1^N *$ in case of non-drastic innovation.

[14] $\Lambda(\theta)$ is the solution of $\Pi_1^V * = \Pi_1^N *$ in case of quasi-drastic innovation.

it may be better for the patentee not to license its innovation to avoid an excessive increase of its own price and a reduction of its own demand. On the other hand, also the size of the innovation matters when deciding between not licensing or licensing through ad valorem royalties: the more profitable is the innovation, the narrower is the parameter set inducing the patentee to sell the innovation instead of keeping it. This is in line with previous results concerning the decision whether to license through per-unit royalties or fixed fee (or combination of them), or not to license (see, for example, Wang, 1998 and 2002, Erkal, 2005, and Filippini, 2005).

3.5. *Welfare*

In this section we compare the social welfare under the two licensing contracts and in the case of no-licensing. In line with Faullì-Oller and Sandonìs (2002), the welfare is defined as $W = U(.) - cq_2$ in case of no-licensing and as $W = U(.)$ in case of licensing. By using the equilibrium quantities, we define the following welfare functions: $W^N = U(q_1^N*, q_2^N*) - cq_2^N*$, $W^U = U(q_1^U*, q_2^U*)$ and $W^V = U(q_1^V*, q_2^V*)$.[15] By comparing welfare under the three regimes, we can state the following proposition:[16]

Proposition 4. *Welfare is higher when licensing by means of an ad valorem contract than under a per-unit contract and than without licensing.*

Proof. See the Technical Appendix. ■

[15] To save on space, the equilibrium welfare in the different cases is reported in the Technical Appendix.
[16] For sake of precision, Proposition 4 is valid under the restriction $\theta \leq 0.99$.

The above proposition shows that when firms are price-setting, the ad valorem royalty licensing scheme yields higher welfare than the per-unit royalty licensing scheme. This differs from the analogous finding in the Cournot set-up, where the per-unit royalty licensing scheme is shown to be better for welfare than the ad valorem licensing mechanism (San Martín and Saracho, 2010). Indeed, the patentee chooses the licensing mechanism that, given the nature of the competition, is more efficient in allowing the patentee to be less aggressive (see footnote 9): under Bertrand this coincides with the per-unit mechanism, due to the fact that the quantity effect dominates over the profits effect, while in Cournot the more anticompetitive mechanism is ad valorem licensing, as the profits effect dominates over the quantity effect. As a consequence, the overall quantity produced by both firms under the per-unit royalties licensing scheme is lower than under the ad valorem royalties licensing scheme in the Bertrand set-up; on the contrary, in a homogenous Cournot context, the overall quantity produced by both firms is higher under the per-unit royalty mechanism than under the ad valorem royalties mechanism. As welfare in case of licensing coincides with the utility of the representative consumer, the licensing scheme chosen by the patentee is welfare-inferior to the licensing scheme not chosen in equilibrium. It follows that under Bertrand (Cournot) welfare is higher under the ad valorem (per-unit) royalty scheme. Moreover, in the Bertrand context, licensing through an ad valorem mechanism unambiguously increases the welfare even with respect to the case of no-licensing. In this sense, the welfare-reducing licensing result by Faullì-Oller and Sandonìs (2002) does not apply when an ad valorem licensing scheme is considered instead than a per-unit mechanism.

4. The Stackelberg case

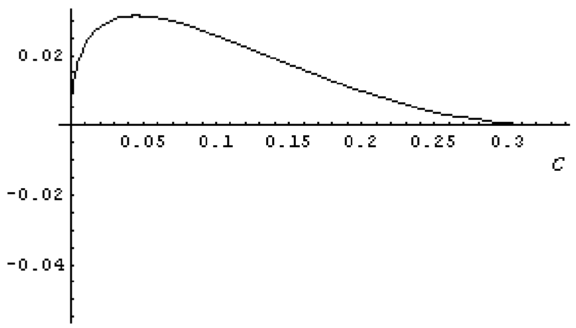
In this section we analyse a standard Stackelberg model, with homogeneous products and where the patentee is the leader and the licensee is the follower. Consistently with our previous analysis, we should expect the ad valorem licensing scheme to be superior to the per-unit licensing scheme. In fact, with respect to a homogenous Cournot model, both the quantity and the profits of the licensee are lower, but the decrease of the quantity is stronger than the reduction of the profits (see Figure 1). Therefore, the incentive for the patentee to adopt the ad valorem licensing scheme in the Stackelberg set-up is stronger than in the homogeneous Cournot set-up: as in the homogeneous Cournot model the ad valorem licensing scheme is preferred by the patentee to the per-unit licensing scheme, the same must hold also in the Stackelberg model. This is precisely what we show in the remaining part of this section.

Suppose that the inverse demand function is given by: $p = 1 - q_i - q_{-i}$, with $i = 1, 2$. With respect to Section 3, in the third stage of the game the firms now move sequentially instead of simultaneously, and set quantities instead of prices. Firm 1 is the leader and Firm 2 is the follower. We only consider a non-drastic innovation, which in this context amounts to require that $c \leq 1/3$.

If Firm 2 does not accept the contract by Firm 1 in the second stage of the game, straightforward calculations yield the following equilibrium profits: $\Pi_{1,S}^{N}* = \frac{(1+c)^2}{8}$ and $\Pi_{2,S}^{N}* = \frac{(1-3c)^2}{16}$, where the subscript S is used here to refer to the Stackelberg model. If Firm 1 adopts a two-part per-unit royalty licensing scheme, it can be shown that the optimal contract is a pure per-unit scheme, such that $F_S^U{}^* = 0$ and $h_S{}^* = c$.[17] The equilibrium profits of the patentee are: $\Pi_{1,S}^U{}^* = \dfrac{1+6c-9c^2}{8}$. Finally, in the case of an ad valorem royalty contract, the fixed fee is set to zero, while *d* solves the incentive-compatibility constraint of the licensee. The optimal contract in this case is therefore: $F_S^V{}^* = 0$ and $d_S^V{}^* = \dfrac{2[9c^2 - 6c + \sqrt{3c(2-3c)}]}{(1-3c)^2}$, while the equilibrium profits of the patentee are: $\Pi_{1,S}^V{}^* = \dfrac{(1-3c)^2}{8[1-\sqrt{3c(2-3c)}]}$. By comparing $\Pi_{1,S}^U{}^*$ with $\Pi_{1,S}^V{}^*$, we have the following proposition, which is consistent with the intuition developed in Section 3:[18]

Proposition 5. *In a Stackelberg model, the patentee prefers licensing a cost reducing innovation by means of an ad valorem scheme instead than by means of a per-unit licensing scheme.*

Proof. Here is the plot of $\Pi_{1,S}^V{}^* - \Pi_{1,S}^U{}^*$ as a function of *c*:



■

[17] Details are available upon request.

[18] Note also that $\Pi_{1,S}^U{}^* \geq \Pi_{1,S}^N{}^*$, that is, the patentee always licenses in equilibrium.

5. Conclusions

This article considers the optimal two-part ad valorem royalties licensing scheme in a differentiated Bertrand duopoly where the innovator is an insider. Ad valorem royalties are a widespread contract in patent licensing. San Martín and Saracho (2010) provide a theoretical justification of their adoption instead of per-unit royalties in a homogenous Cournot model where the innovator is an insider. First, we show that a pure ad valorem royalty contract is preferred to a two-part mechanism. Second and more important, we show that a per-unit royalty contract is preferred to an ad valorem royalty contract when firms compete on prices. The reason for this result can be summarized as follows. When the patentee adopts a per-unit licensing mechanism, it increases its own profits through the quantity produced by the licensee in equilibrium (quantity effect). On the other hand, when the patentee adopts an ad valorem licensing mechanism, it increases its own profits by expropriating a quota of the licensee's profits (profits effect). When the licensee's equilibrium quantity is high, the additional profits for the patentee are high too if it adopts a per-unit license. On the other hand, when the licensee's (pre-license) equilibrium profits are high, the additional profits for the patentee are high too if it adopts an ad valorem licensing scheme. In case of quantity competition, the licensee produces less and gets greater profits than in case of price competition: this explains why under Cournot the patentee prefers adopting an ad valorem licensing scheme, while under Bertrand the patentee prefers adopting a per-unit licensing scheme. In this article, we also show that, in contrast to traditional findings, no-licensing may be preferred (as a second best option) to licensing through ad valorem royalties, even if the innovation is non-drastic. This happens when the degree of substitutability between the products of the firms is high enough. Finally, we show that the welfare is higher under the ad valorem royalties licensing scheme than under the per-unit royalties licensing scheme and than under no-licensing. Therefore, the welfare-reducing licensing result in a Bertrand game may apply to the case of per-unit royalties but not to the case of ad valorem royalties. The intuition behind these results can be generalized also to other contexts. In this article, we have tested the intuition in the case of firms competing in a Stackelberg game, while we have discussed the implications for the case of oligopolistic frameworks and differentiated Cournot markets.

Our findings show that the ranking between the ad valorem licensing scheme and the per-unit licensing mechanism depends on the conditions of competition between the firms: preference goes to per-unit licensing when competition occurs on prices, while preference goes to ad valorem licensing when competition occurs on quantities (and product are sufficiently homogenous). This might help to explain why the evidence of the two types of licensing scheme is still mixed.[19] Moreover, our discourse can be directly applied to the case of international licensing.[20] For example, our model could be interpreted also as a two-country two-firm (home and foreign) duopolistic trade model where the home firm owns a cost-reducing patent.

However, it should also be noted that our analysis relies on a number of restrictive assumptions. For example, we considered the case of a duopoly. Let us suppose that instead of having just one potential licensee, an oligopolistic industry with $N \geq 2$ potential licensees is considered. In this case, two different approaches are possible. The first assumes that the patentee offers the patent to all the firms operating in the industry. In this case, we should expect that per-unit royalties tend to be preferred to ad valorem

[19] The coexistence of per-unit and ad valorem royalties in licensing schemes is documented also in the past (see Burhop and Lubbers, 2009, documenting licensing contracts in Imperial Germany).
[20] Recent contributions in this area are for example Kabiraj and Marjit (2003), Mukherjee and Pennings (2006), and Mukherjee (2010). Mizuno et al. (2011) also consider the possibility of ad valorem tariffs and licensing contract.

royalties when N increases. In fact, as N increases the industry is more competitive. It follows that the overall quantity produced by the licensees increases, while the profits decrease. In line with the explanation provided in the article, we expect that the patentee prefers increasing its own profits by relying on the quantity produced by the licensees via the per-unit royalty instead than relying on the profits of the licensees via the ad valorem royalty. The second case is when the patentee offers the patent only to a fraction of potential licensees.[21] In this case, it is possible in principle that the optimal fraction of firms receiving the offer from the patentee is different when the patentee adopts a per-unit mechanism and when it adopts an ad valorem mechanism. Therefore, with a different number of licensees, it may be possible that the suggested ranking between the two licensing schemes does not hold. The validity of our claim when the patentee fractionates the offer of the license deserves further research.

Technical appendix

Due to the highly non-linear nature of the equations involved in the comparison yielding to Proposition 3 and Proposition 4, an analytical derivation of the sign of the relations between profits or welfare under the different licensing regimes is not possible. Therefore, we perform the comparison between profits and welfare levels by adopting the Mathematica software. In what follows we provide the Mathematica worksheet. For each comparison, we check first the "wrong" relation. Then, we double-check that the "correct" relation exists for the admissible values of the parameters at any given situation. For example, suppose we want to check that, under the set of the assumptions C, A is always higher than B. First, we characterize the set C. Then, we ask the software the following question: "is A lower than B, when C holds?". The expected answer in this case is FALSE. Then, we ask the software the following question: "is A higher than B, when C holds?". The expected answer now is C. When the sign of A–B varies within C, both questions are asked to the software.

[21] Erutku and Richelle (2000) analyse a similar situation.

Proof of Proposition 3

** ** **

Define :

$$R = \sqrt{\left(64 - 64\theta - 16\left(9 - 8c + 4c^2\right)\theta^2 - 64(-2 + c)\theta^3 + 4\left(31 - 48c + 24c^2\right)\theta^4 + (-84 + 64c)\theta^5 + \left(-51 + 96c - 48c^2\right)\theta^6 - 2(-9 + 8c)\theta^7 + \left(9 - 16c + 8c^2\right)\theta^8\right)}$$

$$K = 64 - 64\theta - 112\theta^2 + 64c\theta^2 - 32c^2\theta^2 + 96\theta^3 - 32c\theta^3 + 76\theta^4 - 80c\theta^4 + 40c^2\theta^4 - 44\theta^5 + 24c\theta^5 - 25\theta^6 + 32c\theta^6 - 16c^2\theta^6 + 6\theta^7 - 4c\theta^7 + 3\theta^8 - 4c\theta^8 + 2c^2\theta^8$$

** ** **

Non drastic innovation : $c < \frac{2 - \theta - \theta^2}{2 - \theta^2}$

Firm 1 profits under non - license (equation 1)

$$NL = -\frac{(-2 + \theta - c\theta + \theta^2)^2}{(-4 + \theta^2)^2 (-1 + \theta^2)}$$

Firm 1 profits under optimal per - unit license if $\frac{\theta(2 + \theta)^2}{8 + 10\theta^2} < c$ (equation 5)

$$PU1 = \frac{1}{4(-4 + \theta^2)^2(-4 - \theta^2 + 5\theta^4)} \left(4c^2\left(-2 + \theta^2\right)^2\left(4 + 5\theta^2\right) + (2 + \theta)^2\left(-16 + 32\theta - 40\theta^2 + 36\theta^3 - 17\theta^4 + 4\theta^5 + \theta^6\right) - 8c\left(16 - 8\theta + 4\theta^2 - 6\theta^3 - 16\theta^4 + 5\theta^5 + 5\theta^6\right)\right)$$

Firm 1 profits under optimal per - unit license if $\frac{\theta(2 + \theta)^2}{8 + 10\theta^2} > c$ (equation 6)

$$PU2 = \frac{-(-2 + \theta + \theta^2)^2 + c^2(8 - 3\theta^2 - 4\theta^4) + c(-8 + 4\theta^2 + \theta^3 + 2\theta^4 + \theta^5)}{(-4 + \theta^2)^2(-1 + \theta^2)}$$

Firm 1 profits under optimal ad valorem license (equation 11)

$$AV = \frac{(-1 + \theta)\theta^4(-2 + 2c + \theta + \theta^2 - c\theta^2)^4\left(-4(2 + \theta)^2 - \frac{2\left(-4 - 4\theta + \theta^2 + \theta^3\right)\left(K - R(2 + \theta)^2\left(2 - 3\theta + \theta^2\right)\right)}{\theta^4\left(-2 + \theta + \theta^2 - c\left(-2 + \theta^2\right)\right)^2} + \frac{(1 + \theta)\left(-K + R(2 + \theta)^2\left(2 - 3\theta + \theta^2\right)\right)^2}{\theta^6\left(-2 + \theta + \theta^2 - c\left(-2 + \theta^2\right)\right)^4}\right)}{(1 + \theta)(2 + \theta)^4(2 - 3\theta + \theta^2)^2(8 - R - 4\theta - 6\theta^2 + \theta^3 + \theta^4)^2}$$

Clear[K, R]

** COMPARISON **

Parameter sets :

$$X1 = \text{Simplify}\left[\text{Reduce}\left[0 < \theta < 1 \,\&\&\, 0 < c < \frac{2 - \theta - \theta^2}{2 - \theta^2}\right]\right]$$

$$c < 1 \,\&\&\, c > 0 \,\&\&\, \frac{1}{-1 + c} + \sqrt{\frac{9 - 16c + 8c^2}{(-1 + c)^2}} > 2\theta \,\&\&\, \theta > 0$$

$$X1A = \text{Simplify}\left[\text{Reduce}\left[0 < \theta < 1 \,\&\&\, 0 < c < \frac{2 - \theta - \theta^2}{2 - \theta^2} \,\&\&\, 1 > c > \frac{\theta(2 + \theta)^2}{8 + 10\theta^2}\right]\right]$$

$$0 < \theta < \text{Root}[8 - 12\#1 + 8\#1^2 - 8\#1^3 + \#1^4 \&, 1] \,\&\&\, \frac{\theta(2 + \theta)^2}{8 + 10\theta^2} < c < \frac{-2 + \theta + \theta^2}{-2 + \theta^2}$$

X1B = Simplify[Reduce[$0 < \theta < 1$ && $0 < c < \frac{2-\theta-\theta^2}{2-\theta^2}$ && $\frac{\theta(2+\theta)^2}{8+10\theta^2} > c$]]

$0 < \theta \leq$ Root[8 - 12 #1 + 8 #1^2 - 8 #1^3 + #1^4 &, 1] && $0 < c < \frac{\theta(2+\theta)^2}{8+10\theta^2}$ ||
Root[8 - 12 #1 + 8 #1^2 - 8 #1^3 + #1^4 &, 1] $< \theta < 1$ && $0 < c < \frac{-2+\theta+\theta^2}{-2+\theta^2}$

°° per - unit license vs non - license

Simplify[Reduce[PU1 < NL && X1A]]

False

Simplify[Reduce[PU1 > NL && X1A]]

$0 < \theta <$ Root[8 - 12 #1 + 8 #1^2 - 8 #1^3 + #1^4 &, 1] && $\frac{\theta(2+\theta)^2}{8+10\theta^2} < c < \frac{-2+\theta+\theta^2}{-2+\theta^2}$

Simplify[Reduce[PU2 < NL && X1B]]

False

Simplify[Reduce[PU2 > NL && X1B]]

$0 < \theta \leq$ Root[8 - 12 #1 + 8 #1^2 - 8 #1^3 + #1^4 &, 1] && $0 < c < \frac{\theta(2+\theta)^2}{8+10\theta^2}$ ||
Root[8 - 12 #1 + 8 #1^2 - 8 #1^3 + #1^4 &, 1] $< \theta < 1$ && $0 < c < \frac{-2+\theta+\theta^2}{-2+\theta^2}$

°° per - unit license vs ad valorem license

Simplify[Reduce[PU1 < AV && X1A]]

False

Simplify[Reduce[PU1 > AV && X1A]]

$0 < \theta <$ Root[8 - 12 #1 + 8 #1^2 - 8 #1^3 + #1^4 &, 1] && $\frac{\theta(2+\theta)^2}{8+10\theta^2} < c < \frac{-2+\theta+\theta^2}{-2+\theta^2}$

Simplify[Reduce[PU2 < AV && X1B]]

False

Simplify[Reduce[PU2 > AV && X1B]]

$0 < \theta \leq$ Root[8 - 12 #1 + 8 #1^2 - 8 #1^3 + #1^4 &, 1] && $0 < c < \frac{\theta(2+\theta)^2}{8+10\theta^2}$ ||
Root[8 - 12 #1 + 8 #1^2 - 8 #1^3 + #1^4 &, 1] $< \theta < 1$ && $0 < c < \frac{-2+\theta+\theta^2}{-2+\theta^2}$

°° **ad valorem license vs non - license**

Simplify[Reduce[AV < NL && X1, c]]

$$\frac{1}{4}\left(-1+\sqrt{17}\right)<\theta<1\;\&\&$$
$$\text{Root}[-128+256\theta+160\theta^2-608\theta^3+88\theta^4+552\theta^5-246\theta^6-232\theta^7+154\theta^8+42\theta^9-40\theta^{10}-2\theta^{11}+4\theta^{12}+64\#1-64\theta\#1-144\theta^2\#1+160\theta^3\#1+124\theta^4\#1-196\theta^5\#1-15\theta^6\#1+120\theta^7\#1-32\theta^8\#1-33\theta^9\#1+15\theta^{10}\#1+3\theta^{11}\#1-2\theta^{12}\#1+16\theta^5\#1^2-16\theta^6\#1^2-16\theta^7\#1^2+16\theta^8\#1^2+4\theta^9\#1^2-4\theta^{10}\#1^2+4\theta^6\#1^3-4\theta^8\#1^3+\theta^{10}\#1^3\;\&,1]<c<\frac{-2+\theta+\theta^2}{-2+\theta^2}$$

Simplify[Reduce[AV > NL && X1, c]]

$$0<\theta\leq\frac{1}{4}\left(-1+\sqrt{17}\right)\;\&\&\;0<c<\frac{-2+\theta+\theta^2}{-2+\theta^2}\;||\;\frac{1}{4}\left(-1+\sqrt{17}\right)<\theta<1\;\&\&$$
$$0<c<\text{Root}[-128+256\theta+160\theta^2-608\theta^3+88\theta^4+552\theta^5-246\theta^6-232\theta^7+154\theta^8+42\theta^9-40\theta^{10}-2\theta^{11}+4\theta^{12}+64\#1-64\theta\#1-144\theta^2\#1+160\theta^3\#1+124\theta^4\#1-196\theta^5\#1-15\theta^6\#1+120\theta^7\#1-32\theta^8\#1-33\theta^9\#1+15\theta^{10}\#1+3\theta^{11}\#1-2\theta^{12}\#1+16\theta^5\#1^2-16\theta^6\#1^2-16\theta^7\#1^2+16\theta^8\#1^2+4\theta^9\#1^2-4\theta^{10}\#1^2+4\theta^6\#1^3-4\theta^8\#1^3+\theta^{10}\#1^3\;\&,1]$$

Clear[NL, PU1, PU2, AV]

** ** **

Quasi drastic innovation : $\frac{2-\theta-\theta^2}{2-\theta^2}<c<\frac{2-\theta}{2}$

Firm 1 profits under non - license (equation 3) :

$$NL=-\frac{(-1+c)\,(-1+c+\theta)}{\theta^2}$$

Firm 1 profits under optimal per - unit license if $\frac{\theta\,(2+\theta)^2}{8+10\,\theta^2}<c$ **(equation 7) :**

$$PU1=\frac{8+9\,\theta^2+\theta^3}{4\,(1+\theta)\,(4+5\,\theta^2)}$$

Firm 1 profits under optimal per - unit license if $\frac{\theta\,(2+\theta)^2}{8+10\,\theta^2}>c$ **(equation 8) :**

$$PU2=\frac{-2\,(-1+\theta)\,(2+\theta)^2+c\,\theta\,(1+\theta)\,(2+\theta)^2-c^2\,(4+4\,\theta+5\,\theta^2+5\,\theta^3)}{(1+\theta)\,(-4+\theta^2)^2}$$

Firm 1 profits under optimal ad valorem license (equation 12) :

$$AV=-\frac{-8+9\,\theta^2+\theta^3-2\,\theta^4}{4\,(1+\theta)\,(-2+\theta^2)^2}$$

Parameter sets :

$$X2=\text{Simplify}\left[\text{Reduce}\left[0<\theta<1\;\&\&\;\frac{2-\theta-\theta^2}{2-\theta^2}<c<\frac{2-\theta}{2}\right]\right]$$

$$c\leq\frac{1}{2}\;\&\&\;c>0\;\&\&\;\frac{1}{-1+c}+\sqrt{\frac{9-16\,c+8\,c^2}{(-1+c)^2}}<2\,\theta\;\&\&\;\theta<1\;||$$
$$c<1\;\&\&\;c>\frac{1}{2}\;\&\&\;\frac{1}{-1+c}+\sqrt{\frac{9-16\,c+8\,c^2}{(-1+c)^2}}<2\,\theta\;\&\&\;2\,c+\theta<2$$

X2A = Simplify[Reduce[$0<\theta<1$ && $\frac{2-\theta-\theta^2}{2-\theta^2}<c<\frac{2-\theta}{2}$ && $1>c>\frac{\theta(2+\theta)^2}{8+10\theta^2}$]]

$0<\theta\leq$ Root[8 − 12 #1 + 8 #1^2 − 8 #1^3 + #1^4 &, 1] && $\frac{\theta^3}{-4+2\theta^2}<-1+c+\frac{\theta}{2}<0$ ||
Root[8 − 12 #1 + 8 #1^2 − 8 #1^3 + #1^4 &, 1] $<\theta<1$ && $\frac{\theta(2+\theta)^2}{8+10\theta^2}<c<1-\frac{\theta}{2}$

X2B = Simplify[Reduce[$0<\theta<1$ && $\frac{2-\theta-\theta^2}{2-\theta^2}<c<\frac{2-\theta}{2}$ && $\frac{g(2+g)^2}{8+10g^2}>c$]]

$c\leq$ Root[54 − 269 #1 + 466 #1^2 − 350 #1^3 + 98 #1^4 &, 1] && $c>0$ && $\frac{1}{-1+c}+\sqrt{\frac{9-16c+8c^2}{(-1+c)^2}}<2\theta$ && $\theta<1$ || $c<\frac{1}{2}$ &&
$c>$ Root[54 − 269 #1 + 466 #1^2 − 350 #1^3 + 98 #1^4 &, 1] && $\theta<1$ && $\theta>$ Root[−8 c + 4 #1 + 4 #1^2 − 10 c #1^2 + #1^3 &, 1]

**** COMPARISON ****

°° per − unit license vs non − license

Simplify[Reduce[PU1 < NL && X2A]]

False

Simplify[Reduce[PU1 > NL && X2A]]

$0<\theta\leq$ Root[8 − 12 #1 + 8 #1^2 − 8 #1^3 + #1^4 &, 1] && $\frac{\theta^3}{-4+2\theta^2}<-1+c+\frac{\theta}{2}<0$ ||
Root[8 − 12 #1 + 8 #1^2 − 8 #1^3 + #1^4 &, 1] $<\theta<1$ && $\frac{\theta(2+\theta)^2}{8+10\theta^2}<c<1-\frac{\theta}{2}$

Simplify[Reduce[PU2 < NL && X2B]]

False

Simplify[Reduce[PU2 > NL && X2B, g]]

$c\leq$ Root[54 − 269 #1 + 466 #1^2 − 350 #1^3 + 98 #1^4 &, 1] && $c>0$ && $\frac{1}{-1+c}+\sqrt{\frac{9-16c+8c^2}{(-1+c)^2}}<2\theta$ && $\theta<1$ || $c<\frac{1}{2}$ &&
$c>$ Root[54 − 269 #1 + 466 #1^2 − 350 #1^3 + 98 #1^4 &, 1] && $\theta<1$ && $\theta>$ Root[−8 c + 4 #1 + 4 #1^2 − 10 c #1^2 + #1^3 &, 1]

°° per − unit license vs ad valorem license

Simplify[Reduce[PU1 < AV && X2A]]

False

Simplify[Reduce[PU1 > AV && X2A]]

$0<\theta\leq$ Root[8 − 12 #1 + 8 #1^2 − 8 #1^3 + #1^4 &, 1] && $\frac{\theta^3}{-4+2\theta^2}<-1+c+\frac{\theta}{2}<0$ ||
Root[8 − 12 #1 + 8 #1^2 − 8 #1^3 + #1^4 &, 1] $<\theta<1$ && $\frac{\theta(2+\theta)^2}{8+10\theta^2}<c<1-\frac{\theta}{2}$

Simplify[Reduce[PU2 < AV && X2B]]

```
False
```

Simplify[Reduce[PU2 > AV && X2B]]

$c \le \text{Root}[54 - 269\,\#1 + 466\,\#1^2 - 350\,\#1^3 + 98\,\#1^4\,\&, 1]\ \&\&\ c > 0\ \&\&\ \frac{1}{-1+c} + \sqrt{\frac{9 - 16c + 8c^2}{(-1+c)^2}} < 2\theta\ \&\&\ \theta < 1\ ||\ c < \frac{1}{2}\ \&\&$

$c > \text{Root}[54 - 269\,\#1 + 466\,\#1^2 - 350\,\#1^3 + 98\,\#1^4\,\&, 1]\ \&\&\ \theta < 1\ \&\&\ \theta > \text{Root}[-8c + 4\,\#1 + 4\,\#1^2 - 10c\,\#1^2 + \#1^3\,\&, 1]$

°° ad valorem license vs non - license

Simplify[Reduce[AV < NL && X2, c]]

$\theta < 1\ \&\&\ 1 + 4\theta > \sqrt{17}\ \&\&\ 2c + \theta < 2\ \&\&\ c + \frac{2}{-2+\theta^2} > \frac{\theta(1+\theta)}{-2+\theta^2}\ ||$

$1 + 4\theta \le \sqrt{17}\ \&\&\ \theta > \text{Root}[-4 + 4\,\#1 + 5\,\#1^2 - 3\,\#1^3 - \#1^4 + \#1^5\,\&, 1]\ \&\&$

$2c + \theta < 2\ \&\&\ 2c + \theta + \sqrt{\frac{\theta^2(-4 + 4\theta + 5\theta^2 - 3\theta^3 - \theta^4 + \theta^5)}{(1+\theta)(-2+\theta^2)^2}} > 2$

Simplify[Reduce[AV > NL && X2, c]]

$0 < \theta \le \text{Root}[-4 + 4\,\#1 + 5\,\#1^2 - 3\,\#1^3 - \#1^4 + \#1^5\,\&, 1]\ \&\&\ \frac{\theta^3}{-4 + 2\theta^2} < -1 + c + \frac{\theta}{2} < 0\ ||$

$\text{Root}[-4 + 4\,\#1 + 5\,\#1^2 - 3\,\#1^3 - \#1^4 + \#1^5\,\&, 1] < \theta < \frac{1}{4}\left(-1 + \sqrt{17}\right)\ \&\&$

$\frac{-2 + \theta + \theta^2}{-2 + \theta^2} < c < \frac{1}{2}\left(2 - \theta - \sqrt{\frac{\theta^2(-4 + 4\theta + 5\theta^2 - 3\theta^3 - \theta^4 + \theta^5)}{(1+\theta)(-2+\theta^2)^2}}\right)$

Clear[NL, PU2]

**** ** ****

Drastic innovation : c > $\frac{2-\theta}{2}$

Firm 1 profits under non - license (equation 4) : $\frac{1}{4}$

NL = $\frac{1}{4}$

Firm 1 profits under optimal per - unit license : as equation 7 $\left(\textbf{only}\ \frac{\theta(2+\theta)^2}{8 + 10\theta^2} < \textbf{c is possible}\right)$

Firm 1 profits under optimal ad valorem license : as equation 12

Parameter sets :

X3 = Simplify[Reduce[$0 < \theta < 1\ \&\&\ 1 > c > \frac{2-\theta}{2}$**]]**

$c < 1\ \&\&\ c > \frac{1}{2}\ \&\&\ \theta < 1\ \&\&\ 2c + \theta > 2$

X3A = Simplify[Reduce[$0 < \theta < 1$ && $1 > c > \frac{2-\theta}{2}$ && $1 > c > \frac{\theta (2+\theta)^2}{8+10\theta^2}$]]

$c < 1$ && $\theta < 1$ && $\theta > 0$ && $2c + \theta > 2$

X3B = Simplify[Reduce[$0 < \theta < 1$ && $1 > c > \frac{2-\theta}{2}$ && $\frac{\theta (2+\theta)^2}{8+10\theta^2} > c$]]

False

**** COMPARISON ****

°° per - unit license vs non - license

Simplify[Reduce[PU1 < NL && X3A]]

False

Simplify[Reduce[PU1 > NL && X3A]]

$c < 1$ && $c > \frac{1}{2}$ && $\theta < 1$ && $2c + \theta > 2$

°° ad valorem license vs non - license

Simplify[Reduce[AV < NL && X3, c]]

$c < 1$ && $\theta < 1$ && $\theta >$ Root[$-4 + 4\#1 + 5\#1^2 - 3\#1^3 - \#1^4 + \#1^5$ &, 1] && $2c + \theta > 2$

Simplify[Reduce[AV > NL && X3, c]]

$c < 1$ && $\theta <$ Root[$-4 + 4\#1 + 5\#1^2 - 3\#1^3 - \#1^4 + \#1^5$ &, 1] && $\theta > 0$ && $2c + \theta > 2$

Clear[NL, PU1, PU2, AV]

** ** **

Proof of Proposition 4

** ** **

Non drastic innovation : $c < \frac{2-\theta-\theta^2}{2-\theta^2}$

Welfare under non - license :

$$WNL = \frac{-2(2+\theta)^2(3-5\theta+2\theta^2) + 2c(2+\theta)^2(3-5\theta+2\theta^2) + c^2(-12+9\theta^2-2\theta^4)}{2(-4+\theta^2)^2(-1+\theta^2)}$$

Welfare under under optimal per - unit license if $\frac{\theta(2+\theta)^2}{8+10\theta^2} < c$:

$$WPU1 = \frac{24+4\theta+23\theta^2+3\theta^3}{8(1+\theta)(4+5\theta^2)}$$

Welfare under under optimal per - unit license if $\frac{\theta (2+\theta)^2}{8+10\theta^2} > c$ **:**

$$WPU2 = -\frac{2 (2+\theta)^2 (-3+2\theta) - 2 c (2+\theta)^2 (-1+\theta^2) + c^2 (4+4\theta+5\theta^2+5\theta^3)}{2 (1+\theta) (-4+\theta^2)^2}$$

Welfare under under optimal ad valorem license :

$$
\begin{aligned}
WAV = &\Big(-(K-8R)^2 + \left(K^2 - 24KR + 128R^2\right)\theta + 2\left(K^2 - 18KR + 72R^2\right)\theta^2 + \\
&\left(-16R\left(16-32c+16c^2+17R\right) + K\left(32-64c+32c^2+30R\right)\right)\theta^3 + \\
&4\left(2K\left(6-16c+10c^2+3R\right) - R\left(64-192c+128c^2+35R\right)\right)\theta^4 - \\
&2\left(-32R\left(19-32c+12c^2+3R\right) + K\left(52-72c+16c^2+3R\right)\right)\theta^5 + \\
&\left(1536 - 6144c^3 + 1536c^4 - 32c^2(-288+3K-34R) + 320R + 79R^2 - 4K(15+R) + 16c(11K - 96(4+R))\right)\theta^6 + \\
&\left(-2560 + 1024c^3 + 512c^4 + 64K - 8c(-896+9K-292R) + 8c^2(-768+K-94R) - 1456R - 53R^2\right)\theta^7 - \\
&\left(2432 - 15872c^3 + 3712c^4 - 32K + 304R + 21R^2 + c^2(23040 - 36K + 880R) + 8c(9K - 2(832+83R))\right)\theta^8 + \\
&\left(4608 - 768c^3 - 1280c^4 - 8K + 740R + 5R^2 + 64c^2(168+5R) + 8c(K - 2(832+69R))\right)\theta^9 + \\
&2\left(1104 - 8192c^3 + 1792c^4 - 2K - 2c^2(-6096+K-84R) + 4c(-1712+K-71R) + 94R + R^2\right)\theta^{10} + \\
&4\left(-224c^3 + 320c^4 - 40(21+R) - 3c^2(512+5R) + 8c(286+7R)\right)\theta^{11} - \\
&4\left(-2112c^3 + 432c^4 + 15c^2(224+R) + 6(63+2R) - 4c(512+7R)\right)\theta^{12} - \\
&4\left(-304c^3 + 160c^4 - 3(92+R) + 4c(154+R) - c^2(192+R)\right)\theta^{13} + 4(-1+c)\left(-156 - 440c^2 + 104c^3 - R + c(508+R)\right)\theta^{14} + \\
&32(-1+c)^2\left(-3-5c+5c^2\right)\theta^{15} - 8(-1+c)^3(-13+5c)\theta^{16} - 16(-1+c)^4\theta^{17}\Big) \Big/ \\
&\Big(2\theta^2(1+\theta) \\
&\left(-K + 8R - 4R\theta - 2\left(16-32c+16c^2+3R\right)\theta^2 + (32-32c+R)\theta^3 + \left(32-80c+40c^2+R\right)\theta^4 + 24(-1+c)\theta^5 - \right. \\
&\left.2\left(7-16c+8c^2\right)\theta^6 - 4(-1+c)\theta^7 + 2(-1+c)^2\theta^8\right)^2\Big)
\end{aligned}
$$

**** COMPARISON ****

Parameter sets :

WX1 = Simplify[Reduce[$0 < \theta < \frac{99}{100}$ **&&** $0 < c < \frac{2-\theta-\theta^2}{2-\theta^2}$ **, c]]**

$0 < \theta < \frac{99}{100}$ && $0 < c < \frac{-2+\theta+\theta^2}{-2+\theta^2}$

WX1A = Simplify[Reduce[$0 < \theta < \frac{99}{100}$ **&&** $0 < c < \frac{2-\theta-\theta^2}{2-\theta^2}$ **&&** $1 > c > \frac{\theta (2+\theta)^2}{8+10\theta^2}$ **, c]]**

$0 < \theta <$ Root[8 - 12 #1 + 8 #1^2 - 8 #1^3 + #1^4 &, 1] && $\frac{\theta (2+\theta)^2}{8+10\theta^2} < c < \frac{-2+\theta+\theta^2}{-2+\theta^2}$

WX1B = Simplify[Reduce[$0 < \theta < \frac{99}{100}$ **&&** $0 < c < \frac{2-\theta-\theta^2}{2-\theta^2}$ **&&** $\frac{\theta (2+\theta)^2}{8+10\theta^2} > c$ **, c]]**

$0 < \theta \le$ Root[8 - 12 #1 + 8 #1^2 - 8 #1^3 + #1^4 &, 1] && $0 < c < \frac{\theta (2+\theta)^2}{8+10\theta^2}$ ||

Root[8 - 12 #1 + 8 #1^2 - 8 #1^3 + #1^4 &, 1] $< \theta < \frac{99}{100}$ && $0 < c < \frac{-2+\theta+\theta^2}{-2+\theta^2}$

°° **ad valorem license vs non - license**

Simplify[Reduce[WAV < WNL && WX1, c]]

False

Simplify[Reduce[WAV > WNL && WX1, c]]

$0 < \theta < \frac{99}{100}$ && $0 < c < \frac{-2+\theta+\theta^2}{-2+\theta^2}$

°° **ad valorem license license vs per - unit** $\left(\text{case } c > \frac{\theta (2+\theta)^2}{8+10\theta^2}\right)$

Simplify[Reduce[WAV < WPU1&& WX1A]]

False

Simplify[Reduce[WAV > WPU1&& WX1A]]

$0 < \theta <$ Root[8 - 12 #1 + 8 #1^2 - 8 #1^3 + #1^4 &, 1] && $\frac{\theta (2+\theta)^2}{8+10\theta^2} < c < \frac{-2+\theta+\theta^2}{-2+\theta^2}$

°° **ad valorem license license vs per - unit** $\left(\text{case } c < \frac{\theta (2+\theta)^2}{8+10\theta^2}\right)$**: LATER**

Clear[WNL, WAV]

** ** **

Quasi drastic innovation : $\frac{2-\theta-\theta^2}{2-\theta^2} < c < \frac{2-\theta}{2}$

Welfare under non - license :

$$WNL = -\frac{(-1+c)(-1+c+2\theta)}{2\theta^2}$$

Welfare under under optimal per - unit license if $\frac{\theta (2+\theta)^2}{8+10\theta^2} < c$ **: (AS ABOVE)**

Welfare under under optimal per - unit license if $\frac{\theta (2+\theta)^2}{8+10\theta^2} > c$ **: (AS ABOVE)**

Welfare under under optimal ad valorem license :

$$WAV = \frac{24+4\theta-21\theta^2-3\theta^3+4\theta^4}{8(1+\theta)(-2+\theta^2)^2}$$

** **COMPARISON** **

Parameter sets :

WX2 = Simplify$\left[\textbf{Reduce}\left[0 < \theta < \frac{99}{100} \;\&\&\; \frac{2-\theta-\theta^2}{2-\theta^2} < c < \frac{2-\theta}{2}, c\right]\right]$

$0 < \theta < \frac{99}{100}$ && $\frac{\theta^3}{-4+2\theta^2} < -1+c+\frac{\theta}{2} < 0$

WX2A = Simplify$\left[\textbf{Reduce}\left[0 < \theta < \frac{99}{100} \,\&\&\, \frac{2-\theta-\theta^2}{2-\theta^2} < c < \frac{2-\theta}{2} \,\&\&\, 1 > c > \frac{\theta\,(2+\theta)^2}{8+10\,\theta^2},\, c\right]\right]$

$0 < \theta \le \text{Root}[8 - 12\,\#1 + 8\,\#1^2 - 8\,\#1^3 + \#1^4\,\&,\, 1]\ \&\&\ \frac{\theta^3}{-4+2\,\theta^2} < -1 + c + \frac{\theta}{2} < 0\ ||$

$\text{Root}[8 - 12\,\#1 + 8\,\#1^2 - 8\,\#1^3 + \#1^4\,\&,\, 1] < \theta < \frac{99}{100}\ \&\&\ \frac{\theta\,(2+\theta)^2}{8+10\,\theta^2} < c < 1 - \frac{\theta}{2}$

WX2B = Simplify$\left[\textbf{Reduce}\left[0 < \theta < \frac{99}{100} \,\&\&\, \frac{2-\theta-\theta^2}{2-\theta^2} < c < \frac{2-\theta}{2} \,\&\&\, \frac{\theta\,(2+\theta)^2}{8+10\,\theta^2} > c,\, c\right]\right]$

$\text{Root}[8 - 12\,\#1 + 8\,\#1^2 - 8\,\#1^3 + \#1^4\,\&,\, 1] < \theta < \frac{99}{100}\ \&\&\ \frac{-2+\theta+\theta^2}{-2+\theta^2} < c < \frac{\theta\,(2+\theta)^2}{8+10\,\theta^2}$

°° ad valorem license vs non - license

Simplify[Reduce[WAV < WNL && WX2, c]]

False

Simplify[Reduce[WAV > WNL && WX2, c]]

$0 < \theta < \frac{99}{100}\ \&\&\ \frac{\theta^3}{-4+2\,\theta^2} < -1 + c + \frac{\theta}{2} < 0$

°° ad valorem license vs per - unit $\left(\textbf{case } c > \frac{\theta\,(2+\theta)^2}{8+10\,\theta^2}\right)$

Simplify[Reduce[WAV < WPU1&& WX2A, c]]

False

Simplify[Reduce[WAV > WPU1&& WX2A, c]]

$0 < \theta \le \text{Root}[8 - 12\,\#1 + 8\,\#1^2 - 8\,\#1^3 + \#1^4\,\&,\, 1]\ \&\&\ \frac{\theta^3}{-4+2\,\theta^2} < -1 + c + \frac{\theta}{2} < 0\ ||$

$\text{Root}[8 - 12\,\#1 + 8\,\#1^2 - 8\,\#1^3 + \#1^4\,\&,\, 1] < \theta < \frac{99}{100}\ \&\&\ \frac{\theta\,(2+\theta)^2}{8+10\,\theta^2} < c < 1 - \frac{\theta}{2}$

°° ad valorem license vs per - unit $\left(\textbf{case } c < \frac{\theta\,(2+\theta)^2}{8+10\,\theta^2}\right)$

Simplify[Reduce[WAV < WPU2 && WX2B, c]]

False

Simplify[Reduce[WAV > WPU2 && WX2B, c]]

$\text{Root}[8 - 12\,\#1 + 8\,\#1^2 - 8\,\#1^3 + \#1^4\,\&,\, 1] < \theta < \frac{99}{100}\ \&\&\ \frac{-2+\theta+\theta^2}{-2+\theta^2} < c < \frac{\theta\,(2+\theta)^2}{8+10\,\theta^2}$

** ** **

Drastic innovation : $c > \frac{2-\theta}{2}$

Welfare under non - license :

$WNL = \frac{3}{8}$

Welfare under under optimal per - unit license if $\frac{\theta (2+\theta)^2}{8+10\theta^2} < c$: AS ABOVE

Welfare under under optimal ad valorem license : AS ABOVE

** COMPARISON **

Parameter sets :

$WX3 = Simplify\left[Reduce\left[0 < \theta < \frac{99}{100} \;\&\&\; 1 > c > \frac{2-\theta}{2}, c\right]\right]$

$c < 1 \;\&\&\; \theta < \frac{99}{100} \;\&\&\; \theta > 0 \;\&\&\; 2c + \theta > 2$

$WX3A = Simplify\left[Reduce\left[0 < \theta < \frac{99}{100} \;\&\&\; 1 > c > \frac{2-\theta}{2} \;\&\&\; 1 > c > \frac{\theta (2+\theta)^2}{8+10\theta^2}, c\right]\right]$

$c < 1 \;\&\&\; \theta < \frac{99}{100} \;\&\&\; \theta > 0 \;\&\&\; 2c + \theta > 2$

$WX3B = Simplify\left[Reduce\left[0 < \theta < \frac{99}{100} \;\&\&\; 1 > c > \frac{2-\theta}{2} \;\&\&\; \frac{\theta (2+\theta)^2}{8+10\theta^2} > c, c\right]\right]$

False

°° ad valorem license vs non - license

Simplify[Reduce[WAV < WNL && WX3, c]]

False

Simplify[Reduce[WAV > WNL && WX3, c]]

$c < 1 \;\&\&\; \theta < \frac{99}{100} \;\&\&\; \theta > 0 \;\&\&\; 2c + \theta > 2$

°° ad valorem license license vs per - unit $\left(\text{case } c > \frac{\theta (2+\theta)^2}{8+10\theta^2}\right)$: AS ABOVE

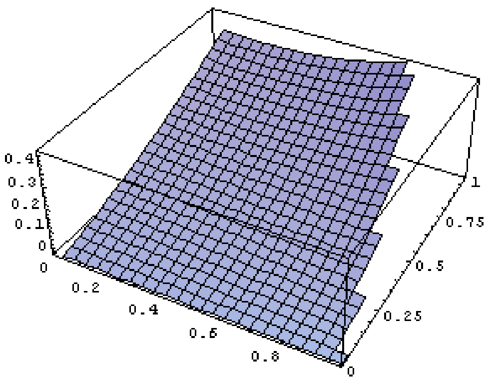
Clear[WNL]

** ** **

GRAPHICAL PROOF OF :

ad valorem license license vs per - unit $\left(\text{case } c < \frac{\theta (2+\theta)^2}{8+10\theta^2}\right)$ under X1B

```
Plot3D[WAV - WPU2, {θ, 0, 99/100}, {c, 0, 1}]
```




```
- SurfaceGraphics -
```

References

Boccard, N., 2010, On royalty licensing, *working paper*, Universitat de Girona.

Bousquet, A., Cremer, H., Ivaldi, M. and M. Wolkovicz, 1998, Risk sharing in licensing, *International Journal of Industrial Organization* 16: 535-554.

Burhop, C. and T. Lubbers, 2009, The historical market for technology licenses: chemicals, pharmaceuticals, and electronic engineering in Imperial Germany, Preprint Max-Planck-Institute for Research on Collective Goods, No. 2009/25.

Bulow, J., Geanakoplos, J. and P. Klemperer, 1985, Multimarket oligopolies: strategic substitutes and complements, *Journal of Political Economy* 93: 488-511.

Das Varma, G., 2003, Bidding for a process innovation under alternative modes of competition, *International Journal of Industrial Organization* 21: 15-37.

Erkal, N., 2005, Optimal licensing policy in differentiated industries, *Economic Record* 81: 51-64.

Erutku, C. and Y. Richelle, 2000, Optimal licensing contract and the value of a patent, *Journal of Economics and Management Strategy* 16: 407-436.

Fan, C., B.H. Jun and E. Wolfstetter, 2011, Licensing process innovations when losers' bids determine royalty rates, *working paper*, Humboldt University.

Faulli-Oller, R. and J. Sandonìs, 2002, Welfare reducing licensing, *Games and Economic Behavior* 41: 192-205.

Filippini, L., 2005, Licensing contract in a Stackelberg model, *The Manchester School* 73: 582-598.

Gallini, N. and B. Wright, 1990, Technology transfer under asymmetric information, *RAND Journal of Economics* 21: 147-160.

Giobe, T. and E. Wolfstetter, 2008, License auctions with royalty contracts for (winner and) losers, *Games and Economic Behavior* 63: 91-106.

Kabiraj, T. and S. Marjit, 2003, Protecting consumers through protection: the role of tariff-induced technology transfer, *European Economic Review* 47: 113-124.

Kamien, M. and Y. Tauman, 1984, The private value of a patent: a game theoretic analysis, *Journal of Economics* 4: 93-118.

Kamien, M. and Y. Tauman, 1986, Fees versus royalties and the private value of a patent, *Quarterly Journal of Economics* 101: 471-491.

Kamien, M., Omen, S. and Y. Tauman, 1992, Optimal licensing of cost reducing innovation, *Journal of Mathematical Economics* 21: 483-508.

Katz, M. and C. Shapiro, 1986, How to license intangible property, *Quarterly Journal of Economics* 101: 567-589.

Lim, T. and R. Veugelers, 2002, On equity as a payment device in technology licensing with moral hazard, *working paper,* Katholieke Universiteit Leuven.

Mizuno, T., K. Takauchi and T. Iida, 2011, Better technology may be sold for a lower fee: the ad valorem tariff and licensing contract, Discussion paper no. 1109, Graduate School of Economics, Kobe University.

Mukherjee, A., 2010, Competition and welfare: the implications of licensing, *The Manchester School* 78: 20-40.
Mukherjee, A. and E. Pennings, 2006, Tariffs, licensing and market structure, *European Economic Review* 50: 1699-1707.
Muto, S., 1993, On licensing policies in Bertrand competition, *Games and Economic Behavior* 5: 257-267.
Poddar, S. and U.B. Sinha, 2004, On patent licensing in spatial competition, *Economic Record* 80: 208-218.
Saracho, A.I., 2002, Patent licensing under strategic delegation, *Journal of Economics and Management Strategy* 11: 225-251.
San Martìn, M. and A.I. Saracho, 2010, Royalty licensing, *Economics Letters* 107: 284-287.
San Martìn, M. and A.I. Saracho, 2011, Two-part tariff licensing mechanisms, *working paper*, University of Basque Country.
Sen, D., 2005, Fee versus royalty reconsidered, *Games and Economic Behavior* 53: 141-147.
Sen, D. and Y. Tauman, 2007, General licensing schemes for a cost-reducing innovation, *Games and Economic Behavior* 59: 163-186.
Singh, N. and X. Vives, 1984, Price and quantity competition in a differentiated industry, *RAND Journal of Economics* 15: 546-554.
Stamatopulos, G. and T. Tauman, 2009, On the superiority of fixed fee over auction in asymmetric markets, *Games and Economic Behavior* 67: 331-333.
Tauman, Y. and M. Weng, 2012, Selling patent rights and the incentive to innovate, *Economics Letters* 114: 241-244.
Wang, X.H., 1998, Fee versus royalty in a Cournot duopoly model, *Economics Letters* 50: 55-62.
Wang, X.H., 2002, Fee versus royalty in a differentiated Cournot model, *Journal of Economics and Business* 54: 253-266.
Wang, X.H. and B. Yang, 1999, On licensing under Bertrand competition, *Australian Economic Papers* 38: 106-119.