Optimal Contracts for Endogenous Information *

Jia Xie†

November 30, 2009

*The author is indebted to Professor Eric Rasmusen for pertinent comments and suggestions. Needless to say, Any mistakes are mine.

†Contact Information: Department of Economics, Indiana University, Bloomington, IN 47405; email: jiaxie@indiana.edu.

Abstract

This paper studies how a principal should design contracts for project selection and monetary transfer with an agent in an information communication context. A distinct assumption is that the agent's information about the true state of nature is acquired rather than endowed, and the only way to increase the signal's informativeness is to undertake a costly effort to acquire information.

We first assume a general information structure, in which effort can affect the joint distribution of the state and the signal in a desirable way, but not their marginal distributions. We show that if the signal is uninformative at zero effort then the optimal project contract is centralized and the agent exerts no effort to acquire information. As a consequence, no information is revealed. On the other hand, if the signal is informative at zero effort, the selected project strictly increases in reported message, and the agent is induced to acquire and reveal information at the optimum.

To explicitly characterize the optimal contracts, I impose a specific information structure, which is characterized by two parameters designed to measure the informativeness (of the state) and responsiveness (to the effort) of the signal. This structure yields the following results: (1) the optimal project contract is more sensitive to the reported message if the signal is more informative or more responsive, or if the agent's action is more expensive. (2) The agent exerts more effort on information acquisition if the signal is more informative or more responsive, or if the effort is less expensive. (3) The principal's welfare increases with the informativeness and responsiveness of the signal. Likewise, welfare decreases with the costliness of the agent's effort, the preference bias between the two parties, and the variance of the prior of the true state. (4) Signals that are more informative and responsive are more efficient. As the effort becomes more expensive, informativeness plays more important role in determining relative efficiencies. In the limit, relative efficiencies of signals are determined exclusively by their informativeness.

Keywords: Principal-agent model, information communication, moral hazard, adverse selection, endogenous information.

JEL Numbers: C70, D82, D83, D86.

1 Introduction

There is a large literature on information communication analyzing the consequences of asymmetric information on the efficiency of decision making.[1] A standard assumption in the literature is that the information possessed by the agent is exogenously given by nature. Real life situations, however, often seem to be different. The agent's private information of the true state of nature is usually acquired rather than endowed. When information acquisition is endogenous, the contracts proposed by the principal affects not only the agent's incentive to reveal the information they observed ex post, but also his incentives to acquire information ex ante. In this context the ex post optimal contract characterized by Krishna and Morgan (2008) may not be optimal ex ante.

The purpose of this paper is to study how a principal should design contracts for project selection and monetary compensation with the agent in an information communication context when information acquisition is endogenous and costly to the agent. The contract for project selection is referred to as the project contract, while the contract for monetary compensation is referred to as the compensation contract. Assuming endogenous information acquisition introduces considerable technical difficulties, and one of the contributions of this paper is to demonstrate an elegant way over these obstacles.

Firstly, endogenous information acquisition causes both moral hazard and adverse selection problems. Contracts are used for the purposes of providing the agent with the right incentives to collect information at the information acquisition stage (moral hazard) and motivating truthful information revelation at the information communication stage (adverse selection). The mixture of moral hazard and adverse selection makes the general model very complex. Therefore, we focus on the standard *quadratic-linear* case, in which the two parties' preferences are quadratic in the deviation of the selected project from the true state

[1] See Crawford and Sobel (1982) for the no commitment (cheap talk) case; Holmstrom (1984), melamud and Shibano (1991), and Alonso and Matouschek (2007, 2008) for the partial commitment on project selection (delegation) case; and Krishna and Morgan (2008) for the partial commitment on compensation (compensation contract) and full commitment cases.

of nature and are linear in monetary compensations, and the standard *ex ante participation constraint* that the agent has a one-time choice to reject the contract before acquiring information. As in Holmstrom and Milgrom (1991), these two assumptions imply that given any incentive compatible contracts, the utility possibility frontier of the two parties represented by a straight line in the R^2 space. This simplifies some technical points, but mainly allows us to use surplus analysis.

A second difficulty lies in that the agent's private information is two-dimensional consisting of his hidden action and his observation of the realized signal. This suggests that the contracting problem here is multi-dimensional and could potentially be very complicated. A common approach in the literature is to focus on a single sufficient statistic, which is usually the agent's posterior estimate of the state given the effort level and realized signal. This approach is justified by a common assumption that the agent's utility is linear in the state.[2] Because we focus on the quadratic-linear case, this approach does not apply here. Instead, I prove a strong result that the agent's private information is essentially one-dimensional because his effort is fully controlled ex ante and perfectly known ex post to the principal, so there is nothing to be gained by contracting on the reported effort level. Another advantage of this approach is that by focusing on the original signal instead of a sufficient statistic, it circumvents the potential moving support problem of the sufficient statistic.[3]

A third difficulty comes from the infinite number of incentive compatibility constraints caused by the moral hazard problem. A common solution is the first-order approach by replacing the information acquisition constraints by the first-order condition of the agent's maximization problem (Rogerson (1985), Jewitt (1988), Colon (2009), and Xie (2009)). I justify the first-order approach for any incentive compatible contracts that ensures truthful revelation of information.

A second contribution of this paper lies in how we assume the agent's effort can improve

[2]See Szalay (2009) and Shi (2009).

[3]The moving support problem indeed exists for the "truth-or-noise" information structure if the sufficient statistic is used. It doesn't exist, however, for the Gaussian learning information structure.

informativeness of the signal. I first establish a general information structure, in which the agent's effort can only affect the joint distribution of the signal and state of nature, but not their marginal distributions. Moreover, it is assumed that the agent revises his posterior estimate of the state upwards if he observes a signal higher than ex ante expected, and downwards if he observes a downward surprise. The upward (downward) revision for surprisingly high (low) signal is larger, the higher is his effort. These assumptions are more general than the first order stochastic dominance (FOSD) condition and the mean reversing first order stochastic dominance (MRFOSD) condition in Szalay (2009). Finally, the posterior estimate is assumed to be convex in effort for signals below the prior expected signal value, and concave in effort for signals above the prior expected signal value.

The main insight arising from this general information structure is that the two roles of contracts, i.e., motivating information acquisition and ensuring information revelation, are complementary in that the optimal contracts serve either both or neither of them. More specifically, if the signal is uninformative of the state of nature at zero effort, the optimal project contract (as well as the compensation contract) is centralized: the principal won't take any suggestions from the agent, but instead follows her own prior. In order to economize on compensations, however, the principal compromises on project-selection by agreeing to the middle-point project between the two parties' ex ante first bests. In this case, the contract is used merely to keep the agent remaining in the relationship, but induces no information acquisition and revelation. This is because information is too costly for the principal to contract for it, even in the first best circumstance, where both parties' interests are perfectly aligned in nature.

If, on the other hand, the signal is informative at zero effort, then the optimal project contract is strictly increasing in the agent's reported message, and the agent is induced to acquire and truthfully communicate information at the optimum. Similar to the exogenous information case, the principal economizes on transfers by agreeing to decisions that are never optimal for her given the states. In other words, the alignment principle, which says that the alignment of incentives and the delegation of authority are complementary tools, does not hold. Different from the exogenous information case, the optimal project contract here has no capping on the top.[4] This is because with exogenous information structure, the project contract is used for the purpose of motivating information revelation. By putting a capping on the top of the project contract, the principal economizes on the monetary compensation at lower state. Whereas in current context with endogenous information structure, the project contract is used for the purpose of motivating information acquisition, and the most efficient contract for this purpose necessarily has no capping on the top. Indeed, imposing a capping on the top of the project contract weakens incentives to information acquisition.

To explicitly characterize the optimal contracts, I impose a specific information structure, which is characterized by two parameters designed to measure the informativeness (about the state of nature) and responsiveness (to the agent's effort) of the signal. The standard "truth-or-noise" information technique[5] fits this structure. This structure yields the following results: (1) the optimal project contract is more sensitive to the reported message if the signal is more informative or more responsive, or if the agent's action is more expensive. (2) The agent exerts more effort on information acquisition if the signal is more informative or more responsive, or if the effort is less expensive. (3) The principal's welfare increases with the informativeness and responsiveness of the signal. Likewise, welfare decreases with the costliness of the agent's effort, the preference bias between the two parties, and the variance of the prior of the true state. (4) Signals that are more informative and more responsive are more efficient. As the agent's effort becomes more expensive, informativeness plays more important role in determining relative efficiencies. In the limit, relative efficiencies of signals are determined exclusively by their informativeness.

The remainders of this paper proceeds as follows. We review the related literature in Section 2, set up the model in Section 3, analyze the general model in Section 4, characterize

[4] Krishna and Morgan (2008) shows in the full commitment model with exogenous information that the optimal project contract has a capping on the top so the project chosen becomes unresponsive to the reported message.

[5] See Lewis and Sappington (1994), Johnson and Myatt (2006), Ottaviani (2000), and Shi (2009)

the analytical solution in Section 5, and briefly conclude in Section 6.

2 Literature Review

Much of the information communication literature is built upon the classic *cheap-talk* model of Crawford and Sobel (1982) which analyzes the strategic interaction between an informed agent and an uninformed principal. In the cheap talk model the principal has no commitment power at all on either the project selection or the monetary compensation.

Holmstrom (1984), melamud and Shibano (1991), and Alonso and Matouschek (2007, 2008) examines the benefits of *delegation*, in which the principal is assumed to be able to commit to the project chosen by the agent. Again, in the delegation model, the principal has no commitment power on the monetary compensation.

On the other hand, Krishna and Morgan (2008) examines how the use of transfers can improve information communication between the agent and principal. Krishna and Morgan (2008) analyze two cases: the *compensation contract* case and the *full commitment* case. In the compensation contract case, the principal is assumed to be able to commit to monetary compensation, but not the project chosen by the agent, whereas in the full commitment case, the principal has full commitment power on both the project selected by the agent and the monetary compensation to the agent.

Table 1: Assumptions of the principal's commitment power in the literature

		Commit to monetary compensations	
		Yes	No
Commit to project selection	Yes	Full commitment	Delegation
	No	Compensation contract	Cheap talk

The different combinations of assumptions on the principal's commitment power is shown in Table 1. In this paper, we focus on the full commitment case.

Another strand of literature is concerned with the economic implications of endogenous information acquisition in different contexts. Among them are Szalay (2009) in the procurement framework, and Shi (2009) in the auction context. Our model differs from theirs in three important dimensions: (1) we study the information communication model, (2) we assume an ex ante participation constrain instead of the interim participial constraint, and (3) the information structure assumed in current study is more general.

3 The Model

We combine the Krishna and Morgan (2008) model structure, and a generalized version of the Szalay (2009) endogenous information structure. The fundamental structure is a variant of the standard quadratic-linear model in which I allow for general, endogenous information structures.

3.1 The model structure

A principal has the legal right to choose a project $y \in \mathbf{R}$, but only the agent has the ability (or channel) to acquire information about the true state of nature which is necessary to choose the "right" project.

The payoff function of the principal is $u(y, \theta) \equiv -(y-\theta)^2$, where θ is the underlying state of nature. With a little abuse of notation, the agent's payoff from implementing project y is $u(y, \theta, b) \equiv -(y - \theta - b)^2$, where b is a bias parameter measuring the the congruence of the agent's and the principal's preferences and is commonly known. The principal and the agent do not know the precise value of θ, but share a common prior about it.

The agent may acquire additional information about θ. Information acquisition is modeled as a costly choice of effort a, that increases the informativeness (or precision) of a signal S he receives. The cost of effort to the agent is $c(a)$, which satisfies $c_a(a) > 0$ for $a > 0$, $c_a(0) = 0$, and $c_{aa}(a) > 0$ for all a.

With perfect commitment, the principal is able to contract with the agent on both project-selection and monetary compensation. The timing is displayed in Figure 1

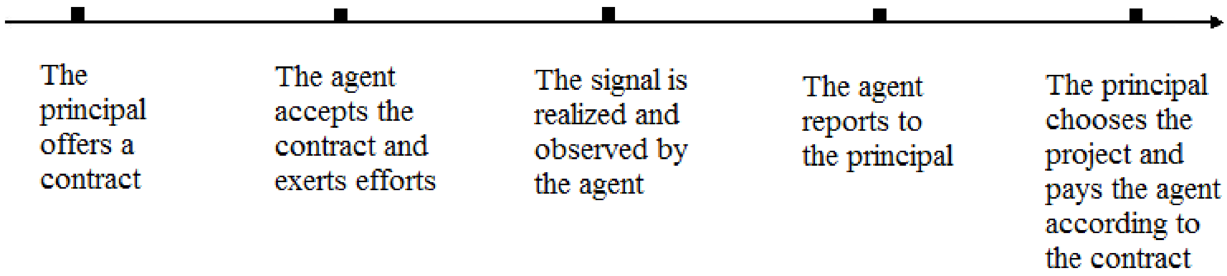

Figure 1: The time line of the events.

First, the principal offers to the agent a take-it-or-leave-it contract, $\big(M, y(m), t(m)\big)$, specifying the message space (M) and both the project $(y(\cdot))$ and the compensation $(t(\cdot))$ as functions of the message sent by the agent. We refer to $y(\cdot)$ the project contract, and $t(\cdot)$ the compensation contract. If the contracts is accepted, the agent then chooses an effort level, a, that determines the informativeness of the signal S. The realized signal s is then observed by the agent. Given s the agent reports to the principal in the form of a costless message m chosen from M. Upon receiving m, the principal chooses the project $y(m)$ and pays the agent $t(m)$ as specified by the contract.

3.2 The information structure

I assume that effort influences only the joint distribution of θ and S but not their marginal distributions. This assumption has an economic content; it means that the signal is non-informative of the effort, so the compensation contract can not affect the agent's effort on information acquisition. That is, the compensation contract plays no information acquisition role. Note, however, that because the effort influences the joint distribution of θ and S, the project contract provides incentives to acquire information.

S has typical realization $s \in [\underline{s}, \bar{s}]$. The marginal pdf and cdf of S are denoted $k(s)$ and $K(s)$ respectively. I assume that $k(s) > 0$, $\forall s \in [\underline{s}, \bar{s}]$. That is, the distribution of S has full support. Thus $K(S)$ contains the same information as S does, but is much more convenient to work with, as $K(S)$ is uniformly distributed on $[0, 1]$. Thus, with a little abuse of notation, I use $S = K(S)$ as the signal.

Let $f(\theta|s, a)$ denote the conditional density function of θ given s and a. Assume that $f(\theta|s, a)$ is differentiable in θ s and a to the order needed. Let $E[\theta|s, a] \equiv \int \theta f(\theta|s, a) d\theta$. We make a series of assumptions on the properties of $E[\theta|s, a]$, which are similar to, but weaker than, those made in Szalay (2009). The following Assumptions 1, 2, and 3 are sufficient for our current analysis. A more specific information structure, Assumption 4, is imposed in Section 5 for exact characterization of the analytical solutions.

Assumption 1.

$$E_s[\theta|s, a] \geq 0, \quad \forall s,\ a, \quad \textit{and} \quad E_s[\theta|s, a] > 0, \quad \forall s,\ a > 0.$$

Assumption 1 implies that θ and S are positively affiliated so that high values of s indicate high expected values of the state. Assumption 1 corresponds to and is weaker than the First Order Stochastic Dominance Condition (Formula (1)) in Szalay (2009).

Next, I introduce a precise sense in which higher effort corresponds to more informative signal by making the following mean reversing monotone expectation (MRME) assumption, which corresponds to and is weaker than the MRFOSD condition (Formula (16)) in Szalay (2009). Let $\tilde{s} \equiv E[S]$.

Assumption 2.

$$E_a[\theta|s, a] \leq 0, \quad \forall a, \ \ s \in [\underline{s}, \tilde{s}],$$

$$E_a[\theta|s, a] \geq 0, \quad \forall a, \ \ s \in [\tilde{s}, \bar{s}].$$

Assumption 2 says that a impacts the posterior estimate of the state given a and the realized signal, and the direction of this impact depends on whether the realized signal is above or below its ex ante mean. More precisely, the direction of the impact of an increase in a on the posterior is reversed as s is increased above its ex ante mean value.

In sum, Assumption 1 implies that relative to $E[\theta]$ the agent revises his posterior estimate of the state upwards if he receives a signal higher than ex ante expected, and downwards if he receives a downward surprise. Assumption 2 says that the upward (downward) revision for surprisingly high (low) signal is larger, the larger is a.

At last, we make the following mean reversing concave/convex expectation condition (MRCEC) which is similar to the MRCDFC condition (Formula (18)) in Szalay (2009). This assumption is important for justifying the first-order approach.

Assumption 3.

$$E_{aa}[\theta|s,a] \geq 0, \quad \forall a, \quad s \in [\underline{s}, \tilde{s}],$$

$$E_{aa}[\theta|s,a] \leq 0, \quad \forall a, \quad s \in [\tilde{s}, \bar{s}].$$

4 The Optimal Contracts

4.1 The Format of the Contract

In our setting, the agent's "private information" is two-dimensional, consisting of his effort on information acquisition, a, and the realized signal, s. Biais, Martimort, and Rochet (2002), Szalay (2009), and Shi (2009) argue that "this suggests that the design problem here is multi-dimensional and could potentially be very complicated."[6] A common approach in the literature is to focus on a single sufficient statistic, $E[\theta|s,a]$. This approach is justified by a common assumption that the agent's utility is linear in the true state of nature. Because linearity is preserved under expectation, the single variable, $E[\theta|s,a]$, completely captures

[6]See Page 8 in Shi (2009).

the dependence of the agent's utility on the two-dimensional information.

In out setting, however, the agent's utility has the quadratic format, so a single sufficient statistic does not exist. Nevertheless, we can prove a stronger result that the agent's effort is actually a public information, so there is nothing to be gained by contracting on the reported effort level, whatever format the agent's utility is.

Proposition 4.1. *For any contract contingent on a and S, there exits a simple contract contingent only on S that yields the same equilibrium outcome.*

Proof. See the appendix. □

Proposition 4.1 says that contracting on the reported signal is sufficient, because there is no such a contract that can improve over the optimal simple contract contingent on the reported signal. The intuition behind this result is simple: because the agent's effort in the equilibrium is controlled by and perfectly known to the principal, it is actually not a private information but a public information, and there is no need to consider complicated mechanisms.

Another advantage of our approach is that we avoid the non-moving support problem of the distribution of the the single sufficient statistic $E[\theta|s, a]$. In general, the non-moving support of S can not guarantee a non-moving support of $E[\theta|s, a]$. To this end, Szalay (2009) makes a strong assumption, condition (2) in that paper, that the conditional distribution of the true state of nature given the lowest or the highest realized signal is independent of the effort level. This is such a strong assumption that the broadly used "truth-or-noise" information structure does not fit this scheme.[7] In current paper, since we are dealing directly with the original signal S, whose distribution is not affected by the effort, this problem does not exist.

[7]See Lewis and Sappington (1994), Johnson and Myatt (2006), and Shi (2009) for the detail of the "truth-or-noise" information structure.

4.2 The Program

Under perfect commitment, the standard revelation principal applies. So we can restrict attention to direct contracts-those in which $M = [0,1]$-that are incentive compatible. A direct contract $(y(\cdot), t(\cdot))$ specifies for each message $s \in [0,1]$, a project $y(s)$ and a compensation $t(s)$. A direct contract (y, t) is incentive compatible if for all s, it is best for the agent to report the signal value truthfully. In particular, the principal chooses a triple of $(y(\cdot), t(\cdot), a)$ to solve the following program.

P1:

$$U_p \equiv \max_{y(s),t(s),a\geq 0} \int_0^1 \left[\int u(y(s),\theta) f(\theta|s,a) d\theta - t(s) \right] ds, \tag{1}$$

$$s.t. \quad U_A \equiv \int_0^1 \left[\int u(y(s),\theta,b) f(\theta|s,a) d\theta + t(s) \right] ds - c(a) \geq 0, \tag{2}$$

$$a = \text{argmax}_{\hat{a}} \int_0^1 \left[\int u(y(s),\theta,b) f(\theta|s,\hat{a}) d\theta + t(s) \right] ds - c(\hat{a}), \tag{3}$$

$$\forall s,\ \xi : \int u(y(s),\theta,b) f(\theta|s,a) d\theta + t(s) \geq \int u(y(\xi),\theta,b) f(\theta|x,a) d\theta + t(\xi). \tag{4}$$

U_P and U_A are the principal's and the agent's expected utilities at the optimum. Constraint (2) is the individual rationality (IR) constraint, which says that the agent's expected utility must be no less than his reserve utility, which is normalized to be zero. Constraint (3) is an information acquisition (IA) constraint, saying that facing the contract $(y(\cdot), t(\cdot))$ the agent chooses his effort level to maximize his expected utility. Constraint (4) is an incentive compatibility (IC) constraint with respect to the agent's reporting strategy. It says that given the contract $(y(\cdot), t(\cdot))$ it is optimal for the agent to truthfully report the signal value.

Program $P1$ is extremely difficult to solve as it is subject to a infinite number of constraints. With the above assumptions on the information structure, however, we can rewrite each of the three constraints in $P1$ to simplify it.

Proposition 4.2. *The IC constraint* (4) *is equivalent to the following two conditions:*

(i) $y(s)$ *is non-decreasing:* $y'(s) \geq 0$, *a.e.* s, *and*

(ii) $t'(s) = 2y'(s)(y(s) - E[\theta|s,a] - b)$.

Proof. See Appendix □

The IA constraint involves a continuum of inequalities, which make the analysis extremely difficult. The main analytical approach in the contract theory literature is the first-order approach, which replaces the IA constraint with the weaker requirement that only the necessary first-order condition for optimality be satisfied. There is a large literature on the sufficient conditions justifying the first-order approach. (See Rogerson (1985), Jewitt (1988), Conlon (2009 a,b), and Xie (2009)) The next proposition says that under our current assumptions, the first-order approach is justified if the project contract is non-decreasing in reported message.

Proposition 4.3. *The first-order approach is valid if $y(s)$ is non-decreasing. So the IA constraint* (3) *can be replaced by its first-order condition that*

$$\iint u(y(s),\theta,b) f_a(\theta|s,a) d\theta\, ds - c_a(a) = 0.$$

Proof. See the appendix. □

Proposition 4.4. *At the optimum, The IR constraint* (2) *must be binding, i.e.,*

$$\int t(s) ds = c(a) - \iint u(y(s),\theta,b) f(\theta|s,a) d\theta\ ds.$$

Proof. See the appendix □

Recall that the principal's objective function is $\iint u(y(s),\theta) f(\theta|s,a) d\theta\, ds - \int t(s) ds$. Therefore, by Proposition 4.4 the joint surplus of both parties is $\iint [u(y(s),\theta) + u(y(s),\theta,b)] f(\theta|s,a) d\theta\, ds - c(a)$, which is independent of the monetary compensation, $t(s)$. As in Holmstrom and Milgrom (1991), this implies that given any incentive compatible contracts, the utility possibility frontier of the two parties is a straight line in the R^2 space.

Note also that different from the standard adverse selection model, the principal gets all the social surplus, and pays no information rent to the agent even if the agent observes a large value of the signal.[8] This is because the standard model assumes an interim participation constraint, by which the agent is able to reject the contract ex post. Therefore to avoid an ex post rejection of the contract, the principal has to pay an information rent to the agent if he observes a large value of the signal. In current model, however, we assume an ex ante participation constraint, by which the agent only has a one-time choice to accept or reject the contract before acquiring any information. Thereby the agent is not able to gain the information rent by a credible threat of rejecting the contract ex post.

Proposition 4.5. *The principal's problem* **P1** *is equivalent to the following program:*

$$\textbf{\textit{P2:}} \qquad \max_{y(s),a\geq 0} \iint [u(y(s),\theta) + u(y(s),\theta,b)] f(\theta|s,a) d\theta\, ds - c(a) \tag{5}$$

$$s.t. \iint u(y(s),\theta,b) f_a(\theta|s,a) d\theta\, ds - c_a(a) = 0, \tag{6}$$

$$y(s) \quad \textit{being non-decreasing.} \tag{7}$$

Proof. It follows directly from Propositions 4.2, 4.3, and 4.4. □

Constraint (6) is the relaxed IA constraint which is justified by Proposition 4.3; Constraint (7) is the simplified IC constraint by Proposition 4.2; The IR constraint in Program $P1$ disappears in $P2$ as we have used it to substitute out $\int t(s)ds$ in the objective function, and this is justified in Proposition 4.4.

It will be convenient to have conditions that guarantee that the monotonicity constraint (7) is not binding at the optimum. Note that Constraint (7) represents the adverse selection problem. Slackness of this constraint means that the moral hazard and adverse selection problems are "consistent" in the sense that the solution (the optimal contract) to the moral hazard problem also solves the adverse selection problem. Despite its lack of generality, this

[8] In other words, the virtual surplus equals to the social surplus. See Salanie (1997) for the analysis of the standard adverse selection model.

special case renders results consistent with our intuition and observation in practice and avoids unnecessary difficulties that are potentially unsolvable.

Therefore, in our following analysis, we will temporarily ignore Constraint (7), and we need to develop sufficient conditions for the solution $y(s)$ in Program **P2** (without Constraint (7)) to be non-decreasing. It will be shown in the next section that in some cases this desired monotonicity is guaranteed by our current assumption, while in some other cases we need impose more information structures. We will discuss these additional information structures whenever needed.

Let λ be the multiplier for the constraint (6), then the Lagrangian function of **P2** without constraint (7) is

$$\begin{aligned}\iint [u(y(s),\theta) + u(y(s),\theta,b)] f(\theta|s,a) d\theta\, ds - c(a) \\ + \lambda[\iint u(y(s),\theta,b) f_a(\theta|s,a) d\theta\, ds - c_a(a)].\end{aligned} \tag{8}$$

The pointwise optimization of the Lagrangian with respect to $y(s)$ yields the following characterization of an optimal project contract,

$$\int [u_1(y(s),\theta) + u_1(y(s),\theta,b)] f(\theta|s,a) d\theta + \lambda \int u_1(y(s),\theta,b) f_a(\theta|s,a) d\theta = 0. \tag{9}$$

Maximizing the Lagrangian with respect to a yields

$$\begin{aligned}\iint u(y(s),\theta) f_a(\theta|s,a) d\theta ds + \lambda[\iint u(y(s),\theta,b) f_{aa}(\theta|s,a) d\theta ds - c_{aa}(a)] \leq 0, \\ a \geq 0, \\ a \cdot \left[\iint u(y(s),\theta) f_a(\theta|s,a) d\theta ds + \lambda[\iint u(y(s),\theta,b) f_{aa}(\theta|s,a) d\theta ds - c_{aa}(a)]\right] = 0.\end{aligned} \tag{10}$$

4.3 Properties of the Optimal Contracts

(9) (10) and (6) are the necessary conditions for $\big(y(\cdot), a, \lambda\big)$ to be a solution to Program **P2** without constraint (7). If $y(s)$ happens to be non-decreasing, then $\big(y(\cdot), a, \lambda\big)$ is also a solution to **P2** and the original Program **P1**.

To proceed, we will have to distinguish between two cases by whether the signal is noninformative of the state of nature if the agent exerts no effort. i.e., whether $E[\theta|s, 0] = E[\theta]$ or not. As we will see, the solutions are dramatically different in these two cases.

Proposition 4.6. *If $E[\theta|s, 0] = E[\theta]$, then the optimal contract has the following properties:*

$$y(s) = E[\theta] + \frac{b}{2}, \quad t(s) = V(\theta) + \frac{b^2}{4}, \quad a = 0, \quad \lambda = 0, \quad \textit{and} \;\; \mathbf{U_P} = -2V(\theta) - \frac{b^2}{2}. \tag{11}$$

Proof. The Kuhn-Tucker conditions (9), (10), and (6) are also sufficient because the relevant concavity conditions are satisfied.[9] By substituting (11) into (9), (10) and (6), one can verify that (11) is a solution to Program **P2** without constraint (7). Because $y(s)$ in (11) is non-decreasing, (11) is also the solution to Programs **P2** and **P1**. This completes the proof. □

This proposition says that if the signal is noninformative at zero effort, then the optimal contracting has the following properties: the principal does not take any suggestions from the agent, but instead follows her own prior. In order to economize on compensations, however, the principal compromises on projects by agreeing to the middle-point project between the principal's and the agent's ex ante first bests. In this case, the contract is used merely to keep the agent remaining in the relationship, but induces no information acquisition, and thereby no information is revealed. The reason for this result is simply: information is too costly for the principal to contract for it.

Why is information too costly in this case? Let's look at an extreme case without moral

[9] That is, the objective function and equality constraints are both indirectly concave. See Page 273 in Birchenhall and Grout (1984).

hazard and adverse selection. We assume that the principal and the agent are perfectly aligned in their preferences, so that we can take the two parties as the same individual, who solves the following unconstrained optimization problem

$$\max_{y(s),\, a\geq 0} - \iint (y(s)-\theta)^2 f(\theta|s,a) d\theta\, ds - c(a).$$

The optimal project selection rule, $y(s) = E[\theta|s,a]$, is increasing in s. Therefore the objective function is concave in a as proven in Proposition 4.3, and is maximized at $a = 0$. So even if the principal and the agent are perfectly aligned in preference, information is too costly to acquire. When moral hazard and adverse selection are involved, information is simply more costly to the principal.

Proposition 4.7. *If $E[\theta|\xi, 0] \neq E[\theta]$ for some ξ, and $E_{sa}[\theta|s,a] > 0$ for all s and a, then the optimal contract has the following properties:*

$$\lambda > 0, \quad y'(s) > 0 \ \forall s, \quad \textit{and} \quad a > 0. \tag{12}$$

Proof. see the Appendix. □

The condition that $E_{sa}[\theta|s,a] > 0$ is a more restrictive version of Assumption 2. Instead of requiring $E_a[\theta|s,a]$ to be negative in lower region of S and positive in higher region of S, this condition says that $E_a[\theta|s,a]$ is increasing in s at all $s \in S$. This condition is the additional structure which we mentioned in the end of the last section that guarantees a monotonic project contract.

Proposition 4.7 says that if the signal is informative at zero effort, then it is optimal for the principal to give incentives to acquire more information: the optimal project contract is strictly increasing in the agent's reported message ($y'(s) > 0$), and the agent exerts a positive effort level on information acquisition and reveals information truthfully ($a > 0$). In addition, an increase in effort at the optimum benefits the principal ($\lambda > 0$).

Note the similarity and distinction between the optimal project contract in the endogenous information case as studied in this paper, and that in the exogenous information case as studied in Krishna and Morgan (2008).

The similarity is that decisions are systematically distorted to favor the agent's preferences. That is, the principal economizes on transfers by agreeing to decisions that are never optimal for her given the state of nature. Again, the alignment principal, which says that the alignment of incentives and the delegation of authority are complementary tools, does not hold.

The difference is that the optimal contract in current study has no capping on the top. With an exogenous information structure, the only role served by the contract is to ensure truthful communication of information. The optimal project contract for that purpose has been proven to have a capping on the top so as to economize on transfers at lower states. In current model, however, the project contract is used for the purpose of motivating information acquisition, as information revelation is taken care by the associated compensation contract. A binding capping on the top of the project contract necessarily weakens the incentives to acquire information.

5 Analytical Solutions

To obtain an explicit characterization of the optimal contracts, we have to impose more detailed information structures on the stochastic relationship between the signal and the state of nature. To this end, we assume the posterior estimate of the state given the realized signal and effort level has the following explicit format.

Assumption 4.

$$E[\theta|s,a] = (\alpha \cdot a + \beta)(2s-1) + E[\theta],$$

where $\alpha \geq 0$ and $\beta \geq 0$ are nonnegative constant coefficients.

The graph of $E[\theta|s,a]$ as a function of s and a is displayed in Figure 2.

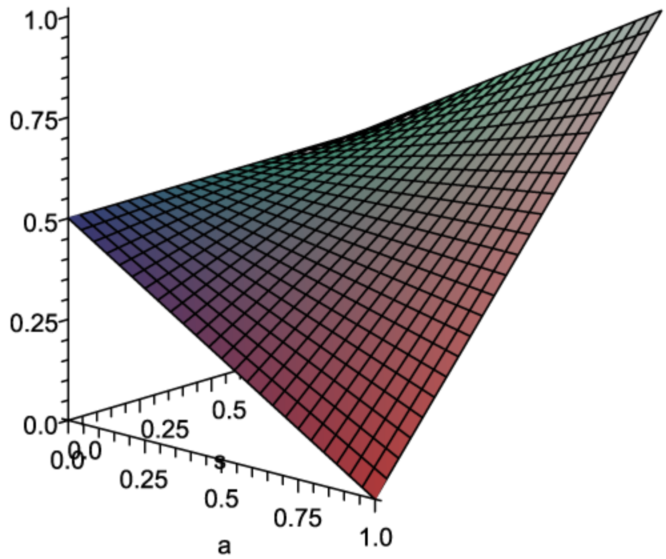

Figure 2: The 3-D graph of $E[\theta|s, a]$, with $\alpha = 1/2$, $\beta = 0$, and $E[\theta] = 1/2$.

The information structure (4) is fully characterized by two parameters, α and β, therefore it is important to understand the economic meaning of these two parameters: α measures the *informativeness* of the signal about the state of nature, while β is a measure of the *responsiveness* of the signal to the agent's effort.

Informativeness of the signal S is represented by the sensitivity (i.e., the slope) of $E[\theta|s, a]$ to s. Then it is clear from (4) that β is a measure of *informativeness* of the signal S.

Like β, a also increases the informativeness of the signal. The strength of this impact of a is measured by α. Specifically, a larger value of α means informativeness of the signal is more responsive to a. In this sense, α measures the responsiveness of the signal S to a.

Note that (4) satisfies the general information structure we imposed in Subsection 3.2, namely Assumptions 1, 2 and 3. This result is clear from the following equations,

$$E_s[\theta|s,a] = 2(\alpha \cdot a + \beta).$$

$$E_a[\theta|s,a] = \alpha(2s-1).$$

$$E_{aa}[\theta|s,a] = 0.$$

$$E_{as}[\theta|s,a] = 2\alpha.$$

$$E[\theta|a] = E[\theta].$$

The first equation says that Assumption 1 is satisfied if $\alpha > 0$ or $\beta > 0$. The second and third equations imply that Assumptions 2 and 3 hold. In addition, The fourth equation implies that the assumption that $E_{as}[\theta|s,a] > 0$, which is assumed in Proposition 4.7, is satisfied if and only if $\alpha > 0$. The last equation says that the marginal distribution of the signal is indeed independent of the agent's effort, a.

The information structure imposed by (4), however, is still more general than the standard "truth-or-noise" information structure.[10] Using current notation, the truth-or-noise structure assumes that the agent has perfect information with probability a, and a signal drawn from the prior with probability $1-a$. Ottaviani (2000) calculates that the posterior expectation of θ given the realized signal s and effort level a is $E[\theta|s,a] = as + \frac{1-a}{2}$, therefore the "true-or-noise" information structure is a special case of (4) with $\alpha = \frac{1}{2}$ and $\beta = 0$.

We also assume that the cost of effort has a quadratic format. Specifically, let $c(a) = \kappa \cdot a^2$, where κ is a parameter measuring the costliness of the agent's effort: the larger is κ, the more expensive is the agent's effort. We assume that $\kappa > \frac{\alpha^2}{2}$.[11]

[10]For example, the "truth-or-noise" information structure is used in Lewis and Sappington (1994), Ottaviani (2000), Johnson and Myatt (2006), and Shi (2009).

[11]This assumption is used to avoid extreme solutions.

The Kuhn-Tucker conditions (9), (10) and (6) become:

$$y(s) = E[\theta|s,a] + \frac{b}{2} + \frac{\lambda}{2}E_a[\theta|s,a]. \tag{13}$$

$$2\int y(s)E_a[\theta|s,a]ds + \lambda[2\int y(s)E_{aa}[\theta|s,a]ds - c_{aa}(a)] \leq 0,$$

$$a \geq 0,$$

$$a \cdot \Big[2\int y(s)E_a[\theta|s,a]ds + \lambda[2\int y(s)E_{aa}[\theta|s,a]ds - c_{aa}(a)]\Big] = 0, \tag{14}$$

$$2\int y(s)E_a[\theta|s,a]ds - c_a(a) = 0. \tag{15}$$

Proposition 5.1. *With endogenous information structure* (4)*, the equilibrium has the following properties,*

$$a = \frac{2\alpha\beta}{6\kappa - 3\alpha^2}. \tag{16}$$

$$y(s) = \frac{2\beta\kappa}{2\kappa - \alpha^2}(2s-1) + E[\theta] + \frac{b}{2}. \tag{17}$$

$$\mathbf{U_P} = \frac{4\beta^2\kappa}{3(2\kappa - \alpha^2)} - 2V(\theta) - \frac{b^2}{2}. \tag{18}$$

$$\mathbf{U_A} = 0. \tag{19}$$

Proof. See the Appendix. □

Consistent with Propositions 4.6 and 5.2, Proposition 5.1 says that if the signal is non-informative at zero effort ($\beta = 0$), then the optimal project contract is centralized and the agent exerts no effort, while if the signal is informative at zero effort ($\beta > 0$), then the optimal project contract is strictly increasing in s, and positive effort is exerted. What is new in Proposition 5.1, if the signal is irresponsive to the agent's effort ($\alpha = 0$), then $a = 0$ also. Indeed, if the agent is not capable of acquiring information, there is no point for the principal to induce him to exert any effort on it.

Comparative statics further provide the following intuitive properties:

(i) the optimal contract is more sensitive to the reported message, if the signal is more informative (β is larger) or more responsive (α is larger), or if the agent's action is more costly (κ is larger).

(ii) The optimal effort level on information acquisition is increasing in the informativeness (β) and responsiveness (α) of the signal, but decreasing in the cost of effort (κ).

(iii) The principal's welfare is increasing in the informativeness (β) and responsiveness (α) of the signal, but decreasing in the cost of the agent's effort (κ), the preference bias (b), and the variance of the prior ($V(\theta)$).

5.1 Comparison of Information

Another practice we are able to do under the specific information structure (4) is comparison of signals. Recall that within this specific information structure, any signal is fully characterized by a set of two parameters, (α, β).

Proposition 5.2. *Given two signals, $S_1 \equiv (\alpha_1, \beta_1)$ and $S_2 \equiv (\alpha_2, \beta_2)$, S_1 is more efficient than S_2 in that the principal gets higher welfare from contracting on S_1 than on S_2 if*

$$\frac{\beta_1^2}{2\kappa - \alpha_1^2} \geq \frac{\beta_2^2}{2\kappa - \alpha_2^2}.$$

Proof. The result is straightforward from (18). □

Proposition 5.2 implies that the efficiency of the signal is increasing in the informativeness (β) and responsiveness (α) of the signal. Interestingly, any signal with $\beta = 0$ is dominated by any signal with $\beta > 0$. As the cost of effort approaches infinity ($\kappa \to \infty$), β plays more and more important roles in comparison of signals. In the extreme case of $\kappa = \infty$, efficiency of the signal is exclusively determined by β: signals with are more informative about the state are more efficient.

6 Conclusion

This paper studies how a principal should design contracts for project selection and monetary compensation with an agent in an information communication context when information acquisition is endogenous and costly to the agent. Making the information acquisition process endogenous causes the moral hazard and adverse selection problems: contracts are used for the purposes of providing the right incentives for information acquisition (moral hazard) and ensuring truthful information revelation (adverse selection).

In this paper, I focus on a case in which the two roles (or purposes) of contracting are not in conflict: the optimal contract in the moral hazard problem also solves the adverse selection problem. This is the reason why we can ignore the monotonicity constraint (7) in the simplified program **P2**. However, it is still not clear what the optimal contracts are if the two roles of contracting is in conflict with each other, so Constraint (7) is binding at the optimum. This is the well-known "bunching" problem.[12]

In this paper, we analyze the full commitment case, in which the principal contracts with the agent for both the project selection and monetary compensation. In many circumstances, however, the principal does not have so much commitment power. As already shown in Table 1, in the delegation case, the principal has no commitment power on the compensations, in the compensation contract case, the principal has no commitment power on the project selection, and in the cheap talk case, the principal has no commitment power at all on either the compensation or the project selection. It remains for future research to analyze how imposing endogenous information structure alters the current results in these imperfect commitment cases.

[12] See Salanie (1997).

7 References

Alonso, R., Matouschek, N., 2007. Relational Delegation. The RAND Journal of Economics. 38, 1070-1089.

Alonso, R., Matouschek, N., 2008. Optimal Delegation. Review of Economic Studies. 75, 259-293.

Biais, B., Martimort, D., Rochet, J., 2002. Competing Mechanisms in a Common Value Environment. Econometrica, 68. 799-837.

Birchenhall, C., Grout, P., 1984. Mathematics for Modern Economics. Barnes and Noble, Totowa.

Conlon, J., 2009. Two New Conditions Supporting the First-Order Approach to Multisignal Principal-Agent Problems. Econometrica. 77, 249-278.

Crawford, V., Sobel, J., 1982. Strategic Information Transmission. Econometrica. 50, 1431-1451.

Holmstrom, B., 1984. On the Theory of Delegation. in M. Boyer and R. Kihlstrom, eds., Bayesian Models in Economic Theory. New York: North-Holland.

Holmstrom, B., Milgrom, P., 1991. Multitask Principal-Agent Analyses: Incentive Contracts, Asset Ownership, and Job Design. Journal of Law, Economics, and Organization. 7, 24-52.

Jewitt, I., 1988. Justifying the First Order Approach to Principal-Agent Problems. Econometrica. 56, 1177-1190.

Johnson, J., Myatt, D., 2006. On the Simple Economics of Advertising, Marketing, and Product Design. American Economic Review. 93, 756-784.

Krishna, V., Morgan, J., 2008. Contracting for Information under Imperfect Commitment. RAND Journal of Economics. 39, 905-925.

Lewis, T., Sappington, D., 1994. Supplying Information to Facilitate Price Discrimination. International Economic Review. 35, 309-327.

Melamud, N., Shibano, T., 1991. Communication in Settings with No Transfers. RAND journal of Economics. 22, 173-198.

Mirrlees, J., 1999. The Theory of Moral Hazard and Unobservable Behavior: Part 1. The Review of Economic Studies. 66, 3-21.

Ottaviani, M., 2000. The Economics of Advice. Working Paper, London Business School.

Rogerson, W., 1985. The First Order Approach to Principal-Agent Problems. Econometrica. 53, 1357-1368.

Shi, X., 2009. Optimal Auctions with Information Acquisition. Working paper, University of Toronto.

Szalay, D., 2009. Contracts with Endogenous Information. Games and Economic Behavior. 65, 586-625.

Xie, J., 2009. Contracts with an Unobservable Outcome and a Risk-Averse Principal. Working Paper, Indiana University.

8 Appendix

Proof of Proposition 4.1

Proof. We prove a general version of Proposition 4.1. Let $U(y, t, \theta, a)$ denote the von Neumann-Morganstern utility of the agent, where y denotes the selected project, t the transfer from the principal to the agent, θ the true state of nature, and a the agent's effort on information acquisition.

In the full commitment case, we can invoke the Revelation Principal to focus on the incentive-compatible direct-revelation mechanisms. Assume the original direct revelation contract, contingent on both the reported message and effort, m and n, is $(y^*(m,n), t^*(m,n))$, and assume a perfect Bayesian equilibrium involves strategies $(a^*, m^*(a,s), n^*(a,s))$ for the agent, where $m^*(a,s)$ and $n^*(a,s)$ is the agent's reported message and effort respectively after he's taken effort a and observed realized signal s. By the Revelation Principal, the agent is induced to tell the truth in the equilibrium, therefore $m^*(a,s) = s$ and $n^*(a,s) = a$.

Let us now consider a simple contract contingent on m only as follows: For all m,

$$\hat{y}(m) \equiv y^*(m, n^*(a^*, s)) = y^*(m, a^*),$$
$$\hat{t}(m) \equiv t^*(m, n^*(a^*, s)) = t^*(m, a^*),$$

where the second equality in both formulas follow from the fact that $n^*(a, s) = a$ by the Revelation Principal. Suppose that, facing this contract, the agent chooses an effort level and reporting strategy $(\hat{a}, \hat{m}(a, s))$ different from $(a^*, m^*(a, s))$. Then this implies that

$$\begin{aligned} &\iint u(\hat{y}(\hat{m}(\hat{a}, s)), \hat{t}(\hat{m}(\hat{a}, s)), \theta, \hat{a}) f(\theta, s|\hat{a}) d\theta \, ds \\ > &\iint u(\hat{y}(m^*(a^*, s)), \hat{t}(m^*(a^*, s)), \theta, a^*) f(\theta, s|a^*) d\theta \, ds, \end{aligned}$$

or, substituting in the formulas of $\hat{y}$ and $\hat{t}$,

$$\begin{aligned} &\iint u(y^*(\hat{m}(\hat{a}, s), n^*(a^*, s)), t^*(\hat{m}(\hat{a}, s), n^*(a^*, s)), \theta, \hat{a}) f(\theta, s|\hat{a}) d\theta \, ds \\ > &\iint u(y^*(m^*(a^*, s), n^*(a^*, s)), t^*(m^*(a^*, s), n^*(a^*, s)), \theta, a^*) f(\theta, s|a^*) d\theta \, ds, \end{aligned}$$

Since in the original equilibrium, the optimality of $m^*(\cdot)$ and $n^*(\cdot)$ requires that

$$\begin{aligned} &\iint u(y^*(m^*(\hat{a}, s), n^*(\hat{a}, s)), t^*(m^*(\hat{a}, s), n^*(\hat{a}, s)), \theta, \hat{a}) f(\theta, s|\hat{a}) d\theta \, ds \\ \geq &\iint u(y^*(\hat{m}(\hat{a}, s), n^*(a^*, s)), t^*(\hat{m}(\hat{a}, s), n^*(a^*, s)), \theta, \hat{a}) f(\theta, s|\hat{a}) d\theta \, ds \end{aligned}$$

it follows that

$$\begin{aligned} &\iint u(y^*(m^*(\hat{a}, s), n^*(\hat{a}, s)), t^*(m^*(\hat{a}, s), n^*(\hat{a}, s)), \theta, \hat{a}) f(\theta, s|\hat{a}) d\theta \, ds \\ > &\iint u(y^*(m^*(a^*, s), n^*(a^*, s)), t^*(m^*(a^*, s), n^*(a^*, s)), \theta, a^*) f(\theta, s|a^*) d\theta \, ds. \end{aligned}$$

This contradicts the fact that a^* is the optimal effort level with respect to the original contract $(y^*(m,n), t^*(m,n))$. Hence the simple contract $(\hat{y}(m), \hat{t}(m))$ gives rise to the same effort level and reporting strategy, $(a^*, m^*(\cdot))$, and therefore the same outcome as the original contract does. $\square$

Proof of Proposition 4.2

Proof. Proof of Necessity

The IC constraint (4) is equivalent to

$$\int [u(y(s), \theta, b) - u(y(\xi), \theta, b)][f(\theta|s, a) - f(\theta|\xi, a)]d\theta \geq 0.$$

Substituting the formulas of $u(y, \theta)$ and $u(y, \theta, b)$ into the above inequality, we get

$$[y(s) - y(\xi)]\big(E[\theta|s, a] - E[\theta|\xi, a]\big) \geq 0.$$

Under Assumption 1, the above inequality implies that $y(s)$ is non-decreasing in s. And any non-decreasing function has non-negative derivative almost everywhere. This proves the necessity of Condition(i).

Define $U(\xi; s, a) \equiv \int u(y(\xi), \theta, b)f(\theta|s, a)d\theta + t(\xi)$. Necessity of Condition (ii) follows directly from the fact that the local first-order necessary condition for truth-telling is

$$\begin{aligned} \frac{\partial U(\xi; s, a)}{\partial \xi}|_{\xi=s} &= \int u_1(y(s), \theta, b)y'(s)f(\theta|s, a)d\theta + t'(s) \\ &= -2y'(s)(y(s) - E[\theta|s, a] - b) + t'(s) = 0. \end{aligned}$$

Proof of Sufficiency

It suffices to prove that under Conditions (i) and (ii), the agent's expected utility function $U(\xi; s, a)$ is quasi-concave in ξ and is maximized at $\xi = s$.

For $\xi > s$, we have

$$\begin{aligned}&\int u_1(y(\xi), \theta, b) y'(\xi) f(\theta|s, a) d\theta + t'(\xi) \\ =&\int u_1(y(\xi), \theta, b) y'(\xi) f(\theta|s, a) d\theta - \int u_1(y(\xi), \theta, b) y'(\xi) f(\theta|\xi, a) d\theta \\ =&2\big(E[\theta|s, a] - E[\theta|\xi, a]\big) y'(\xi) \leq 0,\end{aligned}$$

where the first equality follows from Condition (ii), and the last inequality follows from Assumption 1 and Condition (i).

Similarly, for $\xi < s$, we can prove that

$$\int u_1(y(\xi), \theta, b) y'(\xi) f(\theta|s, a) d\theta + t'(\xi) \geq 0.$$

Thus we have proved that under Conditions (i) and (ii), $U(\xi; s, a)$ is quasi-concave in ξ and is maximized at $\xi = s$. This finishes the proof. □

Proof of Proposition 4.3

Proof. It suffices to prove that if $y(s)$ is non-decreasing, then (i) the agent's expected utility is concave in a, and (ii) the agent's marginal utility in a is non-negative at $a = 0$. In mathematics,

$$\iint u(y(s), \theta, b) f_{aa}(\theta|s, a) d\theta\, ds - c_{aa}(a) \leq 0,\ \forall a, \tag{20}$$

$$\iint u(y(s), \theta, b) f_a(\theta|s, 0) d\theta\, ds - c_a(0) \geq 0. \tag{21}$$

Proof of (20). Let $y(s) \equiv y(1/2) + \epsilon(s)$. Then by the condition that $y(s)$ is non-decreasing, we have $\epsilon(s) \leq 0$ for $s \in [0, 1/2]$, and $\epsilon(s) \geq 0$ for $s \in [1/2, 1]$.

$$\begin{aligned}
&\iint u(y(s), \theta, b) f_{aa}(\theta|s, a) d\theta \, ds - c_{aa}(a) \\
=&\iint -(y(s) - \theta - b)^2 f_{aa}(\theta|s, a) d\theta \, ds - c_{aa}(a) \\
=&2 \int y(s) E_{aa}[\theta|s, a] ds - c_{aa}(a) \\
=&2 \left[\int_0^{\frac{1}{2}} \epsilon(s) E_{aa}[\theta|s, a] ds + \int_{\frac{1}{2}}^1 \epsilon(s) E_{aa}[\theta|s, a] ds \right] - c_{aa}(a) \leq 0,
\end{aligned}$$

where the second equality follows from the assumption that the marginal distribution of θ (and thus its expectation) is independent with a, and the inequality follows from Assumption 3.

Proof of (21). Similarly we get

$$\begin{aligned}
&\iint u(y(s), \theta, b) f_a(\theta|s, 0) d\theta \, ds - c_a(0) \\
=&2 \left[\int_0^{\frac{1}{2}} \epsilon(s) E_a[\theta|s, 0] ds + \int_{\frac{1}{2}}^1 \epsilon(s) E_a[\theta|s, 0] ds \right] \geq 0,
\end{aligned}$$

where the equality follows from the assumption that the marginal distribution of θ (and thus its expectation) is independent with a, and the inequality follows from Assumption 2. □

Proof of Proposition 4.4

Proof. Let $(y(s), t(s), a)$ be the solution to $P1$. Suppose for contradiction that $\exists \varepsilon > 0$ such that

$$\int \left[\int u(y(s), \theta, b) f(\theta|s, a) d\theta + t(s) \right] ds - c(a) = \varepsilon > 0.$$

Then one can verify that $(y(s), t(s) - \varepsilon, a)$ also satisfies the IR, IA, IC constraints, and generates strictly higher expected utility for the principal than $(y(s), t(s), a)$ does, thus $(y(s), t(s), a)$ can not be a solution to $P1$. we thus arrive at a contradiction. □

Proof of Proposition 4.7

Proof. Proof of $\lambda > 0$.

We define the following program as *the doubly-relaxed program*[13]:

$$\mathbf{P3}: \quad \max_{y(s),a\geq 0} \iint [u(y(s),\theta) + u(y(s),\theta,b)] f(\theta|s,a) d\theta\, ds - c(a), \tag{22}$$
$$s.t. \iint u(y(s),\theta,b) f_a(\theta|s,a) d\theta\, ds - c_a(a) \geq 0. \tag{23}$$

let $\hat{\lambda}$ be the lagrange multiplier for Constraint (23) in Program **P3**. Then by the Kuhn-Tucker condition, $\hat{\lambda} \geq 0$. To prove that $\lambda > 0$ it suffices to prove that $\hat{\lambda} > 0$, because if $\hat{\lambda} > 0$, then Constraint (23) is binding at the optimum, therefore **P3** is equivalent to Program **P2** and $\lambda = \hat{\lambda} > 0$.

Suppose for contradiction that $\hat{\lambda} = 0$, then the Kuhn-Tucker conditions of **P3** can be written as

$$\int [u_1(y(s),\theta) + u_1(y(s),\theta,b)] f(\theta|s,a) d\theta = 0, \tag{24}$$
$$\iint [u(y(s),\theta) + u(y(s),\theta,b)] f_a(\theta|s,a) d\theta\, ds - c_a(a) \leq 0,$$
$$a \geq 0,$$
$$a \cdot \Big[\iint [u(y(s),\theta) + u(y(s),\theta,b)] f_a(\theta|s,a) d\theta\, ds - c_a(a) \Big] = 0, \tag{25}$$
$$\iint u(y(s),\theta,b) f_a(\theta|s,a) d\theta\, ds - c_a(a) \geq 0. \tag{26}$$

(25) and (26) imply that

$$\iint u(y(s),\theta) f_a(\theta|s,a) d\theta\, ds \leq 0. \tag{27}$$

By taking the first order derivative of (24) with respect to s and rearranging, we get

$$y'(s) = \frac{\int [u_1(y(s),\theta) + u_1(y(s),\theta,b)] f_s(\theta|s,a) d\theta}{-\int [u_{11}(y(s),\theta) + u_{11}(y(s),\theta,b)] f(\theta|s,a) d\theta} = E_s[\theta|s,a] \geq 0,$$

[13]See Rogerson (1985) for more detail of the doubly-relaxed program.

where the second equality follows from the assumption that the marginal distribution of θ is independent with a, and the inequality follows from Assumption 1. Moreover, because $E_\xi[\theta|\xi, 0] > 0$ for some ξ, $y'(\xi) > 0$.

Therefore, there exists a function $\epsilon(s)$ with $\epsilon(s) \le 0$ for $s \in [0, 1/2]$ and $\epsilon(s) \ge 0$ for $s \in [1/2, 1]$ such that $y(s) = y(1/2) + \epsilon(s)$. Thus we have

$$\iint u(y(s), \theta) f_a(\theta|s, a) d\theta \, ds = 2\Big[\int_0^{1/2} \epsilon(s) E_a[\theta|s, a] ds + \int_{1/2}^{1} \epsilon(s) E_a[\theta|s, a] ds\Big] > 0, \quad (28)$$

where the inequality follows from Assumption 2. (28) is in contradiction with (27). Thus it must be true that $\hat{\lambda} > 0$, Constraint (23) in Program **P3** must be binding, any solution to Program **P3** is also a solution to Programs **P2**, and finally $\lambda = \hat{\lambda} > 0$.

Proof of $y'(s) > 0, \forall s$.

$y(s)$ must satisfy (9). Taking the first order derivative of (9) with respect to s and rearranging, we get

$$y'(s) = E_s[\theta|s, a] + \frac{\lambda}{2} E_{sa}[\theta|s, a] > 0, \quad \forall s, \quad (29)$$

where the inequality follows from the condition that $E_{sa}[\theta|s, a] > 0$ for all s and a.

Proof of $a > 0$.

Suppose for contradiction that $a = 0$, then with the same argument as in (28), we get

$$\iint u(y(s), \theta, b) f_a(\theta|s, 0) d\theta \, ds - c_a(0) > 0,$$

which is in contradiction with (6). Thus we arrive at a contradiction and it must be true that $a > 0$ at the optimum. □

Proof of Proposition 5.1

Proof. By solving (13), (14), and (15), we get

$$\lambda = \frac{2\alpha\beta}{6\kappa - 3\alpha^2}. \tag{30}$$

$$a = \frac{2\alpha\beta}{6\kappa - 3\alpha^2}. \tag{31}$$

$$\begin{aligned} y(s) &= \frac{3\kappa}{\alpha} a[2s-1] + E[\theta] + \frac{b}{2} \\ &= \frac{2\beta\kappa}{2\kappa - \alpha^2}(2s-1) + E[\theta] + \frac{b}{2}. \end{aligned} \tag{32}$$

Then we substitute (31) and (32) into the following formula to calculate the principal's expected utility in the equilibrium,

$$\begin{aligned} \mathbf{U_P} &= \iint -\Big[(y(s)-\theta)^2 + (y(s)-\theta-b)^2\Big] f(\theta|s,a)d\theta ds - c(a) \\ &= -2\int y^2(s)ds - 2E[\theta^2|a] + 4\int y(s)E[\theta|s,a]ds + 2b\int y(s)ds - 2bE[\theta|a] - c(a) - b^2, \end{aligned}$$

and we get

$$\mathbf{U_P} = \frac{4\beta^2\kappa}{3(2\kappa - \alpha^2)} - 2V(\theta) - \frac{b^2}{2}.$$

As in the standard principal-agent model with full commitment, because the principal gets all the welfare surplus, the agent only gets his reserve utility, which is 0, at the optimum.

□