A "Reduced-Form" Nonparametric Test of Common Values in First-Price Auctions with a Binding Reserve Price

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Abstract

We develop a new nonparametric test of common values in first-price auctions with a binding reserve price. The test is based on the behavior of the CDF of bids near the reserve price. We show that this behavior is drastically different under private values (PV) and common values (CV). Next, we show that the problem of discriminating between PV and CV is equivalent to the problem of estimating the lower tail index of the bids distribution. Our approach allows for unobserved auction heterogeneity of an arbitrary form, and in particular doesn't require the number of potential bidders to be observable. Drawing on the existing and recent literature on tail index estimation, we characterize the B. Hill (1975) tail index estimator for panels with stochastic dimension and develop a non-parametric estimator of the asymptotic variance for inference. We implement the test using this new estimator on a sample of BC timber auctions, and find strong support for CV.

1 Introduction

Private and common values are probably the two most important paradigms in auction theory. In a private value environment, bidders perfectly know their willingness to pay for the object, while in a common values environment, only imperfect information is available. Many theory-based policy recommendations, e.g. how to set reserve prices optimally, how many bidders to invite to the auction, what entry fee to impose etc. differ depending on whether the values are private or common. In addition, as Laffont and Vuong (1996) have shown, models with common values are often nonparametrically non-identified, while private value models are often identified. Because of all this, a test of the auction environment is clearly an important and practically relevant issue.

In this paper, we develop a "reduced-form" nonparametric test of common versus private values in first-price auctions when there is a binding reserve price \( r \). We consider the general environment of Milgrom and Weber (1982) that allows for correlation between the private information (signals) of the bidders and potentially a common value component in their
valuations. Our approach is structural in the sense that it is based on auction theory, but at the same time is "reduced-form" in the sense that it tests the prediction of the theory directly. We show that, when values are private (PV), the slope of the bidding strategy at the reserve price is zero, and the CDF of bids behaves like $\sqrt{b - r}$ as $b \downarrow r$. But if there is a common value component in bidders' valuations, the slope of the bidding strategy is positive and the CDF of bids has a bounded derivative (PDF) at $r$.

In order to distinguish between the PV and CV environments, we propose a novel nonparametric asymptotically most powerful test based on the above property of the CDF of bids. The test is based on the results in the literature on nonparametric tail index estimation (Hill (1975), Hsing (1993) and more recently Hill (2005)). The tail index of a distribution describes the manner in which the CDF approaches 0 as the variable approaches the lower bound of the support. We show that both hypotheses can be recast in terms of the tail index $\kappa$ of the distribution of the bids near the reserve price. We show that under PV, $\kappa = 1/2$ and under CV, $\kappa = 1$.

A version of Hill’s (1975) estimator $\hat{\kappa}$ can then be used to estimate the tail index and perform tests. The estimator is remarkably easy to implement. The asymptotically most powerful test of CV versus PV based on an estimator of the tail index is a one-sided t-test. Our testing approach can be implemented even when there is unobserved auction heterogeneity, and even when the number of potential bidders is unobservable. This is because the tail index of a distribution is preserved when the distribution is aggregated along any dimension. Another attractive feature is that it can be implemented even when the only information available is the winning bid and the reserve price.

Our approach is inspired by Hendricks, Pinkse, and Porter (2003) (see also a more detailed discussion in Hendricks and Porter (2007)) who noted that the behavior of bids around the reserve price is different under PV and CV. Specifically, the lower bound of the support of the distribution of pseudo values (Guerre, Perrigne, and Vuong (2000); Athey and Haile (2002)) is equal to $r$ under PV but is strictly greater than $r$ under CV. No formal test along these lines is currently available. Our approach is different in that it does not require nonparametric estimation of pseudo values.

An early approach to testing for common values was to check if the expected value of the bid increases monotonically with the number of potential bidders; a non-monotonic pattern was believed to provide evidence for common values. This approach was initiated by Gilley and Karels (1981), and applied to second-price sealed-bid and English auctions by Paarsch (1991) and Bajari and Hortacsu (2004). However, Pinkse and Tan (2005) have shown that in first-price auctions, this pattern can also arise if values are private and affiliated.

Haile, Hong, and Shum (2003) propose a nonparametric test of PV versus CV. Their approach is entirely different from ours and is based on the variation in the number of bidders across auctions. They implement their test on a sample of US Forest Service (USFS) timber auctions and obtain mixed results. Haile, Hong and Shum’s approach does not require a binding reserve price, but requires the number of potential bidders to be

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1 The PDF behaves like $1/\sqrt{b - r}$. With independent private values, this is known from Guerre, Perrigne, and Vuong (2000).

2 Krasnokutskaya (2003) argues for the importance of accounting for the unobserved heterogeneity in the estimation of auction markups. See also the discussion in Paarsch, Hong, and Haley (2006).

3 See the discussion in Athey and Haile (2002).
Our approach, on the other hand, requires a binding reserve price, but doesn’t require the number of potential bidders to be observable, and doesn’t require variation in the number of bidders.

Recently, Hortacsu and Kastl (2008) proposed a test of common values when some bidders have information about rivals’ bids, and applied it to Canadian Treasury Bill auctions. (In Canadian Treasury bill auctions, bidders naturally fall into two groups - dealers and customers, and the former have an informational advantage over the latter.) Their approach is tailored to multi-unit auctions and is entirely different from ours.

An econometric contribution of this paper is a detailed theoretical and experimental treatment of the Hill (1975) tail index estimator for imbalanced panels with stochastic dimension. In our environment the data generating process is a panel where bids are non-linearly dependent within auctions of random size. Thus, the sample size is itself a random variable that is dependent on bids in an unknown way. The literature is silent concerning extremal statistics with stochastic sample size, and there are only a few applications of tail index estimation for panel data (e.g. Mikosch and C. de Vreis (2006); Jongen, Verschoor, Wolff, and Zwinkels (2006)).

By exploiting theory developed in Hsing (1991) and Hill (2005), Hill (2008), the celebrated Hill-estimator is shown to be asymptotically normal, where the stochastic nature of bid counts is irrelevant. We then characterize and deliver an estimator of the asymptotic variance under the assumption bids are independent across auctions. Finally, the generic kernel variance estimator developed in Hill (2005), a consistent estimator under far more general conditions than those encountered in this paper, is shown to be extraordinarily sharp based on simulation work. This estimator is robust to any degree of within-auction dependence and substantial cross-auction dependence, should the latter prove to be true. Indeed, the fact that Hill’s (2005) kernel estimator strongly dominates the new estimator tailored to panels of independent blocks, and leads to different test results when applied to timber auction bids, suggests bids may be dependent across auctions.

We implement our tests on a sample of BC timber auctions, part of its newly established timber sales program. An important feature of these data is the presence of a binding reserve price (the upset rate). This application is also interesting because the format adopted by BCTS is the one of scaled sales. BCTS estimates the amount (and composition) of timber on the tract to be sold, sets the upset rate, and then solicits "bonus" bids per cubic foot of timber. The actual price paid by the winner (the bidder with the highest bid) is equal to the product of the per unit price (the upset rate plus the bonus bid) times the actual volume of timber on the tract. Since a large portion of the uncertainty is eliminated from the price, one should expect a strong private value component. Indeed, the papers that studied the USFS scaled sales often adopted a private values framework. But the issue is far from being resolved; Baldwin (1995) and Athey and Levin (2001) argue for the presence of a common value component. For BCTS auctions, Paarsch (1992) attempted to discriminate

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4In USFS auctions, the reserve price is typically nonbinding and the number of potential bidders is observable. See Baldwin, Marshall, and Richard (1997), Haile (2001) and Haile and Tamer (2003).


6Another type of sales is lumpsum sales. These sales are used in USFS auctions, and Haile, Hong, and Shum (2003) find more support for common values for this type of sales.

between common and private values within a parametric framework, and obtained some support for common values. Our nonparametric test strongly rejects private values in favor of a model with a common value component.

2 The Model

We consider a first-price, sealed-bid auction with \( N \geq 2 \) risk-neutral bidders. Following Milgrom and Weber (1982), we consider a model where bidders may have a common as well as private value components in their valuations of the object. Specifically, we assume that prior to the auction, each bidder receives a signal \( S_i \) informative about his value of the object \( U_i \). Also, there may be a pure common value component \( V \). We assume that the vector \( (V, S_1, \ldots, S_N) \) is drawn from some joint distribution \( F \) with density \( f \) that satisfies the affiliation property. We assume that the support of \( F \) is \([v, \bar{v}] \times [s, \bar{s}]^N\), where \( v < \bar{v}, s < \bar{s} \). The bidders’s valuations \( U_i = u(V, S_i, \{S_j\}_{j \neq i}) \), where \( u \) is a nonnegative, continuous and nondecreasing function.

The model is symmetric: the function \( u \) is the same for all bidders, and the distribution \( F \) is symmetric in bidders’ signals.

We wish to discriminate between two types of the informational environment of the bidders. If \( u(V, S_i, \{S_j\}_{j \neq i}) \) does not depend on \( V \) and \( \{S_j\}_{j \neq i} \), the model is private values (generally affiliated private value, or PV model). Otherwise the model has a common value component (CV model). Let \( Y_i = \max_{j \neq i} S_j \) and consider

\[
v(s, y) = E\{U_i | S_i = s, Y_i = y\},
\]

the value of the object conditional on "just" winning the auction. As in Milgrom and Weber (1982), we make a non-degeneracy assumption that \( v(s, y) \) is (strictly) increasing in \( s \). In addition, we assume that \( v(s, s) \) is a differentiable function, and that \( dv(s, s)/ds > 0 \) for all \( s \in [s, \bar{s}] \). When there is a common value component, we make a further non-degeneracy assumption: the value of the object conditional on winning the auction,

\[
w(s, y) = E\{U_i | S_i = s, Y_i < y\},
\]

is (strictly) increasing in own signal \( S_i = s \) and maximal rival signal \( Y_i = \max_{j \neq i} S_j \).

Theorem 14 in Milgrom and Weber (1982) implies that there is a unique Bayesian-Nash equilibrium, with bidding strategies satisfying the differential equation

\[
(v(s, s) - B(s)) f_{Y_i|S_i}(s|s) - B'(s) F_{Y_i|S_i}(s|s) = 0,
\]

and the boundary condition \( B(s^*) = r \). With our assumptions, the screening level \( s^* \in \)

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8 Our exposition follows Hendricks and Porter (2007).
9 I.e., \( f(\min\{x, y\}) \cdot f(\max\{x, y\}) \geq f(x) \cdot f(y) \). See Milgrom and Weber (1982).
10 In many empirical applications, it is sufficient to assume that bidders’ valuations depend only on \( V \) and their own signal \( S_i \), see e.g. Hendricks and Porter (2007). We consider a more general model, as in Milgrom and Weber (1982).
(s, \tilde{s}) is determined as the unique solution of the equation\(^{11}\)

\[ w(s_*, s*) = r. \]

This is assuming that the reserve price \( r \) is binding. Only the bidders with signals \( S_i \geq s_* \) enter; these are called actual bidders.

Differentiating equation (1) with respect to \( s \) and evaluating at \( s = s_* \) gives:

\[
\left( \frac{dv(s_*, s*)}{ds} - B'(s_*) \right) f_{Y_1|S_1}(s_*|s*) - B''(s_*) F_{Y_1|S_1}(s_*|s*) = 0, \tag{2}
\]

Equations (1) and (2) together imply the following facts that are at cornerstone of our approach. Under PV,

\[ v(s^*, s^*) = w(s^*, s^*) (= r), \]

and therefore

\[ B'(s_*) = 0, \tag{3} \]

\[ B''(s_*) = \frac{f_{Y_1|S_1}(s_*|s*) dv(s^*, s^*) / ds}{F_{Y_1|S_1}(s_*|s*)} > 0. \tag{4} \]

(here the derivatives are understood as right-derivatives). But under CV, since

\[ v(s^*, s^*) > w(s^*, s^*), \]

equation (1) implies

\[ B'(s_*) = \frac{(v(s^*, s^*) - r) f_{Y_1|S_1}(s^*|s^*)}{F_{Y_1|S_1}(s^*|s^*)} > 0. \]

We now show that the difference in the slopes of the bidding strategy (\( B'(s_*) = 0 \) under PV but \( B'(s_*) > 0 \) under CV) implies different behavior of the CDF of the bidder’s bids around the reserve price.

We need the notion of a tail index of a distribution. For any random variables \( Z \in \mathbb{R} \) and \( X \in \mathbb{R}^d \) such that the lower bound of the support of \( Z|X = x \) is \( \underline{z}(x) = -\infty \), we say that the distribution of \( Z|X = x \) has (left) tail index \( \kappa(x) \) if, cf. Resnick (1987),

\[
\lim_{\varepsilon \searrow 0} \varepsilon^{-\kappa(x)} P \{ Z \leq \underline{z}(x) + \varepsilon|X = x \} > 0,
\]

or equivalently\(^{12}\)

\[ P \{ Z \leq \underline{z}(x) + \varepsilon|X = x \} \sim A \varepsilon^{\kappa(x)} \text{ for some } A > 0. \]

The tail index describes the manner in which the CDF approaches 0 as the random variable approaches the lower bound of its distribution. For example, if \( Z \) has a density at \( \underline{z}(x)\),

\(^{11}\)Of course, we assume that the reserve price is sufficiently low so that a solution to this equation exists.

\(^{12}\)In the proposition below, the notation \( a_n \sim b_n \) means that \( \lim_{n \to \infty} a_n/b_n = 1 \).
then it is easy to show that $\kappa(x) = 1$. (This will be the case in our model under CV.) But
the tail index is defined more generally. For example, we show that under PV, the density
of bids converges to $+\infty$ as bids get closer to the reserve price, even though the CDF is
continuous and there is no mass point at the reserve price.

We now establish our main result: the tail index of the distribution of bids of bidder 1,

$$G(b|s_{-1}) \equiv P\{B(S_1) \leq b|S_{-1} = s_1, S_1 \geq s_1\},$$

is different under PV and CV.

**Proposition 1** The tail index $\kappa$ of $G(b|s_{-1})$ as $b \searrow r$ is equal to 1/2 under PV and is
equal to 1 under CV.

**Proof.** Under PV, we have

$$\lim_{\varepsilon \searrow 0} \varepsilon^{-1/2} P\{B(S_1) \leq r + \varepsilon|S_{-i} = s_{-i}, S_i \geq s_*\}$$

$$= \frac{1}{P\{S_1 \geq s_*|S_{-1} = s_{-1}\}} \lim_{\varepsilon \searrow 0} P\{r \leq B(S_1) \leq r + \varepsilon|S_{-1} = s_{-1}\} \varepsilon^{\frac{1}{2}}$$

and

$$\lim_{\varepsilon \searrow 0} \frac{P\{r \leq B(S_1) \leq r + \varepsilon|S_{-1} = s_{-1}\}}{\varepsilon^{\frac{1}{2}}} = \lim_{\varepsilon \searrow 0} \frac{P\{s_* \leq S_1 \leq s_* + \tilde{\varepsilon}|S_{-1} = s_{-1}\}}{\tilde{\varepsilon}^{\frac{1}{2}}} \left(\frac{(B(s_* + \tilde{\varepsilon}) - r)}{\tilde{\varepsilon}}\right)^{\frac{1}{2}}$$

$$= \frac{1}{\tilde{\varepsilon}} \frac{P\{s_* \leq S_1 \leq s_* + \tilde{\varepsilon}|S_{-1} = s_{-1}\}}{\tilde{\varepsilon}^{\frac{1}{2}}} \left(\frac{(B(s_* + \tilde{\varepsilon}) - r)}{\tilde{\varepsilon}}\right)^{\frac{1}{2}}$$

where $\tilde{\varepsilon} = B^{-1}(r + \varepsilon) - s_*$. Now the limit in the numerator

$$\lim_{\tilde{\varepsilon} \searrow 0} \frac{P\{s_* \leq S_1 \leq s_* + \tilde{\varepsilon}|S_{-1} = s_{-1}\}}{\tilde{\varepsilon}} = f_{S_1|S_{-1}}(s_*|s_{-1}),$$

and from (3) and (4), the limit in the denominator simplifies to

$$\lim_{\tilde{\varepsilon} \searrow 0} \left(\frac{(B(s_* + \tilde{\varepsilon}) - r)}{\tilde{\varepsilon}}\right)^{\frac{1}{2}} = \lim_{\tilde{\varepsilon} \searrow 0} \left(\frac{(B(s_*) + B'(s_*) \tilde{\varepsilon} + \frac{1}{2} B''(s_*) \tilde{\varepsilon}^2 - r)}{\tilde{\varepsilon}}\right)^{\frac{1}{2}}$$

$$= \left[\frac{1}{2} B''(s_*)\right]^\frac{1}{2}.$$}

Therefore

$$\lim_{\varepsilon \searrow 0} \varepsilon^{-1/2} P\{B(S_1) \leq r + \varepsilon|S_{-i} = s_{-i}, S_i \geq s_*\} = \frac{f_{S_1|S_{-1}}(s_*|s_{-1})}{P\{S_1 \geq s_*|S_{-1} = s_{-1}\}} \left[\frac{1}{2} B''(s_*)\right]^\frac{1}{2} > 0,$$

i.e. the tail index is indeed equal to 1/2.
Under CV, $B$ is (left) differentiable at $s = s_*$,
\[
\lim_{\varepsilon \searrow 0} \frac{B(s_* + \varepsilon) - r}{\varepsilon} = B'(s_*),
\]
and therefore
\[
\lim_{\varepsilon \searrow 0} \varepsilon^{-1/2} \mathbb{P} \{ B(S_1) \leq r + \varepsilon|S_{-1} = s_{-1}, S_{i} \geq s_* \} = \frac{\int_{S_1|s_{-1}}(s_*|s_{-1})}{\mathbb{P}\{S_1 \geq s_*|S_{-1} = s_{-1}\} B'(s_*)} > 0.
\]
The tail index is equal to 1. Q.E.D.

We have so far assumed the setting of a single auction. But it is easily seen that the above results generalize to a population of auctions, each with its own reserve price $R$ and the number of potential bidders $N$, and also its own auction characteristic $X$. The testable restrictions in Proposition 1 carry over for the distribution of bids conditional on $(R, X, N)$,
\[
G(b|s_{-1}, r, x, N_0) \equiv \mathbb{P} \{ B(S_1, R, X, N) \leq b|S_{-1} = s_1, S_1 \geq s_*, R = r, X = x, N = N_0 \},
\]
assuming the rest of the properties (affiliation, etc.).

In general, the vector of characteristics $X$ can be partitioned as $X = (X^o, X^u)$, where $X^o$ is the observable part of $X$, and $X^u$ is the unobservable (to the econometrician) part. The power of the tail index approach is that, as long as the tail index doesn’t depend on $X = x$, $\kappa(x) \equiv \kappa \forall x$, the tail index is preserved if we do not condition on $X^u$. Similarly, it is preserved if we do not condition on $N$. More generally, the following fact is true.

**Claim 2** Let $Z$ and $X$ be two random variables defined on some common probability space. Consider a r.v. $\sigma(X_0) \in \sigma(X)$, i.e. $X_0$ is measurable with respect to the sigma-algebra generated by $X$. Assume further that the lower bound of the support $Z|X = x$ is $\underline{z}(x) > -\infty \forall x \in \text{supp}\{X\}$, and that the tail index $\kappa(x) \equiv \kappa \forall x \in \text{supp}\{X\}$. Then the tail index of the distribution $Z|X_0 = x_0$ is also equal to $\kappa \forall x_0 \in \text{supp}\{X_0\}$.

**Proof.** By the law of iterated expectations.
\[
\lim_{\varepsilon \searrow 0} \varepsilon^{-\kappa} \mathbb{P} \{ Z \leq \underline{z}(x) + \varepsilon|X_0 = x_0 \} = \lim_{\varepsilon \searrow 0} \varepsilon^{-\kappa} \mathbb{E} \{ P \{ Z \leq \underline{z}(x) + \varepsilon|X, X_0 = x_0 \} |X_0 = x_0 \}
\]
\[
= \lim_{\varepsilon \searrow 0} \varepsilon^{-\kappa} \mathbb{E} \{ P \{ Z \leq \underline{z}(x) + \varepsilon|X \} |X_0 = x_0 \}
\]
\[
= \lim_{\varepsilon \searrow 0} \varepsilon^{-\kappa} \mathbb{E} \{ P \{ Z \leq \underline{z}(x) + \varepsilon|X \} |X_0 = x_0 \}
\]
\[
= \mathbb{E} \left\{ \lim_{\varepsilon \searrow 0} \varepsilon^{-\kappa} P \{ Z \leq \underline{z}(x) + \varepsilon|X \} |X_0 = x_0 \right\} > 0.
\]
Q.E.D.

If we normalize the bidding strategy by dividing the "bonus" bid by the reserve price,
\[
B^* = \frac{B - R}{R}, \quad (5)
\]
then Claim 2 implies that the PV and CV environments can be distinguished according to the tail index of the distribution \( P\{B^* (S_1, R, X, N) \leq b | S_1 \geq s_\star \} \).

### 3 Estimation and Testing Framework

1. **DATA GENERATING PROCESS (DGP)**

   We assume that a sample of \( L \) auctions is available, and index the auctions by \( l = 1, \ldots, L \). Each auction is characterized by a reserve price \( r_l \) and covariates \( x_l \in \mathcal{X} \). We assume that the covariates \( x_l \) are drawn independently for each auction. The data generating process is further specified as follows.

   1. The vector \((r_l, x_l, N_l)\) is drawn independently across \( l \) from some joint distribution \( \Lambda \).
   2. Conditional on \( x_l = x \), \( r_l = r \) and \( N_l = N \), the signals \( S_{i,l} \) of potential bidders \( i = 1, \ldots, N_l \) and the common value component \( V_l \) are drawn from some distribution with joint density \( f_{V_l, (S_{i,l})_{i=1,\ldots,N_l}, r, x_l, N_l} \) symmetric in its \( s_i \) arguments. (We assume that this conditional distribution doesn’t depend on \( N \).) The vectors \((V_l, (S_{i,l})_{i=1,\ldots,N_l}, r_l, x_l, N_l)\) are independent across auctions \( l \).
   3. Only bidders whose signals exceed the screening level, \( S_{i,l} \geq s^* (r_l, x_l, N_l) \), submit bids. The number of actual bidders in auction \( l \) is denoted as \( n_l \), so that the total number of observations in the sample is \( n = \sum_l n_l \). The bid count is bounded \( 0 \leq n_l \leq \bar{n} \) with probability one for some fixed finite \( \bar{n} \in \mathbb{N} \). The bid \( b_{i,l} \) corresponding to the signal \( S_{i,l} \) is generated according to the bidding strategy
      \[
      b_{i,l} = B (S_{i,l}, r_l, x_l, N_l) \quad (S_{i,l} \geq s^* (r_l, x_l, N_l))
      \]

   The data available to the econometrician consists of an (independent and identically distributed across \( l \)) sample of observations
   \[
   \left\{ \{b_{i,l}\}_{i=1,\ldots,n_l, r_l, x_l, n_l} \right\}_{l=1,\ldots,L}.
   \]

   Note that the observations within each auction are not assumed to be independent. Even in the PV environment, the effect of unobservable \( N \) as well as other kinds of unobserved heterogeneity can make bids statistically dependent within the auction.

2. **THE HILL ESTIMATOR**

   In view of the normalization (5) and Claim 2, we work with normalized bids \( b_{i,l}^* = (b_{i,l}/r_l) - 1 \). Only these bids, and no other information, from the sample will be used to estimate the tail index. Denote as \( G^* (b) \) the CDF of \( b_{i,l}^* \), where from Proposition 1 and Claim 2,
   \[
   G^* (b) = Ab^\kappa \times (1 + o(1)) \quad \text{as} \quad b \searrow 0
   \]
   for some \( A > 0 \). The Hill estimator of the tail index is based on the following easily proved representation:
   \[
   \kappa^{-1} = E \left\{ \ln q - \ln b_{i,l}^* | b_{i,l}^* \leq q \right\} + o (1) \quad \text{as} \quad q \searrow 0. \tag{6}
   \]
That is, $\kappa^{-1}$ is the mean distance of the log-normalized bid $\ln b_{i,l}^*$ below some low threshold $\ln q$ as $q \to 0$. To see why this is true, use $G^* (b) = b^\kappa \times (1 + o(1))$ to write as $q \to 0$

$$E \{ \ln q - \ln b_{i,l}^* | \ln b_{i,l}^* \leq \ln q \} = \ln q - \frac{\int_{-\infty}^{\ln q} (\ln x) \, dG^* (x)}{G^* (q)}$$

$$= \frac{\int_{-\infty}^{\ln q} G^* (e^x) \, dx}{G^* (q)}$$

$$= \frac{\int_{-\infty}^{\ln q} e^{\kappa x} \, dx}{q^\kappa} + o (1)$$

$$= \frac{1}{q^\kappa} + o (1)$$

$$= \kappa^{-1} + o (1).$$

where the equality in the fourth line follows from integration by parts.

Equation (6) suggests a natural way of estimating the (inverse of) the tail index $\kappa^{-1}$ by a sample analogue, with an appropriately chosen sequence of $q \to 0$ as the sample size goes to infinity.\footnote{In the following we write $(z)_+$ to denote $\max\{z, 0\}$.} Namely, let $\{ b_{i,l}^*_t \}_{t=1}^n$ be the sample stacking all $b_{i,l}^*$, let $b^*_1 \geq b^*_2 \geq \ldots \geq b^*_n$, and let $\{ m_n \}$ be any intermediate order sequence conditional on $\{ n_l \}_{l=1}^L$; $1 \leq m_n < n$, $m_n \to \infty$ and $m_n / n \to 0$ as $L \to \infty$ (e.g. Leadbetter et al 1983). Then the sequence of $q$ is chosen as $b^*_{(m_n+1)}$, and the Hill (1975) estimator of $\kappa^{-1}$ is simply the sample average

$$\hat{\kappa}_{m_n}^{-1} = \frac{1}{m_n} \sum_{t=1}^n (\ln b^*_{(m_n+1)} - \ln b^*_t)_+. $$

Other estimators exist but none have been shown to be as robust to unknown forms of dependence and heterogeneity. See the literature reviews in Hill (2005) and Hill (2008).

Note $\hat{\kappa}_{m_n}$ and associated asymptotic arguments are typically based from the $m_n$ largest observations, e.g. $\hat{\kappa}_{m_n}^{-1} = 1/m_n \sum_{t=1}^n (\ln z_t / z_{(m_n+1)})_+$ where $z_t := 1/b^*_t$. Since the two are trivially identical in the sequel we do not translate between the two representations (see Hall 1982, and recently Chernozhukov and Du 2008).

It is convenient to denote

$$F_L := \sigma \left( \{ n_l \}_{l=1}^L \right)$$

to be the sigma-algebra for conditioning on the realizations of bid counts $n_l$. In view of Proposition 1 and Claim 2, we have the following condition satisfied for the tails of the distribution of $b_{i,l}^*$ conditional on $F_L$.

**A1** The left-tail behavior of $b_{i,l}^*$ satisfies for every $i$ and $l$ as $b \to 0$

$$P \left( b_{i,l}^* < b | F_L \right) = d_l \times b^{-\kappa} (1 + o(1)) , \kappa > 0, d_l > 0, \quad (7)$$

where $d_l$ is an $F_L$-measurable random variable.
Next, since the bidders are assumed to be symmetric, our DGP implies that the distribution of \( b_{i,l} \) is strictly stationary.

A2 \( b_{i,l} \) is strictly stationary over \( i \) and \( l \), possibly dependent over \( i \) within auction \( l \), and independent across auctions \( l \). The covariates \( x_{i,l} \) are strictly stationary.

Define the lower \( m/n \)-th conditional quantile sequence \( \{q_{m,n}\} \) by (see Leadbetter et al (1983: Theorem 1.7.13))

\[
\frac{n}{m_n} P \left( b_i^* < q_{m,n} | F_L \right) \to 1 \text{ as } L \to \infty.
\]

(8)

In the conventional setting where \( \{n,m_n,q_{m,n}\} \) are deterministic (i.e. \( F_L \) is trivial so that \( (n/m_n)P(b_i^* < q_{m,n}) \to 1 \)), the form of \( \hat{\kappa}^{-1}_{m_n} \) is best understood as a method of moments estimator since, cf. (6) and Hsing (1991: p. 1548),

\[
\lim_{n \to \infty} \frac{n}{m_n} E \left( \ln \frac{q_{m,n}}{b_i^*} \right) \to \kappa^{-1}.
\]

Thus, \( 1/m_n \sum_{t=1}^{n} (\ln q_{m,n}/b_i^*)^+ \) naturally estimates \( \kappa^{-1} \) since \( E[1/m_n \sum_{t=1}^{n} (\ln q_{m,n}/b_i^*)^+] = (n/m_n)E[(\ln q_{m,n}/b_i^*)^+] \sim \kappa^{-1} \). In the conventional setting \( b_i^* \) estimates \( q_{m,n} \) since \( 1/m_n \sum_{t=1}^{n} I(b_i^* < b_{(m_n+1)}^*) \sim m_n/n \) and \( P(b_i^* < q_{m,n}) \sim m_n/n \) by construction, and under remarkably general conditions \( \ln q_{m,n}/b_i^* \) converges in probability to 0 (Hill (2005), Hill (2008)).

In the present setting, however, \( \hat{\kappa}^{-1}_{m_n} \) is not only a function of random order statistics \( b_i^* \) but also the random sample size \( n \) that may be dependent on \( b_i^* \) in an unknown way. As far as we know this environment has never been explored in the extreme value theory literature. We therefore carefully explore what the Hill-estimator represents, and how randomized bid counts \( n_l \) affect the limit distribution, if at all. Our first task is to show random counts \( \{n_l\}_{L=1}^{L} \) do not affect the interpretation of \( \hat{\kappa}^{-1}_{m_n} \) as a method of moments estimator.

**LEMMA 2.1** Under A1 and A2

\[
\lim_{L \to \infty} E \left( \frac{n}{m_n} \left( \ln \frac{q_{m,n}}{b_i^*} \right)^+ \right) \to \kappa^{-1}.
\]

Remark: Notice we cannot simply factor \( n/m_n \) out of the expectations since i. \( \{n,m_n\} \) are stochastic sequences, and ii. \( \{q_{m,n}\} \) depends on \( \{n,m_n\} \) nonlinearly since (7) and (8) imply \( q_{m,n} \sim d_i^{-1/\kappa}(m_n/n)^{1/\kappa} \).

Now define a zero mean conditional tail array

\[
T^{(L)}_{m_n,t}(u) := T^{(L)}_{m_n,t}(u) - \kappa^{-1} f^{(L)}_{m_n,t}(u)
\]
where
\[
U_{m,n,t}^{(L)} := (\ln q_{m,n}/b^*_t)_+ - E \left[ (\ln q_{m,n}/b^*_t)_+ | \mathcal{F}_L \right]
\]
\[
I_{m,n,t}^{(L)} (u) := I (b^*_t < q_{m,n} e^u) - P (b^*_t < q_{m,n} e^u | \mathcal{F}_L), \ u \geq 0,
\]
and compactly write
\[
T_{m,n,t}^{(L)} := T_{m,n,t}^{(L)} \left( u/m^{1/2}_n \right).
\]
Throughout we write interchangeably the stacked \(\{U_{m,n,t}^{(L)}, I_{m,n,t}^{(L)}, T_{m,n,t}^{(L)}\}\) or the auction/bid array \(\{U_{m,n,i,l}^{(L)}, I_{m,n,i,l}^{(L)}, T_{m,n,i,l}^{(L)}\}\). Define conditional and unconditional variances
\[
v^2_{m,n|L} := E \left[ \left( m^{1/2}_n (\hat{\kappa}^{-1}_{m,n} - \kappa^{-1}) \right)^2 | \mathcal{F}_L \right] \quad \text{and} \quad v^2_{m,n} := E \left[ v^2_{m,n|L} \right].
\]
A characterization of the asymptotic variance of \(\hat{\kappa}^{-1}_{m,n}\) for a stochastic dimensional panel is expedited by a useful decomposition.

**THEOREM 2.2** Under A1 and A2, for any \(m_n = o(n)\) and \(m_n \to \infty\)
\[
m^{1/2}_n (\hat{\kappa}^{-1}_{m,n} - \kappa^{-1}) | \mathcal{F}_L = \frac{1}{m^{1/2}_n} \sum_{t=1}^{n} T_{m,n,t}^{(L)} + o_p(1). \quad (9)
\]
Further,
\[
\frac{m^{1/2}_n}{v^2_{m,n}} (\hat{\kappa}^{-1}_{m,n} - \kappa^{-1}) \overset{d}{\to} N(0,1), \quad v^2_{m,n} = O(1) \quad (10)
\]
and
\[
\frac{v^2_{m,n}}{v^2_{m,n|L}} \overset{p}{\to} 1. \quad (11)
\]

**Remark 1:** Equality (9) implies the scaled and centered Hill estimator, conditional on bid counts, is asymptotically equivalent to a partial sum of tail arrays that are comparatively easy to work with (Hill (2005), Hill (2008)).

**Remark 2:** Limit (10) is arguably supported by Billingsley’s (1999: Theorem 14.4) weak limit theorem for cadlag functionals with stochastic index. In that context it is assumed there exists a deterministic sequence \(\{g(L)\}_{L \geq 1}\) such that \(n/g(L) \to \theta\) where \(\theta\) is a constant, or a random variable under sharp regulatory conditions. We do not require any information on the nature of \(n\) although \(\text{plim}_{L \to \infty} n/L = \infty\) exists since \(n_l\) is iid and bounded. The limit is identical to the ones established in Hill (2005: Theorem 5; 2008: Theorem 5.1), hence random bid counts are non-influential in the limit.

**Remark 3:** Property (11) means variance estimation can proceed as if \(\{n_l\}_{l=1}^L\) were deterministic.

**Remark 4:** In the iid case \(v^2_{m,n} \to \kappa^{-2}\) (Hill (1975), Hall (1982)).
Inference crucially relies on the availability of a robust estimator of $v_{m_n}^2$. We tackle this issue next.

3. ASYMPTOTIC VARIANCE Since $b_n^*$ stacks independent blocks of bid data with non-constant block size, use (6) to deduce

\[ v_{m_n|L}^2 \sim E \left( \frac{1}{m_n^{1/2}} \sum_{l=1}^L \sum_{i=1}^{n_l} T_{m_n,i,l}^{(L)} | F_L \right)^2 = \frac{1}{m_n} \sum_{l=1}^L \sum_{i,j=1}^{n_l} E \left[ T_{m_n,i,l}^{(L)} \times T_{m_n,j,l}^{(L)} | F_L \right]. \]

Stationarity within auctions implies $E[T_{m_n,i,l}^{(L)} \times T_{m_n,j,l}^{(L)} | F_L]$ depends only on the bid count $n_l$ and bid displacement $|i - j|$. Therefore

\[ v_{m_n|L}^2 \sim \frac{1}{m_n} \sum_{l=1}^L \sum_{i=1}^{n_l} E \left[ \left( T_{m_n,i,l}^{(L)} \right)^2 | F_L \right] + 2 \frac{1}{m_n} \sum_{l=1}^L \sum_{i=1}^{n_l-1} (n_l - i) \times E \left[ T_{m_n,1,l}^{(L)} \times T_{m_n,i+1,l}^{(L)} | F_L \right], \]

where by Lemma A.1 of Appendix 2 $(n/m_n)E[(T_{m_n,1,l}^{(L)})^2 | F_L] \rightarrow \kappa^{-2}$. Along with (11) this proves the following claim.

**LEMMA 3.1** Under A1 and A2 the unconditional variance satisfies

\[ \lim_{L \rightarrow \infty} v_{m_n}^2 = \kappa^{-2} + 2 \lim_{L \rightarrow \infty} \frac{1}{m_n} \sum_{l=1}^L \sum_{i=1}^{n_l-1} (n_l - i) \times E \left[ T_{m_n,1,l}^{(L)} \times T_{m_n,i+1,l}^{(L)} | F_L \right]. \]

Remark: If bids are everywhere independent then $E[T_{m_n,i,l}^{(L)} \times T_{m_n,j,l}^{(L)} | F_L] = E[T_{m_n,i,l}^{(L)}] \times E[T_{m_n,j,l}^{(L)} | F_L] = 0$ hence the classic result $\lim_{L \rightarrow \infty} v_{m_n}^2 = \kappa^{-2}$. Synonymously and trivially, if all auctions have one bid $n_l = 1$ with probability one then $\lim_{L \rightarrow \infty} v_{m_n}^2 = \kappa^{-2}$. As long as asymptotically there are infinitely many auctions with more than one bid ($n_l > 1$), the iid asymptotic variance $\kappa^{-2}$ is wrong, and without more information on the nature of bid dependence within auctions $E[T_{m_n,i,l}^{(L)} \times T_{m_n,j,l}^{(L)} | F_L]$ cannot be simplified. From a theoretical point of view this is irrelevant since the blocks have finite dimension, are independent of each other, and $|E[T_{m_n,i,l}^{(L)} \times T_{m_n,j,l}^{(L)} | F_L]| \leq E[(T_{m_n,i,j}^{(L)})^2 | F_L] \leq K \kappa^{-4}((m_n/n)^2)$ hence the conditional covariances are bounded.

4. ASYMPTOTIC VARIANCE ESTIMATOR We now propose an estimator of $v_{m_n}^2$ tailored to our auction DGP a la Lemma 3.1. We must characterize how often a bid displacement $d = |i - j|$ occurs. Let $L_d$ denote the number of auctions with bid displacement $d$. For example, $L_0 = L$ since all auctions used to compute $\kappa_{m_n}^{-1}$ have at least one bid; $L_1 = L$ if all auctions have at least 2 bids, otherwise $L_1 < L$; $L_{\bar{n}} = 0$ since the maximum bid displacement is $\bar{n} - 1$ (first bid to the maximum last bid); $L_{\bar{n}-1} \geq 1$, and $L_{\bar{n}-1} = 1$ if there is a single auction with the maximum bid count $\bar{n}$.
Construct sample covariances across auctions

\[ \hat{c}_{mn}(i) = \frac{1}{L_i} \sum_{l=1}^{L_i} \hat{T}_{mn,1,l} \times \hat{T}_{mn,i+1,l} \text{ if } L_i \geq 2, \text{ } i = 1 \ldots n - 1, \]

\[ = 0 \text{ if } L_i = 1, \]

where \( \hat{T}_{mn,i,l} := \hat{U}_{mn,i,l} - \hat{\kappa}_{mn}^{-1} \hat{I}_{mn,i,l} \), and

\[ \hat{U}_{mn,i,l} := \left( \ln b_{(m+1)}^* / b_{i,l}^* \right) + \frac{m_n}{n} \hat{\kappa}_{mn}^{-1} \text{ and } \hat{I}_{mn,i,l} := I \left( b_l^* < b_{(m+1)}^* \right) - \frac{m_n}{n}. \]

This gives the most direct estimator of the asymptotic variance of the Hill-estimator for our auction data:

\[ \hat{v}_{mn}^2 = \hat{\kappa}_{mn}^{-2} + 2 \frac{1}{m_n} \sum_{l=1}^{L} \sum_{i=1}^{n_l-1} (n_l - i) \times \hat{c}_{mn}(i). \]

Our assumptions on the DGP imply the following condition will be needed for the derivations below.

**A3** Asymptotically every bid displacement occurs infinitely often: \( L_d \to \infty \forall d = 0 \ldots n - 1. \)

**THEOREM 4.1** Under A1-A3 and \( m_n / n^{1/2} \to \infty, \left| \hat{v}_{mn}^2 - v_{mn}^2 \right| \overset{p}{\to} 0. \)

Unfortunately variance estimators like \( \hat{v}_{mn}^2 \) are not guaranteed to be positive, a well known dilemma (e.g. Newey and West (1986)). A kernel estimator has the great advantage of ensuring positivity.

5. **KERNEL VARIANCE ESTIMATOR** Hill (2005) proposes the following non-parametric variance estimator of \( v_{mn}^2, \)

\[ \hat{\sigma}_{mn}^2 = \frac{1}{m_n} \sum_{s,t=1}^{n} k((s-t)/\gamma_n) \hat{U}_{mn,s} \hat{U}_{mn,t}, \]

where \( \hat{U}_{mn,t} := \left[ \ln b_{(m+1)}^* / b_t^* \right] - (m_n / n) \hat{\kappa}_{mn}^{-1}, \) and \( k \) denotes a standard kernel function with bandwidth \( \gamma_n \to \infty. \)

Unlike \( \hat{v}_{mn}^2 \), the kernel estimator \( \hat{\sigma}_{mn}^2 \) does not distinguish between auction blocks. It is not based on the "asymptotic variance" per se, but on the direct small sample serial dependence structure within \( \left( \ln b_{(m+1)}^* / b_t^* \right)^+, \) the stochastic components of \( \hat{\kappa}_{mn}^{-1}. \)

Intuitively a kernel \( k \) is chosen to asymptotically negligibly trim cross products \( \hat{U}_{mn,s} \hat{U}_{mn,t} \) so that \( \hat{\sigma}_{mn}^2 \geq 0 \) with probability one for all \( n \geq 1 \) while retaining consistnecy (Newey and West 1987), and a non-parametric approach allows the analyst to have only a vague idea about cross auction dependence and heterogeniety. In theory this will be helpful if auctions take place sequentially and bidders apply information gleaned from past auctions to present
auctions. By comparison $\hat{\gamma}^2_{m,n} < 0$ has non-negligible probability for any finite sample, and $|\hat{\alpha}^2_{m,n} - v^2_{m,n}| \xrightarrow{p} 0$ only if bids are blockwise independent.

Although a large class of kernels can be considered in the following arguments, including Parzen, Tukey-Hanning, and Quadratic-Spectral, by far the Barlett kernel $k(z) = (1-|z|)_+$ is the most popular in the economics literature. It trims cross products of auctions. By comparison the order statistic $Z_{m,n}$ under cross-auction independence since "distant events" are simply bids that occur in every $\hat{U}_{m,n,t}$ does not affect the limit of $\hat{\alpha}^2_{m,n}$, cf. Hill (2005).

**THEOREM 5.1** Under A1-A4 $\hat{\alpha}^2_{m,n} \geq 0$ with probability one $\forall n \geq 1$, and $|\hat{\alpha}^2_{m,n} - v^2_{m,n}| \xrightarrow{p} 0$.

**Remark 1:** Notice $\gamma_n = o(n)$, $\gamma_n = o(m_n/n^{1/2})$ and $m_n = o(n)$ imply $\gamma_n = o(n^{1/2})$ must hold. The fact that $\gamma_n = o(n)$ ensures $\hat{\alpha}^2_{m,n} \geq 0$, while specifically $\gamma_n = o(n^{1/2})$ promotes consistency by reducing the effects of persistence between distant events. This is trivially satisfied under cross-auction independence since "distant events" are simply bids in different auctions.

**Remark 2:** We require sufficiently many tail observations $m_n/n^{1/2} \to \infty$ to ensure the order statistic $b^*_{(m_n+1)}$ that occurs in every $\hat{U}_{m,n,t}$ does not affect the limit of $\hat{\alpha}^2_{m,n}$, cf. Hill (2005).

6. TEST OF PV AGAINST CV Since $m_n^{1/2} (\hat{\kappa}_{m,n}^{-1} - \kappa^{-1})$ is asymptotically Gaussian under Theorem 2.2, an Asymptotically Most Powerful [AMP] test of PV $H_0 : \kappa = 1/2$ against CV $H_1 : \kappa = 1$ is equivalent to a one-sided test of PV against $H_1 : \kappa > 1/2$. By an extension of the Neyman-Pearson Lemma, cf. Wald (1941) and Karlin and Rubin (1956), a one-sided AMP test of $H_0$ at nominal significance level $\theta \in [0,1]$ characterizes a critical region $(c, \infty)$ satisfying

$$\lim_{L \to \infty} P \left( \frac{\exp \left\{ -0.5 \left( \hat{\kappa}_{m,n}^{-1} - 2 \right)^2 / v^2_{m,n} \right\}}{\exp \left\{ -0.5 \left( \hat{\kappa}_{m,n}^{-1} - 1 \right)^2 / v^2_{m,n} \right\}} \leq c \right) = \theta.$$ 

This easily reduces to

$$\lim_{L \to \infty} P \left( m_n^{1/2} (\hat{\kappa}_{m,n}^{-1} - 2) \leq -Z_\theta \right) = \theta$$

where $Z_\theta$ is the upper $\theta^{th}$-quantile of a standard normal distribution. A $t$-ratio can be constructed with any consistent estimator $\hat{v}^2$ of $v^2_{m,n}$:

$$t_{m,n} = m_n^{1/2} (\hat{\kappa}_{m,n}^{-1} - 2) / \hat{v}.$$ 

By Proposition 1 we know $\kappa = 1/2$ under PV, hence from Theorem 2.2 $t_{m,n} \xrightarrow{d} N(0,1)$. Similarly under CV $\kappa = 1$ hence $|t_{m,n}| \to \infty$ with probability one.

**Remark:** Since this is an asymptotic result and power convergences to one, there is no way to improve upon it within the class of tests based on $\hat{\kappa}_{m,n}^{-1}$. Similarly, since the
small sample distribution of $\hat{\kappa}_{m_n}^{-1}$ is unknown an exact Uniformly Most Powerful test is not available. In fact, simulation evidence reveals for auction samples $L = 1000$ and counts $E[n_l] = 3$ and at least $m_n \geq 50$ empirical power for $t_{m_n}$ is roughly one for one sided tests of PV when the true DGP is CV\(^{14}\).

4 A Monte-Carlo Study

1. Data Generating Processes

We consider a model where bidders may have a common as well as private value components in their valuations of the object. A simple example along these lines is the following model due to Wilson (1998). Suppose the log of bidder $i$'s true valuation $u_i$ is a sum of a common value component $v$ and an idiosyncratic component $a_i$: $u_i = v + a_i$, where $v$ is normally distributed with mean $\mu_v$ and variance $\sigma_v^2$, while $a_i$ is normally distributed with mean 0 and variance $\sigma_a^2$. Generally, the bidders do not observe their valuations, but observe signals $s_i$ that are informative about the valuations: $s_i = u_i + \varepsilon_i$, where the "noise" term $\varepsilon_i$ is also mean zero normally distributed, with variance $\sigma_\varepsilon^2$. This model nests naturally a private values environment within a common values one. If $\sigma_\varepsilon = 0$, then the environment is PV, and the private values are correlated to the extent that $\sigma_a > 0$. (If also $\sigma_v = 0$, then the environment is independent private values, IPV.) But if $\sigma_\varepsilon > 0$, then the true valuations are unobservable, and we have a model with a common value component.

Figure 1 shows numerically computed bidding strategies $B(s)$ and bid densities $g(b)$ for two examples of the above model.\(^{15}\) In the first example (the CV example, on the left panel), we set $\sigma_v = \sigma_a = \sigma_\varepsilon = 0.3$. In the second example (the PV example, on the right panel) we set $\sigma_v = \sigma_a = 0.3$ and $\sigma_\varepsilon = 0$. In both examples, $N = 3$, the mean log valuation $\mu_v = \log 100$ and the reserve price is $80$. A barely noticeable, but important difference between the graphs in the top panel is that under PV, the bidding strategy has zero slope at $s = r$ (the right graph), while it has a positive slope under CV (the left graph). This behavior of the bidding strategy translates into a profoundly different behavior of the density of bids $g(b)$ around the reserve price, illustrating the power of our Proposition 1 (see the graphs on the lower panel). When values are common, $g(b)$ is continuous around $r$. But in the PV case, the density around the reserve price has a "spike". The fact that the slope of $B(s)$ is zero implies that the density is unbounded.

The randomly generated PV and CV samples are $\{b_{i,l}\}$ over bid $i = 1, \ldots, n_l$ with uniformly randomized count $n_l$, $0 \leq n_l \leq 6$ and auction $l = 1, \ldots, L$ for $L = 250,000$. The reserve price is $r_l = 8$. The samples are broken into $R = 250$ subsamples of $L = 1000$ auctions. Thus, each subsample has an average sample size $L \times E[n_l] = 3000$.

For each PV and CV subsample the index $\hat{\kappa}_{m_n}^{-1}$, new variance estimator $\hat{\sigma}_{m_n}^2$ and kernel

\(^{14}\)Theorem 2.2 implies a far more subtle result then required here: asymptotic power is one against any simple alternative in the region $(1/2, \infty)$. Simulation evidence not reported here reveals empirical power is nearly one when $E[n_l] = 3000$ for a test of PV $\kappa^{-1} = 2$ against small deviations from PV like $\kappa^{-1} = 1.9$.

\(^{15}\)A Mathematica notebook used to compute these examples is available at http://artyom239.googlepages.com.
Figure 1: Bidding strategies and bid densities for CV (left panel) and PV (right panel) models
estimator $\hat{\sigma}_{m_n}^2$ are computed over a tail fractile $m_n$ window given below\textsuperscript{16}. The kernel estimator $\hat{\sigma}_{m_n}^2$ is constructed with a Bartlett kernel with bandwidth\textsuperscript{17} $\gamma_n = n^{-2.25}$. Additionally, we compute the variance $\hat{V}_{m_n}$ of $\hat{k}_{m_n}^{-1}$ across the $R = 250$ simulated samples: if the simulation sample of index estimates is $\{\hat{k}_{m_n,i}^{-1}\}_{i=1}^R$ then

$$\hat{V}_{m_n} = \frac{1}{R-1} \sum_{i=1}^R \left( \hat{k}_{m_n,i}^{-1} - \frac{1}{R} \sum_{i=1}^R \hat{k}_{m_n,i}^{-1} \right)^2.$$  

Trivially $E[\hat{V}_{m_n}] = \nu^2$ and $|\hat{V}_{m_n} - \nu^2| \xrightarrow{P} 0$ as $R \to \infty$ since the samples are independently and identically drawn.

Nevertheless the asymptotic 95%-band $1.96 \hat{V}_{m_n}^{1/2} / m_n^{1/2}$ based on the asymptotic Gaussian distribution need not approximate the true small sample 95%-band well. We therefore report the 2.5% and 97.5% quantiles of $\{\hat{k}_{m_n,i}^{-1}\}_{i=1}^R$.

2. Results

Throughout we use $\hat{\nu}$ to denote any $\hat{\nu}_{m_n}^2$, $\hat{\sigma}_{m_n}^2$, and $\hat{V}_{m_n}$. We plot the simulation sample mean of $\hat{k}_{m_n}^{-1}$ over tail fractiles $m_n \in \{10, ..., 200\}$ and the 95% asymptotic confidence bands for each $\hat{\nu}$. We plot separately the confidence band with $\hat{V}_{m_n}$ and the 2.5% and 97.5% quantiles of $\hat{k}_{m_n}^{-2}$. We also plot AMP test p-values and rejection frequencies at the 5% level. See Figures 3 and 4.

The Hill estimator hovers near $\hat{k}_{m_n}^{-1} = 2$ when bids are generating under PV, and near $\hat{k}_{m_n}^{-1} = 1$ under CV, both supporting Proposition 1. Using subsample variation $\hat{V}_{m_n}$ as a benchmark, the most accurate confidence band is derived from $\hat{\nu}_{m_n}^2$. We will see below, however, that use of $\hat{V}_{m_n}$ as a benchmark leads to over rejection of the PV null hypothesis when it is true: a comparatively conservative confidence band like the one generated from the kernel estimator $\hat{\sigma}_{m_n}^2$ is actually preferred. Indeed, the non-parametric $\hat{\sigma}_{m_n}^2$ will ultimately trump both $\hat{\nu}_{m_n}^2$ and the benchmark $\hat{V}_{m_n}$.

Inference in general is quite sharp\textsuperscript{18}. When the true DGP is PV all AMP test p-values for a one-sided test of $H_0 : PV$ are above .13 in all cases for all $m_n$, and in most cases are above .30. Resulting p-values from t-ratios based on the kernel estimator $\hat{\sigma}_{m_n}^2$ and simulation sample variance $\hat{V}_{m_n}$ are nearly identical.

Rejection frequencies of the one-sided test of $H_0 : PV$ at the 5% level tell a far more nuanced story. They are significantly above .05 for all t-ratios at $m_n \geq 25$ except the ratio based on the kernel estimator $\hat{\sigma}_{m_n}^2$. By using asymptotic Gaussian critical values as a decision rule for rejection, the kernel estimator results in the best approximation of the nominal test size .05. Both $\hat{\nu}_{m_n}^2$ and $\hat{V}_{m_n}$ over reject the null, where $\hat{\nu}_{m_n}^2$ is by far the

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\textsuperscript{16}In a few cases $\hat{\nu}_{m_n}^2 < 0$ (roughly .5% of all samples). We do not compute t-ratios based on $\hat{\nu}_{m_n}^2$ in this case.

\textsuperscript{17}Recall $\gamma_n = o(n^{1/2})$ must hold under A4. In this simulation study, and numerous others performed by one author, we consistently find $\gamma_n \sim n^2$ for $\delta \in [2, .25]$ to be superior. If too many or too few cross products are trimmed $\hat{\sigma}_{m_n}^2$ becomes erratic based on base-line comparisons with $\hat{k}_{m_n}^{-2}$ for iid data, and with the simulation variance in general.

\textsuperscript{18}Since we are interested in testing $H_0 : PV$ against $H_1 : CV$ in lieu of the prevailing literature, we focus on the null $H_0 : PV$ in the following. Simulations not reported here reveal tests of $H_0 : CV$ are qualitatively similar.
worst. Over rejection of the null should not be surprising since i. we use the asymptotic Gaussian critical values for the rejection decision; and ii. the p-values are merely composite averages over all simulations while the rejection frequencies tabulate the decision (reject or not) based on each p-value. An average p-value well above .05 (which is good) can be matched with a rejection frequency well over 5% for a test at a 5% nominal level (which is bad). By using asymptotic critical values the only acceptable ratio is the one based on the non-parametric $\hat{\sigma}^2_{mn}$.

When the true DGP is CV then p-values of the test of $H_0 : \text{PV}$ are below .01 in all cases for all $m_n$, and rejection frequencies above .95 in all cases for all $m_n \geq 45$.

RULE OF THUMB There are non-negligible differences between t-ratios based on different variance estimators. Since the analyst does not know if the data are generated by a PV or CV bidding strategy, a rule of thumb for selecting the variance estimator and tail fractile $m_n$ emerges. Based on the above simulation study, as well as extensive simulations not reported here\textsuperscript{19}, the strongest performing t-ratio over all is the one based on Hill’s (2005) $\hat{\sigma}^2_{mn}$ for $m_n \in \{50, 150\}$. The rejection frequency for this ratio in this fractile range under the null of PV is below the nominal 5% when PV is true, and above 95% when CV is true.

5 Empirical Application: BC Timber Auctions

The province of British Columbia (BC) is in possession of a massive forested area. This lucrative natural resource encompasses more than 48 million hectares; an area over four times the size of the United Kingdom. In March 2003, the Ministry of Forests created an independent organization, BC Timber Sales (BCTS). BCTS aims to generate the best possible financial return to the Crown from publicly-owned timber, provide timber harvesting opportunities, and set a credible reference point for the price of timber harvested from Crown land. Since February 29th 2004 stumpage prices in BC have been set using the new “Market Pricing System” (MPS). The underlying principle of the MPS is that the value of timber is evaluated based on auctions of standing timber and this value is then used to set stumpage prices for long term tenures.

An important part in the MPS operations is the system of reserve prices. Technically, the MPS is a “transaction evidence pricing” system and the results of the auction sales are used to determine the reserve price for auctions of other stands of timber. Roise (2005) explains in detail the workings of the MPS. For the purposes of this study, an important aspect of the MPS is that the reserve prices are binding. The firms submit bids in terms of the bonus bids (i.e. markups over the reserve price), and only non-negative bonus bids are allowed.

The format adopted by BCTS is the one of scaled sales. The actual price paid by the winner (the bidder with the highest bid) is equal to the product of the per unit price (the

\textsuperscript{19}These include the above noted tests of $H_0 : \text{CV}$, as well as a study of empirical power. Although in theory the only logical bidding strategies are PV ($\kappa^{-1} = 2$) and CV ($\kappa^{-1} = 1$), we analyzed the power function of the t-ratio for very small deviations from either PV and CV null denoted $H_0 : \kappa^{-1} = \kappa_0^{-1}$, including $\kappa_0^{-1} \pm .1$ and $\kappa_0^{-1} \pm .01$. The AMP t-ratio with the kernel estimator $\hat{\sigma}^2_{mn}$ uniformly trumps all others in all cases, and delivers power near %100 for even small deviations from $\kappa_0^{-1}$.  

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upset rate plus the bonus bid) times the actual volume of timber on the tract. Since a large portion of the uncertainty is eliminated from the price, one should expect a dominant private value component. Indeed, the papers that studied scaled sales in timber auctions often adopted a private values framework. But the issue is not without controversy. For the US Forest Service scaled sales, Baldwin, Marshall and Richard (1995) and Athey and Levin (2001) argue for the presence of a common value component. The reason is the species composition effect. The Forest Service estimates the proportions of different species, but the actual proportions are likely to differ from their estimated values. In the US scaled sales, bidders can insure against this uncertainty by equalizing profit margins for bids on different species. This possibility is absent in BCTS, because bidders are only allowed to submit one "total" bid. Thus the common value component could be even more pronounced.

The BC Ministry of Forests created a database of BCTS auction sales for the use of price determination. The database is publicly available and contains, for each auction sale, the winning bid, the identities of the bidders and also data on the characteristics of the sale (e.g. location, species of trees present, slope etc.). A sample printout of letting information is provided in Appendix 3.

Our sample contains all auctions conducted over the period Jan 14, 2004 to Dec 14, 2006. The sample contains 611 auctions, and a total of 1874 bids. For each auction, the reserve price as well as the bonus bids are available for all bidders. (The reserve prices and bids are quoted per 1m$^3$ of timber.) Theory predicts that, if there is a single potential bidder in the auction and he knows it, he will bid the reserve price $r$. Strictly speaking, our testable restrictions apply to the case of 2 or more potential bidders. In other words, we implicitly condition on there being 2 or more potential bidders. In the sample, there was one instance of a single bidder in the auction who in fact submitted a bid equal to the reserve price. According to the theory, this observation corresponds exactly to the case of a single potential bidder. For the purposes of our test, we eliminated this observation.

Figure 2 shows the histogram of bonus bids divided by the reserve price. The histogram exhibits an overall declining pattern consistent with our findings in the numerical examples (for both models). There is some evidence of bid clustering around the reserve price. If anything, this points to the PV environment.

We follow Paarsch (1992), Paarsch (1997) and Haile, Hong, and Shum (2003), and treat firms symmetrically. Figure 5.1 contains 95% confidence band plots of $\hat{\kappa}_{mn}^{-1}$ over $mn \in \{5, \ldots, 200\}$, and Figure 5.2 contains plots of each t-ratio $t_{mn} = mn^{1/2}(\hat{\kappa}_{mn}^{-1} - 1)/\hat{v}$. There is no evidence from any t-ratio at any $mn$ for PV: p-values are no larger than .004 in all cases. In view of simulation evidence revealing how the various t-ratios perform under a true DGP of PV or CV, the evidence overwhelmingly points to CV.

6 Concluding Remarks

In this paper, we have developed a new "reduced-form" test of common values in first-price auctions and applied it to BC timber sales, an important institution in the economy of British Columbia. The test is based on auction theory, but is reduced-form in the sense that it is based on the properties of bids distribution, which is directly observable. The test exploits the difference in the behavior of bids near the reserve price. More specifically,
the tail index of the bids distribution is equal to one-half under private values, but is equal to one if there is a common-value component in bidders’ valuations. The estimation of the tail index is a well-studied problem in econometrics, where the Hill estimator is by far the most widely-used method. But the available asymptotic results do not cover our setting of imbalanced panels with stochastic dimension. Our econometric contribution is to provide the asymptotic inference framework for the Hill estimator in this setting. These methods are potentially useful in other contexts.

A limitation of our approach is that we consider a somewhat stylized auction model. This model has been adopted by the majority of papers in the literature, but its limitations (no dynamics, symmetry, no resale opportunities etc.) are well recognized. An advantage is simplicity and well-understood theoretical underpinnings. Also, we believe that the difference in the behavior of bids near the reserve price is a fundamental phenomenon, and not simply an artifact of a narrow class of models. As the theoretical literature on auctions with these features matures, we hope our methods can be appropriately generalized.

We apply our test to timber auctions, which have received considerable attention in the literature. But the question of which model, private or common values, is more appropriate, has not been settled. Our results strongly point towards a model with a common value component.
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Appendix 1: Proofs

Proof of Lemma 2.1. A variation on Hsing’s (1991: p.1548) argument suffices. Under A1 and Proposition 1 \( b^*_t \) has a regularly varying conditional tail (7) with index \( \kappa > 0 \). Therefore, by dominated convergence, Fubini’s Theorem, (7) and (8) as \( L \to \infty \)

\[
E \left[ \frac{n}{m_n} \left( \ln \frac{b^*_t}{q_{mn}} \right)_+ \right] = \int_0^\infty P \left( b^*_t < q_{mn} e^{um_n/n} \right) du \\
= \int_0^\infty E \left[ P \left( b^*_t < q_{mn} e^{um_n/n} | F_L \right) \right] du \\
\sim E \left[ P \left( b^*_t < q_{mn} | F_L \right) \right] \int_0^\infty \frac{P \left( b^*_t < q_{mn} e^{um_n/n} | F_L \right)}{P \left( b^*_t < q_{mn} | F_L \right)} du \\
\sim E \left[ \frac{m_n}{n} \int_0^\infty e^{-\alpha um_n/n} du \right] = \kappa^{-1} E \left[ \frac{m_n}{n} \frac{n}{m_n} \right] = \kappa^{-1}.
\]

Proof of Theorem 2.2.

Step 1: Consider approximation (9) and write the conditional bid \( b^*_t(L) := b^*_t | F_L \). We need only show the conditions of Lemma A.7 of Hill (2008) are satisfied. In particular, it suffices if \( b^*_t(L) \) has tail (7), and \( \left\{ b^*_t(L) \right\} \) is \( L^2 \)-Extremal Near Epoch Dependent of arbitrary size on some strong mixing base \( \{ \epsilon_t \} \) with arbitrary size. Condition A1 ensures the regularly varying tail (7) by Proposition 1. Since blockwise dependent data with finite blocks under A2 is trivially strong mixing, \( b^*_t \) is strong mixing with any mixing size (Ibragimov and Linnik (1971)). Simply put \( b^*_t(L) = \epsilon_t \) such that \( b^*_t(L) \) is adapted to \( \mathcal{G}_t := \sigma(\epsilon_T : T \leq t) \). But this means \( \left\{ b^*_t(L) \right\} \) is trivially \( L^2 \)-NED with arbitrary size on itself as a strong mixing with arbitrary mixing size (e.g. Ibragimov and Linnik (1971); Gallant and White (1988)). This proves the conditional asymptotic approximation (9) is valid.

Step 2: In order to prove (10) we will first verify the conditional distribution limit

\[
\frac{m_n^{1/2}}{v_{mn} | L} (\hat{\kappa}_{mn} - \kappa^{-1}) | F_L \overset{d}{\to} N(0, 1) \text{ where } v_{mn}^2 | L = O_p(1) \tag{12}
\]

Limit (12) follows from approximation (9) since under Corollary 3.3 of Hill (2008)

\[
\frac{1}{\sigma_{mn} | L} \frac{1}{m_n^{1/2}} \sum_{t=1}^n T_{mn,t}^{(L)} \overset{d}{\to} N(0, 1), \tag{13}
\]

where \( \sigma_{mn}^2 | L := E(1/m_n^{1/2} \sum_{t=1}^n T_{mn,t}^{(L)})^2 \). The conditions of Corollary 3.3 are easily satisfied since \( b^*_t(L) \) has tail (7) by Proposition 1, and \( \left\{ b^*_t(L) \right\} \) is \( L^2 \)-NED with arbitrary size on a strong mixing base with arbitrary size by Step 1. Notice (9), (12) and (13) imply \( v_{mn}^2 | L / \sigma_{mn}^2 | L \overset{p}{\to} 1 \).
Step 3: Now consider (10). Approximation (9) and limit (13) imply

$$\lim_{L \to \infty} P \left( \frac{m^{1/2}_{n_L}}{v_{m_n}} \left( \hat{\kappa}^{-1}_{m_n} - \kappa^{-1} \right) \leq z | F_L \right)$$

$$= \lim_{L \to \infty} P \left( \frac{1}{\sigma_{m_n|L}} \frac{1}{m^{1/2}_n} \sum_{t=1}^{n} T^{(L)}_{m_n,t} + o_p(1) \leq z | F_L \right) = P(Z \leq z),$$

where $Z$ is an unconditional Gaussian law with zero mean and unit variance. Therefore, by bounded convergence

$$\lim_{L \to \infty} P \left( \frac{m^{1/2}_{n_L}}{v_{m_n}} \left( \hat{\kappa}^{-1}_{m_n} - \kappa^{-1} \right) \leq z \right)$$

$$= \lim_{L \to \infty} E \left[ P \left( \frac{m^{1/2}_{n_L}}{v_{m_n}} \left( \hat{\kappa}^{-1}_{m_n} - \kappa^{-1} \right) \leq z | F_L \right) \right]$$

$$= E \left[ \lim_{L \to \infty} P \left( \frac{m^{1/2}_{n_L}}{v_{m_n}} \left( \hat{\kappa}^{-1}_{m_n} - \kappa^{-1} \right) \leq z | F_L \right) \right]$$

$$= P(Z \leq z).$$

Since $Z$ is a $N(0,1)$-law it follows instantly $v_{m_n}^2 - v_{m_n}^2 \overset{p}{\to} 1$, which completes the proof. □

**Proof of Theorem 4.1.** There is no improvement in generality by allowing for randomized bid counts. In order to reduce notation assume $L_i = L$ for each $i = 1,...,\bar{n} - 1$ (i.e. all auctions have the same bid count $n_i = \bar{n}$). Condition A3 is now trivial. Write

$$c_{m_n}(i) = \frac{1}{L} \sum_{l=1}^{L} T_{m_n,1,l} T_{m_n,i+1,l}. \quad (1)$$

Now use Theorem 2.2 $\hat{\kappa}^{-2}_{m_n} = \kappa^{-2} + O_p(1/m^{1/2}_n)$, Lemma A.2 and $m_n/n^{1/2} \to \infty$ to conclude

$$\hat{\sigma}^2_{m_n} - v^2_{m_n} \leq \frac{1}{m_n} \sum_{l=1}^{L} \sum_{i=1}^{n_i-1} (n_l - i) \times |\hat{c}_{m_n}(i) - c_{m_n}(i)|$$

$$+ \frac{1}{m_n} \sum_{l=1}^{L} \sum_{i=1}^{n_i-1} (n_l - i) \times |c_{m_n}(i) - E[c_{m_n}(i)]| + O_p\left(1/m^{1/2}_n\right)$$

$$= o_p\left(\frac{1}{m_n^2} \sum_{l=1}^{L} \sum_{i=1}^{n_i-1} (n_l - i)\right) = o_p\left(n/m^2_n\right) = o_p(1). \quad (2)$$

□

**Proof of Theorem 5.1.** Under the stated assumptions and the line of proof of Theorem 2.2, all conditions of Hill’s (2005) Theorem 6 hold, hence $|\hat{\sigma}^2_{m_n} - v^2_{m_n}| \overset{p}{\to} 0. \quad \blacksquare$
Appendix 2: Supporting Lemmata

**Lemma A.1** Under A1 and A2 \( (n/m_n)E[(T_{m_{a,i,t}}^{(L)})^2|F_L] \to \kappa^{-2} \).

**Lemma A.2** Under A1 and A2, \( m_n/n^{1/2} \to \infty \), and \( L_i = L \) \( \forall i \), \( \sup_{1 \leq i \leq n} |c_{m_n}(i) - c_{m_n}(i)| = o_p(1/m_n) \) and \( \sup_{1 \leq i \leq n} |c_{m_n}(i) - E[c_{m_n}(i)]| = o_p(1/m_n) \).

**Proof of Lemma A.1.** It can be shown by multiplying out all parts of \( T_{m_{a,i,t}}^{(L)} \), and canceling opposing terms,

\[
E \left[ \left( T_{m_{a,i,t}}^{(L)} \right)^2 |F_L \right] = E \left[ (\ln q_{m_n}/b_i^*)^2 |F_L \right] + E \left[ (\ln q_{m_n}/b_i^*)_+ |F_L \right]^2 \\
+ \kappa^{-2} P \left( b_i^* < q_{m_n} e^{u/m_n^{1/2}} |F_L \right) - \kappa^{-2} P \left( b_i^* < q_{m_n} e^{u/m_n^{1/2}} |F_L \right)^2 \\
- 2\kappa^{-1} E \left[ (\ln q_{m_n}/b_i^*)_+ I \left( b_i^* < q_{m_n} e^{u/m_n^{1/2}} \right) |F_L \right] \\
+ 2\kappa^{-1} E \left[ (\ln q_{m_n}/b_i^*)_+ |F_L \right] \times P \left( b_i^* < q_{m_n} e^{u/m_n^{1/2}} |F_L \right).
\]

Use (7) and (8) to deduce

\[
P \left( b_i^* < q_{m_n} e^{u/m_n^{1/2}} |F_L \right) \sim (m_n/n) e^{\kappa u/m_n^{1/2}} = (m_n/n) \times (1 + o(1)). \quad (14)
\]

Further, by Fubini’s theorem and bounded convergence

\[
E \left[ (\ln q_{m_n}/b_i^*)_+ |F_L \right] = \int_0^\infty P \left( b_i^* < q_{m_n} e^{u/2} |F_L \right) du \\
= P \left( b_i^* < q_{m_n} |F_L \right) \int_0^\infty e^{\kappa u/2} du \sim (m_n/n) 2\kappa^{-2}, \quad (15)
\]

and

\[
E \left[ (\ln q_{m_n}/b_i^*)_+ |F_L \right] = \int_0^\infty P \left( b_i^* < q_{m_n} e^{u} |F_L \right) du \\
\sim P \left( b_i^* < q_{m_n} |F_L \right) \int_0^\infty e^{\kappa u} du \sim (m_n/n) \kappa^{-1}, \quad (16)
\]

and

\[
E \left[ (\ln x_1/q_{m_n}) + I \left( x_1 > q_{m_n} e^{u/m_n^{1/2}} \right) |F_L \right] \\
= E \left[ \int_{u/m_n^{1/2}}^\infty P \left( x_1 > q_{m_n} e^{u} |F_L \right) dv |F_L \right] \\
\sim (m_n/n) \kappa^{-1}. \quad (17)
\]

The claim now follows from (13)-(17): \( (n/m_n)E[(T_{m_{a,i,t}}^{(L)})^2|F_L] \to 2\kappa^{-2} + \kappa^{-2} - 2\kappa^{-2} = \kappa^{-2}. \)
Proof of Lemma A.2. We must show
\[ \hat{c}_{mn}(i) - c_{mn}(i) = \frac{1}{L} \sum_{l=1}^{L} (\hat{T}_{mn,1,l}T_{mn,i+1,l} - T_{mn,1,l}T_{mn,i+1,l}) = o_p(1/m_n) \]
and
\[ \hat{c}_{mn}(i) - c_{mn}(i) = \frac{1}{L} \sum_{l=1}^{L} (T_{mn,1,l}T_{mn,i+1,l} - E[T_{mn,1,l}T_{mn,i+1,l}]) = o_p(1/m_n) \]
where \( o_p(1/m_n) \) does not depend on \( i \). Since \( v_{mn}^2/v_{mn}^2 \rightarrow 1 \) by Theorem 2.2 we implicitly operate conditionally on \( \{n_l\}_{l=1}^{L} \).

Proof of the first and second approximations are simply special cases of Lemmas A.8 and A.9 of Hill (2005) since by blockwise independence \( \{\hat{b}_i^{(L)}\} \) is \( L_2 \)-Extremal NED with arbitrary size on a strong mixing process with arbitrary size (see the proof of Theorem 2.2 above). The remaining conditions hold under A2 and \( m_n/n^{1/2} \rightarrow \infty \). ■
Figure 3.1
Simulated PV Data : $\hat{\kappa}_{mn}^{-1}$

$b(\cdot)$ denotes the 95% confidence band. The top plot uses all variance estimators. The bottom plot uses only the simulation variance $\hat{V}_{mn}$ and 2.5% and 97.5% quantiles of $\hat{\kappa}_{mn}^{-1}$. 

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Figure 3.2
Simulated PV Data: AMP Test p-values

\[ H_0 : PV \kappa = 1/2 \text{ against } \kappa > 1/2 \]

\[
\begin{array}{cccccccccccc}
0.6 & 0.5 & 0.4 & 0.3 & 0.2 & 0.1 & 0.0 & 0.00 & 0.05 & 0.10 & 0.15 & 0.20 & 0.25 & 0.30
\end{array}
\]

\[
\begin{array}{cccccccccccc}
5 & 25 & 45 & 65 & 85 & 105 & 125 & 145 & 165 & 185
\end{array}
\]

Tail Fractile m(n)

\[ p(\text{asym}) \]

\[ p(\text{sim}) \]

\[ p(\text{kern}) \]

\[ p(\text{asym}) \]

Figure 3.3
Simulated PV Data: Test Rejection Frequencies at 5% Level

\[ H_0 : PV \kappa = 1/2 \text{ against } \kappa > 1/2 \]

\[
\begin{array}{cccccccccccc}
0.00 & 0.05 & 0.10 & 0.15 & 0.20 & 0.25 & 0.30
\end{array}
\]

\[
\begin{array}{cccccccccccc}
5 & 25 & 45 & 65 & 85 & 105 & 125 & 145 & 165 & 185
\end{array}
\]

Tail Fractile m(n)

\[ r(\text{asym}) \]

\[ r(\text{sim}) \]

\[ r(\text{kern}) \]

\[ r(\text{asym}) \]

p(\cdot) is the p-value and r(\cdot) is the rejection frequency.
Figure 4.1
Simulated CV Data : $\hat{\kappa}_{mn}^{-1}$

$b(\cdot)$ denotes the 95% confidence band. The left plot uses all variance estimators. The right plot uses only the simulation variance $\hat{V}_{mn}$ and 2.5% and 97.5% quantiles of $\hat{\kappa}_{mn}^{-1}$. 
Figure 4.2
Simulated CV Data: Test p-values
$H_0 : PV \ k = 1/2 \ against \ k > 1/2$

Figure 4.3
Simulated CV Data: Test Rejection Frequencies at 5% Level
$H_0 : PV \ k = 1/2 \ against \ k > 1/2$

$p(\cdot)$ is the p-value and $r(\cdot)$ is the rejection frequency.
Figure 5.1: BC Timber Auction Bids
Tail Index 95% Confidence Bands

Figure 5.2: BC Timber Auction Bids
Test p-value: $H_0: PV k = 1/2$ against $k > 1/2$
Timber Sale Status for A81303

Auction Date Thursday, December 20, 2007

Sec/Cat:.... A  
Location:.... Shesta Lake  
Volume:..... 20,524 /m3  
Upset Rate:.. $18.77 /m3

Auction results have been approved

Status:..... AWARDED  
Client Name: FOREST HILL CONTRACTORS LTD.  
Bonus Bid:... $6.03 /m3  

Client Name: MCKEOWN, COLIN JAMES  
Bonus Bid:... $4.81 /m3

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BC Timber Sales