On The Manipulability of The Fuzzy Social Choice Functions

Fouad Ben Abdelaziz a,∗, 1, José Rui Figueira b, Olfa Meddeb a

a LARODEC, Institut Supérieur de Gestion, 41, Rue de la liberté, Cité Bouchoucha, Le Bardo 2000, Tunisia.

b CEG-IST, Center for Management Studies, Instituto Superior Técnico, Departamento de Engenharia e Gestão, IST - Tagus Park, Av. Cavaco Silva, 2780-990, Porto Salvo, Portugal

Abstract

In the social decision making context, a manipulator attempts to change the social choice in his favor by misrepresenting his preferences. Gibbard and Satterthwaite [7,10] established independently that a strategic manipulation can be observed when using any non-dictatorial voting choice procedure. In this paper, we are interested in the manipulation of the social choice when individuals reveal their preferences as fuzzy binary relations on the set of the alternatives. We address three ways for an individual to obtain the best set of alternatives with respect to his fuzzy preference relation. In each case, we define and prove the manipulability of non-dictatorial fuzzy social choice functions for a fuzzy individual relation domain.

Key words: Fuzzy preference relations, Fuzzy social choice functions, Strategy-proofness.

1 Introduction

In a social decision-making context, voting choice procedures can be used to obtain the social choice from a finite set of alternatives. Each individual

∗ Corresponding author. Tel.: (+961) 01 374 444 extension 3748, fax: (961) 750 214
Email addresses: fb12@aub.edu.lb (Fouad Ben Abdelaziz), figueira@ist.utl.pt (José Rui Figueira), Olfa.Meddeb@isg.rnu.tn (Olfa Meddeb).
1 Visiting at Olayan School of Business, American University of Beirut (AUB) P.O. BOX 11-0236 Riad El Solh, Beirut 1107-2020, Lebanon.

Preprint submitted to Elsevier Science 23 March 2006
expresses his preference as an exact binary relation on the set of alternatives. Then, a preference profile is aggregated into a social choice using a voting choice procedure.

Gibbard and Satterthwaite (henceforth G-S) [7,10] focused their attention on situations where a manipulator knows all individual preference orderings, as well as the used voting choice procedure. When the manipulator expresses his sincere preferences, we will obtain the sincere social choice. If the manipulator prefers another alternative to the sincere choice, he can reveal a non-sincere preference relation in order to obtain such an alternative as a social choice. In such situations, a voting choice procedure is known to be subject to strategic manipulation. G-S prove independently that the strategic manipulation of non-dictatorial voting choice procedures is always possible wherever the model contains at least three alternatives.

In many situations, individual preferences can be represented as fuzzy binary relations on the set of alternatives. The social choice function aggregating these fuzzy preference relations is known in the literature (e.g. [2,5]) as fuzzy social choice functions (FSCF). There exist two types of FSCF decompositions:

- **Aggregation and defuzzification**
  - Apply a fuzzy social welfare function that leads to a social fuzzy relation.
  - Generate from the comprehensive fuzzy relation the best alternative by applying a choice function that leads to a collective choice.

- **Defuzzification and aggregation**
  - Apply a choice function for each individual fuzzy relation to obtain an individual choice set.
  - Aggregate the individual choices using a voting choice procedure.

Our concern, in this paper, is to define the manipulability of an FSCF and to generalize the G-S result to the fuzzy context. An existing attempt to define the manipulability of FSCF is the one by Tang [13]. He assumes a specific kind of fuzzy relations belonging to a specific domain, called non-narrow domain and concludes that the manipulation is possible for any non-dictatorial FSCF.

In this paper, our basic idea is to consider the concept of the best alternative set for an individual with respect to his fuzzy preference relation. We consider here three possible ways to generate a best alternative set. The first corresponds to the Tang’s manipulability. The second is based on a fixed confidence threshold. The best alternative set contains the alternatives with a minimum preference degree over any other alternative at least greater than the fixed threshold. The third best alternative set selects the alternatives with a maximum dominance degree. A dominance degree is defined as the minimum preference degree of an alternative over any other alternative [4]. It follows
that we have two new definitions of fuzzy manipulability, as well as the dictatorship of an $FSCF$. In this paper we propose a generalization of $G-S$ result for three different domains of fuzzy individual preference relations.

Our paper is organized as follows. Section 2 is devoted to elementary definitions related to $FSCF$. Section 3 deals with the different definitions of fuzzy manipulability of an $FSCF$, as well as the dictatorship. Section 4 introduces new results related to the fuzzy manipulability. They can be perceived as the generalization of $G-S$ manipulation theorem. Section 5 is devoted to the interpretation of the strategy-proofness of an $FSCF$.

2 Mathematical background

The focus of the paper is on the context where several individuals have to choose an alternative from a finite set of alternatives. Consider that the preferences of each individual are modelled by using a fuzzy binary relation.

2.1 Elementary concepts

This section is addressed to the basic data of the model, the standard definitions of fuzzy binary relations, and some elementary properties.

2.1.1 Fuzzy relations

Let,

- $X = \{x, x', \ldots, y, y', \ldots, z, z'\}$, denote a finite set of alternatives, with $|X| \geq 3$;
- $N = \{1, 2, \ldots, i, \ldots, n\}$, denote a group of $n$ individuals, with $n \geq 2$.

A fuzzy relation can be introduced to model vagueness or imprecision. Generally, the *imprecision* or *vagueness* is detected when there are some difficulties to express clearly our knowledge (is the turquoise color green or blue?) [6]. In our settings, the vagueness affects the preferences of an individual. It can be considered as a fuzzy set in $X \times X$ with a membership function $R$.

**Definition 1. (fuzzy binary relation)**

A fuzzy binary relation (FBR), over $X$ is a function $R : X \times X \rightarrow [0, 1]$. 
The value of $R(x, y)$ can be considered as the degree to which the crisp weak preference “the alternative $x$ is at least as preferred as the alternative $y$” [14]. Consequently, for each pair of alternatives $x$ and $y$ belonging to $X$, we have a number $R(x, y) \in [0, 1]$ interpreted as the degree of preferences of $x$ over $y$.

2.1.2 Some properties of fuzzy relations

An $FBR$ satisfies

- **Crispness:**
  
  if $R(x, y) \in \{1, 0\}, \forall x, y \in X$.

  This property implies that $R$ corresponds to an exact preference relation.

- **Reflexivity:**
  
  if $R(x, x) = 1, \forall x \in X$.

  This assumption is considered in order to have a weak preference relation. $R(x, y)$ is the degree to which “$x$ is at least as preferred to $y$”.

- **Connectedness:**
  
  if $R(x, y) + R(y, x) \geq 1, \forall x, y \in X$.

  A connected fuzzy relation satisfies the following condition:
  If $R(x, y) \leq 1/2$ then $R(y, x) \geq 1/2$, and for all $x \in X, R(x, x) \geq 1/2$.

- **Type 1 transitivity:**
  
  if $\forall x, y, z \in X, R(x, z) \geq \min\{R(x, y), R(y, z)\}$.

  The type 1 transitivity is known in [4] as the max-min transitivity.

- **Type 2 transitivity:**
  
  if for all $x, y, z \in X$,
  
  $[(R(x, y) \geq R(y, x)) \land (R(y, z) \geq R(z, y))] \Rightarrow R(x, z) \geq \min[R(x, y), R(y, z)]$.

  The type 2 transitivity is known in [11] as the weak max-min transitivity. It should be noted that transitivity of type 1 implies transitivity of type 2[11].
3 The manipulability of fuzzy social choice functions

When a group of \( n \) individuals has to choose an alternative from a set of alternatives \( X \), an FSCF can be used starting with fuzzy individual preference relations.

**Definition 1. (fuzzy social choice function)**

A fuzzy social choice function (FSCF) is a function \( \nu : D^n \rightarrow X \),

\[
R_N = (R_1, R_2, \ldots, R_i, \ldots, R_n) \mapsto x = \nu(R_N).
\]

where, \( D \) is a non-empty set of fuzzy preference relations.

The preference relation \( R_i \) of each individual \( i \in N \), is assumed to be belonging to the set of relations \( D \).

3.1 The circumstances of the manipulation of fuzzy social choice functions

To begin with, we have to look into the necessary circumstances for the manipulation of any procedure in the deterministic G-S’s model. As a matter of fact, each individual has to express his preferences as an exact relation over the alternative set. On the light of his preference relation, the manipulator checks whether there exists an alternative preferred to the sincere outcome social choice. Therefore, he tries to obtain such an alternative by misrepresenting his preference relation. Yet, it should be remarked that the previous assumptions are not always be respected in several real situations. In practice, there can exist several individuals trying to manipulate the procedure independently. It follows that each manipulator can not anticipate the sincere outcome choice social of the procedure.

In the fuzzy context, however, the manipulator can not say if an alternative is preferred to another in an exact way. Let us see if we can find some individual’s interest to modify the sincere social choice. Indeed, the concept of the manipulation is essentially based on how to define the manipulator’s interest.

The key solution is to consider the set of the best alternative(s) for an individual on the basis of his fuzzy preference relation. In this paper we introduce three ways to define such a set as it can be seen as follows.
3.2 The best alternative set with fuzzy preferences

Starting with an individual’s fuzzy relation $R$ over $X$, some possible manners to make an exact choice can be classified as follows (see [4]).

1. The individual considers that the alternative $x$ is better than the alternative $y$, if $R(x, y) \geq R(y, x)$. Thus, for such an individual, the best set of alternatives can be defined as follows.

$$P_1(X, R) = \{x \in X \mid R(x, y) \geq R(y, x), \forall y \in X\}.$$  

It should be remarked that this best set of alternatives is always non-empty whenever the individual relation fulfills connectedness, reflexivity, and transitivity of type 1 [4].

2. In some cases, a confidence threshold $\alpha$ is fixed by the individual. The alternative $x$ is considered better than $y$, for the individual, if $R(x, y) \geq \alpha$. Thus, for such an individual, the best set of alternatives can be defined as follows.

$$P_2(X, R) = \{x \in X \mid R(x, y) \geq \alpha, \forall y \in X\}$$

It should be remarked that this best set of alternatives is always non-empty whenever the individual relation fulfills connectedness, reflexivity, and transitivity of type 2 and $\alpha$ belongs in $]0, 1/2]$ [11].

3. A dominance degree of an alternative $x$ is considered as the minimum value of the preference degree of $x$ over any other alternative. Formally, on the basis of a fuzzy relation $R$, with respect to $X$, the degree of dominance of a fixed alternative $x$, is

$$d(R, X)(x) = \min_{y \in X} \{R(x, y)\}.$$  

Thus, the best set of alternatives can be defined as follows.

$$P_3(X, R) = \{x \in X \mid d(R, X)(x) = \max_{y \in X} \{d(R, X)(y)\}, \forall y \in X\}$$

It should be remarked that this best set of alternatives is always non-empty whenever the individual relation fulfills connectedness, reflexivity, and transitivity of type 1 [4].
3.3 Definitions of the fuzzy social choice function manipulability

Let us now consider the following necessary notation as in [13].

- \( \mathcal{R}_N = (R_1, R_2, \ldots, R_i, \ldots, R_n) \) denotes the sincere preference relations of all individuals.

- \( \mathcal{R}_N | R'_i = (R_1, \ldots, R_{i-1}, R'_i, R_{i+1}, \ldots, R_n) \) denotes the fuzzy preference profile when the individual \( i \) changes his sincere fuzzy relation \( R_i \) by a fuzzy relation \( R'_i \).

- \( \nu : \mathcal{D}^n \rightarrow X \), an \( FSCF \) where \( \mathcal{D} \) is the domain of individual fuzzy relations.

- \( \nu(\mathcal{R}_N) \) is the sincere social choice of the \( FSCF \), \( \nu \), i.e. the outcome of \( \nu \) when all individuals give their sincere fuzzy relations.

- \( \nu(\mathcal{R}_N | R'_i) \) is the outcome of the \( FSCF \), \( \nu \), when the fuzzy preference profile corresponds to \( \mathcal{R}_N | R'_i \).

- \( X(\mathcal{R}_N | R'_i) = \{ \nu(\mathcal{R}_N | R'_i), \forall R'_i \in \mathcal{D} \} \) where, \( \nu \) is an \( FSCF \), represents the set of outcomes of the \( FSCF \), \( \nu \), that can be obtained when the individual \( i \) varies his fuzzy preference relation in the domain \( \mathcal{D} \).

In addition, we consider that the domain of individual fuzzy relations, \( \mathcal{D} \), can correspond to the following sets:

- \( \mathcal{D}_1 \), the set of all fuzzy relations fulfilling connectedness, reflexivity, and transitivity of type 1.
- \( \mathcal{D}_2 \), the set of all fuzzy relations fulfilling connectedness, reflexivity, and transitivity of type 2.
- \( \mathcal{D}_3 \), the set of all fuzzy relations fulfilling connectedness, reflexivity, and transitivity of type 1, i.e., \( \mathcal{D}_3 = \mathcal{D}_1 \).
- \( \mathcal{D}^F_\ell \), all crisp relations in \( \mathcal{D}_\ell \), \( \ell \in \{1, 2, 3\} \).

The basic idea is that the manipulation of an \( FSCF \), \( \nu \), can be made only if the manipulator can change the sincere social choice, \( \nu(\mathcal{R}_N) \), to an alternative in his favor. Such an alternative is the outcome of the \( FSCF \) when he reveals a fuzzy strategic relation \( R'_i \), it is noted by \( \nu(\mathcal{R}_N | R'_i) \). Indeed, when the best set of alternative of the manipulator does not contains the sincere social choice \( \nu(\mathcal{R}_N) \), a sufficient condition for the manipulation of an \( FSCF \) is that there exists an outcome \( \nu(\mathcal{R}_N | R'_i) \) in the best set of alternative contains\(^2\). Therefore, we can introduce three manipulability concepts when the domain \( \mathcal{D}_\ell \) is considered, for \( \ell \in \{1, 2, 3\} \). The choice of \( \mathcal{D}_\ell, \ell \in \{1, 2, 3\} \) is

\(^2\) See Corollary page 9.
made such that the best set of alternatives of each individual is non-empty. In each case, we express a choice way used by the manipulator \(i\) to select the outcome \(\nu(\mathcal{R}_N | R'_i)\) face to the sincere social choice \(\nu(\mathcal{R}_N)\).

**Definition 2. (fuzzy manipulability)**

An FSCF, \(\nu : \mathcal{D}_n^\ell \rightarrow X\) is said to be manipulable in the sense \(\ell, \ell \in \{1, 2, 3\}\) by an individual \(i\) at \(\mathcal{R}_N \in \mathcal{D}_n^\ell\), if there is an \(R'_i \in \mathcal{D}_\ell\) such that the condition \((m-\ell)\) holds.

\[
\begin{align*}
(m-1) & \quad R_i(\nu(\mathcal{R}_N | R'_i), \nu(\mathcal{R}_N)) \geq R_i(\nu(\mathcal{R}_N), \nu(\mathcal{R}_N | R'_i)) \\
(m-2) & \quad \forall \alpha \in [0, 1/2], R_i(\nu(\mathcal{R}_N | R'_i), \nu(\mathcal{R}_N)) \geq \alpha \\
(m-3) & \quad d(R_i, X(\mathcal{R}_N | R'_i))(\nu(\mathcal{R}_N | R'_i)) \geq d(R_i, X(\mathcal{R}_N | R'_i))(\nu(\mathcal{R}_N))
\end{align*}
\]

Let the condition \((m-\ell)\) correspond to the following equation \((m-\ell)\), \(\ell \in \{1, 2, 3\}\)

Each definition of the manipulability allows to introduce the concept of strategy-proofness in the fuzzy context as follows.

**Definition 3. (fuzzy strategy-proofness)**

An FSCF, \(\nu : \mathcal{D}_n^\ell \rightarrow X\) is said to be strategy-proof in the sense \(\ell, \ell \in \{1, 2, 3\}\), if there exists no \(\mathcal{R}_N \in \mathcal{D}_n^\ell\) at which \(\nu\) is manipulable in the sense \(\ell, \ell \in \{1, 2, 3\}\).

Now, we can define the dictatorship of an FSCF in the same way of the G-S impossibility result.

**3.4 Definitions of the fuzzy social choice function dictatorship**

If there exists an individual \(k\) belonging to group \(N\), such that the social choice depends only on his fuzzy preferences, the FSCF, \(\nu\), is said to be dictatorial. In this case, the social choice \(\nu(\mathcal{R}_N)\) is always better than any other alternative \(x\) for the dictator \(k\) having the fuzzy relation \(R_k\). Every time we adopt one definition of the best alternative set to express the previous idea. Thus, relying up on the same reasoning applied to define the manipulability, the dictatorship of an FSCF is also introduced in three possible ways.

**Definition 4. (dictator with fuzzy preferences)**

Let \(\nu : \mathcal{D}_n^\ell \rightarrow X\) be an FSCF. An individual \(k\) is said to be dictator for \(\nu\)
in the sense \( \ell, \ell \in \{1, 2, 3\} \), if for every \( \mathfrak{R}_N \in \mathcal{D}_\ell^n \) such that the condition (d-\( \ell \)), \( \ell \in \{1, 2, 3\} \) is satisfied.

\[(d-1) \quad \forall x \in X, x \neq \nu(\mathfrak{R}_N), R_k(\nu(\mathfrak{R}_N), x) \geq R_k(x, \nu(\mathfrak{R}_N))\]

\[(d-2) \quad \forall x \in X, x \neq \nu(\mathfrak{R}_N), R_k(\nu(\mathfrak{R}_N), x) \geq \alpha\]

\[(d-3) \quad \forall x \in X, x \neq \nu(\mathfrak{R}_N), d(R_k, X)(\nu(\mathfrak{R}_N)) \geq d(R_k, X)(x)\]

Let condition (d-\( \ell \)) correspond to the following equation (d-\( \ell \)), \( \ell \in \{1, 2, 3\} \)

**Definition 5. (dictatorship of FSCF)**

An FSCF, \( \nu : \mathcal{D}_\ell^n \to X \) is said to be dictatorial in the sense \( \ell, \ell \in \{1, 2, 3\} \), if there exists a dictator \( k \) belonging to \( N \) in the sense \( \ell, \ell \in \{1, 2, 3\} \).

**4 Impossibility results**

**4.1 The Manipulability of fuzzy social choice functions**

**Theorem (manipulability of an FSCF)**

If an FSCF, \( \nu : \mathcal{D}_\ell^n \to X \) is strategy-proof in the sense \( \ell \), then it is dictatorial in the sense \( \ell \), \( \ell \in \{1, 2, 3\} \).

Before introducing the proof of this theorem, we have to present the following corollary.

**Corollary**

Let \( \nu : \mathcal{D}_\ell^n \to X \) be an FSCF. If there exists an \( R'_i \in \mathcal{D}_\ell \), such that \( \nu(\mathfrak{R}_N \mid R'_i) \) is in \( P_\ell(R_i, X) \) and \( \nu(\mathfrak{R}_N) \) is not in \( P_\ell(R_i, X) \), then the FSCF, \( \nu \), is manipulable in the sense \( \ell \), \( \ell \in \{1, 2, 3\} \) by the individual \( i \) at \( \mathfrak{R}_N \).

**Proof of corollary**

Consider \( i \) in \( N \) and \( \mathfrak{R}_N \) as a sincere preference profile of group \( N \).

For \( \ell = 1 \), suppose that \( \nu(\mathfrak{R}_N \mid R'_i) \) is in \( P_1(R_i, X) \) and \( \nu(\mathfrak{R}_N) \) is not in \( P_1(R_i, X) \). It follows that \( R_i(\nu(\mathfrak{R}_N \mid R'_i), y) \geq R_i(y, \nu(\mathfrak{R}_N \mid R'_i)), \forall y \in X \). In particular for \( y = \nu(\mathfrak{R}_N) \), we have \( R_i(\nu(\mathfrak{R}_N \mid R'_i), \nu(\mathfrak{R}_N)) \geq R_i(\nu(\mathfrak{R}_N), \nu(\mathfrak{R}_N) \mid R'_i) \).
For \( \ell = 2 \), suppose that \( \nu(\mathcal{R}_N \mid R_i') \) is in \( P_2(R_i, X) \) and \( \nu(\mathcal{R}_N) \) is not in \( P_2(R_i, X) \). It follows that \( R_i(\nu(\mathcal{R}_N \mid R_i'), y) \geq \alpha, \forall y \in X \). In particular for \( y = \nu(\mathcal{R}_N) \), we have \( R_i(\nu(\mathcal{R}_N \mid R_i'), \nu(\mathcal{R}_N)) \geq \alpha \). This is equivalent to the manipulability in the sense 2 of \( \nu \) by \( i \) at \( \mathcal{R}_N \).

For \( \ell = 3 \), suppose that \( \nu(\mathcal{R}_N \mid R'_i) \) is in \( P_3(R_i, X) \) and \( \nu(\mathcal{R}_N) \) is not in \( P_3(R_i, X) \). It follows that \( d(R_i, X)(\nu(\mathcal{R}_N \mid R'_i)) \geq d(R_i, X)(y), \forall y \in X \). In particular for \( y = \nu(\mathcal{R}_N) \), we have \( d(R_i, X)(\nu(\mathcal{R}_N \mid R'_i)) \geq d(R_i, X)(\nu(\mathcal{R}_N)) \). This is equivalent to the manipulability in the sense 3 of \( \nu \) by \( i \) at \( \mathcal{R}_N \).

Here, we introduce the proof of the main theorem with the help of the previous corollary.

**Proof of Theorem**

For all \( \ell \in \{1, 2, 3\} \), consider a strategy-proof \( FSCF, \nu : \mathcal{D}_\ell^n \subseteq \mathcal{D}_\ell^n \rightarrow X \) in the sense \( \ell \). Note that for \( \mathcal{D}_\ell^n \subseteq \mathcal{D}_\ell \), G-S manipulation theorem can be applicable. Hence, there exists an individual, assume individual 1, as the dictator for \( \nu \) restricted to the domain \( (\mathcal{D}_\ell^n)^n \).

For any \( \mathcal{R}_N \in \mathcal{D}_\ell^n \), let \( P_\ell^1 \) denote the best set of alternatives \( P_\ell(X, R_1) \) for the individual 1, i.e.,

\[
\begin{align*}
P_1^1 &= \{ x \in X \mid R_1(x, y) \geq R_1(y, x), \forall y \in X \}. \\
P_2^1 &= \{ x \in X \mid R_1(x, y) \geq \alpha, \forall y \in X \}. \\
P_3^1 &= \{ x \in X \mid d(R_1, X)(x) \geq d(R_1, X)(y), \forall y \in X \}.
\end{align*}
\]

where \( d(R_1, X)(x) = \min_{y \in X} \{ R_1(x, y) \} \).

It should be remarked that \( P_\ell^1 \) is non-empty for \( \ell \in \{1, 2, 3\} \) (see [4] and [11]). Let \( \mathcal{R}'_N \) be in \( (\mathcal{D}_\ell^n)^n \) such that for all \( x \in P_\ell^1 \) and all \( y \in X - P_\ell^1 \):

- if \( \ell = 1 \) or \( 2 \), \( R_1'(x, y) = 1 \) and \( R_1'(y, x) = 1 \), \( (i \neq 1) \).
- if \( \ell = 3 \), \( R_1'(x, y) = 1 \) and \( R_1'(y, z) = 1 \), \( \forall z \in X \), \( (i \neq 1) \).

In addition, we suppose that \( \mathcal{R}'_N \) satisfies the following condition:

- if \( R_i(x, y) = 1 \), then \( R_i(y, x) = 0, \forall i \in N, \forall x \neq y \).
Since $\mathfrak{R}'_N \in (\mathcal{D}^F_\ell)^n$, 1 is dictator. Thus,

$$\forall x \in X, x \neq \nu(\mathfrak{R}'_N), R'_1(\nu(\mathfrak{R}'_N), x) = 1 \Rightarrow \nu(\mathfrak{R}'_N) \in P^1_\ell, \forall \ell \in \{1, 2, 3\}.$$ 

Let $w_0$ be $\nu(\mathfrak{R}_N)$ and $w_k = \nu(\mathfrak{R}_N | R'_1, R'_2, \ldots, R'_i, \ldots, R'_k)$ with $1 \leq k \leq n$.

Let $j$ denote the least $k$ in $\{0, 1, \ldots, i, \ldots, n\}$ such that $w_k \in P^1_\ell$ for $\ell \in \{1, 2, 3\}$.

If $j = 1$, i.e. $w_1 = \nu(\mathfrak{R}_N | R'_1) \in P^1_\ell$ and $w_0 = \nu(\mathfrak{R}_N), w_0 \notin P^1_\ell$. Thus, according to the corollary, $\nu$ is manipulable in the sense $\ell$, by 1 at $\mathfrak{R}_N$.

If $j > 1$, i.e., $w_j = \nu(\mathfrak{R}_N | R'_1, R'_2, \ldots, R'_j) \in P^1_\ell$, and $w_{j-1} = \nu(\mathfrak{R}_N | R'_1, \ldots, R'_i, \ldots, R'_{j-1}), w_{j-1} \notin P^1_\ell$. Thus, for $j$, $R'_j(w_{j-1}, w_j) = 1$. Consequently, the function $\nu$ is manipulable in the sense $\ell$, by $j$ at $(\mathfrak{R}_N | R'_1, \ldots, R'_i, \ldots, R'_j)$.

This leads to conclude that $j$ must be equal to 0. Thus, $w_0 = \nu(\mathfrak{R}_N) \in P^1_\ell$, i.e. the equation $(d-\ell)$ holds. It follows that 1 is also the dictator in the sense $\ell$, for $\nu$.

5 An interpretation of the impossibility results

5.1 The manipulability in the sense 1

It should be remarked that the manipulability and the dictatorship in the sense 1, was introduced by Tang [13]. He also established that any strategy-proof an $FSCF$ in the sense 1, is dictatorial in the sense 1, if it has a non-narrow domain. For more details, see [13]. Our contribution differs from the one of Tang by the consideration of a different domain for individual fuzzy relations.

5.2 The manipulability in the sense 2

Our concern is to verify the following propositions:

(1) If the sincere social choice is in the best set of alternatives of an manipulability in the sense 1

Note that if $\ell = 3, R'_j(w_{j-1}, x) = 1, \forall x \in X$, then $d(R'_j, X)(w_{j-1}) = 1$. 

11
ulator $i$ with a threshold $\alpha$, then it can not be manipulated by such a manipulator.

Indeed, if $\nu(\mathcal{R}_N)$ is in $P_2(X, R_i)$, then $\forall x \in X, R_i(\nu(\mathcal{R}_N), x) \geq \alpha$. Since $R_i$ is in $D_2$, we can deduce that $R_i(x, \nu(\mathcal{R}_N)) \leq 1 - \alpha, \forall x \in X$. This result can not affirm that there exists no alternative $x$ in $X(\mathcal{R}_N | R'_i)$, such that $R_i(x, \nu(\mathcal{R}_N)) \geq \alpha$ since $\alpha$ is supposed to be belonging to $[0, 1/2]$. We can conclude that the first proposition is not always verified.

(2) If a manipulator $i$ can not manipulate in the sense 2 the FSCF at $\mathcal{R}_N$), the sincere social choice is in the best choice of alternatives of $i$ with the confidence threshold $\alpha$.

The hypotheses of this proposition implies that no outcome $\nu(\mathcal{R}_N | R'_i)$ can be considered better than the sincere social choice $\nu(\mathcal{R}_N)$, when $R'_i$ varies in $D_2$. This amounts that any alternative of the set $X(\mathcal{R}_N | R'_i)$ is not better than the sincere social choice $\mathcal{R}_N$, i.e., $\forall x \in X(\mathcal{R}_N | R'_i), R_i(x, \nu(\mathcal{R}_N | R'_i)) < \alpha$.

Since $R_i$ is connected, i.e. $R(x, y) + R(y, x) \geq 1, \forall x, y \in X$, we have $\forall x \in X(\mathcal{R}_N | R'_i), R_i(\nu(\mathcal{R}_N), x) > 1 - \alpha$.

We can concluded that if a manipulator $i$ can not manipulate in the sense 2 the FSCF at $\mathcal{R}_N$), the sincere social choice is in the best choice of alternatives of $i$ with the confidence threshold $(1 - \alpha)$.

(3) The sincere social choice $\nu(\mathcal{R}_N)$ is in the best alternative set for a manipulator $i$ with a confidence threshold $(1 - \alpha)$, then the FSCF, $\nu$, can not be manipulable in the sense 2 by this manipulator.

A negative answer to this question can be given with the help of the following example.

Example

Let $X = \{x, y, z\}$ be the set of alternatives. We are interested in a context where a group of 3 individuals, $N = \{1, 2, 3\}$, has to choose an alternative from $X$. We consider that the preferences of each individual $i$ over $X$ are modeled by using a fuzzy binary relation $R_i$ as in the table 4.

We apply the arithmetic mean function, as an example of aggregation operators, to obtain the collective fuzzy relation $R^*$ (see Table 2). The exact choice of $\nu$, is represented by the best set of alternatives $P_1(X, R^*)$. We obtain $x$ as the sincere social choice.

We suppose that the confidence threshold of the individual $3, \alpha$, is

\footnote{For all tables cited in this example, see page 15.}
equal to 0.45. We have the set $X(\mathfrak{R}_N \mid R'_3) = X$, i.e. when the individual 3 varies his fuzzy relation in $D_2$, he can obtain each alternative from $X$ as an outcome of the FSCF.

In addition, the best alternative set of the individual 3, with the confidence threshold 0.55, $P_2(X, R_3)$, coincides with the set $\{x, y\}$. According to the fuzzy preference relation of the individual 3, the preference degree $R_3(y, x)$ is greater than $\alpha$.

Thus, the individual 3 has the opportunity to manipulate the used FSCF, through the sincere social choice $x$ is in the best alternative set for the individual 3 with a confidence threshold $(1 - \alpha)$. He can declare for example the fuzzy relation $R'_3$ (see Table 3) to obtain the alternative $y$ as the outcome of the FSCF.

5.3 The manipulability in the sense 3

According to the definition of the manipulability in the sense 3, we can deduce that an FSCF is not manipulable in the sense 3, by the individual $i$ at $\mathfrak{R}_N \in D^3_3$, if

$$d(R_i, X(\mathfrak{R}_N \mid R'_i))(\nu(\mathfrak{R}_N \mid R'_i)) < d(R_i, X(\mathfrak{R}_N \mid R'_i))(\nu(\mathfrak{R}_N)), \forall R'_i \in D_3.$$ 

Thus, an FSCF is not manipulable by $i$ at $\mathfrak{R}_N$, if $\forall x \in X(\mathfrak{R}_N \mid R'_i)$,

$$d(R_i, X(\mathfrak{R}_N \mid R'_i))(x) < d(R_i, X(\mathfrak{R}_N \mid R'_i))(\nu(\mathfrak{R}_N)).$$

This is equivalent to have $\nu(\mathfrak{R}_N)$ in $P_3(X(\mathfrak{R}_N \mid R'_i), R_i)$.

6 Conclusion

In the case when individual preferences are expressed as a fuzzy relation, there are many FSCF domains that can be considered. In this paper, we addressed the main domains for fuzzy individual preference relations. We defined the fuzzy manipulability for the considered domains, as well as the fuzzy dictatorship of FSCF. Our results can be stated as generalizations of G-S results to a fuzzy context.

Acknowledgements This work has benefited from the Luso-Tunisian bilateral co-operation (2004-2005). The second author was also supported by the MONET research project (POCTI/GES/37707 /2001) at INESC-Coimbra.
References


<table>
<thead>
<tr>
<th>$R_1$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$R_2$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$R_3$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>1</td>
<td>0.7</td>
<td>0.8</td>
<td>$x$</td>
<td>1</td>
<td>0.5</td>
<td>0.8</td>
<td>$x$</td>
<td>1</td>
<td>0.6</td>
<td>0.7</td>
</tr>
<tr>
<td>$y$</td>
<td>0.4</td>
<td>1</td>
<td>0.6</td>
<td>$y$</td>
<td>0.7</td>
<td>1</td>
<td>0.75</td>
<td>$y$</td>
<td>0.55</td>
<td>1</td>
<td>0.8</td>
</tr>
<tr>
<td>$z$</td>
<td>0.4</td>
<td>0.4</td>
<td>1</td>
<td>$z$</td>
<td>0.6</td>
<td>0.5</td>
<td>1</td>
<td>$z$</td>
<td>0.4</td>
<td>0.4</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1

<table>
<thead>
<tr>
<th>$R^*$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>1</td>
<td>0.6</td>
<td>0.766</td>
</tr>
<tr>
<td>$y$</td>
<td>0.55</td>
<td>1</td>
<td>0.71</td>
</tr>
<tr>
<td>$z$</td>
<td>0.46</td>
<td>0.43</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2

<table>
<thead>
<tr>
<th>$R'_3$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>1</td>
<td>0.6</td>
<td>0.7</td>
</tr>
<tr>
<td>$y$</td>
<td>0.9</td>
<td>1</td>
<td>0.8</td>
</tr>
<tr>
<td>$y$</td>
<td>0.4</td>
<td>0.4</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 3