

# SIGN TESTS FOR DEPENDENT OBSERVATIONS

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## Abstract

New sign tests for testing equality of conditional distributions of two (arbitrary) adapted processes as well as for testing conditionally symmetric martingale-difference assumptions are introduced. The analysis is based on results that demonstrate that randomization over ties in sign tests for equality of conditional distributions of two adapted sequences produces a stream of i.i.d. symmetric Bernoulli random variables. This reduces the problem of evaluating the critical values of the tests to computing the quantiles or moments of Binomial or normal distributions. Similar properties also hold under randomization over zero values of signs of a conditionally symmetric martingale-difference sequence.

*Key words and phrases:* Sign tests, dependence, adapted processes, martingale-difference sequences, conditional symmetry, Bernoulli random variables, exact tests, conservative tests

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# 1 Introduction

The paper proposes new sign tests for testing equality of conditional distributions of two (arbitrary) adapted sequences of random variables (r.v.'s) as well as for testing conditionally symmetric martingale-difference assumptions. The analysis is based on results that demonstrate randomization over ties in sign tests for equality of conditional distributions of two adapted processes produces a stream of i.i.d. symmetric Bernoulli r.v.'s. This reduces the problem of estimating the critical values of the tests to computing the quantiles or moments of Binomial or normal distributions (see Theorem 2.4 and Corollary 2.4). A similar proposition holds for randomization over zero values of three-valued r.v.'s in a conditionally symmetric martingale-difference sequence (Theorem 2.1 and Corollary 2.1). One should point out that the obtained results can also be used to test the hypothesis that the conditional median of a sequence of r.v.'s  $Z_t$  adapted to a filtration  $(\mathfrak{S}_t)$  equals some constant  $\mu \neq 0$ . More precisely, the results can be used to test the null hypothesis that the conditional distributions  $\mathcal{L}(Z_t|\mathfrak{S}_{t-1})$  are symmetric about  $\mu$  or conditional medians  $m_t = \text{med}(Z_t|\mathfrak{S}_{t-1})$  equal to  $\mu$ :  $m_t = \mu$ ) using tests based on  $\text{sign}(X_t - \mu)$ .

The above results allow one to obtain general estimates for the tail probabilities of sums of signs of random variables forming a conditionally symmetric martingale-difference sequence or signs of differences of the components of two adapted sequences. Such estimates give sharp (i.e., attainable either in finite samples or in the limit) bounds for the tail probabilities in terms of (generalized) moments of sums of i.i.d. Bernoulli r.v.'s (or corresponding moments of Binomial distributions) and standard normal r.v.'s (see Corollaries 2.2, 2.3, 2.6 and 2.7). Similar estimates hold as well for expectations of arbitrary functions of the signs that are convex in each of their arguments (Theorem 2.2 and Corollary 2.5).

Importantly, the testing procedures proposed in the paper are applicable under general assumptions of heterogeneity and heavy-tailedness in observations and broad classes of time series, including GARCH and related processes, broadly used in economic and financial applications.

The analysis in this paper is based, in a large part, on general characterization results for two-valued martingale difference sequences and multiplicative forms obtained in [Sharakhmetov and Ibragimov \(2002\)](#) (see also [de la Peña et al., 2006](#)). The characterization results allow one to reduce the study of many problems for three-valued martingales to the case of i.i.d. symmetric Bernoulli r.v.'s and provide the key to the development of sign tests for dependent observa-

tions.

There are many studies focusing on procedures for dealing with ties in sign tests if observations are independent (see [Coakley and Heise, 1996](#), for a review and comparisons of sign tests in the presence of ties). Using the conclusions derived from a size and power study, [Coakley and Heise \(1996\)](#) recommend using the asymptotic uniformly most powerful nonrandomized (ANU) test due to [Putter \(1955\)](#) if ties occur in the sign test. The results obtained by [Putter \(1955\)](#) show that randomization over ties reduces the exact power of the sign test and the asymptotic efficiency of the sign test. It is known, however, that the exact version of the ANU test is conservative for small samples compared to both its randomized conditional version as well as to the ANU (see [Coakley and Heise, 1996](#); [Wittkowski, 1998](#)). The estimates obtained in this paper shed new light on the sign tests' comparisons and suggest that randomization over ties leads, in general, to more conservative unconditional sign tests since it provides bounds for the tail probabilities of signs in terms of generalized moments of i.i.d. Bernoulli r.v.'s. An advantage of randomization over ties or zero observations is that it allows one to use sign tests in the presence of dependence while nonrandomized sign tests can only be used in the case of independent data. In this regard, the results in the paper demonstrate that, in addition to their other appealing properties, sign tests also have the important property of robustness to dependence.

An appealing property of sign tests is that a simple linear transformation of a test statistic based on signs leads to a Binomial distribution, and, thus, its distribution can be computed exactly. This is in contrast to other commonly used test statistics for which the exact distributions are frequently unknown. Even if known, the exact distributions of such test statistics are usually difficult to compute and their critical values have to be obtained by relying on computationally intensive algorithms or Monte-Carlo techniques.

A further important property of sign tests is that they are applicable under minimal assumptions: the tests can be used in the case of a small number of observations and also even if observations under consideration are not identically distributed or do not have finite moments, e.g., generated by heavy-tailed distributions (see the discussion in [Dufour and Hallin, 1993](#), [Dufour and Hallin, 2003](#), [de la Peña and Ibragimov, 2017](#) and references therein). This is important since large sample approximations, e.g., those based on central limit theorems, require special regularity assumptions on the distribution of the observations such as existence of the second or higher moments or identical distributions.

Several papers in the literature have focused on applications of sign tests in econometrics and

time series analysis. [Campbell and Dufour \(1995\)](#) and [Campbell and Dufour \(1997\)](#) provide sign tests and related nonparametric tests of orthogonality and random walks. [So and Shin \(2001\)](#), [So and Shin \(2006\)](#) and [Luger \(2003\)](#) consider several extensions of the approaches proposed in those works, including the robust analysis of unit root models.

One of key advantages of sign tests is their robustness to moment assumptions and the property that they can be employed under heavy-tailedness that is exhibited by many key economic and financial variables, including financial returns and income and wealth distributions (see the review in [Ibragimov, Ibragimov and Walden \(2015\)](#) and references therein). It is also important to note that the sign testing procedures proposed in the paper can be used in the case of conditional heteroscedastic and GARCH processes widely used in modelling the dynamics of key economic and financial indicators such as financial returns and foreign exchange rates. They provide robust tests of symmetry of innovations of such processes and equality of their distributions (see Remark 3.1).

The paper is organized as follows. Section 2 presents the main results of the paper on the distributional properties of sign tests for dependent r.v.'s that provide the key to the development of statistical procedures based on signs of dependent observations. Section 3 describes the new sign tests based on the results obtained in Section 2. The sign tests provide approaches to testing for conditionally symmetric martingale-difference assumptions as well as for testing that conditional distributions of two (arbitrary) adapted sequences are the same. Section 4 concludes and discusses suggestions for further research. Section 5 is an appendix that recalls the relevant results from [Sharakhmetov and Ibragimov \(2002\)](#) (Propositions 5.1 and 5.2) that give the basis for the analysis in the paper.

## 2 Distributions of sign test statistics for dependent observations

This section presents the results on the distributional properties of the sign tests for adapted processes that provide the basis for the development of statistical procedures based on signs of dependent observations in the Section 3.

Let  $(\Omega, \mathfrak{F}, P)$  be a probability space equipped with a filtration  $\mathfrak{F}_0 = (\Omega, \emptyset) \subseteq \mathfrak{F}_1 \subseteq \dots \mathfrak{F}_t \subseteq \dots \subseteq \mathfrak{F}$ .

Let  $Z_t, t = 1, 2, \dots$ , be an  $(\mathfrak{S}_t)$ –conditionally symmetric martingale-difference sequence (mds, so that  $P(Z_t > x | \mathfrak{S}_{t-1}) = P(Z_t < -x | \mathfrak{S}_{t-1}), t = 1, 2, \dots$ , for all  $x > 0$ ; thus, throughout, symmetry refers to symmetry about zero, if not stated otherwise) consisting of r.v.'s each of which takes three values  $\{-a_t, 0, a_t\}$ . Further, let, for  $z \in \mathbf{R}$ ,  $sign(z)$  denote the sign of  $z$  defined by  $sign(z) = 1$ , if  $z > 0$ ,  $sign(z) = -1$ , if  $z < 0$ , and  $sign(0) = 0$ .

Throughout Sections 2 and 3,  $\epsilon_t, t = 1, 2, \dots$ , stand for a sequence of i.i.d. symmetric Bernoulli r.v.'s independent of  $Z_t, t = 1, 2, \dots$ ; in addition to that, in what follows, we denote by  $\mathcal{N}$  the standard normal r.v.

**Theorem 2.1** *The r.v.'s  $\eta_t = sign(Z_t) + \epsilon_t I(Z_t = 0)$  are i.i.d. symmetric Bernoulli r.v.'s.*

Proof. The theorem follows from Proposition 5.1 since, as it is easy to see, the r.v.'s  $(\eta_t)$  form an  $(\mathfrak{S}_t)$ –martingale-difference sequence and each of them takes two values  $-1$  and  $1$ . ■

**Corollary 2.1** *The statistic  $S_n = (\sum_{t=1}^n sign(Z_t) + \epsilon_t I(Z_t = 0) + n)/2$  has Binomial distribution  $Bin(n, 1/2)$  with parameters  $n$  and  $p = 1/2$ .*

Proof. The corollary is an immediate consequence of Theorem 2.1. ■

**Theorem 2.2** *For any function  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  convex in each of its arguments,*

$$Ef(sign(Z_1), sign(Z_2), \dots, sign(Z_n)) \leq Ef(\epsilon_1, \epsilon_2, \dots, \epsilon_n).$$

Proof. The theorem follows from Proposition 5.2 applied to the martingale-difference sequence  $\eta_t = sign(Z_t), t = 1, 2, \dots$ , consisting of r.v.'s each of which takes three values  $\{-1, 0, 1\}$ . ■

**Corollary 2.2** *For any  $x > 0$ ,*

$$P\left(\sum_{t=1}^n sign(Z_t) > x\right) \leq \inf_{0 < c < x} \frac{E \max\left(\sum_{t=1}^n \epsilon_t - c, 0\right)}{(x - c)}. \quad (2.1)$$

Proof. The corollary is an immediate consequence of Markov's inequality and Theorem 2.2 applied to the functions  $f_c(x_1, x_2, \dots, x_n) = \max\left(\sum_{t=1}^n x_t - c, 0\right), 0 < c < x$ . ■

**Remark 2.1** *Similar to Section 2 in Eaton (1974), for a fixed  $x > 0$ , consider the class of functions  $\phi = \phi_x$  satisfying  $\phi(y) = \int_0^y \max(y - u, 0) dF(u), y \geq 0, \phi(y) = 0, y < 0$ , and  $\phi(x) = \int_0^x \max(x - u, 0) dF(u) = 1$ , for a nonnegative bounded nondecreasing function  $F(x)$*

on  $[0, +\infty)$  with  $F(0) = 0$ . As in the proof of Corollary 2.2 we obtain

$$P\left(\sum_{t=1}^n \text{sign}(Z_t) > x\right) \leq E\phi\left(\sum_{t=1}^n \epsilon_t\right) \quad (2.2)$$

for all  $\phi$ . It is not difficult to show, similar to Proposition 4 in Eaton (1974) (see also the discussion following Theorem 5 in de la Peña, Ibragimov and Jordan, 2004, for related optimality results for bounds on the expected payoffs of contingent claims in the binomial model) that bound (2.1) is the best among all estimates (2.2), that is,

$$\inf_{\phi} E\phi\left(\sum_{t=1}^n \epsilon_t\right) = \inf_{0 < c < x} \frac{E \max\left(\sum_{t=1}^n \epsilon_t - c, 0\right)}{(x - c)}.$$

The following result gives sharp bounds for the tail probabilities of the normalized sum of signs of the r.v.'s  $Z_t$  in terms of (generalized) moments of the standard normal r.v.

**Corollary 2.3** For any  $x > 0$ ,

$$P\left(\frac{\sum_{t=1}^n \text{sign}(Z_t)}{\sqrt{n}} > x\right) \leq \inf_{0 < c < x} \frac{E \max\left(\frac{\sum_{t=1}^n \epsilon_t}{\sqrt{n}} - c, 0\right)}{(x - c)} \leq \inf_{0 < c < x} \frac{(E[\max(\mathcal{N} - c, 0)]^3)^{1/3}}{x - c}. \quad (2.3)$$

Proof. Using Markov's inequality and Theorem 2.2 applied to the functions

$$f_c(x_1, x_2, \dots, x_n) = \max\left(\frac{\sum_{t=1}^n x_t}{\sqrt{n}} - c, 0\right),$$

$0 < c < x$ , we get the first estimate in (2.3). From Jensen's inequality we obtain

$$E \max\left(\frac{\sum_{t=1}^n \epsilon_t}{\sqrt{n}} - c, 0\right) \leq \left\{E\left[\max\left(\frac{\sum_{t=1}^n \epsilon_t}{\sqrt{n}} - c, 0\right)\right]^3\right\}^{1/3}, \quad (2.4)$$

$0 < c < x$ . The second bound in (2.3) is a consequence of estimate (2.4) and the inequality

$$E\left[\max\left(\frac{\sum_{t=1}^n \epsilon_t}{\sqrt{n}} - c, 0\right)\right]^3 \leq E[\max(\mathcal{N} - c, 0)]^3 \quad (2.5)$$

for all  $c > 0$  implied by the results in [Eaton \(1974\)](#). ■

Let  $(X_t)$ ,  $t = 1, 2, \dots$ , and  $(Y_t)$ ,  $t = 1, 2, \dots$ , be two  $(\mathfrak{S}_t)$ -adapted sequences of r.v.'s.

The following results provide analogues of Theorem 2.1 and Corollaries 2.1-2.3 that concern the distributional properties of sign tests for equality of conditional distributions of  $(X_t)$  and  $(Y_t)$ . The results follow from Theorem 2.1 and Corollaries 2.1-2.3 applied to the r.v.'s  $Z_t = X_t - Y_t$  that form a conditionally symmetric martingale-difference sequence under the assumption that the conditional distributions of  $(X_t)$  and  $(Y_t)$  are the same.

**Theorem 2.3** *If the conditional (on  $\mathfrak{S}_{t-1}$ ) distributions of  $(X_t)$  and  $(Y_t)$  are the same:*

*$\mathcal{L}(X_t|\mathfrak{S}_{t-1}) = \mathcal{L}(Y_t|\mathfrak{S}_{t-1})$ , then the r.v.'s  $\tilde{\eta}_t = \text{sign}(X_t - Y_t) + \epsilon_t I(X_t = Y_t)$  are i.i.d. symmetric Bernoulli r.v.'s*

**Remark 2.2** *Theorem 2.1 (Theorem 2.3) provides necessary conditions for conditional symmetry of the process  $Z_t$  (resp., equality of conditional distributions of the processes  $X_t$  and  $Y_t$ ). Evidently, these conditions are not sufficient. Indeed, Theorem 2.1 continues to hold for all  $(\mathfrak{S}_t)$ -processes  $(Z_t)$  whose conditional medians  $m_t = \text{med}(Z_t|\mathfrak{S}_{t-1})$  equal to zero:  $m_t = \mu = 0$ . For instance, starting from any  $(\mathfrak{S}_t)$ -adapted processes  $(V_t)$ ,  $t = 1, 2, \dots$ , that is not conditionally symmetric (e.g., from i.i.d. r.v.'s  $V_t$ ,  $t = 1, 2, \dots$ , with an asymmetric distribution), one can define  $Z_t = V_t - \text{med}(V_t|\mathfrak{S}_{t-1})$ . Then the process  $Z_t$  is not conditionally symmetric ( $Z_t$  is a sequence of i.i.d. asymmetric r.v.'s with zero median in the case of i.i.d. r.v.'s  $V_t$  with an asymmetric distribution) but its conditional (and unconditional) medians are zero, and Theorem 2.1 still applies. In a similar way, we get that Theorem 2.3 continues to hold, in particular, in the case where  $X_t = Y_t + Z_t$  (a.s.) with any (possibly not conditionally symmetric)  $(\mathfrak{S}_t)$ -adapted process  $Z_t$  whose conditional median is zero:  $m_t = \text{med}(Z_t|\mathfrak{S}_{t-1}) = \mu = 0$  (for instance,  $Z_t$  can be defined as the process in the above example that is independent of  $Y_t$ ).*

**Corollary 2.4** *If the conditional (on  $\mathfrak{S}_{t-1}$ ) distributions of  $(X_t)$  and  $(Y_t)$  are the same:*

*$\mathcal{L}(X_t|\mathfrak{S}_{t-1}) = \mathcal{L}(Y_t|\mathfrak{S}_{t-1})$ , then the statistic  $\tilde{S}_n = (\sum_{t=1}^n \text{sign}(X_t - Y_t) + \epsilon_t I(X_t = Y_t) + n)/2$  has the Binomial distribution  $\text{Bin}(n, 1/2)$  with parameters  $n$  and  $p = 1/2$ .*

**Corollary 2.5** *If the conditional (on  $\mathfrak{S}_{t-1}$ ) distributions of  $(X_t)$  and  $(Y_t)$  are the same:*

*$\mathcal{L}(X_t|\mathfrak{S}_{t-1}) = \mathcal{L}(Y_t|\mathfrak{S}_{t-1})$ , then, for any function  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  that is convex in each of its arguments,*

$$Ef(\text{sign}(X_1 - Y_1), \text{sign}(X_2 - Y_2), \dots, \text{sign}(X_n - Y_n)) \leq Ef(\epsilon_1, \epsilon_2, \dots, \epsilon_n).$$

**Corollary 2.6** *If the conditional (on  $\mathfrak{S}_{t-1}$ ) distributions of  $(X_t)$  and  $(Y_t)$  are the same:*

$\mathcal{L}(X_t|\mathfrak{S}_{t-1}) = \mathcal{L}(Y_t|\mathfrak{S}_{t-1})$ , then, for any  $x > 0$ ,

$$P\left(\sum_{t=1}^n \text{sign}(X_t - Y_t) > x\right) \leq \inf_{0 < c < x} \frac{E \max\left(\sum_{t=1}^n \epsilon_t - c, 0\right)}{(x - c)}.$$

**Corollary 2.7** *If the conditional (on  $\mathfrak{S}_{t-1}$ ) distributions of  $(X_t)$  and  $(Y_t)$  are the same:  $\mathcal{L}(X_t|\mathfrak{S}_{t-1}) = \mathcal{L}(Y_t|\mathfrak{S}_{t-1})$ , then, for any  $x > 0$ ,*

$$P\left(\frac{\sum_{t=1}^n \text{sign}(X_t - Y_t)}{\sqrt{n}} > x\right) \leq \inf_{0 < c < x} \frac{E \max\left(\frac{\sum_{t=1}^n \epsilon_t}{\sqrt{n}} - c, 0\right)}{(x - c)} \leq \inf_{0 < c < x} \frac{(E[\max(Z - c, 0)]^3)^{1/3}}{x - c}.$$

**Remark 2.3** *Bounds for the tail probabilities of sums of bounded r.v.'s forming a conditionally symmetric martingale-difference sequence implied by the results in the present section provide better estimates than many inequalities implied, in the trinomial setting, by well-known estimates in martingale theory. In particular, from Markov's inequality and Theorem 2.2 applied to the function  $f(x_1, x_2, \dots, x_n) = \exp(h \sum_{i=1}^n u_i x_i)$ ,  $h > 0$ , it follows that the tail probability  $P\left(\sum_{t=1}^n X_t > x\right)$ ,  $x > 0$ , of the sum of r.v.'s  $X_t$  that take three values  $\{-u_t, 0, u_t\}$  is bounded from above by  $\exp(-hx) E \exp\left(h \sum_{t=1}^n u_t \epsilon_t\right)$ ,  $h > 0$ :*

$$P\left(\sum_{t=1}^n X_t > x\right) \leq \inf_{h > 0} \exp(-hx) E \exp\left(h \sum_{t=1}^n u_t \epsilon_t\right). \quad (2.6)$$

From estimate (2.6) it follows that Hoeffding-Azuma inequality for martingale-differences in the above setting

$$P\left(\sum_{t=1}^n X_t > x\right) \leq \exp\left(-\frac{x^2}{2 \sum_{t=1}^n u_t^2}\right) \quad (2.7)$$

is implied by the corresponding bounds on the expectation of exponents of weighted i.i.d. Bernoulli r.v.'s  $E \exp\left(h \sum_{t=1}^n u_t \epsilon_t\right)$  (see [Hoeffding, 1963](#); [Azuma, 1967](#)). More generally, Markov's inequality and Theorem 2.2 imply the following bound for the tail probabilities of three-valued r.v.'s form-



ing a conditionally symmetric martingale-difference sequence with the support on  $\{-u_t, 0, u_t\}$  :

$$P\left(\sum_{t=1}^n X_t > x\right) \leq \inf_{\phi} \frac{\phi\left(\sum_{t=1}^n u_t \epsilon_t\right)}{\phi(x)}, \quad (2.8)$$

where the infimum is taken over convex increasing functions  $\phi : \mathbf{R} \rightarrow \mathbf{R}_+$ . It is easy to see that estimate (2.8) is better than Hoeffding-Azuma inequality (2.7) since the latter follows from choosing a particular (close to optimal)  $h$  in estimates for the right-hand side of (2.6) which is a particular case of (2.8) (see [Hoeffding, 1963](#), and also Remark 2.1 on the optimality of bound (2.2))

### 3 Sign tests under dependence

As follows from the results in the previous section, sign tests for testing the null hypothesis that the conditional distributions of two  $(\mathfrak{F}_t)$ -adapted processes  $(X_t)$  and  $(Y_t)$  are the same:  $\mathcal{L}(X_t|\mathfrak{F}_{t-1}) = \mathcal{L}(Y_t|\mathfrak{F}_{t-1})$  for all  $t$ , or that  $(Z_t)$  is an  $(\mathfrak{F}_t)$ -conditionally symmetric martingale-difference sequence:  $P(Z_t > x|\mathfrak{F}_{t-1}) = P(Z_t < -x|\mathfrak{F}_{t-1})$ ,  $x > 0$ , for all  $t$ , can be based on the procedures described below. As most of the testing procedures in statistics and econometrics, they can be classified as falling into one of the following classes: exact tests, conservative tests or testing procedures based on asymptotic approximations. The exact tests are based on the fact that, according to Corollaries 2.1 and 2.4, the distributions of the transformation of signs in the model is known precisely to be Binomial and thus the statistical inference can be based on critical values for the sum of i.i.d. Bernoulli r.v.'s (the case of exact randomized ER tests below). The asymptotic tests use approximations for the quantiles of the Binomial distribution in terms of the limiting normal distribution (the case of asymptotic randomized AR tests). The conservative testing procedures in the present section are based on sharp estimates for the tail probabilities of sums of dependent signs in the model in terms of sums of i.i.d. Bernoulli or normal r.v.'s implied by Corollaries 2.2, 2.3, 2.6 and 2.7 and corresponding estimates for the critical values of the sign tests for dependent observations in terms of quantiles of the Binomial or Gaussian distributions (Binomial conservative non-randomized BCN and normal conservative non-randomized NCN testing procedures). The classification of the sign tests in the present section as non-randomized or randomized refers, respectively, to whether the inference is based on the original (three-valued) signs  $sign(X_t - Y_t)$  (resp.,  $sign(Z_t)$ ) in the model with dependent observations or the

r.v.'s  $\text{sign}(X_t - Y_t) + \epsilon_t I(X_t = Y_t)$  (resp.,  $\text{sign}(Z_t) + \epsilon_t I(Z_t = 0)$ ) that form, according to the results in the previous section, a sequence of symmetric i.i.d. Bernoulli r.v.'s.

Let  $(X_t)$  and  $(Y_t)$  be two  $(\mathfrak{F}_t)$ -adapted sequences of r.v.'s. The following are statistical procedures for testing the null hypothesis that conditional distributions of components of the processes  $(X_t)$  and  $(Y_t)$  are the same:  $\mathcal{L}(X_t|\mathfrak{F}_{t-1}) = \mathcal{L}(Y_t|\mathfrak{F}_{t-1})$  for all  $t$ .

We describe the tests for the two-sided alternative since this is usually the case of interest in most of the applications.

1. The exact randomized (ER) sign test with the test statistic  $\tilde{S}_n^{(1)} = (\sum_{t=1}^n \text{sign}(X_t - Y_t) + \epsilon_t I(X_t = Y_t) + n)/2$  rejects the null hypothesis  $\mathcal{L}(X_t|\mathfrak{F}_{t-1}) = \mathcal{L}(Y_t|\mathfrak{F}_{t-1})$  for all  $t$  in favor of the (two-sided) hypothesis  $\mathcal{L}(X_t|\mathfrak{F}_{t-1}) \neq \mathcal{L}(Y_t|\mathfrak{F}_{t-1})$  for all  $t$  at the significance level  $\alpha \in (0, 1/2)$ , if  $\tilde{S}_n^{(1)} < B_{\alpha/2}^{(1)}$  or  $\tilde{S}_n^{(1)} > B_{\alpha/2}^{(2)}$  where  $B_{\alpha/2}^{(1)}$  and  $B_{\alpha/2}^{(2)}$  are, respectively, the  $(\alpha/2)$ - and  $(1 - \alpha/2)$ -quantiles of the Binomial distribution  $\text{Bin}(n, 1/2)$ .

2. The asymptotic randomized (AR) sign test with the test statistic  $\tilde{S}_n^{(2)} = (\sum_{t=1}^n \text{sign}(X_t - Y_t) + \epsilon_t I(X_t = Y_t))/\sqrt{n}$  rejects the null hypothesis  $\mathcal{L}(X_t|\mathfrak{F}_{t-1}) = \mathcal{L}(Y_t|\mathfrak{F}_{t-1})$  for all  $t$  in favor of  $\mathcal{L}(X_t|\mathfrak{F}_{t-1}) \neq \mathcal{L}(Y_t|\mathfrak{F}_{t-1})$  for all  $t$  at the significance level  $\alpha \in (0, 1/2)$ , if  $|\tilde{S}_n^{(2)}| > z_{\alpha/2}$ , where  $z_{\alpha/2}$  is the  $(1 - \alpha/2)$ -quantile of the standard normal distribution  $\mathcal{N}(0, 1)$ .

3. The binomial conservative non-randomized (BCN) sign test with the test statistic  $\tilde{S}_n^{(3)} = \sum_{t=1}^n \text{sign}(X_t - Y_t)$  rejects the null hypothesis  $\mathcal{L}(X_t|\mathfrak{F}_{t-1}) = \mathcal{L}(Y_t|\mathfrak{F}_{t-1})$  for all  $t$  in favor of  $\mathcal{L}(X_t|\mathfrak{F}_{t-1}) \neq \mathcal{L}(Y_t|\mathfrak{F}_{t-1})$  for all  $t$  at the significance level  $\alpha \in (0, 1/2)$ , if  $|\tilde{S}_n^{(3)}| > B_{\alpha/2}$ , where  $B_{\alpha/2}$  is such that

$$\inf_{0 < c < B_{\alpha/2}} \frac{E \max\left(\sum_{t=1}^n \epsilon_t - c, 0\right)}{(B_{\alpha/2} - c)} < \alpha/2.$$

4. The normal conservative non-randomized (NCN) sign test with the test statistic  $\tilde{S}_n^{(4)} = \sum_{t=1}^n \text{sign}(X_t - Y_t)$  rejects the null hypothesis  $\mathcal{L}(X_t|\mathfrak{F}_{t-1}) = \mathcal{L}(Y_t|\mathfrak{F}_{t-1})$  for all  $t$  in favor of  $\mathcal{L}(X_t|\mathfrak{F}_{t-1}) \neq \mathcal{L}(Y_t|\mathfrak{F}_{t-1})$  for all  $t$  at the significance level  $\alpha \in (0, 1/2)$ , if  $|\tilde{S}_n^{(4)}| > z_{\alpha/2}$ , where  $z_{\alpha/2}$  is such that

$$\inf_{0 < c < z_{\alpha/2}} \frac{(E[\max(\mathcal{N} - c, 0)]^3)^{1/3}}{z_{\alpha/2} - c} < \alpha/2.$$

Let  $(Z_t)$  be an  $(\mathfrak{F}_t)$ -adapted sequence of r.v.'s. The following are the analogues of the above procedures for testing the null hypothesis that  $(Z_t)$  is an  $(\mathfrak{F}_t)$ -conditionally symmetric martingale-difference sequence, that is,  $P(Z_t > x|\mathfrak{F}_{t-1}) = P(Z_t < -x|\mathfrak{F}_{t-1})$ ,  $x > 0$ , for all  $t$ .

1. The exact randomized (ER) sign test with the test statistic  $S_n^{(1)} = (\sum_{t=1}^n \text{sign}(Z_t) + \epsilon_t I(Z_t = 0) + n)/2$  rejects the null hypothesis  $P(Z_t > x | \mathfrak{S}_{t-1}) = P(Z_t < -x | \mathfrak{S}_{t-1})$ ,  $x > 0$ , for all  $t$  in favor of  $P(Z_t > x | \mathfrak{S}_{t-1}) > P(Z_t < -x | \mathfrak{S}_{t-1})$ ,  $x > 0$ , for all  $t$  at the significance level  $\alpha \in (0, 1/2)$ , if  $S_n > B_\alpha$ , where  $B_\alpha$  is the  $(1 - \alpha)$ -quantile of the Binomial distribution  $\text{Bin}(n, 1/2)$ .

Using the central limit theorem for the statistic  $(\sum_{t=1}^n \text{sign}(Z_t) + \epsilon_t I(Z_t = 0))/\sqrt{n}$ , in the case of large sample sizes  $n$  one can also use the following asymptotic version of the previous testing procedure.

2. The asymptotic randomized (AR) sign test with the test statistic  $S_n^{(2)} = (\sum_{t=1}^n \text{sign}(Z_t) + \epsilon_t I(Z_t = 0))/\sqrt{n}$  rejects the null hypothesis  $P(Z_t > x | \mathfrak{S}_{t-1}) = P(Z_t < -x | \mathfrak{S}_{t-1})$ ,  $x > 0$ , for all  $t$  in favor of  $P(Z_t > x | \mathfrak{S}_{t-1}) > P(Z_t < -x | \mathfrak{S}_{t-1})$ ,  $x > 0$ , for all  $t$  at the significance level  $\alpha \in (0, 1/2)$ , if  $S_n > z_\alpha$ , where  $z_\alpha$  is the  $(1 - \alpha)$ -quantile of the standard normal distribution  $\mathcal{N}(0, 1)$ .

3. The binomial conservative non-randomized (BCN) sign test with the test statistic  $S_n^{(3)} = \sum_{t=1}^n \text{sign}(Z_t)$  rejects the null hypothesis  $P(Z_t > x | \mathfrak{S}_{t-1}) = P(Z_t < -x | \mathfrak{S}_{t-1})$ ,  $x > 0$ , for all  $t$  in favor of  $P(Z_t > x | \mathfrak{S}_{t-1}) > P(Z_t < -x | \mathfrak{S}_{t-1})$ ,  $x > 0$ , for all  $t$  at the significance level  $\alpha \in (0, 1/2)$ , if  $S_n > B_\alpha$ , where  $B_\alpha$  is such that

$$\inf_{0 < c < B_\alpha} \frac{E \max\left(\sum_{t=1}^n \epsilon_t - c, 0\right)}{(B_\alpha - c)} < \alpha.$$

4. The normal conservative non-randomized (NCN) sign test with the test statistic  $S_n^{(4)} = \sum_{t=1}^n \text{sign}(Z_t)$  rejects the null hypothesis  $P(Z_t > x | \mathfrak{S}_{t-1}) = P(Z_t < -x | \mathfrak{S}_{t-1})$ ,  $x > 0$ , for all  $t$  in favor of  $P(Z_t > x | \mathfrak{S}_{t-1}) > P(Z_t < -x | \mathfrak{S}_{t-1})$ ,  $x > 0$ , for all  $t$  at the significance level  $\alpha \in (0, 1/2)$ , if  $S_n > z_\alpha$ , where  $z_\alpha$  is such that

$$\inf_{0 < c < z_\alpha} \frac{(E[\max(\mathcal{N} - c, 0)]^3)^{1/3}}{z_\alpha - c} < \alpha.$$

The analogues of the above tests in the case of the two-sided alternative  $P(Z_t > x | \mathfrak{S}_{t-1}) \neq P(Z_t < -x | \mathfrak{S}_{t-1})$  are completely similar.

**Remark 3.1** *As discussed in the introduction, similar to other inference procedures based on signs, the tests described in this section can be used under minimal assumptions on distributions of observations. The tests are exact and can be applied in the case of a small number of observations; further, they do not require that r.v.'s in consideration (r.v.'s  $Z_t$  or the differences*

$X_t - Y_t$ ) are identically distributed or have finite moments, and, thus, can be used in heterogeneous and heavy-tailed settings commonly observed in economic and financial applications. Randomization over zero values of the process  $Z_t$  or over ties of the processes  $X_t$  and  $Y_t$  allows one to use the testing approaches under dependence and mds assumptions, including conditional heteroscedastic and GARCH processes widely used in modelling the dynamics of key economic and financial indicators such as financial returns and foreign exchange rates.

In particular, the conditional mds assumptions are satisfied for GARCH-type processes  $Z_t = \sigma_t \nu_t$ , where  $\nu_t$  is a sequence of i.i.d. symmetric (e.g., standard Normal or Student's  $t$ -) r.v.'s,  $\sigma_t$  is an  $(\mathfrak{S}_{t-1})$ -adapted sequence of r.v.'s, and  $\mathfrak{S}_t = \sigma(\nu_1, \dots, \nu_t)$ . More generally, the zero conditional median assumptions are satisfied for GARCH processes with zero median innovations  $\nu_t$ . This implies that the sign tests proposed in the paper can be used in the analysis of zero median conditions for innovations of GARCH processes. They can also be used for the analysis of the hypotheses that the innovations of two GARCH processes have the same distributions (e.g., using the tests applied to truncations of the processes dealt with, see the next remark).

**Remark 3.2** . As follows from Remark 2.2, similar to the case of other sign tests, the i.i.d. Bernoulli properties hold not only for (conditionally) symmetric processes  $Z_t$  but under more general assumptions: more precisely, they hold if (conditional) medians of the processes are zero. As indicated in **introduction**, the inference procedures described in this section can thus be used in testing of this more general null hypothesis (the processes constructed in Remark 2.2 using zero-median assumptions also provide examples of alternative hypothesis against which the tests of conditional symmetry or equality of conditional distributions based on signs and randomization have no power and are inconsistent).

One should also note that further inference on conditional symmetry (resp., equality of conditional distributions) of the process  $(Z_t)$  (resp., processes  $(X_t)$  and  $(Y_t)$ ) in consideration can be based on sign tests applied to truncations  $Z_t I(|Z_t| < z)$  ( $(X_t - Y_t) I(|X_t - Y_t| < z)$ ) for different truncation levels  $z > 0$ . Rejection of the null hypothesis of zero (conditional) median for different truncation levels  $z$  for the latter processes would provide further evidence on (conditional) symmetry hypothesis of the original time series  $(Z_t)$ , with similar conclusions for tests of zero (conditional) medians of truncated processes  $(X_t - Y_t) I(|X_t - Y_t| < z)$  and their implications for the hypothesis of equality of (conditional) distributions of the original processes  $X_t$  and  $Y_t$ . Comparisons of (size and power) properties of the sign tests applied to the original processes and those for their truncations depend on the shape of (conditional) distributions of time series  $(Z_t, X_t$  and  $Y_t)$  dealt with, including, e.g., whether asymmetry in (conditional) distributions of  $Z_t$

(resp.,  $X_t - Y_t$ ) is concentrated in the tails or in the center of the distributions. The analysis of theoretical and finite-sample properties of the sign tests for truncations vs. original processes is an interesting problem that is left for further research.

Table 1 illustrates the finite sample properties of testing procedures described in this section. The table provides the numerical results on finite sample size and power properties of the asymptotic randomized (AR) test of conditional symmetry (the calculations imply similar results on finite sample properties of tests of equality of conditional distributions of stochastic processes discussed in the section). In the numerical analysis, we assume that  $Z_t$  is a GARCH-type process discussed in Remark 3.1 with i.i.d. Bernoulli innovations:  $Z_t = \sigma_t \nu_t$ , where  $\nu_t$  is a sequence of i.i.d. Bernoulli r.v.'s such that, for  $a > 0$  and  $p \in [1/2, 1)$ ,  $P(\nu_t = a) = p$  and  $P(\nu_t = -ap/(1-p)) = (1-p)$  (so that  $E(\nu_t) = 0$ , and  $\nu_t$  is symmetric if and only if  $p = 1/2$ );  $\sigma_t$  is an  $(\mathfrak{S}_{t-1})$ -adapted sequence of r.v.'s, and  $\mathfrak{S}_t = \sigma(\nu_1, \dots, \nu_t)$ . Evidently,  $(Z_t)$  is an  $(\mathfrak{S}_t)$ -m.d.s that is conditionally symmetric:  $P(Z_t > x | \mathfrak{S}_{t-1}) = P(Z_t < -x | \mathfrak{S}_{t-1})$ ,  $x > 0$ , if and only if  $p = 1/2$ ). As indicated above, the null hypothesis of conditional symmetry of the process  $(Z_t) : P(Z_t > x | \mathfrak{S}_{t-1}) = P(Z_t < -x | \mathfrak{S}_{t-1})$ ,  $x > 0$ , holds if and only if the distribution of i.i.d. r.v.'s  $\nu_t$  is symmetric:  $p = 1/2$ . The alternative corresponds to  $p > 1/2$  (equivalently, to the property that  $P(Z_t > 0) > P(Z_t < 0)$  or  $P(\epsilon_t > 0) > P(\epsilon_t < 0)$ ).

Table 1. The size and power of the asymptotic randomized (AR, see Section 2) sign test for testing  $P(X_t > x | \mathfrak{S}_{t-1}) = P(X_t < -x | \mathfrak{S}_{t-1})$  against the alternative  $P(X_t > x | \mathfrak{S}_{t-1}) = p > 0.5 > 1 - p = P(X_t < -x | \mathfrak{S}_{t-1})$  ( $N$  denotes the number of observations)

$N$	10	20	30	40	50	60	70	80	90	100	150	200	250	300	500	1000
$p$																
0.5	0.05	0.06	0.05	0.04	0.06	0.05	0.06	0.05	0.06	0.05	0.05	0.05	0.04	0.04	0.05	0.05
0.55	0.10	0.13	0.14	0.21	0.20	0.26	0.32	0.29	0.34	0.38	0.44	0.53	0.60	0.66	0.83	0.97
0.60	0.17	0.25	0.29	0.44	0.45	0.56	0.65	0.64	0.71	0.76	0.86	0.93	0.97	0.98	1.00	1.00
0.65	0.26	0.42	0.51	0.69	0.73	0.83	0.89	0.90	0.94	0.96	0.99	1.00	1.00	1.00	1.00	1.00
0.70	0.38	0.61	0.73	0.88	0.92	0.96	0.98	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.75	0.53	0.79	0.89	0.97	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.80	0.68	0.91	0.97	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.85	0.82	0.98	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.90	0.93	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.95	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

According to the numerical results, the testing procedures have very good size and power properties, even in the case of small samples.

## 4 Conclusion and suggestion for further research

The paper introduced new easily implementable sign tests for testing equality of conditional distributions of two adapted processes and conditionally symmetric martingale-difference assumptions. The testing procedures are based on probabilistic results on randomization over ties and zero values of the processes considered that reduce the analysis to the case of i.i.d. symmetric Bernoulli random variables and thus to computing the quantiles or moments of Binomial or normal distributions.

As discussed in the paper (see Remarks 2.2, 3.1 and 3.2), in the context of the framework considered conditional symmetry about the origin implies the zero median conditions, although the converse is not necessarily true. As indicated in Remark 3.2, further inference on conditional symmetry and equality of conditional distributions can be based on sign tests applied to truncations of processes in consideration. Other important problems that are left for further research concern potential extensions and applications of the proposed and related approaches to testing conditionally symmetric martingale-difference assumptions to the analysis of symmetry about (unknown) constant, equality of (unknown) centers, e.g., means of two adapted processes and the related problem of testing for structural breaks. Numerical results on finite sample properties of testing approaches for general hypotheses of symmetry and equality of distributions dealt with in this paper may serve, in particular, as a benchmark for evaluation of performance of and comparisons with traditional (e.g., approaches based on heteroscedasticity and autocorrelation consistent standard errors) and recently proposed robust asymptotic inference methods for the analysis of parameters in economic and financial models, their equality and structural breaks (see [Ibragimov and Müller, 2010, 2016](#)). Further extensions and simulation studies may also focus on the development of robust non-parametric (sign and rank) tests for orthogonality and random walks (see [Campbell and Dufour, 1995](#)) with randomization over zero values (and ties). The results in the above directions of research and the comparative analysis will be presented elsewhere.

## 5 Appendix. Probabilistic foundations for the analysis

Let  $(a_t)_{t=1}^{\infty}$  and  $(b_t)_{t=1}^{\infty}$  be arbitrary sequences of real numbers such that  $a_t \neq b_t$  for all  $t$ .

The key to the analysis in this paper is provided by Proposition 5.1. This proposition is a consequence of more general results obtained in [Sharakhmetov and Ibragimov \(2002\)](#) that show that r.v.'s taking  $k + 1$  values form a multiplicative system of order  $k$  if and only if they are jointly independent (see also [de la Peña, Ibragimov and Sharakhmetov, 2006](#)).

**Proposition 5.1** *If r.v.'s  $X_t, t = 1, 2, \dots$ , form a martingale-difference sequence with respect to a filtration  $(\mathfrak{S}_t)_t$  and each of them takes two (not necessarily the same for all  $t$ ) values  $\{a_t, b_t\}$ , then they are jointly independent.*

Let  $X_t, t = 1, 2, \dots$ , be an  $(\mathfrak{S}_t)$ -martingale-difference sequence consisting of r.v.'s each of which takes three values  $\{-a_t, 0, a_t\}$ . Denote by  $\epsilon_t, t = 1, 2, \dots$ , a sequence of i.i.d. symmetric Bernoulli r.v.'s independent of  $(X_t)_{t=1}^{\infty}$ . The following proposition provides an upper bound for the expectation of arbitrary convex function of  $X_t$  in terms of the expectation of the same function of the r.v.'s  $\epsilon_t$ .

**Proposition 5.2** *If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a function convex in each of its arguments, then the following inequality holds:*

$$Ef(X_1, \dots, X_n) \leq Ef(a_1\epsilon_1, \dots, a_n\epsilon_n). \quad (5.9)$$

Proof. Let  $\tilde{\mathfrak{S}}_0 = \mathfrak{S}_n$ . For  $t = 1, 2, \dots, n$ , denote by  $\tilde{\mathfrak{S}}_t$  the  $\sigma$ -algebra spanned by the r.v.'s  $X_1, X_2, \dots, X_n, \epsilon_1, \dots, \epsilon_t$ . Further, let, for  $t = 0, 1, \dots, n$ ,  $E_t$  stand for the conditional expectation operator  $E(\cdot | \tilde{\mathfrak{S}}_t)$  and let  $\eta_t, t = 1, \dots, n$ , denote the r.v.'s  $\eta_t = X_t + \epsilon_t I(X_t = 0)$ .

Using conditional Jensen's inequality, we have

$$\begin{aligned} Ef(X_1, X_2, \dots, X_n) &= Ef(X_1 + E_0[\epsilon_1 I(X_1 = 0)], X_2, \dots, X_n) \leq \\ &E[E_0 f(X_1 + \epsilon_1 I(X_1 = 0), X_2, \dots, X_n)] = Ef(\eta_1, X_2, \dots, X_n). \end{aligned} \quad (5.10)$$

Similarly, for  $t = 2, \dots, n$ ,

$$\begin{aligned}
& Ef(\eta_1, \eta_2, \dots, \eta_{t-1}, X_t, X_{t+1}, \dots, X_n) = \\
& Ef(\eta_1, \eta_2, \dots, \eta_{t-1}, X_t + E_{t-1}[\epsilon_t I(X_t = 0)], X_{t+1}, \dots, X_n) \leq \\
& E[E_{t-1}f(\eta_1, \eta_2, \dots, \eta_{t-1}, X_t + \epsilon_t I(X_t = 0), X_{t+1}, \dots, X_n)] = \\
& Ef(\eta_1, \eta_2, \dots, \eta_{t-1}, \eta_t, X_{t+1}, \dots, X_n).
\end{aligned} \tag{5.11}$$

From equations (5.10) and (5.11) by induction it follows that

$$Ef(X_1, X_2, \dots, X_n) \leq Ef(\eta_1, \eta_2, \dots, \eta_n). \tag{5.12}$$

It is easy to see that the r.v.'s  $\eta_t, t = 1, 2, \dots, n$ , form a martingale-difference sequence with respect to the sequence of  $\sigma$ -algebras  $\tilde{\mathfrak{S}}_0 \subseteq \tilde{\mathfrak{S}}_1 \subseteq \dots \subseteq \tilde{\mathfrak{S}}_t \subseteq \dots$ , and each of them takes two values  $\{-a_t, a_t\}$ . Therefore, from Proposition 5.1 we get that  $\eta_t, t = 1, 2, \dots, n$ , are jointly independent and, therefore, the random vector  $(\eta_1, \eta_2, \dots, \eta_n)$  has the same distribution as  $(a_1\epsilon_1, a_2\epsilon_2, \dots, a_n\epsilon_n)$ . This and (5.12) implies estimate (5.9). ■

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