Granger causality testing in mixed-frequency VARs with possibly (co)integrated processes*

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Abstract

We analyze Granger causality testing in mixed-frequency VARs with possibly (co)integrated time series. It is well known that conducting inference on a set of parameters is dependent on knowing the correct (co)integration order of the processes involved. Corresponding tests are, however, known to often suffer from size distortions and/or a loss of power. Our approach, which boils down to the mixed-frequency analogue of the one by Toda and Yamamoto (1995) or Dolado and Lütkepohl (1996), works for variables that are stationary, integrated of an arbitrary order, or cointegrated. As it only requires the estimation of a mixed-frequency VAR in levels with appropriately adjusted lag length, after which Granger causality tests can be conducted using simple standard Wald tests, it is of great practical appeal. We show that the presence of non-stationary and trivially cointegrated high-frequency regressors (Götz et al., 2013) leads to standard distributions when testing for causality on a subset of parameters, sometimes even without any need to augment the VAR order. Monte Carlo simulations and two applications involving the oil price and consumer prices as well as GDP and industrial production in Germany illustrate our approach.

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1 Introduction

"The world is mixed-frequent" a young researcher said when presenting his paper on forecasting with a mixed-frequency (MF) time series model.[1] It not only shows that MF models constitute a popular and widely studied topic in time series econometrics, it is simply an omnipresent fact applied and theoretical researchers need to deal with, and they do: by now, it has become standard to properly account for the mismatch in publication frequencies among (macroeconomic) time series, instead of aggregating high-frequency (HF) observations using predetermined aggregation schemes (Silvestrini and Veredas, 2008). The set of MF models ranges from single-regression models (e.g., the, by this time, routinely used MIDAS model; see Ghysels et al., 2004 or Ghysels et al., 2007 and, for the unrestricted version, Foroni et al., 2015b) over factor models (see Mariano and Murasawa, 2003, Marcellino and Schumacher, 2010 and Blasques et al., 2016) to vector autoregressive (VAR) models (see, most notably, Ghysels, 2016, Schorfheide and Song, 2015 and Chiu et al., 2011).[2]

Particularly the latter model class, MF-VAR models, has received a lot of attention recently, predominantly in two related fields of application: forecasting (Schorfheide and Song, 2015 and Götz and Hauzenberger, 2018, among others) and Granger causality testing (Ghysels et al., 2016, Götz et al., 2016 and Ghysels et al., 2017). Both topics are of immense interest to practitioners at, e.g., central banks, who routinely forecast key variables like the gross domestic product (GDP) using a variety of, usually higher-frequent, indicators, or investigate causal patterns between the time series they monitor. We will focus on Granger causality (GC) testing, generally introduced in Granger (1969), whereby both concepts are obviously related when defining GC as predictability of one series conditional on past observations of the other (Dufour and Renault, 1998). Under

[1]Unfortunately, the name of said researcher as well as the conference he presented at slipped the authors' memories; as if choosing the adjective "young" was not enough for proving that one grew older.

[2]There exists a multitude of sub-variants of each model class, e.g., hybrid versions of MIDAS LASSO (Siliverstovs, 2017), Markov-switching MF models (Foroni et al., 2015a) and MF-VARs with time-varying parameters and stochastic volatility (Götz and Hauzenberger, 2018 or Cimadomo and D'Agostino, 2016), among many others.

this definition, GC testing in fact boils down to testing a set of zero restrictions on the parameters of the respective system. The three papers mentioned above cover different aspects of such GC testing within a MF-VAR: Ghysels et al. (2016) discuss the general theory of associated hypothesis tests in detail, which – while asymptotically valid – suffer from size distortions and a loss of power in case the number of HF observations is large relative the low-frequency (LF) period. Götz et al. (2016) and Ghysels et al. (2017) then introduce various ways to overcome these implications of the *curse of dimensionality*. Yet, all three papers have one assumption in common, i.e., they remain in a stationary time series environment after properly transforming the series; in this paper, however, we allow the variables to be integrated or cointegrated.

If we knew the order of integration of the variables under consideration as well as whether the series are cointegrated, we could transform an initial MF-VAR in levels to a model in differences or to an error correction model (ECM). Afterwards, we could test for GC using the methods of Ghysels et al. (2016) or Götz et al. (2016) and Ghysels et al. (2017), depending on the frequency mismatch. In practice, though, we usually do not know the precise (co)integration order, and appropriate tests are required beforehand; tests that – in the case of unit roots – tend to have rather low power or that – in the case of cointegration – may suffer from severe size distortions (Ghysels and Miller, 2015). Instead of testing for GC in a system, that is a-priori transformed based on the outcomes of more or less error-prone pre-tests, we aim for a methodology that allows estimation of the MF-VAR in levels and leads to valid and standard inference procedures.

In this paper, we thus extend the (somewhat classical) methodology of Toda and Yamamoto (1995), Sims et al. (1990), Dolado and Lütkepohl (1996), Toda and Phillips (1993, 1994) as well as Lütkepohl and Reimers (1992) among others to the MF case. Starting from a MF-VAR in levels, in which the series may be stationary, integrated of an arbitrary order, or cointegrated, we propose – like in the common-frequency counterparts – not to take differences or rewrite the model into an ECM format. In order to conduct asymptotically valid inference on the subset of coefficients determining the presence or absence of GC (in the sense of Dufour and Renault, 1998), we could instead apply a suitable adjustment of the lag length (see Toda and Yamamoto, 1995 or Dolado and Lütkepohl, 1996). The price one has to pay for intentionally over-fitting the model is inefficiency due to a loss of power. In the MF case, though, we show that for the stacked, observation-driven MF-VAR system of Ghysels (2016),[3] the necessary adjustment is small at worst and in some cases even entirely superfluous. This causes the corresponding inefficiencies to be lower than in the common-frequency case or to be absent altogether.

To be more precise, and to highlight the most important consequence of our approach, consider testing for GC from a high- to a low-frequency variable, i.e., the arguably more interesting case in terms of nowcasting a LF series (e.g., quarterly GDP) using HF indicators with eventual leading properties (e.g., monthly surveys). Using our simple methodology one can apply a standard Wald test on a MF-VAR estimated in levels without any need to adjust the lag length, irrespective of the (co)integration order of the series involved. Key for this finding is the presence of "trivial" cointegrating relationships among the stacked HF series (Götz et al., 2013) and a suitable application of Theorem 1 of Toda and Phillips (1993). With respect to testing the reverse direction, only a small adjustment of the system suffices to rely on an asymptotic χ^2-distribution.

The rest of the paper is organized as follows. In Section 2 we describe the model, introduce GC testing within the MF-VAR and outline the different augmentations ensuring valid inference in levels, irrespective of the order of (co)integration. We present the theoretical background for our approach and confirm our findings using Monte Carlo simulations in Section 3. An empirical analysis in Section 4 illustrates our approach for German data involving the consumer price index and the oil price as well as GDP and industrial production. Section 5 concludes.

[3]We leave an analysis of the same research question for parameter-driven MF-VAR models à la Schorfheide and Song (2015) for future research. A discussion on these alternative model specifications follows below.

2 GC Testing in MF-VARs

2.1 The Model

Let us assume a two-variable MF system, where y_t represents the LF variable running from $t = 1, \ldots, T$. The HF variable x appears m times as often, implying $m = 3$ for a month/quarter- or $m = 4$ for a quarter/year-example. We write $x^{(m)}_{t-i/m}$ for a specific HF observation, whereby $i = m - 1$ ($i = 0$) represents the first (last) HF observation within LF period t. The LF and HF lag and difference operators are denoted L^i and Δ^i as well as $L^{j/m}$ and $\Delta^{j/m}$, respectively. Hence, $\Delta^i y_t = y_t - L^i y_t = y_t - y_{t-i}$ and $\Delta^{j/m} x_t = x_t - L^{j/m} x_t = x_t - x_{t-j/m}$. Also, $L^{1/m} x_{t-(m-1)/m} = x_{t-1}$. For this rather standard notation in the MF literature, see also Clements and Galvão (2008) or Miller (2014). Unless stated differently, the ordinary least squares (OLS) estimator of a generic process Υ is labelled $\hat{\Upsilon}$. Finally, $I(d)$ represents an integrated process of integer order d, *vec* the operator stacking the columns of a matrix, $\otimes$ the Kronecker product, I_k the identity matrix of dimension k, and $\mathbf{0}_{i \times j}$ an $(i \times j)$-matrix of zeros.

Remark 1 *In principle, we could allow for higher dimensional multivariate systems by, e.g., considering n_l LF and n_h HF series, where the HF series may have different sampling frequencies $m_j, j = 1, \ldots, n_h$. Firstly, though, analyzing GC in a system with more than two variables opens the door for multi-horizon causality and thus to causal chains (see, e.g., Lütkepohl, 1993). Secondly, such an extension would complicate the notation and illustration of results.*

In this paper we lean on the observation-driven MF-VAR of Ghysels (2016). It is constructed by first stacking the HF observations corresponding to one t-period together with the observation for y yielding $Z_t = (y_t, X^{(m)\prime}_t)'$, where $X^{(m)}_t = (x^{(m)}_t, x^{(m)}_{t-1/m}, \ldots, x^{(m)}_{t-(m-1)/m})'$. Then, a dynamic structural equations model for Z can be written as

$$A_c Z_t = A^*_1 Z_{t-1} + \ldots + A^*_p Z_{t-p} + u^*_t \tag{1}$$

for $t = p+1, \dots, T$. Note that coefficients in A_c govern the evolution within the HF process x as well as so-called *nowcasting causality* (Götz and Hecq, 2014). u_t^* is an independently and identically distributed (i.i.d.) error term with $\mathbb{E}(u_t^*) = \mathbf{0}_{(m+1)\times 1}$, $\mathbb{E}(u_t^* u_t^{*\prime}) = \Sigma_{u^*}$, the latter being a diagonal matrix of dimension $m+1$ with σ_y^2 (the variance of y) as the $(1,1)$-element and σ_x^2 (the variance of x) on the remainder of the diagonal. After pre-multiplying (1) by A_c^{-1} we obtain the reduced-form MF-VAR(p):

$$Z_t = A_1 Z_{t-1} + \dots + A_p Z_{t-p} + u_t \tag{2}$$

with $A_j = A_c^{-1} A_j^*$ for $j = 1, \dots, p$, and with $u_t = A_c^{-1} u_t^*$. Summarizing all lags of Z_t up until p in $\underline{Z}_{t-p} = (Z_{t-1}', \dots, Z_{t-p}')'$ and using $A = (A_1, \dots, A_p)$, we may write the model more compactly as

$$Z_t = A\underline{Z}_{t-p} + u_t. \tag{3}$$

Additionally stacking the observations over t by writing $Z = (Z_{p+1}, \dots, Z_T)$, $\underline{Z}_p = (\underline{Z}_1, \dots, \underline{Z}_{T-p})$ and $u = (u_{p+1}, \dots, u_T)$ yields

$$Z = A\underline{Z}_p + u. \tag{4}$$

Note that we exclude an intercept, mostly for ease of notation.[4] Let us make the following assumptions:

Assumption 1 *For the MF-VAR(p) in (2), (3) or (4), y_t and $X_t^{(m)}$ are both $I(d)$. This implies the presence of $m-1$ trivial cointegrating relationships among the HF series (Götz et al., 2013); additional cointegration between both series, however, may or may not be present; u_t is an i.i.d. sequence of $(m+1)$-dimensional random vectors with $\mathbb{E}(u_t) = \mathbf{0}_{(m+1)\times 1}$, $\mathbb{E}(u_t u_t') = \Sigma_u$, where $\Sigma_u > 0$ such that $\mathbb{E}|u_{jt}|^{2+\delta} < \infty$ for some $\delta > 0$.*

Remark 2 *As in Toda and Yamamoto (1995) and Dolado and Lütkepohl (1996), we*

[4] One may think of the processes to be demeaned a-priori. As in Dolado and Lütkepohl (1996), the theory remains valid if deterministic terms of any sort are present.

initially assume the lag order p to be a-priori known or estimated via some standard selection criterion. Indeed, one may expect biases resulting from such pre-tests to affect all approaches more or less equally.

Remark 3 *As far as the difference in frequencies between y and x, captured by the parameter m, is concerned, we primarily focus on rather small values, i.e., $m \leq 4$. Firstly, the corresponding cases are usually of more interest in typical macroeconomic applications (see, e.g., Section 4). Secondly, the size of the MF-VAR grows rapidly with m. Consequently, inefficiencies resulting from the inherent curse of dimensionality will most likely dominate any effects from testing for GC using one or the other approach. Thirdly, small values of m allow us to rely on standard asymptotic theory for the Wald test (Ghysels et al., 2016), even in the benchmark case.*[5]

Remark 4 *Assumption 1 implies that we specify our model "truly" in mixed frequencies. Alternatively, one could base the analysis on a model, in which both processes are initially sampled at the high frequency, yet one of them (y) is only observed on a lower one. Such parameter-driven models (as the ones in Ghysels and Miller, 2015, Zadrozny, 2016 or Koelbl and Deistler, 2018 among others) inherently contain missing observations and are thus often represented in a state space format. The latent observations are then "filled" in the estimation process, for example using the Kalman filter. While these models certainly have their merits, they do not – as we outline in more detail later – lend themselves as easily to a (co)integration-order-robust way of GC testing as the approach we have in mind. Hence, we stick to the observation-driven MF-VAR model in (2), (3) or (4).*

Whether the underlying data generating process (DGP) evolves at mixed or high frequency, though, is of course a different question. One could even think of a situation, in which both series are generated continuously and where the observable MF data emerge as aggregations (or discretizations in the sense of Geweke, 1978) of this process, whereby

[5] One could – as mentioned above – rely on the approaches of Götz et al. (2016) or Ghysels et al. (2017) in case m is comparably large. However, the performance of these tests in a situation, where the (co)integration order of the variables is unknown, is unclear and should be inspected first.

they occur in different time intervals.[6] *Surely, the model underlying our analysis here is tailored more to a DGP operating at mixed frequencies. To get an idea of our method's performance in case the data are generated at HF, we also consider such a scenario as a robustness check (see Section 3.3).*

Before discussing the various approaches to test for GC, let us provide a formal definition of the otherwise potentially ambiguous notion of *causality*. Let Ω_t represent the information set available at moment t. The same information set excluding information about a generic process Υ is denoted by Ω_t^{Υ}. The best linear forecast of y in period $t+h$ based on the information set Ω_t is represented by $P[y_{t+h}|\Omega_t]$. Finally, recall that $X_t^{(m)}$ collects all HF observations corresponding to period t. Then, we may define GC (or, in fact, its absence) as follows (Dufour and Renault, 1998):

Definition 1 *Process x does not Granger cause process y if $P[y_{t+1}|\Omega_t^x] = P[y_{t+1}|\Omega_t]$. Similarly, y does not Granger cause x if $P[X_{t+1}^{(m)}|\Omega_t^y] = P[X_{t+1}^{(m)}|\Omega_t]$.*

In other words, y does not Granger cause x if past information of the LF variable do not help in predicting current (or future) values of the HF variable and vice versa.

2.2 Benchmark Approaches

Using the definition just introduced, testing for Granger non-causality in the MF-VAR above boils down to testing a set of zero restrictions on the coefficient matrices $A = (A_1, \ldots, A_p)$. In particular, it implies the following null hypotheses:

$$H_0^{HF \nrightarrow LF} : A_i^{(1,2)} = A_i^{(1,3)} = \ldots = A_i^{(1,m+1)} = 0 \ \forall i = 1, \ldots, p, \tag{5}$$

$$H_0^{LF \nrightarrow HF} : A_i^{(2,1)} = A_i^{(3,1)} = \ldots = A_i^{(m+1,1)} = 0 \ \forall i = 1, \ldots, p, \tag{6}$$

where $A_i^{(r,c)}$ denotes the (r, c)-element of matrix A_i.[7] Of course, if we knew the process Z to be $I(0)$, we could just estimate the model in levels and apply a standard Wald test:

[6] See also Ghysels et al., 2004 for a discussion on this issue in the context of MIDAS models.

[7] The alternative hypotheses, of course, imply that at least one of the respective coefficients is non-zero.

for $a_p = vec(A_1, \ldots, A_p) = vec(A)$ and a suitably constructed matrix R, we can rewrite both null hypotheses as

$$H_0 : Ra_p = \mathbf{0}_{mp\times 1}.$$

Computing the corresponding OLS residuals $\hat{u}_t$ and recalling the definition for u in (4), we may obtain a consistent estimator of Σ_u as $\hat{\Sigma}_u = (\hat{u}\hat{u}')/(T-p)$. Using, once again from (4), the stacked (over lags and observations) regressor matrix $\underline{Z}_p$ to get the usual consistent estimator of the asymptotic variance of the OLS estimator, we can compute the sample analogue of the Wald statistics as:[8]

$$W = (R\hat{a}_p)'[R((\underline{Z}_p\underline{Z}_p')^{-1} \otimes \hat{\Sigma}_u)R']^{-1}(R\hat{a}_p).$$

For a stationary MF-VAR, Ghysels et al. (2016) show W to be asymptotically $\chi^2(mp)$-distributed. We refer to testing for GC in this way as "standard test".

In case we knew y_t and $X_t^{(m)}$ to be $I(d)$, we could achieve stationarity of the MF-VAR by differencing d times. Here, however, an additional ambiguity is added to the situation due to the presence of mixed frequencies: while LF differences are surely being applied to y, one could either use Δ or $\Delta^{1/m}$ for x. Indeed, both transformations applied d times yield a stationary process, yet have consequences on the dynamics of the system and thereby on conducting inference. Somewhat similarly, in the additional presence of cointegration between y and x we can follow the lines of either Götz et al. (2013) or Ghysels and Miller (2015), who derived alternative specifications of a MF-VECM.[9] Again, which specification is chosen in the end has implications for the construction of GC tests and may affect their performance, especially in finite samples. At the very least, it affects the way in which trivial cointegrating relationships among the HF series themselves enter the model (Götz

[8]For general versions of these standard derivations, see one of the many textbooks, e.g., Lütkepohl (2007) or Greene (2018).

[9]For the single-regression counterpart, Miller (2014) and Götz et al. (2014) developed MF-ECMs, whereas Miller (2016) focused on efficient estimation of a cointegrating vector in a MF scenario.

et al., 2013).

Obviously, the (co)integration orders of the series are not known a-priori and a battery of pre-tests are normally performed. With respect to tests for the order of integration (usually the ones based on Dickey and Fuller, 1979, Phillips, 1987 or Phillips and Perron, 1988),[10] however, power tends to be rather low against a (trend) stationary series. As for tests on cointegration, Ghysels and Miller (2015) show that depending on the (often unknown) aggregation scheme underlying the series, one may have to expect severe size distortions.[11] Given the pitfalls of such pre-tests, an approach for GC testing in levels irrespective of the (co)integration orders of the variables would be highly valuable.

Remark 5 *Like Dolado and Lütkepohl (1996), we focus on* $d = 1$ *in this paper to simplify notation and discussion. On the one hand it is indeed the most important case in practice, on the other hand the approach and theory extend quite straightforwardly to* $d > 1$ *(see Toda and Yamamoto, 1995 for the common-frequency case). We will mention any changes due to larger* d *in footnotes.*

In the common-frequency framework, Toda and Yamamoto (1995) as well as Dolado and Lütkepohl (1996) show the following simple strategy to achieve the desired outcome: instead of transforming the VAR, one should estimate it in levels, but augment the regressor set by an additional lag, i.e., $Z_{t-(p+1)}$. Subsequently, one may test for Granger non-causality on the original coefficients (corresponding to $Z_{t-1}, \ldots, Z_{t-p}$) in the modified model. The reason why this approach leads to valid inference, also for $I(1)$ process that are not cointegrated, goes back to an early contribution by Sims et al. (1990). They showed that parameters, which can be rewritten as coefficients on zero-mean $I(0)$ regressors, have standard asymptotic distributions. Here, it is important to notice that one

[10] Note that as such tests are done for each variable individually, the MF nature of the variables plays a minor role here.

[11] To be precise, if the variables are aggregated with identical schemes (e.g., both end-of-period sampled or averaged), size distortions are null at best and mild at most; if, however, the underlying aggregation schemes differ, size distortions can be very severe (Ghysels and Miller, 2015). One can mitigate these problems by constructing a trace test on a MF-VECM or bootstrap the critical values of a LF residual-based test. However, even then there are instances in which size and power issues – albeit small – may occur.

does not need to rewrite the model accordingly, it is enough if it is theoretically possible to do so.

Providing more practical appeal to this result, let us consider the situation of a common-frequency VAR, where $Z_t^{CF} = (y_t, x_t)'$ with both series being $I(1)$. Additionally, let us assume y and x are cointegrated. A well-known way to rewrite this model in accordance with the finding of Sims et al. (1990) is the ECM format: due to cointegration, all coefficients (which are transformations of the parameters in the original VAR in levels) are assigned to stationary regressors. In the absence of cointegration, however, one of the regressors (depending on how we transform the original model, i.e., which lag $i \in 1, \ldots, p$ we capture the long-run term with) will remain $I(1)$. Now, imagine we add $Z_{t-(p+1)}$ to the VAR in levels and rewrite the model as follows:

$$\begin{aligned} Z_t^{CF} &= \sum_{i=1}^{p} A_i Z_{t-i}^{CF} + A_{p+1} Z_{t-(p+1)}^{CF} + \varepsilon_t \\ \Leftrightarrow \Delta_{p+1} Z_t^{CF} &= \sum_{i=1}^{p} A_i \Delta_{(p+1)-i} Z_{t-i}^{CF} - (I - \sum_{i=1}^{p+1} A_i) Z_{t-(p+1)}^{CF} + \varepsilon_t, \end{aligned}$$

where $\Delta_j Z_t^{CF} = Z_t^{CF} - Z_{t-j}^{CF}$. Hence, no matter whether the series are cointegrated, the standard Wald test applies to the coefficients corresponding to the first p (stationary) regressors.[12] Said differently – using the terminology in Sims et al. (1990) or Toda and Phillips (1994) – in case of an $I(1)$ system, one needs "enough cointegration" or additional lags to account for the non-degenerate stochastic trends.

The MF analogue of this approach is thus to replace Z_t^{CF} by Z_t and estimate the following MF-VAR in levels:

$$Z_t = \sum_{i=1}^{p} A_i Z_{t-i} + A_{p+1} Z_{t-(p+1)} + \epsilon_t = (A, A_{p+1}) \underline{Z}_{t-(p+1)} + \epsilon_t, \tag{7}$$

with $\underline{Z}_{t-(p+1)} = (Z_{t-1}', \ldots, Z_{t-(p+1)}')'$, thereby obtaining the OLS estimator of $a_{p+1} = vec(A, A_{p+1})$ as well as the corresponding residuals $\hat{\epsilon}_t = Z_t - (\hat{A}, \hat{A}_{p+1}) \underline{Z}_{t-(p+1)}$ for $t =$

[12] We refer to Dolado and Lütkepohl, 1996, p. 372 for this illustrating example and Theorem 1 of the same paper for the formal result.

$p+2,\ldots,T$ to get $\hat{\epsilon} = (\hat{\epsilon}_{p+2},\ldots,\hat{\epsilon}_T)$. The modified Wald tests in both test directions (its quantities henceforth being labeled with a "$\sim$") are then still applied on the corresponding mp GC-relevant elements in $A_1,\ldots,A_p$; for a suitably defined selection matrix $\widetilde{R}$ and in terms of null hypotheses and sample Wald statistics:

$$\begin{aligned}\widetilde{H}_0: &\qquad \widetilde{R}a_{p+1} = \mathbf{0}_{mp\times 1},\\ \widetilde{W} = &\quad (\widetilde{R}\hat{a}_{p+1})'[\widetilde{R}((\underline{Z}_{p+1}\underline{Z}'_{p+1})^{-1}\otimes\hat{\Sigma}_\epsilon)\widetilde{R}']^{-1}(\widetilde{R}\hat{a}_{p+1}),\end{aligned}$$

where $\hat{\Sigma}_\epsilon = (\hat{\epsilon}\hat{\epsilon}')/(T-p-1)$ is a consistent estimator of the covariance matrix and where $\underline{Z}_{p+1}$ is the stacked (over p and t) regressor matrix in the adequately modified MF-VAR, i.e., $\underline{Z}_{p+1} = (\underline{Z}_1,\ldots,\underline{Z}_{T-(p+1)})$. We refer to $\widetilde{W}$ based on Toda and Yamamoto (1995) or Dolado and Lütkepohl (1996) as "TY/DL-test".

Of course, the robustness of the TY/DL-test to the (co)integration order of the system does not come freely. Intentionally over-fitting the model in this way leads to inefficiencies in case the adjustment is not necessary. This cost is obviously higher in a MF-VAR as an extra lag adds $(n_l + mn_h)^2$ coefficients to be estimated (in the two-variable system: $(m+1)^2$). As will be shown momentarily, there are, however, ways to decrease these costs or to get rid of them altogether.

2.3 Mixed-Frequency Approach

Under the null hypothesis, it is clear that one cannot do better than designing an asymptotically valid inference method. While the TY/DL-test does so irrespectively of the (co)integration order of the series involved, it may – in small samples – suffer from size distortions and inefficiencies by intentionally over-fitting the model. We aim for an approach that may keep such issues at bay, while still providing an asymptotically valid test. We propose two approaches: the "MF-dep-test", indicating that this procedure depends on the GC testing direction, and an alternative "MF-indep-test".

2.3.1 MF-dep-test

Testing $H_0^{HF\nrightarrow LF}$: We start by considering the test direction from the high- to the low-frequency series, i.e., the arguably more interesting case in practice given that one is often interested in evaluating the effects of a HF indicator (e.g., a survey variable) on a LF aggregate (e.g., GDP). We propose the following procedure:

- Estimate the original MF-VAR in levels and obtain $\hat{a}_p$ as well as $\hat{\Sigma}_u$ as in Section 2.2, i.e., without augmenting the system by an additional lag.

- Consider the mp GC-relevant coefficients, i.e., the ones appearing in (5). Construct two Wald statistics corresponding to (i) the $(1,2)$-element of each autoregressive matrix being equal to zero and (ii) the $(1,3)$ up to $(1,m+1)$-elements of each A-matrix being jointly equal to zero.[13] For accordingly constructed selection matrices R^{HF1} and R^{HF2}, this boils down to

$$\begin{aligned} H_0^{HF1}: &\quad R^{HF1}a_p = \mathbf{0}_{p\times 1}, \\ H_0^{HF2}: &\quad R^{HF2}a_p = \mathbf{0}_{p(m-1)\times 1}, \\ W^{HFj} = &\quad (R^{HFj}\hat{a}_p)'[R^{HFj}((\underline{Z}_p\underline{Z}_p')^{-1}\otimes\hat{\Sigma}_u)R^{HFj\prime}]^{-1}(R^{HFj}\hat{a}_p) \end{aligned}$$

  for $j = 1, 2$.

- Compare the corresponding p-values of the $\chi^2(p)$- and $\chi^2(p(m-1))$-distributed test statistics to $\alpha/2$, where α represents the significance level. In other words, apply a Bonferroni correction to account for the fact that we want to test both null hypotheses jointly (Dunn, 1961).

- Reject $H_0^{HF\nrightarrow LF}$ if either H_0^{HF1}, H_0^{HF2} or both are rejected; otherwise do not reject $H_0^{HF\nrightarrow LF}$.

[13] For $d > 1$, we propose to apply tests on (i) the elements $(1,2)$ up to $(1,d+1)$ and (ii) the elements $(1,d+2)$ up to $(1,m+1)$ of each autoregressive matrix. In case $d \geq m$ one does not get around adding $X_{t-(p+1)}^{(m)}$ to the MF-VAR, i.e., what would be done for the TY/DL-test. But such a case should hardly occur in practice.

In this case, we can hold on to the original model, i.e., no intentional over-fitting is necessary! It also means that we can stick to the usual and simple MF-VAR(p) model in levels, a remarkably convenient outcome, especially for applied work. Avoiding eventual inefficiencies comes at a rather cheap price: all we have to do is compute two Wald statistics instead of one.

The intuition for this finding rests on the fact that the stacked VAR structure provides us with "enough [or] sufficient cointegration", in the sense of Toda and Phillips (1994). To be more precise, the HF variables – provided they are $I(1)$ – are trivially cointegrated with each other, i.e., $m-1$ cointegrating relationships are a-priori known (Götz et al., 2013). Hence, in the absence of additional cointegration between y and x, we need to test at most $m-1$ coefficients at a time.[14] Loosely speaking, one could say that we apply the argument of Toda and Yamamoto (1995) and Dolado and Lütkepohl (1996) "backwards": instead of looking at $m+1$ coefficients, of which m are tested on, we look at m and test on $m-1$ (two times).[15]

Remark 6 *The advantage of our MF-dep-test, which is most pronounced in the situation just outlined, i.e., when testing for GC from x to y, is the absence of any need to intentionally over-fit the model. One has to admit, though, that this advantage is due to the specification of the MF-VAR in its stacked form. On the one hand, the implicit presence of the trivial cointegrating relationships enables us to perform GC testing on the original MF-VAR(p). On the other hand, it assigns a larger cost of over-fitting to the TY/DL-approach than would have been the case in a non-stacked model version. Applying the TY/DL-approach to a bivariate (for y and x) VAR operating at the high frequency would imply just 2^2 additional parameters instead $(m+1)^2$.*

[14] An alternative to splitting up the m coefficients as proposed here would be to test $m-1$ coefficients two times, once elements $(1,2)$ up to $(1,m)$ of each A-matrix and once elements $(1,3)$ up to $(1,m+1)$. However, the presence of overlapping coefficients makes the subsequent Bonferroni correction overly conservative. Likewise, testing each of the m sets of coefficients individually is asymptotically valid, yet complicates a joint consideration of the m respective null hypotheses.

[15] If y and x are cointegrated, the corresponding additional cointegrating relationship compensates for the one missing linear combination among the trivial ones; the standard approach, i.e., testing zero restrictions on all m coefficients per autoregressive matrix jointly, would thus have sufficed (Lütkepohl and Reimers, 1992). See also Theorem 2 and Corollaries 1 to 3 thereafter.

As mentioned already in Remark 4, however, such non-stacked MF-VAR models inherently contain missing observations. Within a corresponding state space representation, merely the state equation is usually specified as a VAR process, allowing to test for GC like we have in mind here. As the interaction between latent observations, the (co)integration order of the series involved and robust ways to test for GC is unclear, it remains to be seen whether the TY/DL-test readily transfers to such a setup (and whether one can improve upon it). While such a research question is of great interest given that parameter-driven MF-VAR models are also heavily used in practice, it is beyond the scope of this paper.

Testing $H_0^{LF \nrightarrow HF}$: Let us now consider the reverse test direction, i.e., the one from the low- to the high-frequency series. Albeit being less common, this situation is still of interest. Apart from being complete on the issue, there are (macro)economic examples such as quarterly capacity utilization rates (e.g., in Germany), which may affect indicators like monthly industrial production. Here is what we propose for this test direction:

- Augment the original MF-VAR by adding the regressor $y_{t-(p+1)}$ to each equation:[16]

$$Z_t = \sum_{i=1}^{p} A_i Z_{t-i} + A_{p+1}^{(\cdot,1)} y_{t-(p+1)} + \upsilon_t = (A, A_{p+1}^{(\cdot,1)}) \underline{Z}_{t-p,y} + \upsilon_t \tag{8}$$

  with $A_{p+1}^{(\cdot,1)}$ being an $(m+1)$-vector (the notation resembling the similarity to the first column of matrix A_{p+1} used for the TY/DL-test) and $\underline{Z}_{t-p,y} = (\underline{Z}'_{t-p}, y_{t-(p+1)})'$. Estimate the model in levels, thereby obtaining the OLS estimator of $a_{p,y} = vec(A, A_{p+1}^{(\cdot,1)})$ and the residuals $\hat{\upsilon}_t = Z_t - (\hat{A}, \hat{A}_{p+1}^{(\cdot,1)}) \underline{Z}_{t-p,y}$ for $t = p+2, \ldots, T$ to get $\hat{\upsilon} = (\hat{\upsilon}_{p+2}, \ldots, \hat{\upsilon}_T)$ and then $\hat{\Sigma}_\upsilon = (\hat{\upsilon}\hat{\upsilon}')/(T-p-1)$.

- Consider the usual mp GC-relevant coefficients, i.e., the ones appearing in (6). For an accordingly constructed selection matrix R^{LF}, the null hypothesis and sample

[16] Generally, i.e., also for $d > 1$, one has to add regressors $y_{t-(p+1)}, \ldots, y_{t-(p+d)}$ to each equation.

Wald statistic read

$$
\begin{aligned}
H_0^{LF}: &\qquad R^{LF} a_{p,y} = \mathbf{0}_{mp\times 1}, \\
W^{LF} = &\ (R^{LF}\hat{a}_{p,y})'[R^{LF}((\underline{Z}_{p,y}\underline{Z}'_{p,y})^{-1}\otimes \hat{\Sigma}_v)R^{LF\prime}]^{-1}(R^{LF}\hat{a}_{p,y})
\end{aligned}
$$

with $\underline{Z}_{p,y}$ stacking the matrices $\underline{Z}_{t-p,y}$ over $t = p+2,\ldots,T$ similarly to before.

- Inspect the p-value of the $\chi^2(mp)$-distributed test statistic to decide upon $H_0^{LF\nrightarrow HF}$.

Here, the situation is a bit different, because the LF variable y is not part of any trivial cointegrating relationship. Hence, we will not get around performing some sort of adjustment along the lines of the TY/DL-test. Yet, in contrast to how the straightforward extension described in the previous subsection works, we want to limit the amount of over-fitting as much as possible. As we only require $y_{t-(p+1)}$ for being able to rewrite the model in an ECM-fashion (see the example in Section 2.2), we propose to merely add $y_{t-(p+1)}$ to each equation of the system. This implies an addition of merely $m+1$ coefficients to be estimated, in contrast to the $(m+1)^2$ for the TY/DL-test.

2.3.2 MF-indep-test

In contrast to the approach of Toda and Yamamoto (1995) or Dolado and Lütkepohl (1996), the MF-dep-test depends on the direction of GC we are interested in. If one aims for GC testing in both directions using the same estimated model, one does not get around an adjustment similar to the TY/DL-test. To be precise, one would need to add at least $y_{t-(p+1)}$ and *one* HF observation from period $t-(p+1)$, e.g., $x_{t-(p+1)}^{(m)}$. Formally, one would have to regress Z_t on $\underline{Z}_{t-p,yx} = (\underline{Z}'_{t-p}, y_{t-(p+1)}, x_{t-(p+1)}^{(m)})'$ for $t = p+2,\ldots,T$ and follow a set of steps very similarly to those of the MF-dep-test for the direction from y to x.[17] Of course, one would sacrifice efficiency as far as testing for GC from x to y is concerned. Still, the amount of over-fitting is smaller than for the TY/DL-test, as one needs to estimate $2(m+1)$ additional coefficients instead of $(m+1)^2$.

[17] Due to this similarity, these steps are not outlined here in detail.

Table 1 provides a small overview of all the testing procedures discussed above.

3 Theoretical Background and Simulations

3.1 Partially cointegrated systems

The validity of the proposed approach in this paper – at least as far as testing the direction from the HF to the LF series is concerned – rests on the theoretical framework in Toda and Phillips (1994), which we revisit here. While our approach for testing the reverse direction may also be validated using similar grounds, the methodology underlying both the MF-dep- and the MF-indep-test in this case is, however, broadly identical to the TY/DL-procedure and we refer to the respective papers.

Recall that we consider a two-variable MF system, where we want to test whether the m series corresponding to the HF variable Granger cause y. Let us adapt assumptions (A2)-(A4) in Toda and Phillips (1994) to our original MF-VAR:[18]

Assumption 2 *Rewrite the model in (2) using the lag polynomial* $A(L) = \sum_{i=1}^{p} A_i L^{i-1}$ *to get* $Z_t = A(L)Z_{t-1} + u_t$*. We assume:*

(i) $|I_{m+1} - A(z)z| = 0$ *implies* $|z| > 1$ *or* $z = 1$*.*

(ii) $A(1) - I_{m+1} = \Pi_\alpha \Pi_\theta'$*, where* Π_α *and* Π_θ *are* $(m+1) \times r$*-matrices of full column rank* r*,* $0 \leq r \leq m$*; if* $r = 0$*, there is no* Π_α *or* Π_θ*, and* $A(1) = I_{m+1}$*.*

(iii) $\Pi_{\alpha,\perp}'[\sum_{i=1}^{p-1}(\sum_{j=i+1}^{p} A_j) - I_{m+1}]\Pi_{\theta,\perp}$ *is nonsingular, where* $\Pi_{\alpha,\perp}$ *and* $\Pi_{\theta,\perp}$ *are* $(m+1) \times (m+1-r)$*-matrices of full column rank such that* $\Pi_{\alpha,\perp}'\Pi_\alpha = 0 = \Pi_{\theta,\perp}'\Pi_\theta$*; if* $r = 0$*, we take* $\Pi_{\theta,\perp} = I_{m+1} = \Pi_{\alpha,\perp}$*.*

Now, for $\Pi_{\theta,g_1:g_2}$ denoting rows g_1 up to g_2 of Π_θ and given Assumptions 1 and 2, the following holds:

[18] An adapted version of assumption (A1) in Toda and Phillips (1994) is, in fact, encompassed by Assumption 1 in this paper; for details on all assumptions, we refer to p. 261-262 of said paper.

Theorem 2 *Suppose we are under the following null hypothesis:*

$$H_0 : A_i^{(1,g_1)} = \ldots = A_i^{(1,g_2)} = 0 \ \forall i$$

for $2 \leq g_1 \leq g_2$*. Then, if* $rank(\Pi_{\theta,g_1:g_2}) = g(\leq n_g = g_2 - g_1 + 1)$*, we obtain the following for the corresponding test statistic:*

$$W^{HF} \to^d \chi^2(n_g(p-1) + g) + \mathbb{1}_{(g<n_g)} TNP,$$

where TNP *denotes a term (nonstandard distribution) that depends on nuisance parameters, which in turn depend on the long-run covariance matrix. Note that* TNP *cancels for* $g = n_g$*, though (labeled using the indicator function).*

This Theorem has a series of implications, directly related to our situation. Corrolary 1 shows that the standard approach works even if the series are $I(1)$, provided y and x are cointegrated. In case cointegration between y and x is absent, though, the test converges to a mixture of a chi-square and and a nonstandard distribution, the latter depending on nuisance parameters (Corrolary 2).

Corrolary 1 *Suppose there is cointegration between the* $I(1)$*-series* y *and* x *and we test the entire set of coefficients corresponding to the HF series, i.e.,* $g_1 = 2$ *and* $g_2 = m + 1$*, implying* $n_g = m$*. Then, as* $rank(\Pi_{\theta,2:m+1}) = m$ *(see Götz et al., 2013) we have that* $W^{HF} \to^d \chi^2(mp)$*.*

Corrolary 2 *Suppose there is no cointegration between the* $I(1)$*-series* y *and* x *and we test the entire set of coefficients corresponding to the HF series, i.e.,* $g_1 = 2$ *and* $g_2 = m+1$*, implying* $n_g = m$*. Then, as* $rank(\Pi_{\theta,2:m+1}) = m - 1$ *(see Götz et al., 2013) we have that* $W^{HF} \to^d \chi^2(mp-1) + TNP$*.*

Now, the MF-dep-test rests on computing two Wald statistics, that are subsequently combined using the Bonferroni approach. For each of the individual tests, we have the following for the cointegration- and the no-cointegration-cases (Corrolary 3):

Corrolary 3 *Suppose the series y and x are $I(1)$ and we test two sets of coefficients corresponding to the HF series: (i) for $g_1 = 2$ and $g_2 = 2$, implying $n_g = 1$ and (ii) for $g_1 = 3$ and $g_2 = m+1$, implying $n_g = m-1$. For (i) $rank(\Pi_{\theta,2:2}) = 1$ s.t. $W^{HF} \rightarrow^d \chi^2(p)$ and for (ii) $rank(\Pi_{\theta,3:m+1}) = m-1$ s.t. $W^{HF} \rightarrow^d \chi^2((m-1)p)$.*

3.2 Monte Carlo study

3.2.1 Setup

In order to investigate size and power properties of the various test versions in finite samples, we conduct a series of simulation experiments. In particular, we assess the sensitivity of the results with respect to the sample size ($T = 50, 150, 250$) and the frequency discrepancy ($m = 2, 3, 4$). All simulations are based on 10,000 replications of the respective DGP and plot the rejection frequencies of the test statistics in Table 1.

We start by describing our MF-DGP, i.e., the data are truly generated at mixed frequencies (see Remark 4), flexibly incorporating the different possible features of the data. Due to the potential presence of cointegration between y and x, through which GC in one direction is present by construction, we need to differentiate the DGP for both test directions. Hence, let y_t and $x_t^{(m)}$ be generated by one of the following systems, depending on whether we inspect...

...GC from x to y, i.e.,

$$y_t = \rho y_{t-1} + \sum_{j=0}^{m-1} \lambda_j \Delta x_{t-1-j/m}^{(m)} + \epsilon_{y,t}, \tag{9}$$

$$x_{t-j/m}^{(m)} = \theta y_t + v_{x,t-j/m}^{(m)}, \text{ where } v_{x,t-j/m}^{(m)} = (\alpha+1) v_{x,t-(j+1)/m}^{(m)} + \epsilon_{x,t-j/m}^{(m)}, \tag{10}$$

..., or GC from y to x, i.e.,

$$y_t = \theta x_t^{(m)} + v_{y,t}, \text{ where } v_{y,t} = (\alpha+1) v_{y,t-1} + \epsilon_{y,t}, \tag{11}$$

$$x_{t-j/m}^{(m)} = \rho x_{t-(j+1)/m}^{(m)} + \delta_j \Delta y_{t-1} + \epsilon_{x,t-j/m}^{(m)}, \tag{12}$$

where $\epsilon_{y,t}, \epsilon^{(m)}_{x,t-j/m} \sim N(0,1)$,[19] $j = 0, \dots, m-1$ and $-2 \leq \alpha \leq 0$.

Note that (9) contains a U-MIDAS-type (Foroni et al., 2015b) impact of the HF series on y, and that (12) features a similar effect of past LF-differences on x.[20] This setup allows us to look at the consequences of $I(1)$-ness as well as cointegration simultaneously or in isolation. To elaborate, let us – for a moment – only consider the test direction from x to y; the situation for the reverse direction is analogous. On the one hand, $-1 < \rho < 1$ in equation (9) implies that y is $I(0)$. The value of α in (10) then determines whether x is $I(0)$ as well (for $-2 < \alpha < 0$) or whether it is $I(1)$ (for $|\alpha + 1| = 1$). On the other hand, and more interestingly, $|\rho| = 1$ in (9) implies a unit root for y making the process $I(1)$. In this case, α in (10) controls the presence (for $-2 < \alpha < 0$) or absence (for $|\alpha + 1| = 1$) of cointegration. In the cointegrated case, θ then governs the cointegrating relationship.

After some manipulations, both DGPs can be rewritten into a reduced-form MF-VAR(2) in levels, i.e.,

$$Z_t = A_1 Z_{t-1} + A_2 Z_{t-2} + u_t,$$

where $u_t = A^* u_t^*$ with $u_t^* = (\epsilon_{y,t}, \epsilon^{(m)}_{x,t}, \epsilon^{(m)}_{x,t-1/m}, \dots, \epsilon^{(m)}_{x,t-(m-1)/m})' \sim N(\mathbf{0}_{m+1\times 1}, I_{m+1})$ such that $\Sigma_u = A^* A^{*\prime}$; precise formulae for A_1, A_2 and A^* under both DGPs are being delegated to Appendix B.

Remark 7 *To show that these DGPs nest the case of a cointegrated system with trivial cointegrating relationships, consider the example of $m = 3$ and $\rho = 1$ such that both y and x are $I(1)$. Using the formulae for A_1, A_2 and A^* in the Appendix, the reduced-form MF-VARs can be rewritten into the following VECMs (Götz et al., 2013 or Ghysels and Miller, 2015).*

[19] Hence, $\sigma_y^2 = \sigma_x^2 = 1$.

[20] Note that one could – as we actually do in the simulations – leave the impact of the lagged series, i.e., λ_j and δ_j, constant over one t-period, significantly simplifying notation. At this stage we opted for a more general description of the DGP, due to its potential use for fellow researchers, especially in light of its relation to the commonly applied U-MIDAS model.

For GC from x to y, i.e., the DGP in (9) and (10):

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t^{(3)} \\ \Delta x_{t-1/3}^{(3)} \\ \Delta x_{t-2/3}^{(3)} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ -\theta\alpha(\alpha^2+3\alpha+3) & \alpha(\alpha^2+3\alpha+3) & 0 & 0 \\ -\theta\alpha(\alpha+2) & \alpha(\alpha+2)+1 & -1 & 0 \\ -\theta\alpha & \alpha+1 & 0 & -1 \end{bmatrix}}_{\Pi} \begin{bmatrix} y_{t-1} \\ x_{t-1}^{(3)} \\ x_{t-4/3}^{(3)} \\ x_{t-5/3}^{(3)} \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & \lambda_0 & \lambda_1 & \lambda_2 \\ 0 & \theta\lambda_0 & \theta\lambda_1 & \theta\lambda_2 \\ 0 & \theta\lambda_0 & \theta\lambda_1 & \theta\lambda_2 \\ 0 & \theta\lambda_0 & \theta\lambda_1 & \theta\lambda_2 \end{bmatrix} \begin{bmatrix} \Delta y_{t-1} \\ \Delta x_{t-1}^{(3)} \\ \Delta x_{t-4/3}^{(3)} \\ \Delta x_{t-5/3}^{(3)} \end{bmatrix} + u_t.$$

For GC from y to x, i.e., the DGP in (11) and (12):

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t^{(3)} \\ \Delta x_{t-1/3}^{(3)} \\ \Delta x_{t-2/3}^{(3)} \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha & -\theta\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}}_{\Pi} \begin{bmatrix} y_{t-1} \\ x_{t-1}^{(3)} \\ x_{t-4/3}^{(3)} \\ x_{t-5/3}^{(3)} \end{bmatrix}$$
$$+ \begin{bmatrix} \theta(\delta_0+\delta_1+\delta_2) & 0 & 0 & 0 \\ \delta_0+\delta_1+\delta_2 & 0 & 0 & 0 \\ \delta_1+\delta_2 & 0 & 0 & 0 \\ \delta_2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta y_{t-1} \\ \Delta x_{t-1}^{(3)} \\ \Delta x_{t-4/3}^{(3)} \\ \Delta x_{t-5/3}^{(3)} \end{bmatrix} + u_t.$$

Indeed then, for $\alpha = 0$ *the matrix* Π *(in both DGPs) merely contains the trivial cointegrating relationships among the HF variable itself. For* $-2 < \alpha < 0$*, though, there is an additional cointegrating relationship between* y *and* x *of the form* $(-\theta, 1)$ *or* $(1, -\theta)$*, respectively. To see this, note that one can write* $\Pi = \Pi_\alpha \Pi_\theta'$*, where* Π_α *and* Π_θ *are both* (4×3)*-matrices and where the columns of* Π_θ *contain the aforementioned cointegrating* *relationships.*

Now, the null hypotheses for both test directions (under their respective DGP) are easily seen to be:

$$H_0^{HF \nrightarrow LF} : \lambda_j = 0 \ \forall j = 0, \dots, m-1,$$

$$H_0^{LF \nrightarrow HF} : \delta_j = 0 \ \forall j = 0, \dots, m-1.$$

We consider three cases: (a) y and x are $I(0)$ by setting $\rho = 0.8$ and $\alpha = -0.5$, (b) y and x are $I(1)$ and cointegrated by setting $\rho = 1$ and $\alpha = -0.5$, (c) y and x are $I(1)$ but not cointegrated by setting $\rho = 1$ and $\alpha = 0$. Throughout the simulations, $\theta = 0.5$. To simplify matters we set the GC-determining coefficients constant, i.e., $\lambda_j = \lambda \ \forall j$, but we let them depend on T and consider three different overall values: $\lambda = T^{-0.5}\lambda^*$ with $\lambda^* = \{0, 1, 2\}$; likewise for δ. Note that values of zero imply no GC (size), whereas non-zero values imply GC (power).[21]

3.2.2 Results – Both series $I(0)$

We start by discussing the results for case (a), in which both series are stationary. The outcomes are summarized in Tables 2 and 3 for the test direction from x to y and vice versa, respectively. This scenario is obviously favourable for the standard test, which is asymptotically $\chi^2(mp)$-distributed. While incurring some size distortions for small T, it leads to the correct size for larger sample sizes. Furthermore, this test is the most powerful one for each $(T, m, \lambda^*/\delta^*)$-combination as it does not feature any alteration to the estimated model. The TY/DL- and the MF-indep-test perform almost identically, the latter having slightly higher rejection rates under the alternative, because $m^2 - 1$ fewer parameters need to be estimated. Compared to the standard test, though, power is clearly inferior. Interestingly, the MF-dep-test seems to cope somewhat better with size distortions arising from small T, particularly when testing the empirically more interesting

[21]This specification of the GC-determining coefficients is derived from the Monte Carlo study in Dolado and Lütkepohl (1996) for the common-frequency case.

GC-direction from x to y (see Table 2). Granted, it still falls short in terms of power compared to the standard test, yet it beats the TY/DL- and MF-indep-tests, clearly so again for testing HF $\nrightarrow$ LF. Overall, the MF-dep-test presents a more than compelling approach for $I(0)$-MF series.

3.2.3 Results – Both series $I(1)$ with cointegration

Turning to the more interesting cases, in which y and x are non-stationary, we first consider case (b), in which both series are cointegrated; Tables Tables 4 and 5 contain the respective set of results. Corollary 1 implies that the standard test is, in fact, still favoured in this situation. Together with the trivial cointegrating relationships among the HF series, the presence of additional cointegration between y and x implies "enough cointegration" (Toda and Phillips, 1994) to yield a $\chi^2(mp)$-distribution of the standard Wald test. As a consequence, the outcomes in Tables 4 and 5 are qualitatively identical to the ones for the stationary case discussed in the previous subsection.

3.2.4 Results – Both series $I(1)$ without cointegration

Finally, we inspect the non-stationary and non-cointegrated case (c), the outcomes of which being summarized in Tables 6 and 7. First, the standard test does not have a $\chi^2(mp)$ under H_0 (see Corrolary 2), explaining why actual size clearly exceeds the nominal level of 5% even for large T. The TY/DL- and MF-indep-tests, on the other hand, constitute asymptotically valid tests, although they are – as before – somewhat oversized for a small sample size. In terms of power, though, they perform rather well, more so given that size-adjusted power of the standard test would surely turn out to be smaller than the size-unadjusted figures presented here.

As far as the MF-dep-test is concerned, let us discuss the outcomes for the two GC test directions separately. For testing GC from y to x (Table 7) the approach is almost identical to the MF-indep-test (and the TY/DL-test for that matter); only that fewer parameters need to be estimated (revisit Section 2.3). In terms of size, the outcomes are thus by and large comparable, whereas power is either as high as or even slightly higher when using the MF-dep-test. For testing GC from x to y (Table 6), asymptotic validity of the MF-dep-test is also confirmed (see Corrolary 3 and, for the Bonferroni correction, Dunn, 1961), but, more noteworthily, the test appears far less oversized than its competitors: even for $T = 150$ and $m = 4$ empirical size coincides with the nominal one. The cost of this presumably controlling effect of the Bonferroni correction is a loss of power, which is most pronounced for cases that are closer to the common-frequency setup, i.e., for small m. Indeed, for larger m the benefits in terms of efficiency counterweigh the conservative nature of the Bonferroni adjustment.

3.2.5 Results – Summary

To sum up, the MF approaches present competitive and easy-to-implement alternatives to the existing TY/DL-approach. Here, the MF-indep-test yields marginally better results throughout the entire analysis, which is not too surprising given the merely slight adjustment in test design. The MF-dep-test is the preferred choice most of the times, although the merits of a correctly sized test – also for small samples – stand against somewhat lower power in case cointegration between y and x is absent. From an empirical and conservative standpoint, though, the MF-dep-test is the dominant strategy.

3.3 Alternative High-Frequency DGP

As already mentioned in Remark 4, the MF-VAR we base our analysis on is somewhat tailored to a mixed-frequency DGP like the one underlying equations (9)-(10) or (11)-(12). It might just as well be the case, though, that the data are generated by a different process. While a common low-frequency and an "infinitely" high-frequent (in other words, continuous) VAR probably represent the two most extreme options, a common-frequency VAR operating at the highest frequency, that at least one of the variables appears in, has become a popular alternative in the MF literature (Ghysels and Miller, 2015, Ghysels et al., 2016 or Miller, 2014). The observed MF data are then obtained by temporally aggregating a subset of the variables, thereby implying the presence of latent HF observations.

Remark 8 *Just like the MF-VAR is tailored to a MF-DGP, such a HF-DGP would favor a parameter-driven MF-VAR model with latent observations (revisit Remark 4 again). As (co)integration-robust GC tests within such MF-VAR models are not derived and evaluated yet, we cannot perform a comparison of the respective tests and models for either DGP, mixed- or high-frequent. Such an analysis would, however, be of great interest, emphasizing again the need to extend the approach of Toda and Yamamoto (1995) or Dolado and Lütkepohl (1996) also to parameter-driven MF-VAR models. Here, we thus only investigate the sensitivity of our various MF-VAR-based tests to an alternative HF-DGP.*

The HF-DGP we consider is directly derived from the setup in Dolado and Lütkepohl (1996) for the common-frequency case; see Appendix C for explicit formulae. As far as temporal aggregation of y, the series that is only observed at the low frequency, is concerned, we apply point-in-time sampling, i.e., $y_t = y_t^{(m)}$ such that we only observe the last HF observation within each t-period.[22] Before going into details on the simulation results, one should note that Granger non-causality is a property that is not generally robust to temporal aggregation (Marcellino, 1999). With respect to VAR models, Ghysels et al. (2016) show that, depending on the aggregation scheme, Granger non-causality will not be preserved when moving from a HF- to a MF-VAR. In particular, "a crucial condition for non-causality preservation is that the information for the [variable that is caused by the other under H_A, i.e., y] is not lost by temporal aggregation" (p. 216).

That being said, Table 8 contains the outcomes for the cointegrated case and the test direction from x to y. The rejection rates corresponding to $\lambda^* = 0$ (size) for all tests are clearly well above the nominal level of 5%. In fact, these results hint towards one example of the aforementioned finding, i.e., Granger non-causality does not seem to be preserved in this specific case. Put differently, the outcomes suggest that "spurious"

[22]Another commonly used aggregation scheme (see, e.g., Silvestrini and Veredas, 2008), not discussed here to save on space, is average sampling, i.e., $y_t = \sum_{j=0}^{m-1} y_{t-j/m}^{(m)}$ such that we observe the average of the m values corresponding to each t-period. The outcomes are very similar, though, and are available upon request.

causality is introduced here due to the temporal aggregation of y. As this fact applies to the underlying model in general, it affects the performance of all tests similarly, making an inspection of the outcomes in this respect superfluous. This changes when considering the results for the reverse direction in Table 9. Here, Granger non-causality appears to be preserved as the standard test has – as expected – correct size; like in the MF-DGP, it dominates its competitors in terms of power. The other test approaches, while being correctly sized, perform virtually identically to one another.

The outcomes for the no-cointegration-scenario (Tables 10 and 11) also deliver a familiar picture: in both test directions, the standard test does not have a $\chi^2(mp)$ under the null hypothesis and is therefore oversized, also for large T. When testing for causality from y to x (Table 11), the other test approaches are correctly sized, but show somewhat lower power than in the cointegrated case. Furthermore, there is again no clear ranking among them. When considering the test direction from x to y (Table 10), however, the MF-dep-test is again outperforming the other approaches in terms of size (also for small T), but falls somewhat short in terms of power. This may, however, be due to the TY/DL- and the MF-indep-tests being oversized to some degree.

All in all, the results obtained using a MF-DGP appear robust to an alternative HF-DGP, provided Granger non-causality is preserved after temporal aggregation is applied to one of the series.

4 Application

To illustrate our approach with actual data, we consider two empirical applications. The first one involves the weekly WTI oil price traded in New York and denoted in US-Dollar per barrel (OIL) as well as the monthly consumer price index in Germany (CPI), seasonally and calender adjusted.[23] Hence, $m = 4$. The series were downloaded from the

[23]The underlying series for OIL is actually based on working days, which have been aggregated to the weekly frequency. Some months get assigned four weeks, whereas some even have five weeks. To simplify notation we delete the first weekly observation in the latter case.

internal database of the Deutsche Bundesbank and refer to the period from January 1991 to May 2018.[24] According to economic theory one would usually expect OIL to Granger cause CPI. First, OIL appears in the consumption basket of German consumers, be it directly (through, e.g., car fuel or energy production) or indirectly (through affecting other goods). Second, being a global indicator OIL may affect several countries and eventual uncertainties may also have implications for the German economy.

With respect to the integration order and especially the presence of a cointegrating relationship between the two series, the situation is, however, not at all clear a-priori. To illustrate this ambiguity we conduct a series of tests one would often pursue in practice. Standard Dickey-Fuller tests for a unit root in the log-levels of the series reveal that both series are in fact $I(1)$, with only marginal (and thus negligible) indications of a potential $I(2)$-ness of CPI. We use the Schwartz information criterion to determine the lag order of $p = 1$. As far as cointegration tests are concerned, the conclusions are somewhat discordant: on the one hand, the trace and maximum-eigenvalue-tests of Johansen (1991) both indicate – using the critical values of MacKinnon et al. (1999) and applied to the MF-VAR(1) – the presence of cointegration between y and x on top of the $m-1 = 3$ trivial long-run terms. On the other hand, the Engle and Granger (1987) two-step procedure – applied to MF data – points toward the no-cointegration scenario.[25] Clearly, a GC testing procedure that is robust to the cointegration order would thus be desirable.

To this end we compare the four different GC test approaches for both test directions.[26] We start by estimating the five-dimensional MF-VAR(1) in log-levels and apply the usual Wald test for the four respective coefficient estimates. It turns out that the p-values corresponding to the test statistics $W_{OIL\nrightarrow CPI}$ and $W_{CPI\nrightarrow OIL}$ are 0.001 and 0.013, respectively. The standard tests thus indicate bi-directional GC between CPI and OIL, a somewhat surprising result. For the TY/DL-test we simply estimate a MF-VAR(2)

[24]The data was downloaded on 21 June, 2018 implying that OIL would have been available for June even. Due to the publication delay of CPI, though, we consider a balanced dataset ending in May.

[25]Hereby, it did not matter which of the weekly observations of OIL we place into a potential cointegrating term.

[26]All calculations are easily doable in a software such as EViews, making the methods very appealing to practitioners, without the need for advanced programming skills.

in log-levels and apply a Wald test on the original (lag-one-)coefficients. The p-values associated with $W^*_{OIL\nrightarrow CPI}$ and $W^*_{CPI\nrightarrow OIL}$ turn out to be 0.0001 and 0.199, respectively. Maybe cointegration between the two series is, in fact, absent, causing the standard test not to deliver asymptotically valid inference for the GC test from LF to HF; power may be fine as far as the detected causal link from OIL to CPI is concerned, however.

Let us see whether the MF-tests, which often tended to be less oversized or even more powerful, back up the finding of the TY/DL-test. For the MF-indep-test we simply add $(CPI_{t-2}, OIL_{t-2})'$ to each equation of the MF-VAR(1), estimate the system and apply a Wald test on the same coefficients as before. The p-values become 0.0001 and 0.122, respectively. Finally, for the MF-dep-test and the test direction from CPI to OIL, we merely add y_{t-2} to each row of the MF-VAR(1) and apply the same procedure as before. The resulting p-value equals not less than 0.275; recalling that this was an instance, in which the MF-dep-test was clearly outperforming its competitors, it puts all the more weight on Granger non-causality from CPI to OIL. Testing the reverse direction boils down to two Wald tests on the original MF-VAR(1) in log-levels: once on the OIL_{t-1}-coefficient and once on the coefficients corresponding to $OIL_{t-5/4}$, $OIL_{t-6/4}$ and $OIL_{t-7/4}$ jointly. The p-values of the two individual tests, which are based on a 0.025 level due to the Bonferroni correction, turn out to be 0.001 each. Overall, we thus find overwhelming evidence for uni-directional GC from OIL to CPI, in line with economic theory.

For the second application we consider quarterly GDP and the monthly industrial production index (IP) in Germany, again seasonally and calender adjusted. This time we thus have $m = 3$. Again, the series originate from the database of the Deutsche Bundesbank and – being downloaded mid-June and achieving balancedness in view of publication lags – cover the period from January 1991 to March 2018.[27] Consequently, $T = 109$ implying a much shorter sample size than before; recall that some approaches suffered from size distortions for small T. Economic theory may actually support bi- or uni-directional (from IP to GDP) causality. On the one hand, the HF series is an

[27]GDP for the second quarter of 2018 gets released in the middle of August.

important determinant of the LF series and, particularly in Germany, has a large share in the country's National Accounts. On the other hand, GDP may cause IP as high economic activity last period may imply a continuously thriving economy with full order books that will keep industrial output expanding as well.

Like before, standard procedures are agreeing fully on the integration order, I(1), as well as the lag length, $p = 1$, but disagree with respect to cointegration: the Johansen (1991) procedures again find m cointegrating relationships, i.e., two trivial ones and one additional long-run term for GDP and IP. The MF Engle and Granger (1987) approach, however, yields no cointegration. Going through the different GC test options draws the following picture: the standard test finds bi-directional GC, whereby a p-value of 0.042 for $W_{GDP \nrightarrow IP}$ puts more doubt on this causal link than in the reverse direction (p-value of 0.0001). In fact, all tests clearly detect GC from IP to GDP. For the reverse direction, however, the MF-dep-test overwhelmingly rejects the null as well, the MF-indep-test is somewhat torn in the middle (p-value of 0.092) and the TY/DL-test concludes no GC (p-value of 0.487). It is to be expected that the small sample size affects the outcomes, as differences between the MF-tests are mainly driven by eventual inefficiencies, particularly for LF-to-HF GC tests. Given that the MF-dep-test is the sparsest one in this respect and that it showed quite robust results, we conclude bi-directional GC between GDP and IP.

5 Conclusion

In this paper we extended the method of Toda and Yamamoto (1995) or Dolado and Lütkepohl (1996) on testing for causality in VAR models to the mixed-frequency scenario. Based on the fact that transformations of a VAR, that result from non-stationarities and/or cointegration among the series, rest on pre-tests, which are often prone to size distortions and/or losses of power, we aimed for an approach that works independently from the (co)integration properties of the variables. Apart from the straightforward application of the test by Toda and Yamamoto (1995) or Dolado and Lütkepohl (1996) to a mixed-frequency VAR, we proposed two further test approaches, that exploit the stacked nature of the VAR vector in the observation-driven model of Ghysels (2016). These methods were shown to come at smaller or even zero costs in terms of intentionally over-fitting the model; when testing for Granger causality in the empirically more commonly investigated direction from the high- to the low-frequency series, one mixed-frequency test approach is even based on the standard MF-VAR(p) in (log-)levels without any adjustment. For the other instances, minor extensions of the model were sufficient to ensure asymptotically valid inference.

A Monte Carlo study revealed that the MF approaches are indeed competitive and easy-to-implement alternatives to the TY/DL-procedure. While one of the mixed-frequency tests (MF-indep) yielded only marginally but consistently better results throughout the entire analysis, the other one (MF-dep) proved to be the overall preferred choice. In the absence of cointegration between y and x, however, the merits of a correctly sized test – also for a small sample size – stood against somewhat lower power. By and the large, these findings applied similarly to a mixed-frequency and an alternative high-frequency data generating process. Two applications involving (i) the consumer price index and the oil price, for which uni-directional causality from the latter to the former was concluded, as well as (ii) GDP and industrial production, where bi-directional causality was detected, illustrated the different test options and their practical appeal. Here, the ambiguous outcomes of standard cointegration tests motivated the use of robust Granger causality tests, whose outcomes were then shown to potentially deviate from the ones of the standard Wald test.

The analysis presented here can and should obviously be extended along a couple of lines. Firstly, of course, the effects of estimating the lag length should be analyzed, i.e., Remark 2 should be relaxed. Secondly, and as mentioned several times throughout the paper, the observation-driven MF-VAR we consider here is not the only way to model a system of time series sampled at varying frequencies. One could investigate the extension of the test by Toda and Yamamoto (1995) or Dolado and Lütkepohl (1996) within a parameter-driven MF-VAR à la Schorfheide and Song (2015). However, one should keep in mind that the absence of trivial cointegrating relationships, due to the non-stacked design of such systems, may diminish the scope for efficiency improvements. Finally, extensions toward systems of larger size – either owing to a larger set of variables or a larger frequency mismatch – may be fruitful. Hopefully, the present paper leads to a well-deserved revival of this literature, which was somewhat pushed into the background in recent years. Sometimes apparently, a small adjustment of a simple model suffices to conduct valid inference; why complicate matters?

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A Tables & Figures

Table 1: Overview of GC tests

Test	Direction	Regress Z_t on...	Tested Coefficients
Standard	HF $\nrightarrow$ LF	$\ldots \underline{Z}_{t-p}$	$A^{HF\nrightarrow LF}$
	LF $\nrightarrow$ HF		$A^{LF\nrightarrow HF}$
TY/DL	HF $\nrightarrow$ LF	$\ldots \underline{Z}_{t-(p+1)}$	$A^{HF\nrightarrow LF}$
	LF $\nrightarrow$ HF		$A^{LF\nrightarrow HF}$
MF-dep	HF $\nrightarrow$ LF	$\ldots \underline{Z}_{t-p}$	(i) $A_1^{(1,2)}, A_2^{(1,2)}, \ldots, A_p^{(1,2)}$ (ii) $A^{HF\nrightarrow LF} \backslash \{A_1^{(1,2)}, A_2^{(1,2)}, \ldots, A_p^{(1,2)}\}$
	LF $\nrightarrow$ HF	$\ldots \underline{Z}_{t-p,y}$	$A^{LF\nrightarrow HF}$
MF-indep	HF $\nrightarrow$ LF	$\ldots \underline{Z}_{t-p,yx}$	$A^{HF\nrightarrow LF}$
	LF $\nrightarrow$ HF		$A^{LF\nrightarrow HF}$

Note: The set of coefficients corresponding to testing for GC from the HF to the LF series in the standard approach, i.e., the ones in (5), are denoted as $A^{HF\nrightarrow LF} = A_1^{(1,2)}, \ldots, A_1^{(1,m+1)}, \ldots, \ldots, A_p^{(1,2)}, \ldots, A_p^{(1,m+1)}$. Likewise for the reverse direction, i.e., the ones in (6): $A^{LF\nrightarrow HF} = A_1^{(2,1)}, \ldots, A_1^{(m+1,1)}, \ldots, \ldots, A_p^{(2,1)}, \ldots, A_p^{(m+1,1)}$. For generic sets of coefficients C and D, "$C \backslash D$" denotes "C without D".

Table 2: Granger Causality Tests; HF $\nrightarrow$ LF; Both series $I(0)$

	Standard			TY/DL			MF-dep			MF-indep		
		$m=2$			$m=2$			$m=2$			$m=2$	
T	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$
50	6.9	13.9	41.4	7.2	11.6	30.1	6.3	11.9	33.7	7.2	11.8	30.9
150	6.0	13.7	41.8	5.9	11.6	34.1	5.6	12.0	34.5	5.9	11.6	34.4
250	5.1	13.7	41.8	5.1	11.7	34.7	5.2	11.8	34.9	5.1	11.7	34.9
		$m=3$			$m=3$			$m=3$			$m=3$	
T	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$
50	7.6	16.8	48.9	7.9	13.0	30.4	6.7	14.0	40.4	7.7	13.2	33.7
150	5.7	15.0	50.2	5.9	11.7	35.5	5.5	12.9	42.7	6.0	11.9	37.0
250	5.5	15.3	50.3	5.6	12.3	37.1	5.2	13.5	42.7	5.6	12.5	37.9
		$m=4$			$m=4$			$m=4$			$m=4$	
T	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$
50	7.9	19.8	61.5	9.0	14.3	29.4	7.2	16.0	49.9	8.7	14.7	35.0
150	5.7	16.4	58.6	5.7	12.0	33.5	5.3	13.5	49.6	5.8	12.1	35.8
250	5.4	16.3	59.3	5.6	11.3	35.8	5.3	13.7	50.7	5.6	11.7	37.7

Note: The figures represent rejection rates of the various Granger non-causality tests described in Sections 2.2 - 2.3 (and summarized in Table 1). The underlying DGP is a reduced-form MF-VAR(2) in levels corresponding to Equations (9) and (10), implying that there is no Granger causality from x to y, i.e., HF $\nrightarrow$ LF. Here, $\rho = 0.8$, $\alpha = -0.5$ and $\theta = 0.5$ such that both series are stationary, i.e., $I(0)$. Moreover, $\lambda = T^{-0.5}\lambda^*$, whereby $\lambda^* = 0$ indicates the size and $\lambda^* = \{1, 2\}$ the power of the test. The nominal level is set to 5%.

Table 3: Granger Causality Tests; LF $\nrightarrow$ HF; Both series $I(0)$

	Standard			TY/DL			MF-dep			MF-indep		
		$m=2$			$m=2$			$m=2$			$m=2$	
T	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$
50	8.2	23.8	69.0	7.7	20.5	60.6	7.5	20.9	63.2	7.6	20.8	61.8
150	5.9	22.6	72.9	5.4	20.7	68.2	5.5	21.2	69.1	5.5	20.8	68.4
250	5.3	21.3	73.5	5.0	19.5	69.0	5.0	19.8	70.0	5.0	19.7	69.2
		$m=3$			$m=3$			$m=3$			$m=3$	
T	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$
50	9.3	30.0	79.6	9.6	26.7	70.6	9.3	27.4	74.4	9.1	27.1	73.0
150	6.4	27.3	84.4	6.4	25.3	79.4	6.5	25.8	80.9	6.4	25.3	80.1
250	5.8	27.1	85.3	5.7	24.9	81.3	5.6	25.7	82.2	5.8	25.2	81.7
		$m=4$			$m=4$			$m=4$			$m=4$	
T	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$
50	13.0	36.7	85.9	13.2	31.6	76.8	12.5	33.0	81.1	12.3	32.2	79.7
150	6.5	33.0	90.9	6.6	29.7	86.4	6.6	30.8	88.1	6.4	30.4	87.2
250	6.5	32.9	92.2	6.6	29.3	89.3	6.4	30.2	90.2	6.5	29.2	89.6

Note: The figures represent rejection rates of the various Granger non-causality tests described in Sections 2.2 - 2.3 (and summarized in Table 1). The underlying DGP is a reduced-form MF-VAR(2) in levels corresponding to Equations (11) and (12), implying that there is no Granger causality from y to x, i.e., LF $\nrightarrow$ HF. Here, $\rho = 0.8$, $\alpha = -0.5$ and $\theta = 0.5$ such that both series are stationary, i.e., $I(0)$. Moreover, $\delta = T^{-0.5}\delta^*$, whereby $\delta^* = 0$ indicates the size and $\delta^* = \{1, 2\}$ the power of the test. The nominal level is set to 5%.

Table 4: Granger Causality Tests; HF $\nrightarrow$ LF; Both series $I(1)$; Cointegration

	Standard			TY/DL			MF-dep			MF-indep		
		$m = 2$			$m = 2$			$m = 2$			$m = 2$	
T	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$
50	6.9	14.2	41.4	6.8	12.4	30.6	6.3	11.7	33.2	6.8	12.4	31.3
150	6.0	14.0	43.3	5.9	11.8	34.4	5.7	11.9	35.2	6.0	11.8	34.5
250	5.2	13.7	42.9	5.2	11.8	34.8	5.0	12.0	35.3	5.2	11.8	35.2
		$m = 3$			$m = 3$			$m = 3$			$m = 3$	
T	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$
50	7.4	17.0	50.0	7.9	13.4	30.8	6.8	14.0	40.8	7.8	13.6	33.6
150	5.7	15.4	54.0	6.0	11.8	35.7	5.5	13.0	45.0	5.9	12.0	36.9
250	5.5	16.4	54.4	5.4	12.2	37.1	5.0	13.8	45.4	5.5	12.5	38.0
		$m = 4$			$m = 4$			$m = 4$			$m = 4$	
T	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$	$\lambda^* = 0$	$\lambda^* = 1$	$\lambda^* = 2$
50	8.3	20.4	64.9	9.4	14.3	30.4	7.3	16.1	51.8	9.0	14.6	35.8
150	5.7	18.1	66.7	5.9	11.8	33.7	5.2	14.4	54.5	5.8	12.1	36.1
250	5.6	18.6	67.9	5.7	11.6	36.2	5.4	14.6	56.8	5.7	11.9	38.0

Note: $\rho = 1$, $\alpha = -0.5$ and $\theta = 0.5$ such that both series are non-stationary, i.e., $I(1)$. Moreover, there is an additional cointegrating relationship – on top of trivial ones among x itself – between y and x of the form $(-\theta, 1)$. For the rest see Table 2.

Table 5: Granger Causality Tests; LF $\nrightarrow$ HF; Both series $I(1)$; Cointegration

	Standard			TY/DL			MF-dep			MF-indep		
		$m = 2$			$m = 2$			$m = 2$			$m = 2$	
T	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$
50	8.5	23.4	68.5	7.6	20.4	60.7	7.6	21.7	63.6	7.6	20.8	62.0
150	5.9	22.4	72.4	5.6	20.4	67.6	5.7	20.7	68.7	5.6	20.5	67.8
250	5.7	22.5	72.5	5.3	20.5	68.2	5.4	20.8	69.2	5.3	20.5	68.4
		$m = 3$			$m = 3$			$m = 3$			$m = 3$	
T	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$
50	10.3	30.6	80.0	10.6	26.5	71.3	9.8	27.6	74.6	9.7	26.9	73.4
150	6.6	28.1	84.2	6.1	24.8	80.0	6.3	25.5	81.4	6.1	25.0	80.8
250	5.5	28.4	85.3	5.3	25.5	81.6	5.4	26.3	82.8	5.3	26.0	82.0
		$m = 4$			$m = 4$			$m = 4$			$m = 4$	
T	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$
50	12.5	36.3	86.3	13.4	31.9	77.5	12.1	33.4	81.7	12.2	32.5	80.5
150	6.7	33.4	90.7	6.7	29.9	86.1	6.8	30.8	87.8	6.7	30.5	86.9
250	5.9	32.0	91.7	6.1	28.9	88.5	6.1	30.0	89.5	6.2	29.4	89.0

Note: $\rho = 1$, $\alpha = -0.5$ and $\theta = 0.5$ such that both series are non-stationary, i.e., $I(1)$. Moreover, there is an additional cointegrating relationship – on top of trivial ones among x itself – between y and x of the form $(1, -\theta)$. For the rest see Table 3.

Table 6: Granger Causality Tests; HF $\nrightarrow$ LF; Both series $I(1)$; No Cointegration

	Standard			TY/DL			MF-dep			MF-indep		
		$m=2$			$m=2$			$m=2$			$m=2$	
T	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$
50	12.4	21.6	49.0	7.6	14.5	38.6	5.7	10.2	24.4	7.5	14.5	39.1
150	11.5	20.2	49.2	5.9	13.6	41.0	4.8	9.5	25.5	6.1	13.8	41.0
250	10.7	20.3	49.2	5.3	13.1	41.0	4.3	9.1	26.2	5.2	13.0	41.3
		$m=3$			$m=3$			$m=3$			$m=3$	
T	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$
50	12.3	24.0	60.0	8.3	17.9	48.6	6.4	11.9	32.9	8.0	17.6	50.6
150	10.3	21.8	62.0	6.0	15.6	53.5	5.3	10.7	36.8	6.1	15.7	54.2
250	9.6	22.6	62.2	5.6	15.6	54.6	4.9	11.4	38.2	5.5	15.9	54.8
		$m=4$			$m=4$			$m=4$			$m=4$	
T	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$
50	12.8	26.2	68.7	10.0	20.2	55.6	7.2	13.6	41.3	9.4	20.7	59.7
150	9.8	24.0	70.3	5.8	18.0	61.8	5.0	12.4	49.0	5.8	18.1	63.0
250	9.7	24.5	71.4	5.9	17.8	63.6	5.3	12.5	50.9	5.9	18.1	64.5

Note: $\rho = 1$, $\alpha = 0$ and $\theta = 0.5$ such that both series are non-stationary, i.e., $I(1)$, but *not* additionally cointegrated. For the rest see Tables 2 and 4.

Table 7: Granger Causality Tests; LF $\nrightarrow$ HF; Both series $I(1)$; No Cointegration

	Standard			TY/DL			MF-dep			MF-indep		
		$m=2$			$m=2$			$m=2$			$m=2$	
T	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$
50	14.0	26.0	61.8	8.5	18.8	52.1	8.5	19.3	53.9	8.4	18.8	52.9
150	11.3	24.8	64.6	5.7	17.0	57.2	5.8	17.6	57.9	5.7	16.8	57.7
250	11.3	24.8	64.6	5.7	17.1	57.7	5.7	17.5	58.3	5.7	17.2	57.9
		$m=3$			$m=3$			$m=3$			$m=3$	
T	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$
50	15.9	31.7	73.1	11.0	24.4	62.8	10.9	25.2	65.2	10.2	24.9	64.3
150	11.0	28.3	75.9	6.2	21.2	69.2	6.2	21.5	70.3	6.3	21.5	69.8
250	10.5	28.4	75.8	5.4	21.3	70.6	5.9	21.7	71.4	5.3	21.4	71.4
		$m=4$			$m=4$			$m=4$			$m=4$	
T	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$
50	17.9	36.9	80.4	13.5	29.8	70.8	12.9	30.2	74.3	12.7	29.7	73.3
150	11.2	31.4	82.8	6.6	24.3	76.7	7.1	25.0	78.3	6.6	24.7	77.9
250	10.7	30.2	83.7	6.2	24.1	79.4	6.3	24.4	80.2	6.4	24.4	80.0

Note: $\rho = 1$, $\alpha = 0$ and $\theta = 0.5$ such that both series are non-stationary, i.e., $I(1)$, but *not* additionally cointegrated. For the rest see Tables 3 and 5.

B MF-VAR Parameters

In the reduced-form MF-VAR(2) levels formulation, the formulae for A_1, A_2 and A^* for both DGPs in Section 3.2 are as follows. For the DGP used to test GC from x to y, i.e., (9) and (10), we have

$$
\begin{aligned}
A_1 &= \begin{bmatrix}
\rho & \lambda_0 & \lambda_1 & \dots & \lambda_{m-1} \\
\theta(\rho - (\alpha+1)^m) & (\alpha+1)^m + \theta\lambda_0 & \theta\lambda_1 & \dots & \theta\lambda_{m-1} \\
\theta(\rho - (\alpha+1)^{m-1}) & (\alpha+1)^{m-1} + \theta\lambda_0 & \theta\lambda_1 & \dots & \theta\lambda_{m-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\theta(\rho - (\alpha+1)^2) & (\alpha+1)^2 + \theta\lambda_0 & \theta\lambda_1 & \dots & \theta\lambda_{m-1} \\
\theta(\rho - \alpha - 1) & \alpha + 1 + \theta\lambda_0 & \theta\lambda_1 & \dots & \theta\lambda_{m-1}
\end{bmatrix}, \\
A_2 &= \begin{bmatrix}
0 & -\lambda_0 & -\lambda_1 & \dots & -\lambda_{m-1} \\
0 & -\theta\lambda_0 & -\theta\lambda_1 & \dots & -\theta\lambda_{m-1} \\
0 & -\theta\lambda_0 & -\theta\lambda_1 & \dots & -\theta\lambda_{m-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -\theta\lambda_0 & -\theta\lambda_1 & \dots & -\theta\lambda_{m-1} \\
0 & -\theta\lambda_0 & -\theta\lambda_1 & \dots & -\theta\lambda_{m-1}
\end{bmatrix}, \\
A^* &= \begin{bmatrix}
1 & 0 & 0 & \dots & \dots & 0 \\
\theta & 1 & \alpha+1 & \dots & \dots & (\alpha+1)^{m-1} \\
\theta & 0 & 1 & \dots & \dots & (\alpha+1)^{m-2} \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\theta & 0 & 0 & \dots & \ddots & \alpha+1 \\
\theta & 0 & 0 & \dots & \dots & 1
\end{bmatrix}.
\end{aligned}
$$

For the DGP used to test GC from y to x, i.e., (11) and (12), we have

$$A_1 = \begin{bmatrix} \alpha+1+\theta\tilde{\delta}^{m-1} & \theta(\rho^m-\alpha-1) & \\ \tilde{\delta}^{m-1} & \rho^m & \\ \tilde{\delta}^{m-2} & \rho^{m-1} & \\ \vdots & \vdots & 0_{(m+1)\times(m-1)} \\ \tilde{\delta}^{1} & \rho^2 & \\ \tilde{\delta}^{0} & \rho & \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -\theta\tilde{\delta}^{m-1} & \\ -\tilde{\delta}^{m-1} & \\ -\tilde{\delta}^{m-2} & \\ \vdots & 0_{(m+1)\times m} \\ -\tilde{\delta}^{1} & \\ -\tilde{\delta}^{0} & \end{bmatrix},$$

$$A^* = \begin{bmatrix} 1 & \theta & \theta\rho & \dots & \dots & \theta\rho^{m-1} \\ 0 & 1 & \rho & \dots & \dots & \rho^{m-1} \\ 0 & 0 & 1 & \dots & \dots & \rho^{m-2} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \ddots & \rho \\ 0 & 0 & 0 & \dots & \dots & 1 \end{bmatrix},$$

where $\tilde{\delta}^{\xi} = \sum_{i=0}^{\xi} \rho^i \delta_{m-1-\xi+i}$.

C High-Frequency DGP

To derive a common HF-DGP, we assume $Z^{(m)}_{t-j/m} = (y^{(m)}_{t-j/m}, x^{(m)}_{t-j/m})'$ with $j = 0, \ldots, m-1$ and $t = 1, \ldots, T$, is generated by one of the following VECMs, depending on whether we inspect...

...GC from x to y, i.e.,

$$\Delta^{1/m} Z^{(m)}_{t-j/m} = \begin{bmatrix} 0 & 0 \\ \theta & -\theta \end{bmatrix} Z^{(m)}_{t-(j+1)/m} + \begin{bmatrix} 0.5 & \lambda \\ 0.3 & 0.5 \end{bmatrix} \Delta^{1/m} Z^{(m)}_{t-(j+1)/m} + v^{(m)}_{t-j/m}, \tag{13}$$

..., or GC from y to x, i.e.,

$$\Delta^{1/m} Z^{(m)}_{t-j/m} = \begin{bmatrix} -\theta & \theta \\ 0 & 0 \end{bmatrix} Z^{(m)}_{t-(j+1)/m} + \begin{bmatrix} 0.5 & 0.3 \\ \delta & 0.5 \end{bmatrix} \Delta^{1/m} Z^{(m)}_{t-(j+1)/m} + v^{(m)}_{t-j/m}, \tag{14}$$

where $v^{(m)}_{t-j/m} \sim N(\mathbf{0}_{2\times 1}, I_2)$. Similar to the MF-case, $\lambda = (mT)^{-0.5}\lambda^*$ and $\delta = (mT)^{-0.5}\delta^*$ for $\lambda^*, \delta^* = \{0, 1, 2\}$, where the sample size is adjusted to the HF, in which the system is specified. θ governs the cointegrating relationship, yet – unlike in the MF-case – it controls the presence (e.g., $\theta = 1$) or absence ($\theta = 0$) of cointegration altogether.

Table 8: High-Frequency DGP; Granger Causality Tests; HF $\nrightarrow$ LF; Both series $I(1)$; Cointegration

	Standard			TY/DL			MF-dep			MF-indep		
	$m=2$			$m=2$			$m=2$			$m=2$		
T	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$
50	11.8	14.1	28.0	11.6	11.3	20.2	10.3	8.7	16.5	11.3	10.9	20.2
150	25.0	23.8	39.7	22.3	16.3	23.1	23.2	14.0	17.4	22.6	16.3	23.5
250	39.5	37.1	51.3	34.7	25.9	29.8	37.2	24.2	22.7	34.8	25.6	29.6
	$m=3$			$m=3$			$m=3$			$m=3$		
T	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$
50	15.4	23.4	37.7	15.2	19.6	28.6	12.5	19.0	30.4	15.2	20.5	30.6
150	35.1	53.0	70.4	32.4	44.7	57.5	30.8	45.1	60.0	33.2	46.0	60.5
250	58.1	75.3	88.0	54.2	67.9	79.0	52.9	68.7	81.0	54.9	69.0	80.7
	$m=4$			$m=4$			$m=4$			$m=4$		
T	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$
50	15.8	25.1	37.4	15.6	22.7	31.8	12.9	20.9	31.7	15.7	23.9	33.7
150	36.3	53.3	71.3	33.9	50.1	65.8	31.4	47.6	65.3	34.6	50.7	67.4
250	59.2	76.6	88.1	56.0	73.0	85.0	53.7	71.8	84.6	56.9	74.0	85.8

Note: The underlying DGP is the HF system in (13) with $\theta = 1$, implying that y and x are $I(1)$ and cointegrated. Moreover, $\lambda = (mT)^{-0.5}\lambda^*$, whereby $\lambda^* = 0$ indicates the size and $\lambda^* = \{1, 2\}$ the power of the test. The nominal level is set to 5%.

Table 9: High-Frequency DGP; Granger Causality Tests; LF $\nrightarrow$ HF; Both series $I(1)$; Cointegration

	Standard			TY/DL			MF-dep			MF-indep		
		$m = 2$			$m = 2$			$m = 2$			$m = 2$	
T	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$
50	6.8	11.6	28.2	7.2	11.5	24.6	7.4	10.8	23.9	7.5	10.8	23.8
150	5.7	11.6	32.6	5.8	9.9	25.8	5.7	9.7	25.1	5.9	9.7	25.2
250	5.2	11.2	32.9	5.1	9.6	25.6	5.2	9.4	25.3	5.1	9.5	25.1
		$m = 3$			$m = 3$			$m = 3$			$m = 3$	
T	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$
50	8.4	11.5	21.9	9.1	11.9	20.0	9.0	11.9	21.1	8.9	12.0	20.7
150	6.1	9.7	20.8	6.2	9.1	19.3	6.2	9.1	20.1	6.1	9.1	19.8
250	5.6	8.9	20.6	5.6	8.5	19.2	5.7	8.6	19.5	5.6	8.3	19.3
		$m = 4$			$m = 4$			$m = 4$			$m = 4$	
T	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$	$\delta^* = 0$	$\delta^* = 1$	$\delta^* = 2$
50	10.8	12.6	20.2	12.1	14.6	21.0	11.4	13.0	20.2	11.5	13.1	20.3
150	6.4	8.2	15.1	6.5	8.2	15.4	6.3	8.4	15.1	6.3	8.3	15.2
250	5.7	7.4	15.2	5.7	7.5	15.4	5.8	7.5	15.2	5.7	7.5	15.2

Note: The underlying DGP is the HF system in (14) with $\theta = 1$, implying that y and x are $I(1)$ and cointegrated. Moreover, $\delta = (mT)^{-0.5}\delta^*$, whereby $\delta^* = 0$ indicates the size and $\delta^* = \{1, 2\}$ the power of the test. The nominal level is set to 5%.

Table 10: High-Frequency DGP; Granger Causality Tests; HF $\nrightarrow$ LF; Both series $I(1)$; No Cointegration

	Standard			TY/DL			MF-dep			MF-indep		
	$m=2$			$m=2$			$m=2$			$m=2$		
T	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$
50	13.9	16.7	31.7	8.2	10.9	23.3	5.3	6.4	16.3	8.4	10.9	24.0
150	13.9	13.0	24.8	6.9	7.7	19.1	5.3	4.7	11.7	6.9	7.9	19.2
250	14.3	11.7	21.7	7.4	7.1	17.5	5.9	3.8	9.6	7.4	7.0	17.4
	$m=3$			$m=3$			$m=3$			$m=3$		
T	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$
50	13.7	14.4	20.7	9.5	9.7	14.9	6.0	6.5	11.6	9.1	9.6	15.2
150	12.0	10.9	16.5	6.9	6.5	11.0	4.8	4.7	8.5	7.0	6.5	11.2
250	13.2	10.5	14.3	7.6	5.8	9.9	5.7	3.8	7.1	7.6	5.9	9.8
	$m=4$			$m=4$			$m=4$			$m=4$		
T	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$	$\lambda^*=0$	$\lambda^*=1$	$\lambda^*=2$
50	12.5	13.9	18.0	9.9	10.2	13.9	6.3	6.8	10.1	9.5	9.7	13.6
150	11.4	11.5	15.1	7.4	7.5	10.3	5.0	5.2	8.0	7.2	7.4	10.2
250	11.9	11.1	14.0	7.3	6.9	9.1	5.1	4.7	6.8	7.5	6.8	8.8

Note: The underlying DGP is the HF system in (13) with $\theta = 0$, implying that y and x are $I(1)$, but not cointegrated. For the rest see Table 8.

Table 11: High-Frequency DGP; Granger Causality Tests; LF $\nrightarrow$ HF; Both series $I(1)$; No Cointegration

	Standard			TY/DL			MF-dep			MF-indep		
		$m=2$			$m=2$			$m=2$			$m=2$	
T	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$
50	15.1	19.0	31.6	9.6	12.5	23.0	9.7	12.4	23.3	9.7	12.7	22.9
150	12.4	16.4	28.7	6.7	9.5	21.1	6.7	9.8	21.5	6.5	9.5	21.3
250	11.3	15.9	29.4	5.5	9.0	21.3	5.5	9.0	21.9	5.5	9.1	21.7
		$m=3$			$m=3$			$m=3$			$m=3$	
T	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$
50	16.5	18.0	23.2	10.7	12.7	17.2	10.1	12.3	16.8	10.4	12.4	16.8
150	11.7	13.2	18.8	6.4	7.6	12.1	6.4	7.7	12.4	6.7	7.8	12.2
250	10.4	12.6	18.0	5.5	7.3	12.0	5.5	7.2	12.2	5.6	7.3	11.9
		$m=4$			$m=4$			$m=4$			$m=4$	
T	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$	$\delta^*=0$	$\delta^*=1$	$\delta^*=2$
50	18.5	19.6	22.9	13.9	15.1	17.4	12.6	13.7	16.6	12.7	14.0	16.8
150	11.0	12.0	14.8	7.0	7.7	9.6	6.8	7.7	9.5	6.8	7.7	9.4
250	9.9	10.8	13.1	5.8	6.6	8.8	5.6	6.7	9.1	5.8	6.7	8.8

Note: The underlying DGP is the HF system in (14) with $\theta = 0$, implying that y and x are $I(1)$, but not cointegrated. For the rest see Table 9.