Estimation of exchangeable distribution when the highest or lowest and another order statistics are observable: Application to first-price auctions[1]

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Summary:

In this paper, we study the estimation of the exchangeable joint distribution when the highest (lowest) and another order statistics are observable. The estimator would be appropriate for the estimation of the valuation distribution of auctions and "order-biased" sampling, such as school student achievement investigations, when the sampling probability is not completely captured by the observed covariates. We present our method in conjunction with an application for a first-price sealed-bid auction with affiliation. The results potentially extend the existing non-parametric identification result for first-price sealed-bid auctions with symmetric affiliation.

Keywords: Order statistics; Semi-parametric estimation; Exchangeable distribution; First-price Auction; Copula

JEL Classification: C40; C57

1 Introduction

In the empirical analyses in social sciences, the available information is often limited due to various reasons. For example, some residents do not reply to questionnaires, some students do not attend the assessment of academic abilities, or some auctioneers do not release full information on the bids of past auctions. If researchers assume respondents to be independent, it is possible to estimate the distribution of the population. However, the available information is often restricted and participation bias (non-response bias) has been reported in every science field related to human behavior (see, e.g., Fowler Jr, 2013). For example, motivated respondents are more likely to reply to questionnaire surveys, students whose scholastic standing is high are more likely to attend assessments of academic ability, or auctioneers often release information on the highest and second highest bids of past auctions. Additionally, individuals are often suspected of interacting with each other. For example, residents living on the same block, students learning in the same classroom, and bidders bidding for the same goods can physically or socially affect each other. Specifically, a rapidly growing body of studies on social networks (e.g., Jackson, Rogers, & Zenou, 2017; Wang et al., 2018) suggest dependence within groups, including for online interactions, such as social networking websites.

When such correlation are suspected within a group, assuming independence can bias estimation. Under this assumption, solutions are typically fully or partially specifying the data generating process or assuming observable covariates to explain response probability. In fact, such problems have been studied by numerous authors (e.g., Heckman, 1977; Qin, 2017). However, estimation using information on order statistics alone is not considered sufficient, while response probability is often ordered with respect to the variables of interest, such as bids in auctions. Here, we consider the case where response probability is ordered with respect to the variable of the researcher's interest, such as academic ability for students. Therefore, we study the estimation of the joint distribution of units (classroom, city block, etc.).

To this end, we apply the copula maximum likelihood estimator of Chen and Fan (2006). Specifically, we utilize the link between the marginal distributions of different order statistics from an N variate exchangeable joint distribution. First, we construct a link for the marginal distribution function of $n(= 1, ..., N-1)$ and Nth order statistics, which extend the study of He (2016), who identifies the Archimedean copula from $N-1$ and Nth order statistics.[1] Second, we propose a likelihood function for the nth order statistics, which is expressed by the copula and empirical marginal distribution of the Nth order statistics. Third, we substitute the nth order statistics into the likelihood function to estimate the copula parameter. Finally, given the copula parameter, we identify the joint distribution by utilizing the link between the marginal distribution and that of the Nth order statistics. The link function between the marginal distribution and that of order statistics is also utilized by Furmańczyk (2016) for maximum likelihood. However, the marginal distribution is not observable in our problem.

We conduct simulations to study finite sample properties. As other plug-in semiparametric estimator, the estimator suffers some finite sample bias, which is not severe in Hubbard, Li, and Paarsch (2012) since their estimator is constructed by $N \times T$ observations. The simulation result suggests that the future development of higher moment expansion would help reduce small sample bias. However, the simulation results also suggest that a small number of observations are sufficient for decreasing finite sample bias. Additionally, we find that strong correlations between variables help with the precise estimation of the joint distribution. Further, the identification of the marginal distribution from that of order statistics is easy when observations are independent. However, our simulation results suggest that strong correlations help estimate the marginal distribution efficiently while weak correlations do not. As such, one can precisely estimate the joint distribution if a strong rank correlation is observed in the first-stage estimation of the copula parameter, even if the true marginal distribution is unknown.

The first contribution of this study is the construction of a likelihood function for the estimation of the joint distribution when particular order statistics are observed. To this end, we propose an equation that links the highest or lowest and another order statistics for the exchangeable distribution. As a result, one can estimate the parameter of the marginal distribution using the proposed equation even if only limited information of order statistics is

[1] He (2016) tries to derive a nonparametric analytical expression for generating the function of the Archimedean copula. Unfortunately, the expression is integral of joint density $f^{(N-1:N,N:N)}(x,x)$, which is always zero. However, the property that can recover the copula from the highest and second highest order statistics of bids is derived from the exchangeable joint distribution. Therefore, one can apply this idea for constructing a link for n and Nth order statistics.

available. When one can assume independence or identify individuals, the link function does not provide additional information. However, participation bias is frequently reported for data related to human behavior.[2] In such cases, it is difficult to believe the sampling is unbiased or individuals are independent. The link function provides a solution for such contaminated sampling problem. If N and nth order statistics of the size N group are missing at random, one can estimate the joint distribution even if another order statistics is not missing at random.

The second contribution is the estimation of the parameter of limited information auctions for the exchangeable distribution. The estimator studied in this paper requires the highest or lowest and another order statistics of valuation for the semiparametric estimation of joint distribution of valuation in the APV paradigm. While joint distributions are often assumed to be common knowledge for bidders, there are few possibilities for bidders to know the distribution when values are correlated because, in many cases, such information is not provided. At the same time, most econometric methods, such as in Campo, Perrigne, and Vuong (2003), Li, Perrigne, and Vuong (2002), and Hubbard et al. (2012), for identifying the joint distribution of valuation for APV require full observation of bids. This fact is an obstacle to elegant results of auction theory being applicable to auction in the real world. Recently, He (2016), following Athey and Haile (2002) and Athey and Haile (2007), identifies the exchangeable joint distribution by deriving the analytical expression of the Archimedean copula by utilizing the highest and second highest (lowest) bids. However, his expression of the Archimedean copula is integral of zero except for special cases. Hence, estimating the joint distribution following this expression is often difficult. Our results on the highest (lowest) and another order statistics suggest that He (2016)'s idea is feasible through maximum likelihood and can be extended to the highest and another order statistics case. As such, the results remove an obstacle to applying auction theory to the real world.

Additionally, as a third contribution, our results allow applying Hubbard et al. (2012)'s two step approach for symmetric APV auctions to the case where reserve price exists. Hubbard et al. (2012)s estimator cannot be applied when there is reserve price because their likelihood

[2] For example, Porter and Umbach (2006) find that students with high grade point averages and self-ratings of academic ability are more likely to respond to surveys in college. Although controversial, there are empirical studies that suggest exam scores are affected by social interaction (e.g., Angrist & Lang, 2004; Abdulkadiroğlu, Angrist, & Pathak, 2014).

function approach requires a likelihood function for the joint distribution. By contrast, our strategy uses limited information. Since the copula parameter is determined by rank correlation and the bid function is strictly monotonic for bidders whose valuation is higher than the reserve price, the strategy can access the copula parameter even if there is a reserve price.

The fourth and final contribution is finding a sufficient condition for deriving semi-analytical expression for the bid function when bidders valuation is related to an exchangeable copula. We utilize the semi- analytical expression of the bid function to simulate the bid under affiliation in a Monte Carlo simulation.[3]

The remainder of this paper is organized as follows. In the next section, we present the estimator based on the link between distributions of order statistics. We provide a description of the copula in this section as well. In section 3, we apply the estimator for a symmetric APV paradigm auction. First, we present the model and, second, we propose a two-step estimator based on Hubbard et al. (2012) and the semiparametric estimator in section 2. In section 4, we present the results of the simulation. In section 5, we discuss the result and conclude the study.

2 Semiparametric Maximum Likelihood Estimator

Here, we first present the link between the marginal distribution of n and the Nth order statistics. Then, we construct the objective function for identifying the joint distribution with the link.

2.1 Likelihood based on order statistic

Throughout the paper, we assume the exchangeability of the random variable. Additionally, we assume the monotonicity of the equicoordinate expression to establish the link between marginal distribution and that of Nth order statistics.

Assumption 2.1 (Exchangeable joint distribution)**.** *Joint distribution $\boldsymbol{F}$ is exchangeable if*

[3] He (2016) derives a similar semi-analytical expression for the Archimedean copula, which is exchangeable.

$$\boldsymbol{F}(x_1, ..., x_N) = \boldsymbol{F}(x_{\sigma(1)}, ..., x_{\sigma(N)}) \tag{1}$$

holds for all permutations $\sigma \ : \{1, ..., N\} \to \{1, ..., N\}$ *and for all* $(x_1, ..., x_N)$*.*

The joint distribution is expressed by the copula function, which is shown by Sklar (1959) as per Lemma 2.1.

Lemma 2.1 (Sklar, 1959)**.** *For any* N *dimensional continuous distribution function* $\boldsymbol{F}(\cdot, ..., \cdot)$*, there exists unique* N *dimensional copula function* $C(\cdot, ..., \cdot)$ *so that*

$$\boldsymbol{F}(x_1, x_2, ..., x_N) = C(F_1(x_1), ..., F_N(x_N)),$$

where, $F_n(\cdot)$ *is the marginal distribution of the* n*th coordinate. Additionally, if* $\boldsymbol{F}(\cdot, ..., \cdot)$ *is exchangeable,*

$$\boldsymbol{F}(x_1, x_2, ..., x_N) = C(F(x_1), ..., F(x_N)).$$

Definition 2.1 (Exchangeable copula with inverse function)**.** *Let exchangeable copula function* $[0, 1]^N \to [0, 1]$ *be expressed as per equation (2):*

$$C(u_1, ..., u_N) = \boldsymbol{\Psi}(\Psi^{\leftarrow}(u_1), ..., \Psi^{\leftarrow}(u_N)), \tag{2}$$

where $\boldsymbol{\Psi} : [0, 1]^N \to [0, 1]$ *is strictly increasing on each coordinate and exchangeable for any permutation* $\{0, ..., N\} \to \{0, ..., N\}$*. Then, the equicoordinate expression of the distribution of the exchangeable random variable is expressed as the univariate function (3). In this paper, we call this function the equicoordinate function of the copula.*

$$C_{eq}(F(x)) = \boldsymbol{\Psi}(\Psi^{\leftarrow}(F(x)), \Psi^{\leftarrow}(F(x)), ..., \Psi^{\leftarrow}(F(x))). \tag{3}$$

If joint distribution $\boldsymbol{F}$ has a positive Lebesgue density almost everywhere on the equicoordinate vector, $C_{eq}(\cdot)$ is a univariate monotone increasing function and one can define inverse function $C_{eq}^{\leftarrow}(\cdot)$. For example, Archimedean, elliptical, and Plackett copulas are exchangeable and they have inverse function $C_{eq}^{\leftarrow}(\cdot)$. In the following, we assume the copula is determined by the link function and parameter θ. We discuss the problem of estimating θ given the family of link functions. Therefore, copula C depends on θ. For simplicity of notation, we omit θ to express the copula if the correspondence is clear from the context of the presentation.

Example 2.1 (Archimedean copula)**.** *The Archimedean copula is expressed as:*

$$C(u_1, ..., u_N; \theta) = \zeta^{\leftarrow}\left(\sum_{i=1}^N \zeta\left(u_i; \theta\right); \theta\right)$$

where $\zeta(\cdot; \theta)$ is a generating function that satisfies $\zeta(1; \theta) = 0$, $\zeta_u(u; \theta) < 0$, $\zeta_{uu}(u; \theta) \geq 0$ for all $u \in (0, 1)$, where ζ_u and ζ_{uu} are the first and second derivatives of ζ with respect to u. The Archimedean copula is exchangeable because of the commutative law of addition. One can easily see the Archimedean copula has $C_{eq}^{\leftarrow}$ if one lets $\Psi^{\leftarrow}(u; \theta) = \zeta(u; \theta)$ and $\mathbf{\Psi}(u_1, ..., u_N; \theta) = \zeta^{\leftarrow}(\sum_{i=1}^N u_i; \theta)$. The popular Clayton, Gumbel, and Frank copula belongs to the Archimedean family. One can find variety in generating the function in Nelsen (2007).

Example 2.2 (Elliptical copula)**.** *The elliptical copula is a copula where $\mathbf{\Psi}$ is a joint distribution of an N variate elliptically contoured distribution and Ψ is a marginal distribution of an elliptically contoured distribution. Whether an elliptical copula is exchangeable is determined by coordinates of the symmetric positive semi-definite matrix. For example, the Normal and t copulas belong to the elliptical copulas. For the Normal and t copula, whether copulas are exchangeable is determined by their covariance functions. These copulas are exchangeable if and only if the off-diagonal elements of the covariance matrix are the same (see, e.g., Theorem 4.2.8 of Harder, 2016.)*

For the Normal copula, $\mathbf{\Psi} = \mathbf{\Phi}\left(\cdot, ...\cdot; \mathbf{\Sigma}(\theta)\right)$ and $\Psi = \Phi$, $C\left(u_1, ..., u_N\right) = \mathbf{\Phi}\left(\Phi^{\leftarrow}(u_1), ...\Phi^{\leftarrow}(u_N); \theta\right)$, where $\mathbf{\Sigma}(\theta) = \theta\mathbf{1}\mathbf{1}' + \mathrm{diag}(1-\theta)$ is the covariance matrix of the multivariate normal distribution and Φ is the distribution function of standard normal distribution. By Theorem 5.3.3 of Tong (2012), $\mathbf{\Phi}\left(\Phi^{\leftarrow}(u_1), ...\Phi^{\leftarrow}(u_N); \theta\right) = \int_{-\infty}^{\infty} \prod_{i=1}^N \Phi\left(\frac{\Phi^{\leftarrow}(u_i)+\sqrt{\theta}z}{\sqrt{1-\theta}}\right)\phi(z)dz$. Therefore, $C_{eq}(F(x)) = \int_{-\infty}^{\infty} \Phi\left(\frac{\Phi^{\leftarrow}(F(x))+\sqrt{\theta}z}{\sqrt{1-\theta}}\right)^N \phi(z)dz$. One can derive the inverse function by solving the nonlinear equation that includes $C_{eq}(F(x))$.

If the copula is an exchangeable t copula, it is expressed by a two-dimensional integral,

$\mathbf{\Psi}\left(\Psi^{\leftarrow}(u_1),...\Psi^{\leftarrow}(u_N);\theta\right) = \int_0^\infty \left[\int_{-\infty}^\infty \prod_{i=1}^N \Phi\left(\frac{\Phi^{\leftarrow}(u_i)s+\sqrt{\theta}z}{\sqrt{1-\theta}}\right)\phi(z)dz\right]\psi_\nu(s)ds$. *The expression is derived from Theorem 5.3.3 and equation 9.0.1 of Tong (2012). Therefore,* $C_{eq}(F(x)) = \int_0^\infty \left[\int_{-\infty}^\infty \Phi\left(\frac{\Phi^{\leftarrow}(u_i)s+\sqrt{\theta}z}{\sqrt{1-\theta}}\right)^N \phi(z)dz\right]\psi_\nu(s)ds$.

To estimate the joint distribution with the copula, there are several alternatives. For example, maximum likelihood and the measure of concordance are frequently applied. These procedures require identification of individuals. Therefore, if missing data occur systematically, these ordinal approaches can result in biased estimation because of the biased sampling population.

Hence, we study the case where specific order statistics are observed without individual identification. Even if all coordinates of the joint distribution are not observed, the underlying correlation structure between coordinates is reflected by the distance between the marginal distributions of the different order statistics. For example, if coordinates have strong positive correlations, the different order statistics of the random variables from the joint distribution become closes. Then, for example, the distribution function of the highest and second highest order statistics become close. Therefore, if one constructs a link between the distribution functions of different order statistics using the copula, one can estimate the strength of dependence through the link equation.

To begin, we present the basic equations that reflect the relationship between the marginal distribution and marginal distributions of order statistics. Given the copula family, if the copula is determined by parameter θ, one can construct the likelihood function for the nth order statistics, which is expressed by the Nth order statistics and the copula parameter. Hereinafter, we assume θ is a finite dimensional vector.

Lemma 2.2. *Let $F(\cdot)$ be the distribution function and $f(\cdot)$ its density. Let $F^{(n:N)}(\cdot)$ be the distribution function of the nth order statistics of size N drawn from $\boldsymbol{F}(\cdot,...,\cdot)$. If* (i) *the random vector is exchangeable,* (ii) *the joint distribution is expressed by the copula, and* (iii) *the joint distribution has positive Lebesgue density on the equicoordinate line, then the distribution functions of n and the Nth order statistics follow equations (4), (5), and (6).*

$$F^{(N:N)}(x) = C_{eq}\left(F(x)\right), \tag{4}$$

$$F(x) = C_{eq}^{\leftarrow}\left(F^{(N:N)}(x)\right) \tag{5}$$

$$F^{(n:N)}(x) = F^{(N:N)}(x) + \sum_{k=1}^{N-n}\binom{N}{k}\left\{(-1)^k\sum_{j=0}^{k}\binom{k}{j}(-1)^j C\left(\underbrace{1,...,1}_{j}, C_{eq}^{\leftarrow}\left(F^{(N:N)}(x)\right),...,C_{eq}^{\leftarrow}\left(F^{(N:N)}(x)\right)\right)\right\}, \tag{6}$$

where copulas $C(\cdot,...,\cdot)$ and $C_{eq}(\cdot)$ depend on parameter θ.

Equations (4) and (5) of Lemma 1 suggest it is possible to identify the marginal distribution by using the marginal distribution of the Nth order statistics, which is identified if Nth order statistics and θ are observable. Once the copula parameter is determined, one can identify the marginal distribution as per the second equation. Equation (6) describes the link between the Nth order statistics and $n(=1,...,N-1)$th order statistics with the copula. Equation (6) is derived from these two equations and the recursive application of exchangeable property. A similar recursive formula is derived for the exchangeable joint distribution(see, e.g., Navarro & Spizzichino, 2010; Bairamov & Parsi, 2011). Our result transforms the equation to link the n and Nth order statistics by utilizing the copula. The equations suggest that, if the marginal distributions of the nth and Nth order statistics are observable, one can estimate copula parameter θ from the likelihood function.

Lemma 2.3. *If* (i), (ii), (iii) *of Lemma 2.2 are satisfied, the marginal densities of the n and Nth order statistics follow equation (7)*

$$\begin{aligned} f^{(n:N)}(x) =& f^{(N:N)}(x)\left\{1 + \sum_{k=1}^{N-n}\binom{N}{k}\left[(-1)^k\sum_{j=0}^{k}\binom{k}{j}(-1)^j\right.\right. \\ & \left.\left.\left(\frac{N-j}{N}\right)\frac{c_{u_N}\left(\overbrace{1,...,1}^{j}, C_{eq}^{\leftarrow}\left(F^{(N:N)}(x)\right),...,C_{eq}^{\leftarrow}\left(F^{(N:N)}(x)\right)\right)}{c_{u_N}\left(C_{eq}^{\leftarrow}\left(F^{(N:N)}(x)\right),...,C_{eq}^{\leftarrow}\left(F^{(N:N)}(x)\right)\right)}\right]\right\}, \end{aligned} \tag{7}$$

where $c_{u_N}(\cdot,...,\cdot)$ is the partial derivative of $C(\cdot,...,\cdot)$ with respect to the Nth coordinate. $c_{u_N}(\cdot,...,\cdot)$ and $C(\cdot,...,\cdot)$ depends on parameter θ.[4]

From the result of Lemma 2.3, one can substitute the nth order statistics for x in equation

[4] When copula $C(\cdot,...,\cdot)$ is expressed by link function $\boldsymbol{\Psi}(\cdot,...,\cdot)$ and $\Psi^{\leftarrow}(\cdot)$, one can rewrite the expression as

(7). Then, one can derive the right-hand side of equation (7) as the likelihood function for $F^{(n:N)}$ and the copula parameter. If the Nth order statistics are observable, $F^{(N:N)}$ is identified. Then, the likelihood function can be used for estimating the copula parameter. When nth and first order statistics are observed, one can multiply (-1) to the data and estimate $P(-X \leq b)$ and copula parameter using $P((-X)^{(N:N)} \leq x)$ and lemma 2.3. Then, one can estimate $P(X \leq x)$ from $F(x) = P(X \leq x) = \int_{-\infty}^{x} \frac{d}{dt} P(-X \leq t)|_{t=-s} ds$. Joint distribution $C(F(x_1), ..., F(x_N))$ is estimated from the estimated copula parameter and $F(x)$. When the copula belongs to the Archimedean family, one can derive a simple analytical expression because the Archimedean copula is constructed using the inverse and sum of differentiable generating functions. While the elliptical copula has complicated expression that includes integrals, the numerical calculation of $C_{eq}^{\leftarrow}$ is possible (see, e.g., Bornkamp, 2018). Additionally, the exchangeable property simplifies the calculation of c_N, as in the following example.

Example 2.3 (exchangeable normal copula)**.** *By interchanging integration and differentiation,*

$$\frac{\partial \mathbf{\Phi}\left(\Phi^{\leftarrow}(u_1), ...\Phi^{\leftarrow}(u_N); \theta\right)}{\partial u_N} = \int_{-\infty}^{\infty} \frac{\phi\left(\frac{\Phi^{\leftarrow}(u_N)+\sqrt{\theta} z}{\sqrt{1-\theta}}\right) \phi(z)}{\phi\left(\Phi^{\leftarrow}(u_N)\right)\sqrt{1-\theta}} \prod_{i=1}^{N-1} \Phi\left(\frac{\Phi^{\leftarrow}(u_i) + \sqrt{\theta} z}{\sqrt{1-\theta}}\right) dz.$$

Since the right-hand side of the above equation is the integral from $-\infty$ to ∞ with standard normal density, the expression is approximated with a Monte Carlo integration such as

$$\frac{1}{R} \sum_{r=1}^{R} \frac{1}{\phi\left(\Phi^{\leftarrow}(u_N)\right)\sqrt{1-\theta}} \prod_{i=1}^{N-1} \Phi\left(\frac{\Phi^{\leftarrow}(u_i) + \sqrt{\theta} z_r^*}{\sqrt{1-\theta}}\right) \phi\left(\frac{\Phi^{\leftarrow}(u_N) + \sqrt{\theta} z_r^*}{\sqrt{1-\theta}}\right),$$

where $z_r^*(r = 1, ..., R)$ are random draws from a standard normal distribution. From this expression, one can calculate c_{u_N}. The representation is also calculated by numerical integration because the integral essentially requires integration with respect to the one-dimensional variable, z.

$\frac{c_{u_N}\left(1,...,1,C_{eq}^{\leftarrow}\left(F^{(N:N)}(x)\right),...,C_{eq}^{\leftarrow}\left(F^{(N:N)}(x)\right)\right)}{c_{u_N}\left(C_{eq}^{\leftarrow}\left(F^{(N:N)}(x)\right),...,C_{eq}^{\leftarrow}\left(F^{(N:N)}(x)\right)\right)} = \frac{\mathbf{\Psi}_{u_N}\left(\Psi^{\leftarrow}(1),...,\Psi^{\leftarrow}(1),\Psi^{\leftarrow}\left(C_{eq}^{\leftarrow}\left(F^{(N:N)}(x)\right)\right),...,\Psi^{\leftarrow}\left(C_{eq}^{\leftarrow}\left(F^{(N:N)}(x)\right)\right)\right)}{\mathbf{\Psi}_{u_N}\left(\Psi^{\leftarrow}\left(C_{eq}^{\leftarrow}\left(F^{(N:N)}(x)\right)\right),...,\Psi^{\leftarrow}\left(C_{eq}^{\leftarrow}\left(F^{(N:N)}(x)\right)\right)\right)}$ for all j, where $\mathbf{\Psi}_{u_N}$ is the partial derivative of $\mathbf{\Psi}$ with respect to the Nth coordinate. From this expression, one can easily calculate the density when the joint distribution is expressed by the Archimedean or elliptical copulas by applying examples 2.1 and 2.2.

Example 2.4 (exchangeable t copula)**.** *By interchanging integration and differentiation,*

$$\frac{\partial \mathbf{\Psi}\left(\Psi^{\leftarrow}(u_1),...\Psi^{\leftarrow}(u_N);\theta\right)}{\partial u_N}$$
$$=\int_0^{\infty}\left[\int_{-\infty}^{\infty}\frac{s\phi(z)\phi\left(\frac{\Phi^{\leftarrow}(u_N)s+\sqrt{\theta}z}{\sqrt{1-\theta}}\right)}{\phi\left(\Phi^{\leftarrow}(u_N)\right)\sqrt{1-\theta}}\prod_{i=1}^{N-1}\Phi\left(\frac{\Phi^{\leftarrow}(u_i)s+\sqrt{\theta}z}{\sqrt{1-\theta}}\right)dz\right]\psi_{\nu}(s)ds.$$

One can calculate c_{u_N} from this expression by numerical integration or Monte Carlo integral because the integral essentially requires integration with respect to two-dimensional variables, z and s.

Corollary 2.1. *Let $F(\cdot)$ be the distribution function and $f(\cdot)$ its density. Let $F^{(n:N)}(\cdot)$ be the distribution function of the nth order statistics of size N drawn from $F(\cdot)$. If* (i), (ii), (iii) *of Lemma 2.2 are satisfied, the marginal distributions of the highest order and second highest order statistics have the following relationship:*

$$\begin{aligned}F^{(N-1:N)}(x)=&(1-N)F^{(N:N)}(x)\\&+NC\left(1,C_{eq}^{\leftarrow}\left(F^{(N:N)}(x)\right),...,C_{eq}^{\leftarrow}\left(F^{(N:N)}(x)\right)\right),\end{aligned}\tag{8}$$

$$\begin{aligned}f^{(N-1:N)}(x)=&(1-N)f^{(N:N)}(x)\\&+f^{(N:N)}(x)\left(\frac{N-1}{N}\right)\frac{c_{u_N}\left(1,C_{eq}^{\leftarrow}\left(F^{(N:N)}(x)\right),...,C_{eq}^{\leftarrow}\left(F^{(N:N)}(x)\right)\right)}{c_{u_N}\left(C_{eq}^{\leftarrow}\left(F^{(N:N)}(x)\right),...,C_{eq}^{\leftarrow}\left(F^{(N:N)}(x)\right)\right)}.\end{aligned}\tag{9}$$

When the copula belongs to the Archimedean family, Corollary 2.1 gives the maximum likelihood counterpart of Theorem 3.3.2 of He (2016) by applying the nonparametric estimation method and shape restriction on the generating function.

2.2 Semiparametric estimator

Here, we present the semiparametric maximum likelihood estimator and its asymptotic property for the copula parameter based on the expression of the density function of order statistics. From Lemma 2.3, one can derive the log likelihood function of nth order statistics $X_t^{(n:N)}$, which includes the copula and $F^{(N:N)}\left(X_t^{(n:N)}\right)$. When the Nth order statistics of each t are observable, one can replace $F^{(N:N)}$ with its nonparametric estimator. Then, the objective function for estimating copula parameter θ is as per equation (10), noting that $c_{u_N}(\cdot, ..., \cdot)$ and $C_{eq}(\cdot)$ depend on parameter θ.

$$Q(\theta) = \frac{1}{T}\sum_{t=1}^{T} \ell(\hat{F}^{(N:N)}(X_t^{(n:N)}); \theta), \tag{10}$$

where

$$\ell(F^{(N:N)}(X_t^{(n:N)}); \theta) = \log\Bigg[1 + \sum_{k=1}^{N-n}\binom{N}{k}(-1)^k$$
$$\times\left[\sum_{j=0}^{k}\binom{k}{j}(-1)^j\left(\frac{N-j}{N}\right)\frac{c_{u_N}\left(\overbrace{1,...,1}^{j}, C_{eq}^{\leftarrow}\left(F^{(N:N)}(X_t^{(n:N)})\right), ..., C_{eq}^{\leftarrow}\left(F^{(N:N)}(X_t^{(n:N)})\right)\right)}{c_{u_N}\left(C_{eq}^{\leftarrow}\left(F^{(N:N)}(X_t^{(n:N)})\right), ..., C_{eq}^{\leftarrow}\left(F^{(N:N)}(X_t^{(n:N)})\right)\right)}\right]\Bigg] \tag{11}$$

and

$$\hat{F}^{(N:N)}(x) = \frac{1}{T}\sum_{t=1}^{T}\mathbf{1}\left[X_t^{(N:N)} \leq x\right]. \tag{12}$$

The estimator for θ is a semiparametric M estimator including the copula given empirical distribution function. Since the score function and its derivatives could reach infinity near boundaries, we impose the assumption of Chen and Fan (2006):

Assumption C1. $\{(X_{1t}, ..., X_{Nt})\}_{t=1}^{T}$ *is a random sample from the joint distribution modeled by the Archimedean copula, where* $F(\cdot)$ *is absolutely continuous with respect to the Lebesgue measure on the real line and* $C(\cdot, ..., \cdot; \theta_0) = \mathbf{\Psi}(\Psi^{\leftarrow}(\cdot; \theta_0), ..., \Psi^{\leftarrow}(\cdot; \theta_0); \theta_0)$ *is the true parametric copula for* $(X_{1t}, ..., X_{Nt})$*, which is absolutely continuous with respect to the Lebesgue measure on* $[0, 1]^N$*, has a equicoordinate inverse function, and does not attain either the lower or upper bounds of Frechet-Hoeffding.*

Assumption C2. $\theta_0 \in \Theta$*, a compact set in* $\mathbf{R}$*, and* $E\left[\ell_\theta(F^{(N:N)}(X_t^{(n:N)}); \theta)\right]$ *equals zero, if and only if* θ *equals* θ_0*.*

Assumption C3. $\ell_\theta(u; \theta)$ *is defined for* $(u, \theta) \in [0, 1] \times \Theta$*; and for all* $\theta \in \Theta$*,* $\ell_\theta(u; \theta)$ *is Lipschitz continuous for* θ *with probability one;* $\ell_{u\theta}(u; \theta)$ *are defined and continuous in* $(u, \theta) \in [0, 1] \times \Theta$*.*

Assumption C4. *$E\left[\sup_{\theta\in\Theta}\left\|\ell_\theta(F^{(N:N)}(X_t^{(n:N)});\theta)\right\|\right] < \infty$.*

Assumption C5. *$E\left[\sup_{\theta\in\Theta, F^{(N:N)}\in\mathcal{F}_\delta^{(N:N)}}\left\|\ell_{u\theta}(F^{(N:N)}(X_t^{(n:N)});\theta)\right\| w(F^{(N:N)}(X_t^{(n:N)}))\right] < \infty$, where $\mathcal{F}_\delta^{(N:N)}$ equals $\{F \in \mathcal{F}^{(N:N)} : \|F - F_0\|_{\mathcal{F}} \leq \delta\}$ with $\mathcal{F}^{(N:N)}$ being the space of the continuous probability distributions of the Nth order statistics over the support of $X_t^{(N:N)}$ and $w(v)$ denotes $[v(1-v)]^{1-\xi}$ for $v \in (0,1)$ and $\xi \in (0,1)$.*

Proposition 1. *Under assumptions C1–C5, $\hat{\theta} \to \theta_0$ in probability.*

Assumption D1.

1. *$E\left[\ell_\theta(F^{(N:N)}(X_t^{(n:N)});\theta)\right] = 0$ if and only if $\theta = \theta_0 \in int(\Theta)$.*
2. *$\mathbf{\Gamma} \equiv -E\left[\ell_{\theta\theta}(F^{(N:N)}(X_t^{(n:N)});\theta_0)\right]$ is positive definite.*
3. *$\mathbf{\Lambda} \equiv \lim_{T\to\infty} Var\left(\sqrt{T}\mathbf{A}_T\right)$ is positive definite, where $\mathbf{A}_T \equiv \frac{1}{T}\sum_{t=1}^{T}\left[\ell_\theta(F^{(N:N)}(X_t^{(n:N)});\theta_0) + \mathcal{W}(X_t^{(n:N)})\right]$ with $\mathcal{W}(X_t^{(n:N)}) = \int_{-\infty}^{\infty}\ell_{u\theta}(F^{(n:N)}(v);\theta_0)\left[\mathbf{1}(X_t^{(n:N)} \leq v) - F^{(n:N)}(v)\right] f^{(n:N)}(v)dv$.*
4. *$\hat{\theta} = \theta_0 + o_p(1)$ and $\sup_x\left|\left[\hat{F}^{(N:N)}(x) - F^{(N:N)}(x)\right]/w\left[F^{(N:N)}(x)\right]\right| = O_p(T^{-\frac{1}{2}})$.*

Assumption D2. *$\ell_\theta(u;\theta)$ and $\ell_{u\theta}(u;\theta)$ are all defined, and continuous in $(u,\theta) \in [0,1] \times int(\Theta)$.*

Assumption D3. *$\ell_\theta(u;\theta_0)$ is defined for $u \in [0,1]\times\Theta$ and, for all $\theta_0 \in \Theta$; $\ell_\theta(u;\theta)$, is Lipschitz continuous in θ almost surely; $\ell_{u\theta}(u;\theta_0)$ is defined and continuous in $(u;\theta_0) \in [0,1]^N \times \Theta$.*

Assumption D4. *It is valid to interchange the order of differentiation and integration of $\ell_\theta(F_\eta^{(N:N)}(x);\theta_\eta)$ with respect to $\eta \in (0,1)$, where $\{(\theta_\eta, F_\eta^{(N:N)}) : \eta \in [0,1]\} \subset \tilde{\mathcal{F}}_\delta^{(N:N)} \equiv \{(\theta, F) \in \Theta \times \mathcal{F}_\delta^{(N:N)} : \|\theta - \theta_0\| \leq \delta\}$ for a small δ is a one-dimensional smooth path in $\mathcal{F}_\delta^{(N:N)}$.*

Assumption D5. *Both $E\left[\sup_{(\tilde{\theta},\tilde{F}^{(N:N)})\in\tilde{\mathcal{F}}_\delta^{(N:N)}}\left\|\ell_\theta(\tilde{F}^{(N:N)}(X_t^{(n:N)});\tilde{\theta})\right\|\right]^2 < \infty$ and $E\left[\left\|\mathcal{W}(X^{(n:N)})\right\|\right]^2 < \infty$.*

Assumption D6. *$E\left[\sup_{(\tilde{\theta},\tilde{F}^{(N:N)})\in\tilde{\mathcal{F}}_\delta^{(N:N)}}\left\|\ell_{\theta\theta}(\tilde{F}^{(N:N)}(X_t^{(n:N)});\tilde{\theta})\right\|\right]^2 < \infty$.*

Assumption D7. $E\left[\sup_{(\tilde{\theta},\tilde{F}^{(N:N)})\in\tilde{\mathcal{F}}_{\delta}^{(N:N)}}\left\|\ell_{u\theta}(\tilde{F}^{(N:N)}(X_t^{(n:N)});\tilde{\theta})\right\|w(X^{(n:N)})\right]^2<\infty.$

Proposition 2. *Under assumptions D1–D7,* $\sqrt{T}(\hat{\theta}-\theta_0)\to N(0,\Gamma^{-1}\Lambda\Gamma^{-1})$ *in the distribution.*

These asymptotic results are an application of Chen and Fan (2006). Here, we assume the independence between different t. The asymptotic result of Chen and Fan (2006) is derived under the assumption that X_t for a $\beta-mixing$ sequence. Hence, they impose slightly stricter moment conditions than assumptions C4, C5, D5, D6, and D7. However, without dependence assumption C4, C5 is enough for consistency and D5, D6,and D7 are enough for normality. These moment conditions are related to Rio (1995)s uniform SLLN and CLT conditions for mixing random variables (polynomial power decay rate). In the same paper, when random variables are independent, the result can be applied if the first and second moments exist. Our estimator utilizes the marginal distribution of order statistics. Hence, to construct a likelihood function, we do not require a higher-order derivative of the copula function. The differences are reflected in assumptions C5, D2, D3, and D7. The asymptotic property is, of course, valid even if the distribution is stationary with respect to t under the original moment conditions of Chen and Fan (2006).

2.3 Extension: neither 1 st nor N th order statistics are observable

Likelihood function (11) is constructed from the information on n and first or Nth order statistics. However, the link function derived in Lemma 2.2 suggests an estimation procedure of the copula parameter for $n(\neq 1,N)$ and $m(\neq 1,m,N)$th order statistics. First, by solving nonlinear equation (6), one can construct estimator $\hat{\hat{F}}^{(N:N)}$ for each copula parameter θ, given $\hat{F}^{(n:N)}$. While we do not present simple analytical expressions here, one can numerically calculate $\hat{\hat{F}}^{(N:N)}(x)$ by solving the one-dimensional nonlinear equation. Then, the true value for θ is the maximizer of likelihood function for the mth order statistics based on equation

(13).

$$\ell(\hat{\hat{F}}^{(N:N)}(X_t^{(m:N)});\theta) = \log\Bigg[1 + \sum_{k=1}^{N-n}\binom{N}{k}(-1)^k$$

$$\times\Bigg[\sum_{j=0}^{k}\binom{k}{j}(-1)^j\left(\frac{N-j}{N}\right)\frac{c_{u_N}\left(\overbrace{1,...,1}^{j}, C_{eq}^{\leftarrow}\left(\hat{\hat{F}}^{(N:N)}(X_t^{(m:N)})\right),...,C_{eq}^{\leftarrow}\left(\hat{\hat{F}}^{(N:N)}(X_t^{(m:N)})\right)\right)}{c_{u_N}\left(C_{eq}^{\leftarrow}\left(\hat{\hat{F}}^{(N:N)}(X_t^{(m:N)})\right),...,C_{eq}^{\leftarrow}\left(\hat{\hat{F}}^{(N:N)}(X_t^{(m:N)})\right)\right)}\Bigg]\Bigg]. \tag{13}$$

3 First-Price Auctions

The identification of value distribution with incomplete data is crucial to determining whether bidders follow economic models of auctions. It is common to assume the value distribution is common knowledge for bidders and is not observable for the econometrician. However, if there is no method for identifying the distribution through the data bidders can access, it is not easy to consider the joint distribution of value as common knowledge for bidders, especially in non-cooperative games such as auctions.

For first-price sealed-bid auctions, recovering the joint distribution through limited information is not completed, while, for the Independent Private Value (IPV) paradigm, it is possible to nonparametrically identify the value distribution if the highest bids are observable (e.g., Athey & Haile, 2007; Athey & Haile, 2002; Guerre, Perrigne, & Vuong, 2000), and estimation without independence such as APV is limited. Lamy (2012) shows the nonparametric identification of asymmetric heterogeneous auctions is not possible if the identification of bidders is not possible. On the nonparametric identification of the joint distribution of value, Athey and Haile (2002) suggest that first or Nth order statistics alone are not enough for identifying the joint distribution. When one can identify bidders, several procedures can be used. For instance, Li et al. (2002) develop an estimation of the pairwise distribution. If the auction is symmetrical, the semiparametric copula solution of Hubbard et al. (2012) is applicable. He (2016) derives an analytical expression for the Archimedean family of copulas, which is exchangeable as in example 2.1 with highest (lowest) and second highest (lowest) bids.

$$\zeta_0(u) = \int_1^u \alpha \exp\left(-\int_0^s \frac{n}{n-1} f^{(N-1:N,N:N)}(q^{(N:N)}(w), q^{(N:N)}(w))(q_w^{(N:N)}(w))^2 dw \right) ds,$$

where $q^{(N:N)}(\cdot)$ is the quantile function of the Nth order statistics. However, $f^{(N-1:N,N:N)}(x,x) = 0$, except for special cases. Therefore, a tractable estimator of copula with highest (lowest) and second highest (lowest) bids is not developed. Our estimator, presented in section 2, is applicable for symmetric APV sealed-bid first-price auctions. Our application, although the estimation is semiparametric, extends the result of He (2016) to the case when the highest (lowest) and nth order statistics of bids are observable, which includes second highest (lowest) bids. Additionally, our estimator does not require $f^{(N-1:N,N:N)}(x,x) > 0$.

In the following, we show the application of our likelihood function based on the strategy of Hubbard et al. (2012). First, we briefly present the model for the APV auction. Then, we apply the results of section 2 to estimate the copula parameter of the joint distribution of bids. Second, given the copula parameter, we demonstrate the estimation of the joint distribution through equation (4).

3.1 Model

For simplicity, assume there are T auctions at which no reserve price exists. $t(=1,...,T)$ is the index of auctions. For each auction, N risk neutral bidders bid without budget constraints. Let $V_{1t}, V_{2t}, ..., V_{Nt}$ be the valuation that follows joint distribution function $\boldsymbol{H}(v_1, v_2, ..., v_N)$. The density of $\boldsymbol{H}(v_1, v_2, ..., v_N)$ is $\boldsymbol{h}(v_1, v_2, ..., v_N)$. Let $B_{1t}, B_{2t}, ..., B_{Nt}$ be the bids that follow joint distribution function $\boldsymbol{G}(b_1, b_2, ..., b_N)$. The density of $\boldsymbol{G}(b_1, b_2, ..., b_N)$ is $\boldsymbol{g}(b_1, b_2, ..., b_N)$. The marginal distribution and density of value are $H(v)$ and $h(v)$, while the marginal distribution and density of bids are $G(b)$ and $g(b)$, respectively.

Our likelihood function approach is applicable if the joint distribution is exchangeable and the equicoordinate function is increasing. In this paper, while our approach is semiparametric, our estimations do not require the copula to belong to the Archimedean family. Additionally, the likelihood function is applicable if

$$\left\{B_t^{(N:N)}, B_t^{(n:N)}\right\}_{t=1}^{T} \qquad for\ n = 1, ..., N-1$$

$$or$$

$$\left\{B_t^{(1:N)}, B_t^{(n:N)}\right\}_{t=1}^{T} \qquad for\ n = 2, ..., N$$

are observable. Finally, both the special case of our approach which applies Corollary 2.1 and He (2016)'s approach utilize the link between the marginal distribution and marginal distribution of the highest bid.

Throughout the first-price auction model, let $H^{(n:N)}(v)$ be the marginal distribution of nth order statistics of value and $G^{(n:N)}(b)$ the marginal distribution of nth order statistics of the bid. Distribution and density $g^{(N:N)}(b)$ and $h^{(n:N)}(b)$ are respectively nonparametrically identifiable when $n = 1, ..., N-1$.

Assumption 3.1 (Risk Neutral Affiliated Private Value Paradigm)**.** *The observed data follow the symmetric Bayesian Nash equilibrium of first-price sealed-bid auctions, which is related to the private value paradigm without reserve price played by risk neutral bidders when the number of bidders is common knowledge.*

Hereinafter, the joint distribution of valuation is assumed to be expressed by its marginal distribution function and the copula that satisfies assumption C1. The copula is the function that relates marginal distribution and joint distribution. The joint distribution of valuation is expressed by $\boldsymbol{H}(v_1, v_2, ..., v_N) = C(H(v_1), H(v_2), ..., H(v_N))$, where $C(\cdot, ..., \cdot)$ is the copula function. In other words, copula function $C(\cdot, ..., \cdot)$ determines the dependence structure of bidders' valuations.

Under assumption 3.1 and when function (16) is increasing, the observed bids follow bid function (14)

$$\beta(v) = \int_0^v y\ dL(y|v), \tag{14}$$

where

$$L(y|v) = \exp\left(-\int_y^v \frac{h(t|t)}{H(t|t)}dt\right), \tag{15}$$

$$\frac{h(\varphi|\varphi)}{H(\varphi|\varphi)} \equiv (N-1)h(\varphi)\frac{c_{u_1u_2}(H(\varphi),...,H(\varphi))}{c_{u_1}(H(\varphi),...,H(\varphi))}. \tag{16}$$

Hence, the bid is a monotone transformation of valuation. Since the copula does not vary if coordinates are transformed without changing the order, joint distribution of the bid and expressed as $\boldsymbol{G}(b_1, b_2, ..., b_N) = C(G(b_1), G(b_2), ..., G(b_N))$ by the same copula function $C(\cdot, ..., \cdot)$ as the joint distribution for valuation.

3.2 Semiparametric two-step estimation

Basically, we follow the strategy of Hubbard et al. (2012), who utilize the monotonicity of the bid function. Monotonicity of the bid function ensures that the copula parameter for the bid and value distributions are the same. Therefore, one can know the correlation structure without estimating the value distribution. Specifically, the two-step strategy of Hubbard et al. (2012) avoids violating the regularity condition related to the support problem. The two-step procedure divides the estimation of copula parameter and marginal distribution of the bidders value. In the first step, we demonstrate the estimation of the copula parameter with the distribution of order statistics of observed bids by applying corollary 2.1 when $\{B_t^{(N:N)}, B_t^{(N-1:N)}\}_{t=1}^{t=N}$ is observable. In the second step, given the estimated correlation structure expressed by the copula parameter, we estimate the bidder's marginal and joint distribution of valuation.

3.2.1 First step

From corollary 2.1, logarithm of $g^{(N-1:N)}(b)$ is expressed as per equation (17).

$$\log g^{(N-1:N)}(b) = \log(N-1) + \log g^{(N:N)}(b) + \log\left[\frac{c_{u_N}\left(1, C_{eq}^{\leftarrow}\left(F^{(N:N)}(b)\right), ..., C_{eq}^{\leftarrow}\left(F^{(N:N)}(b)\right)\right)}{c_{u_N}\left(C_{eq}^{\leftarrow}\left(F^{(N:N)}(b)\right), ..., C_{eq}^{\leftarrow}\left(F^{(N:N)}(b)\right)\right)} - 1\right]. \tag{17}$$

Since $\log(N-1)$ and $\log g^{(N:N)}(b)$ are not functions of copula parameters, it is not required to include these terms into the objective function for estimating θ. Then, the log likelihood function is maximized with respect to the copula parameter when objective function (18) is maximized.

$$Q(\theta) = \sum_{t=1}^{T} \log \left[\frac{c_{u_N}\left(1, C_{eq}^{\leftarrow}\left(G^{(N:N)}(B_t^{(n:N)});\theta\right), ..., C_{eq}^{\leftarrow}\left(G^{(N:N)}(B_t^{(n:N)});\theta\right);\theta\right)}{c_{u_N}\left(C_{eq}^{\leftarrow}\left(G^{(N:N)}(B_t^{(n:N)});\theta\right), ..., C_{eq}^{\leftarrow}\left(G^{(N:N)}(B_t^{(n:N)});\theta\right);\theta\right)} - 1 \right]. \quad (18)$$

However, the distribution functions of the highest bids are unknown. Therefore, we replace $G^{(N:N)}$ with the empirical distribution function of the highest bids in each auction. Then following feasible objective function is derived:

$$\hat{Q}(\theta) = \sum_{t=1}^{T} \log \left[\frac{c_{u_N}\left(1, C_{eq}^{\leftarrow}\left(\hat{G}^{(N:N)}(B_t^{(n:N)});\theta\right), ..., C_{eq}^{\leftarrow}\left(\hat{G}^{(N:N)}(B_t^{(n:N)});\theta\right);\theta\right)}{c_{u_N}\left(C_{eq}^{\leftarrow}\left(\hat{G}^{(N:N)}(B_t^{(n:N)});\theta\right), ..., C_{eq}^{\leftarrow}\left(\hat{G}^{(N:N)}(B_t^{(n:N)});\theta\right);\theta\right)} - 1 \right], \quad (19)$$

where

$$\hat{G}^{(N:N)}(b) = \frac{1}{T} \sum_{t=1}^{T} \mathbf{1}\left[B_t^{(N:N)} \leq b\right]. \quad (20)$$

Hence, one can estimate the joint distribution if the highest and second highest bids are observable. This expression is valid for an exchangeable copula. Therefore, by plugging in the copula generating function, one can derive the specific likelihood function. Here, we provide examples for when the copula is Clayton, Gumbel, and Frank or Normal. Archimedean copulas, which include Clayton, Gumbel, and Frank copulas are also studied by Hubbard et al. (2012) and He (2016).

Example (Clayton copula)**.** *When the joint distribution follows the Clayton copula, the copula generating function is* $\zeta(u;\theta) = u^{-\theta} - 1$ *and* $\zeta^{\leftarrow}(u;\theta) = (1+u)^{-\frac{1}{\theta}}$*. The derivative is* $\zeta_u(u;\theta) = -\theta u^{-\theta-1}$*. From these relationships on copula generating function, one can derive* $\zeta_u\left(\zeta^{\leftarrow}\left(\frac{N-1}{N}\zeta\left(G(b);\theta\right);\theta\right);\theta\right) = -\theta\left(1 - \frac{N-1}{N} + \frac{N-1}{N}G(b)^{-\theta}\right)^{\frac{1+\theta}{\theta}}$*. Combining the expression* *with equation (18), one can derive the objective function for the copula parameter.*

$$Q_{Clayton}(\theta)$$
$$= \sum_{t=1}^{T} \log \left[N^{1+\frac{1}{\theta}} G^{(N:N)\ -\theta-1}(B_t^{(N-1:N)}) \left(1 + (N-1) G^{(N:N)\ -\theta}(B_t^{(N-1:N)})\right)^{-\left(1+\frac{1}{\theta}\right)} - 1 \right].$$

Example (Gumbel copula)**.** *When the joint distribution follows the Gumbel copula, the copula generating function is* $\zeta(u;\theta) = (-\log u)^{\theta}$ *and* $\zeta^{\leftarrow}(u;\theta) = \exp\left(-u^{\frac{1}{\theta}}\right)$*. The derivative is* $\zeta_u(u;\theta) = -\theta\left(-\log u\right)^{\theta-1} u^{-1}$*. From these relationships on the copula generating function, one can derive* $\zeta_u\left(\zeta^{\leftarrow}\left(\frac{N-1}{N}\zeta\left(G(b);\theta\right);\theta\right);\theta\right) = -\theta\left(-\log G(b)^{\frac{N-1}{N}\frac{1}{\theta}}\right)^{\theta-1} G(t)^{-\frac{N-1}{N}\frac{1}{\theta}}$*. Then,*

$$Q_{Gumbel}(\theta) = \sum_{t=1}^{T} \log \left[1 - N + N\left(\frac{N-1}{N}\right)^{\frac{1}{\theta}} G^{(N:N)}(B_t^{(N-1:N)})^{\left(\frac{N-1}{N}\right)^{\frac{1}{\theta}}-1} \right].$$

Example (Frank copula)**.** *When the joint distribution follows the Frank copula, the copula generating function is* $\zeta(u;\theta) = -\log\left(\frac{e^{-\theta t}-1}{e^{-\theta}-1}\right)$ *and* $\zeta^{\leftarrow}(u;\theta) = -\frac{1}{\theta}\log\left[1-\left(1-e^{-\theta}\right)e^{-u}\right]$*. The derivative is* $\zeta_u(u;\theta) = \frac{\theta e^{-\theta u}}{e^{-\theta u}-1}$*. From these relationships on the copula generating function, one can derive* $\zeta_u\left(\zeta^{\leftarrow}\left(\frac{N-1}{N}\zeta\left(G(b);\theta\right);\theta\right);\theta\right) = \frac{\left(e^{-\theta G(b)}-1\right)^{\frac{N-1}{N}} + \left(e^{-\theta}-1\right)^{\frac{N-1}{N}-1}}{\left(e^{-\theta G(b)}-1\right)^{\frac{N-1}{N}}}$*.*

$$Q_{Frank}(\theta) = \sum_{t=1}^{T} \log \left[\frac{-1 + \left(\frac{\exp\left(-\theta G^{(N:N)}(B_t^{(N-1:N)})\right)-1}{\exp(-\theta)-1}\right)^{-\frac{1}{N}}}{1 + \left(\exp\left(-\theta G^{(N:N)}(B_t^{(N-1:N)})\right)-1\right)\left(\frac{\exp\left(-\theta G^{(N:N)}(B_t^{(N-1:N)})\right)-1}{\exp(-\theta)-1}\right)^{-\frac{1}{N}}} \right].$$

Example (Normal copula)**.** *When the joint distribution follows the Normal copula,*

$$Q_{Normal}(\theta)$$
$$= \sum_{t=1}^{T} \log \left[\frac{1}{N} \frac{\int_{-\infty}^{\infty} \Phi\left(\frac{\Phi^{\leftarrow}\left(C_{eq}^{\leftarrow}(G^{(N:N)}(B_t^{(N-1:N)}))\right)+\sqrt{\theta}z}{\sqrt{1-\theta}}\right)^{N-2} \phi\left(\frac{\Phi^{\leftarrow}\left(C_{eq}^{\leftarrow}(G^{(N:N)}(B_t^{(N-1:N)}))\right)+\sqrt{\theta}z}{\sqrt{1-\theta}}\right)\phi(z)\,dz}{\int_{-\infty}^{\infty} \Phi\left(\frac{\Phi^{\leftarrow}\left(C_{eq}^{\leftarrow}(G^{(N:N)}(B_t^{(N-1:N)}))\right)+\sqrt{\theta}z}{\sqrt{1-\theta}}\right)^{N-1} \phi\left(\frac{\Phi^{\leftarrow}\left(C_{eq}^{\leftarrow}(G^{(N:N)}(B_t^{(N-1:N)}))\right)+\sqrt{\theta}z}{\sqrt{1-\theta}}\right)\phi(z)\,dz} - 1 \right].$$

Nonparametric identification for the Archimedean copula using highest (lowest) and second highest (lowest) was also conducted by He (2016). However, the estimation following their analytical expression for generating a function of the Archimedean copula is valid for special cases because their expression relies on the equicoordinate expression of joint density for different order statistics. Our estimator does not use this expression. Additionally, our likelihood function derived in Lemma 2.3 holds true not only for highest and second highest order statistics. Moreover, the result is true even if the copula does not belong to the Archimedean family. For example, an elliptical copula such as the Normal copula is not studied by He (2016). Hence, Lemma 2.3 implies that He (2016)'s result can be extend to a more general setting. In the context of our paper, it is straightforward to construct a likelihood function for the n $(\neq N-1)$th order statistics by applying equation (7).[5] The strategy of using a plugged-in likelihood function for estimating copula parameter θ is conceptually the same as in Hubbard et al. (2012). However, Hubbard et al. (2012) simply use the joint distribution of bids for likelihood, including θ, since they assume all bids are available.

$$\hat{Q}_{HPL}(\theta) = \sum_{t=1}^{T} \log \left[\mathbf{\Psi} \left(\Psi^{\leftarrow} \left(\hat{G}\left(B_{it}\right); \theta \right), ..., \Psi^{\leftarrow} \left(\hat{G}\left(B_{it}\right); \theta \right); \theta \right) \right]. \tag{21}$$

This paper, by contrast, uses the marginal distribution of two order statistics of bids for likelihood, including θ, because our recursive formula is constructed by the marginal distribution of the highest (lowest) and some other order statistics of bids. Moreover, our estimator can be applied even if there is a reserve price. Hubbard et al. (2012)s estimator uses the joint distribution function of bids for the estimation of the copula parameter. Therefore, all bidders valuations need to be observed. If more than one bidders valuation is lower than reserve price, these bidders' valuations cannot be observed. Then, one cannot calculate the likelihood function without assuming missing auctions at random. However, our approach only requires two bidders bids to be observable. The monotonicity of the bid function ensures no effect on the rank correlation of bidders valuation for highest and second highest bids.

[5] θ can be estimated using $m(\neq 1, N)$ and the $n(\neq 1, N, m)$th order statistics through equation (13).

3.2.2 Second step

In the second step, given copula parameter θ estimated in the first step, we construct the estimator of the marginal distribution of value $H(v)$ by utilizing the relationship $C(H(v_1), ..., H(v_N)) = \boldsymbol{H}(v_1, ..., v_N)$. Once θ is estimated in the first step, it is possible to extend to the case where only the Nth and nth ($\neq N$) or first and nth ($\neq 1$) bids are observable.

As a minor modification of Guerre et al. (2000), the pseudo value follows equation (22).

$$\tilde{V}_{it} = \begin{cases} B_{it} + \frac{c_{u_1}(G(B_{it}),...,G(B_{it});\theta_0)}{(N-1)g(B_{it})c_{u_1u_2}(G(B_{it}),...,G(B_{it});\theta_0)} & B_{min} + \frac{\theta_g h_g}{2} \leq B_{it} \leq B_{max} - \frac{\theta_g h_g}{2} \\ \infty & otherwise, \end{cases} \tag{22}$$

where C_{u_1} and $C_{u_1u_2}$ are the derivatives of copula function with respect to the first and first and second elements, respectively. From the expression of the bid function given one's own bid, the pseudo value can be constructed using equation (22) if C_{u_1}, $C_{u_1u_2}$, G, and g are available. The parameters of C_{u_1} and $C_{u_1u_2}$ are estimated in the first step. $G(\cdot)$ and $g(\cdot)$ are constructed using the relationship between the marginal distribution and order statistics as in Lemma 2.2. When the distribution is exchangeable and has equicoordinate inverse,

$$G(b) = C_{eq}^{\leftarrow}\left(G^{(N:N)}(b);\theta_0\right). \tag{23}$$

$$g(b) = \frac{g^{(N:N)}(b)}{Nc_{u_N}\left(C_{eq}^{\leftarrow}\left(G^{(N:N)}(b);\theta_0\right);\theta_0\right)} \tag{24}$$

Since the bid function preserves the order of value, the nth order statistics of bid are produced from the nth order statistics of value. In other words, when $V^{(n:N)}$ is the valuation of individual i in the tth auction and $B^{(n:N)}$ is the bid of individual i in the tth auction. Therefore, from the above identity, one can replace subscript $_i$ with superscript $^{(n:N)}$ as per equation (25).

$$\tilde{V}_t^{(n:N)} = \begin{cases} B_t^{(n:N)} + \frac{c_{u_1}\left(G(B_t^{(n:N)}),...,G(B_t^{(n:N)});\theta_0\right)}{(N-1)g(B_t^{(n:N)})c_{u_1u_2}\left(G(B_t^{(n:N)}),...,G(B_t^{(n:N)});\theta_0\right)} & B_{min} + \frac{\theta_g h_g}{2} \leq B_t^{(n:N)} \leq B_{max} - \frac{\theta_g h_g}{2} \\ \infty & otherwise. \end{cases} \tag{25}$$

This order-preserving property is utilized as quantile preserving by Marmer and Shneyerov (2012). Since $\tilde{V}_t^{(N:N)} \sim H^{(N:N)}$, the distribution function satisfies $H^{(N:N)}(v) = P(\tilde{V}_t^{(N:N)} \leq v)$ by definition. Hence, one can estimate $H^{(N:N)}(v)$ by the empirical distribution function.

$$\hat{H}^{(N:N)}(v) = \frac{1}{T}\sum_{t=1}^{T} \mathbf{1}\left[\tilde{V}_t^{(N:N)} \leq v\right]. \tag{26}$$

From Lemma 2.2,[6].

$$\boldsymbol{H}(v_1, ..., v_N) = C\left(C_{eq}^{\leftarrow}\left(H^{(N:N)}(v_1);\theta_0\right), ..., C_{eq}^{\leftarrow}\left(H^{(N:N)}(v_N);\theta_0\right);\theta_0\right) \tag{27}$$

Then, the empirical joint distribution of valuation is

$$\hat{\boldsymbol{H}}(v_1, ..., v_N) = C\left(C_{eq}^{\leftarrow}\left(\hat{H}^{(N:N)}(v_1);\hat{\theta}\right), ..., C_{eq}^{\leftarrow}\left(\hat{H}^{(N:N)}(v_N);\hat{\theta}\right);\hat{\theta}\right) \tag{28}$$

From Proposition 2, the copula parameter achieves the same rate of convergence as in Hubbard et al. (2012). Additionally, both equation (28) and Hubbard et al. (2012) are constructed by estimated the copula and empirical distribution. Therefore, even if the identification of bidders is not observed or only specific order statistics of bids are observable, one can achieve the following optimal rate of convergence, which is the same rate as in Hubbard et al. (2012).

4 Simulation Studies

Here, we present the results of a Monte Carlo experiment that focuses on the small sample (information) performance of our estimator. We first show the result of the maximum likelihood estimator without auction. Then, we present the result when DGP is subject to a first-price sealed-bid auction.

[6]From this expression one can estimate the joint distribution of bids once the copula parameter is identified. $G(b_1, ..., b_N) = \zeta^{\leftarrow}(\frac{1}{N}\sum_{i=1}^{N}\zeta(G^{(N:N)}(b_i);\theta);\theta)$

4.1 Estimation without first-price auction

Here, we study estimator of the Kendall τ and marginal distribution. τ is estimated by the likelihood function, which is directly derived from equation (9). While Kendall τ does not appear in the discussion above, parameter values of the Archimedean or exchangeable elliptical copulas uniquely correspond to Kendall τ. The marginal distribution is estimated by the same expression as $C_{eq}^{\leftarrow}\left(\hat{F}^{(N:N)}(v);\hat{\theta}\right)$. In the simulation study without auction, sample size T is equal 50, 100, and 200 with N of 3, 5, and 8, respectively. Each sample is replicated 1000 times. Random variables are generated from the exchangeable copula, whose marginal distribution is $U(0,1)$; dependence is determined by pairwise Kendall τ, which is equal to 0.25, 0.5, and 0.75.

Table 1: simulated mean squared error for the estimator of Kendall τ from order statistics

Clayton	τ=0.25			τ=0.50			τ=0.75		
$N\backslash T$	50	100	200	50	100	200	50	100	200
3	0.01608	0.00405	0.00218	0.01011	0.00405	0.00158	0.00438	0.00218	0.00104
5	0.01549	0.00704	0.00286	0.00906	0.00362	0.00186	0.00469	0.00229	0.00108
8	0.01697	0.00743	0.00294	0.00936	0.00416	0.00170	0.00485	0.00239	0.00109
Gumbel	τ=0.25			τ=0.50			τ=0.75		
$N\backslash T$	50	100	200	50	100	200	50	100	200
3	0.01082	0.00583	0.00244	0.00928	0.00428	0.00175	0.00550	0.00197	0.00073
5	0.01253	0.00500	0.00250	0.00996	0.00391	0.00174	0.00581	0.00198	0.00075
8	0.01097	0.00515	0.00240	0.00885	0.00391	0.00180	0.00558	0.00194	0.00079
Frank	τ=0.25			τ=0.50			τ=0.75		
$N\backslash T$	50	100	200	50	100	200	50	100	200
3	0.00977	0.000497	0.00246	0.00544	0.00262	0.00116	0.00218	0.00093	0.00046
5	0.01059	0.00515	0.00257	0.00601	0.00283	0.00144	0.00259	0.00149	0.00075
8	0.01061	0.00529	0.00270	0.00708	0.00329	0.00147	0.00305	0.00161	0.00081
Normal	τ=0.25			τ=0.50			τ=0.75		
$N\backslash T$	50	100	200	50	100	200	50	100	200
3	0.02052	0.00976	0.00423	0.01159	0.00485	0.00207	0.00391	0.00171	0.00116
5	0.01542	0.00687	0.00286	0.01096	0.00364	0.00172	0.00389	0.00162	0.00131
8	0.01155	0.00520	0.00193	0.01024	0.00418	0.00161	0.00495	0.00337	0.00094

The simulation result of estimating τ are presented in Tables 1 and 2. The simulated mean squared error for τ is presented in Table 1 and simulated bias for τ in Table 2. While the MSE presented in Table 1 is basically consistent with $\sqrt{T}$ consistency, realized MSE changes as τ and N changes. MSE for $\tau = 0.75$ is smaller than the MSE for $\tau = 0.5$. MSE for $\tau = 0.5$ is smaller than MSE for $\tau = 0.25$. When τ is close to one, as random draws from different individuals tend to realize similar numbers. Then, the information for comparing

Table 2: simulated bias for estimating Kendall τ from order statistics

Clayton	τ=0.25			τ=0.50			τ=0.75		
$N\backslash T$	50	100	200	50	100	200	50	100	200
3	-0.05736	-0.02765	-0.01274	-0.02344	-0.009941	-0.00295	-0.01033	-0.00419	-0.00283
5	-0.05618	-0.02368	-0.01060	-0.0493	-0.00981	-0.00241	-0.01317	-0.00555	-0.00188
8	-0.05461	-0.02764	-0.01338	-0.01991	-0.01306	-0.00432	-0.01751	-0.00574	-0.00279
Gumbel	τ=0.25			τ=0.50			τ=0.75		
$N\backslash T$	50	100	200	50	100	200	50	100	200
3	-0.03566	-0.01997	-0.00723	-0.04580	-0.02328	-0.00900	-0.05128	-0.02500	-0.01035
5	-0.03928	-0.01930	-0.00811	-0.04710	-0.02402	-0.01021	-0.05349	-0.02679	-0.01227
8	-0.04094	-0.01624	-0.01093	-0.04755	-0.02083	-0.01221	-0.05472	-0.02495	-0.01330
Frank	τ=0.25			τ=0.50			τ=0.75		
$N\backslash T$	50	100	200	50	100	200	50	100	200
3	-0.00996	-0.00590	-0.00302	-0.01332	-0.00485	-0.00351	-0.02691	-0.01404	-0.00823
5	0.00147	-0.00064	0.00028	-0.00244	-0.00258	-0.00191	-0.01230	-0.00420	-0.00013
8	0.00110	0.00088	0.00081	-0.00134	-0.00090	-0.00223	-0.01081	-0.00072	-0.00019
Normal	τ=0.25			τ=0.50			τ=0.75		
$N\backslash T$	50	100	200	50	100	200	50	100	200
3	0.10220	0.07137	0.04391	0.06940	0.04095	0.02355	0.02908	0.01837	0.01197
5	0.07347	0.04872	0.02843	0.05570	0.02614	0.01726	0.02727	0.01464	0.00546
8	0.05987	0.03725	0.02004	0.04413	0.02688	0.01525	0.02669	0.01888	-0.00494

the empirical marginal distribution for different order statistics is large when the true τ is close to one. Next, we refer to the effect of N in estimating τ. As presented in Table 1, the MSE for $N = 8$ is larger than that for $N = 3$. However, the effect of N on simulated MSE is mild compared with the effect of τ. Specifically, the results when $N = 8$ and $\tau = 0.75$ are overall better than those when $N = 3$ and $\tau = 0.25$. Next, we discuss the bias presented in Table 2. Basically, bias decreases as sample size increases, which is naturally predicted by asymptotic theory. However, the sign of the bias does not seem to be random. For example, when the copula is Clayton or Gumbel, the sign of the bias is negative in our experiment. Such bias is reported for semiparametric plug-in estimators such as average treatment effect estimators (see, e.g., Ichimura & Linton, 2005). To remove bias, a bias correction estimator based on higher-order asymptotic expansion is required; however, this is beyond the scope of this study.

Next, we study the finite sample property of the estimated marginal distribution, given the copula and its estimator. The simulation results for the Clayton copula are presented in Table 3, for Gumbel copula in Table 4, for Frank copula in Table 5, and for Normal copula in Table 6. Overall, MSE increases as N increases. This tendency is natural because the estimator

Table 3: Simulated MSE for estimating $F(0.25)$, $F(0.50)$, and $F(0.75)$ from order statistics when the joint distribution follows Clayton copula

	Clayton								
$\tau = 0.25$	N=3			N=5			N=8		
$x\backslash T$	50	100	200	50	100	200	50	100	200
0.25	0.00730	0.00291	0.00116	0.01924	0.00701	0.00265	0.03862	0.01707	0.00696
0.50	0.00529	0.00251	0.00118	0.00765	0.00352	0.00157	0.01647	0.00600	0.00272
0.75	0.00235	0.00113	0.00062	0.00221	0.00110	0.00057	0.00260	0.00139	0.00068
$\tau = 0.50$	N=3			N=5			N=8		
$x\backslash T$	50	100	200	50	100	200	50	100	200
0.25	0.00466	0.00226	0.00107	0.00697	0.00326	0.00164	0.01048	0.00518	0.00261
0.50	0.00475	0.00229	0.00121	0.00624	0.00292	0.00153	0.00901	0.00450	0.00221
0.75	0.00277	0.00156	0.00073	0.00293	0.00151	0.00077	0.00346	0.00183	0.00084
$\tau = 0.75$	N=3			N=5			N=8		
$x\backslash T$	50	100	200	50	100	200	50	100	200
0.25	0.00447	0.00212	0.00102	0.00494	.00236	0.00125	0.00583	0.00278	0.00140
0.50	0.00560	0.00250	0.00128	0.00642	0.00288	0.00145	0.00738	0.00350	0.00174
0.75	0.00355	0.00183	0.00090	0.00366	0.00191	0.00093	0.00390	0.00229	0.00108

Table 4: Simulated MSE for estimating $F(0.25)$, $F(0.50)$, and $F(0.75)$ from order statistics when the joint distribution follows Gumbel copula

	Gumbel								
$\tau = 0.25$	N=3			N=5			N=8		
$x\backslash T$	50	100	200	50	100	200	50	100	200
0.25	0.00955	0.00288	0.00128	0.04025	0.02607	0.01097	0.06135	0.05668	0.05029
0.50	0.00357	0.00176	0.00097	0.00642	0.00231	0.00108	0.04203	0.00898	0.00207
0.75	0.00229	0.00117	0.00055	0.00218	0.00109	0.00056	0.00246	0.00113	0.00059
$\tau = 0.50$	N=3			N=5			N=8		
$x\backslash T$	50	100	200	50	100	200	50	100	200
0.25	0.00423	0.00193	0.00093	0.01094	0.00306	0.00125	0.02716	0.01003	0.00332
0.50	0.00438	0.00227	0.00121	0.00459	0.00220	0.00111	0.00543	0.00266	0.00119
0.75	0.00312	0.00150	0.00074	0.00287	0.00152	0.00069	0.00287	0.00147	0.00072
$\tau = 0.75$	N=3			N=5			N=8		
$x\backslash T$	50	100	200	50	100	200	50	100	200
0.25	0.00424	0.00202	0.00110	0.00502	0.00202	0.00102	0.00588	0.00235	0.00103
0.50	0.00512	0.00261	0.00130	0.00546	0.00264	0.00124	0.00591	0.00299	0.00127
0.75	0.00349	0.00166	0.00083	0.00348	0.00180	0.00080	0.00349	0.00177	0.00087

Table 5: Simulated MSE for estimating $F(0.25)$, $F(0.50)$, and $F(0.75)$ from order statistics when the joint distribution follows Frank copula

	Frank								
$\tau = 0.25$	N=3			N=5			N=8		
$x\backslash T$	50	100	200	50	100	200	50	100	200
0.25	0.00792	0.00250	0.00102	0.03923	0.02193	0.00873	0.05985	0.05637	0.05037
0.50	0.00370	0.00160	0.00089	0.00532	0.00242	0.00135	0.03122	0.00582	0.00242
0.75	0.00245	0.00116	0.00061	0.00241	0.00110	0.00056	0.00276	0.00140	0.00067
$\tau = 0.50$	N=3			N=5			N=8		
$x\backslash T$	50	100	200	50	100	200	50	100	200
0.25	0.00347	0.00140	0.00071	0.00720	0.00203	0.00087	0.02427	0.00774	0.00221
0.50	0.00365	0.00201	0.00101	0.00424	0.00192	0.00101	0.00488	0.00253	0.00111
0.75	0.00309	0.00162	0.00079	0.00324	0.00174	0.00084	0.00345	0.00199	0.00087
$\tau = 0.75$	N=3			N=5			N=8		
$x\backslash T$	50	100	200	50	100	200	50	100	200
0.25	0.00320	0.00167	0.00078	0.00292	0.00147	0.00075	0.00307	0.00145	0.00073
0.50	0.00485	0.00253	0.00115	0.00498	0.00259	0.00135	0.00564	0.00283	0.00153
0.75	0.00369	0.00196	0.00093	0.00385	0.00197	0.00105	0.00453	0.00238	0.00113

Table 6: Simulated MSE for estimating $F(0.25)$, $F(0.50)$, and $F(0.75)$ from order statistics when the joint distribution follows Normal copula

	Normal								
$\tau = 0.25$	N=3			N=5			N=8		
$x\backslash T$	50	100	200	50	100	200	50	100	200
0.25	0.00897	0.00356	0.00183	0.02198	0.00982	0.00390	0.04046	0.02708	0.01208
0.50	0.00723	0.00324	0.00139	0.00971	0.00393	0.00184	0.01662	0.00629	0.00225
0.75	0.00387	0.00166	0.00082	0.00413	0.00184	0.00079	0.00449	0.00210	0.00082
$\tau = 0.50$	N=3			N=5			N=8		
$x\backslash T$	50	100	200	50	100	200	50	100	200
0.25	0.00457	0.00225	0.00111	0.00659	0.00294	0.00137	0.00907	0.00413	0.00167
0.50	0.00585	0.00271	0.00128	0.00661	0.00294	0.00133	0.00774	0.00384	0.00158
0.75	0.00406	0.00174	0.00083	0.00448	0.00189	0.00091	0.00480	0.00229	0.00086
$\tau = 0.75$	N=3			N=5			N=8		
$x\backslash T$	50	100	200	50	100	200	50	100	200
0.25	0.00397	0.00186	0.00100	0.00397	0.00218	0.00134	0.00456	0.00252	0.00191
0.50	0.00505	0.00252	0.00125	0.00532	0.00270	0.00186	0.00566	0.00355	0.00266
0.75	0.00409	0.00173	0.00098	0.00404	0.00190	0.00150	0.00464	0.00241	0.00191

uses the information of only the Nth order statistics. The realized value of the Nth order statistics contains small information around the lower bound of the support of the marginal distribution. Additionally, the estimation when $x = 0.25$ is less precise than that for $x = 0.75$. This is because, when the marginal distribution is $U(0, 1)$, the Nth order statistics are more likely to be close to the 0.75 percentile compared than the 0.25 percentile. Next, we study the magnitude of different Ns for different values of τ. Overall, a large N effect is mentioned just before a decrease when τ is large. In fact, for Clayton, Gumbel, Frank, and Normal copulas, MSE when $\tau = 0.75$ is smaller than MSE when $\tau = 0.50$ and $\tau = 0.50$ is smaller than MSE when $\tau = 0.25$. This is natural because the estimator uses the information of only the Nth order statistics. The realized value of the Nth order statistics contain limited information on the entire population. When random variables are independent, one can observe similar tendencies. Specifically, since independent means a small τ, the MSE using only the $N - 1$ and Nth th order statistics is large when variables are independent.

From the discussion above, it is better to add $n(< N)$th order statistics for the estimation of the marginal distribution function, especially when the estimated value of τ is close to zero and N is large. However, when the estimated value of τ is close to one, the estimation of marginal distribution using the Nth order statistics is more precise compared with the estimation under independence.

4.2 Estimation with first-price auction

Here, we study the estimation when $N-1$ and Nth order statistics of the bid are observable for each auction. Especially, we focus on the magnitude of limited information on the estimator. The estimator studied in this section is constructed in the following section 3.2. Regarding the semiparametric estimator based on an Archimedean copula, Hubbard et al. (2012) study the performance of the copula-based estimator with the nonparametric estimator of Guerre et al. (2000) under the IPV paradigm. Basically, our likelihood function is derived from the joint distribution, which is expressed by the exchangeable copula of Hubbard et al. (2012). Hence, our estimator with the GPV estimator under the IPV paradigm is expected to be the same as in Hubbard et al. (2012). Hubbard et al. (2012)'s estimator's performance is based on observing all bids. However, our estimator is based on the likelihood of order statistics utilizing limited information. Our second-step estimator utilizes the empirical distribution of specific order statistics. Therefore, our estimator's performance could be poor when the dimension of the joint distribution is large compared with available information and $N-1$ and Nth order statistics. To investigate this point, we focus on the effect of size N on the performance of the estimator.

Simulator

To generate bids under the APV paradigm in section 3, we construct the bid function of equation (14). Specifically, we utilize the fact that the integral in equation (14) is approximated by a Monte Carlo integral if distribution $L(y|x)$ has a tractable expression.

Lemma 4.1. *$L(\cdot|v)$ satisfies the condition to be a distribution function. Additionally, if $\frac{\boldsymbol{\Psi}_{11}(u,...,u)}{\boldsymbol{\Psi}_1(u,...,u)} = \frac{d\tilde{\Psi}(u)}{du}$ for some function $\tilde{\Psi}$,*

$$L(y|v) = \left[\frac{\boldsymbol{\Psi}_1\left(\Psi^{\leftarrow}(H(y)), ..., \Psi^{\leftarrow}(H(y))\right)\exp\left(-\tilde{\Psi}(\Psi^{\leftarrow}(H(y)))\right)}{\boldsymbol{\Psi}_1\left(\Psi^{\leftarrow}(H(v)), ..., \Psi^{\leftarrow}(H(v))\right)\exp\left(-\tilde{\Psi}(\Psi^{\leftarrow}(H(v)))\right)}\right]. \tag{29}$$

When the valuation is expressed by marginal distribution H and the Archimedean copula,

$$L(y|v) = \left[\frac{\zeta_u\left(\zeta^{\leftarrow}\left(N\zeta\left(H(v);\theta\right);\theta\right);\theta\right)}{\zeta_u\left(\zeta^{\leftarrow}\left(N\zeta\left(H(y);\theta\right);\theta\right);\theta\right)}\right]^{\frac{N-1}{N}}. \tag{30}$$

From Lemma 4.1, one can simulate the bid from the given marginal distribution, given class of copula, and given value of θ. In this study, we generate v from the predetermined marginal distribution and copula. For the calculation of the integral in the bid function, we generate 1000 y by the inverse function method from $L(\cdot|v)$. Then, the average approximates the integral. The derivation of $L(\cdot|v)$ is presented in the Appendix. For the Archimedean copula, a similar relationship is also derived by He (2016). We employ a truncated Pareto distribution for the marginal distribution following the Monte Carlo experiment of Hubbard et al. (2012). In their study, the marginal distribution is $\frac{9}{8}\left(1-\left(\frac{1}{v}\right)^2\right)$, which has positive density on $[1,3]$; however, we employ $H(v) = \frac{9}{8}\left(1 - \left(\frac{1}{v+1}\right)^2\right)$, which has positive density on $[0,2]$ instead to the apply Monte Carlo integral derived from Lemma 4.1 and equation (14).

In the simulation study, sample sizes of T are equal to 50, 100, and 200 with N of 3, 5, and 8 bidders, respectively. Each sample is replicated 1000 times. Random variables are generated from the Archimedean copula, whose marginal distribution is a truncated Pareto distribution; dependence is measured by pairwise Kendall τ, which is equal to 0.25, 0.5, and 0.75.

Table 7: Simulated MSE for estimating value distribution $H(0.5)$, $H(1.0)$, and $H(1.5)$ from order statistics of the bids when the joint distribution follows Clayton copula

				Clayton					
$\tau = 0.25$		N=3			N=5			N=8	
$v\backslash T$	50	100	200	50	100	200	50	100	200
0.5	0.00202	0.00163	0.00211	0.0231	0.00178	0.00197	0.00433	0.00275	0.00211
1.0	0.00478	0.00335	0.00280	0.00298	0.00207	0.00171	0.00178	0.00127	0.00100
1.5	0.01143	0.01058	0.01032	0.00678	0.00601	0.00549	0.00370	0.00318	0.00290
$\tau = 0.50$		N=3			N=5			N=8	
$v\backslash T$	50	100	200	50	100	200	50	100	200
0.5	0.00526	0.00459	0.00451	0.00460	0.00369	0.00377	0.00465	0.00377	0.00338
1.0	0.00274	0.00201	0.00183	0.00240	0.00153	0.00117	0.00181	0.00126	0.00084
1.5	0.00724	0.00702	0.00695	0.00458	0.00411	0.00375	0.00275	0.00248	0.00222
$\tau = 0.75$		N=3			N=5			N=8	
$v\backslash T$	50	100	200	50	100	200	50	100	200
0.5	0.00698	0.00551	0.00431	0.00662	0.00485	0.00364	0.00663	0.00500	0.00384
1.0	0.00252	0.00154	0.00101	0.00227	0.00123	0.00076	0.00204	0.00121	0.00071
1.5	0.00441	0.00347	0.000327	0.00278	0.00215	0.00187	0.00198	0.00149	0.00131

Here we present simulation result of estimation of marginal value distribution when only the highest and second highest bids are observable. The simulation results for the Clayton copula are presented in Table 7, for Gumbel copula in Table 8, and for Frank copula in Table 9. For Gumbel and Frank copula, local MSE increases as N increases. Additionally, the estimation when $x = 0.25$ is less precise than that for $x = 0.75$. These tendency is also observed in the simulation study without auction. For Gumbel and Frank copulas, MSE when $\tau = 0.75$ is smaller than MSE when $\tau = 0.50$ and $\tau = 0.50$ is smaller than MSE when $\tau = 0.25$. For Clayton copula, the tendency is mild. This difference is related to dependence structure of Clayton copula, which has strong correlation around lower bound and weak correlation around upper bound. Overall, the effect of τ and N on the precision of the estimator does not vary when estimating value distribution given bids are generated from symmetric affiliated value

Table 8: Simulated MSE for estimating value distribution $H(0.5)$, $H(1.0)$, and $H(1.5)$ from order statistics of the bids when the joint distribution follows Gumbel copula

	Gumbel								
$\tau = 0.25$		N=3			N=5			N=8	
$v\backslash T$	50	100	200	50	100	200	50	100	200
0.5	0.00699	0.00437	0.00374	0.02233	0.02069	0.01899	0.09700	0.09724	0.09776
1.0	0.00271	0.00137	0.00078	0.00857	0.00724	0.00641	0.03316	0.03212	0.03187
1.5	0.00117	0.00102	0.00090	0.00127	0.00057	0.00311	0.00416	0.00299	0.00259
$\tau = 0.50$		N=3			N=5			N=8	
$v\backslash T$	50	100	200	50	100	200	50	100	200
0.5	0.00986	0.00733	0.00661	0.00202	0.00298	0.00190	0.02128	0.01789	0.01666
1.0	0.00281	0.00155	0.00091	0.00393	0.00246	0.00161	0.01001	0.00814	0.00702
1.5	0.00097	0.00051	0.00031	0.00139	0.00072	0.00038	0.00244	0.00171	0.00127
$\tau = 0.75$		N=3			N=5			N=8	
$v\backslash T$	50	100	200	50	100	200	50	100	200
0.5	0.00843	0.00640	0.00515	0.00497	0.00252	0.00154	0.00524	0.00282	0.00164
1.0	0.00271	0.00143	0.00075	0.00279	0.00142	0.00074	0.00371	0.00221	0.00152
1.5	0.00105	0.00050	0.00025	0.00133	0.00066	0.00036	0.00156	0.00087	0.00055

Table 9: Simulated MSE for estimating value distribution $H(0.5)$, $H(1.0)$, and $H(1.5)$ from order statistics of the bids when the joint distribution follows Frank copula

	Frank								
$\tau = 0.25$		N=3			N=5			N=8	
$v\backslash T$	50	100	200	50	100	200	50	100	200
0.5	0.00540	0.00293	0.00166	0.02411	0.02047	0.01865	0.07428	0.07372	0.07386
1.0	0.00369	0.00239	0.00173	0.01446	0.01351	0.01227	0.03709	0.03809	0.04030
1.5	0.00102	0.00040	0.00015	0.00279	0.00154	0.00095	0.00603	0.00499	0.00466
$\tau = 0.50$		N=3			N=5			N=8	
$v\backslash T$	50	100	200	50	100	200	50	100	200
0.5	0.00689	0.00459	0.00344	0.00613	0.00398	0.00197	0.01515	0.01124	0.00965
1.0	0.00384	0.00206	0.00146	0.00850	0.00710	0.00541	0.01663	0.01536	0.01389
1.5	0.00213	0.00137	0.00090	0.00286	0.00227	0.00162	0.00566	0.00448	0.00355
$\tau = 0.75$		N=3			N=5			N=8	
$v\backslash T$	50	100	200	50	100	200	50	100	200
0.5	0.00701	0.00449	0.00326	0.00652	0.00341	0.00205	0.00567	0.00291	0.00158
1.0	0.00312	0.00169	0.00089	0.00398	0.00219	0.00132	0.00398	0.00237	0.00174
1.5	0.00251	0.00176	0.00145	0.00244	0.00162	0.00131	0.00257	0.00194	0.00149

auction. If researchers find Kendall τ of bids is close to zero by maximum likelihood using order statistics, they should add information on order statistics to derive precise estimation of value distribution.

5 Summary and Limitations

In this paper, we propose a semiparametric maximum likelihood method for estimating an exchangeable joint distribution when limited information about order statistics is available. Specifically, our approach allows identification of the exchangeable distribution if the highest (lowest) and some other order statistics of the joint distribution are expressed by an exchangeable copula whose equicoordinate has positive Lebesgue density. Additionally, our link function suggests that observing different order statistics is sufficient for estimating the exchangeable joint distribution.

Our approach can be applied for data where the responses of motivated individuals are more likely to be observed and suspected to be correlated between small units such as city blocks, classrooms, and auctions.

However, we have to note the limitations of this approach. We utilize a recursive formula to derive the likelihood function and assume an exchangeable property. The assumption imposes that the dependence within a small group or unit does not change even if the number of members of the unit increases or decreases. Therefore, one should be cautious about the objects to which one applies the recursive formula. Small city blocks, small classrooms, and auctions of non-rare goods for which bidders can easily access information on the goods would be appropriate.

We apply the semiparametric method for first-price auctions when bidders' valuations are affiliated. The estimation of auctions is also studied by Hubbard et al. (2012) and He (2016). Our two-step estimation concept is the same as in Hubbard et al. (2012). Our approach weakens the requirement for observable information. He (2016) derives a nonparametric analytical expression for the Archimedean copula when the highest (or lowest) and second highest (or second lowest) bids are observable, and $f^{(N-1:N,N:N)}(x,x) > 0$. Our recursive likelihood formula suggests one can extend He (2016)'s results to the case where the joint distribution is exchangeable $f^{(N-1:N,N:N)}(x,x) = 0$ and highest (or lowest) and some other order statistics are observed. Additionally, we derive a semi-analytical expression of the equilibrium bid function for exchangeable distribution, and propose to use the expression for simulating bids, given bidders valuation. A similar, semi-analytical expression for Archimedean copula is derived by He (2016). Our method of simulating bids from the semi-analytical expression of the equilibrium bid function suggests that the formula can be applied to simulated moment estimations, including those in Bierens and Song (2012).

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A Proof of Theorems

Proof of Lemma 2.2

Proof. First, we show equations (4) and (5).

$$\begin{aligned} F^{(N:N)}(x) = &\ \mathrm{P}\left(\max_{n=1,\dots,N} X_{nt} \leq x\right) = \mathrm{P}(X_{1t} \leq x, ..., X_{Nt} \leq x) \\ = &\ F(x, ..., x) = C(F(x), ..., F(x)) = C_{eq}(F(x)) \end{aligned} \tag{31}$$

The first equality is from the definition of order statistics and the fourth from Sklar's theorem, while the fifth comes from the definition of C_{eq}. Equation (5) is derived from the monotonicity of C_{eq}.

□

Proof. Here, we show equation (6). We derive equation (6) by utilizing technical lemmas. The proofs of technical Lemmas A.1 and A.2 are presented after the proof of the main result.

From Lemma A.1,

$$F^{(n:N)}(x) = F^{(N:N)}(x) + \sum_{k=1}^{N-n} \mathrm{P}\left(exactly\ N-k\ of\ X_1, ..., X_N\ are\ at\ most\ x\right).$$

In this representation,

$$\mathrm{P}\left(exactly\ N-k\ of\ X_1, ..., X_N\ are\ at\ most\ x\right) = \binom{N}{k}\left\{(-1)^k \sum_{j=0}^{k} \binom{k}{j} (-1)^j \mathrm{P}\left(X_{j+1} \leq x, ..., X_N \leq x\right)\right\}$$

from Lemma A.2. Therefore,

$$\begin{aligned} F^{(n:N)}(x) =& F^{(N:N)}(x) + \sum_{k=1}^{N-n} \binom{N}{k} \left\{(-1)^k \sum_{j=0}^{k} \binom{k}{j} (-1)^j \mathrm{P}\left(X_{j+1} \leq x, ..., X_N \leq x\right)\right\} \\ =& F^{(N:N)}(x) + \sum_{k=1}^{N-n} \binom{N}{k} \left\{(-1)^k \sum_{j=0}^{k} \binom{k}{j} (-1)^j C\left(\underbrace{1, ..., 1}_{j}, F(x), ..., F(x)\right)\right\}. \end{aligned}$$

□

Lemma A.1. *For* $n \in \{1, 2, ..., N-1\}$,

$$F^{(n:N)}(x) = F^{(N:N)}(x) + \sum_{k=1}^{N-n} \mathrm{P}\left(exactly\ N-k\ of\ X_1, ..., X_N\ are\ at\ most\ x\right).$$

Proof. From the definition of order statistics, we derive the following expression between marginal distributions of m and $m+1$th order statistics for all $m \in \{1, 2, ..., N-1\}$.

$$\begin{aligned} F^{(m:N)}(x) = \mathrm{P}\left(X^{(m:N)} \leq x\right) & \\ =\ & \mathrm{P}\left(at\ least\ m\ of\ X_1, ..., X_N\ are\ at\ most\ x\right) \\ =\ & \mathrm{P}\left(exactly\ m\ of\ X_1, ..., X_N\ are\ at\ most\ x\right) + \mathrm{P}\left(at\ least\ m+1\ of\ X_1, ..., X_N\ are\ at\ most\ x\right) \\ =\ & F^{(m+1:N)}(x) + \mathrm{P}\left(exactly\ m\ of\ X_1, ..., X_N\ are\ at\ most\ x\right). \end{aligned} \tag{32}$$

Since equation (32) holds true for all m, we can remove $F^{(n+1)}(x), ..., F^{(N-1)}(x)$ by taking the repeated sum of equation (32).

$$\begin{aligned} F^{(n:N)}(x) &= F^{(N:N)}(x) + \sum_{j=s}^{N-1} \mathrm{P}\left(exactly\ j\ of\ X_1, ..., X_N\ are\ at\ most\ x\right) \\ &= F^{(N:N)}(x) + \sum_{k=1}^{N-s} \mathrm{P}\left(exactly\ N-k\ of\ X_1, ..., X_N\ are\ at\ most\ x\right). \end{aligned} \tag{33}$$

□

Lemma A.2. *For all $0 \le k < l \le N$, the following recursive formula is true:*

$$\begin{aligned} &\mathrm{P}\left(exactly\ N-k\ of\ X_1, ..., X_N\ are\ at\ most\ x\right) \\ =&\binom{N}{k}\left\{(-1)^k \sum_{j=0}^{k} \binom{k}{j} (-1)^j \mathrm{P}\left(X_{j+1} \le x, ..., X_N \le x\right)\right\} \end{aligned} \tag{34}$$

.

Proof. For notational simplicity, we respectively define $p(k,l)$ and $p(k)$ as follows:

$$\begin{aligned} p(k,l) &= \mathrm{P}\left(X_1 > x, ..., X_k > x, X_l \le x, ..., X_N \le x\right), \\ p(k) &= \mathrm{P}\left(X_k \le x, ..., X_N \le x\right),. \end{aligned}$$

Since $\{X_1 > x, ..., X_k > x, X_l \le x, ..., X_N \le x\} \cup \{X_1 > x, ..., X_{k-1} > x, X_k \le x, X_l \le x, ..., X_N \le x\} \Leftrightarrow \{X_1 > x, ..., X_{k-1} > x, X_l \le x, ..., X_N \le x\}$, one can derive

$$\begin{aligned} &p(k,l) = \mathrm{P}(X_1 > x, ..., X_k > x, X_l \le x, ..., X_N \le x) \\ &= -\mathrm{P}(X_1 > x, ..., X_{k-1} > x, X_k \le x, X_l \le x, ..., X_N \le x) + \mathrm{P}(X_1 > x, ..., X_{k-1} > x, X_l \le x, ..., X_N \le x) \\ &= -\mathrm{P}(X_1 > x, ..., X_{k-1} > x, X_{l-1} \le x, X_l \le x, ..., X_N \le x) + \mathrm{P}(X_1 > x, ..., X_{k-1} > x, X_l \le x, ..., X_N \le x) \\ &= -p(k-1, l-1) + p(k-1, l). \end{aligned} \tag{35}$$

The first and last equalities are from the definition of p, and the third equality derives from the exchangeability assumption. From the recursive application of equation (35), one can predict the general formula for the mth order expansion of $p(k,l)$ as

$$p(k,l) = (-1)^m \sum_{j=0}^{m} \binom{m}{j} (-1)^j p(k-m, l-m+j). \tag{36}$$

Noting that $\binom{m}{j} + \binom{m}{j+1} = \binom{m+1}{j+1}$ and equation (35), we can derive

$$\begin{aligned}
&\binom{m}{j} p(k-m, l-m+j) - \binom{m}{j+1} p(k-m, l-m+j+1) \\
&\quad = -\binom{m}{j} p(k-m-1, l-m+j-1) + \binom{m+1}{j+1} p(k-m-1, l-m+j) \\
&\quad\quad - \binom{m}{j+1} p(k-m-1, l-m+j+1).
\end{aligned} \tag{37}$$

$$\begin{aligned}
&(-1)^m \sum_{j=0}^{m} \binom{m}{j} (-1)^j p(k-m, l-m+j) \\
&\quad = (-1)^m \left\{ \binom{m}{0} p(k-m, l-m) - \binom{m}{1} p(k-m, l-m+1) \right. \\
&\quad\quad \left. +, ..., +(-1)^{m-1} \binom{m}{m-1} p(k-m, l-1) + (-1)^m \binom{m}{m} p(k-m, l) \right\} \\
&\quad = (-1)^m \left\{ -\binom{m}{0} p(k-m-1, l-m-1) + \binom{m+1}{1} p(k-m-1, l-m) \right. \\
&\quad\quad +, ..., +(-1)^j \binom{m+1}{j} p(k-m-1, l-m+j-1) + (-1)^{j+1} \binom{m+1}{j+1} p(k-m-1, l-m+j) \\
&\quad\quad \left. +, ..., +(-1)^{m-1} \binom{m}{m-1} p(k-m-1, l-1) + (-1)^m \binom{m}{m} p(k-m-1, l) \right\} \\
&= (-1)^{m+1} \left\{ \binom{m+1}{0} p(k-m-1, l-m-1) - \binom{m+1}{1} p(k-m-1, l-m) \right. \\
&\quad\quad +, ..., +(-1)^{j+1} \binom{m+1}{j} p(k-m-1, l-m+j-1) + (-1)^{j+2} \binom{m+1}{j+1} p(k-m-1, l-m+j) \\
&\quad\quad \left. +, ..., +(-1)^m \binom{m+1}{m} p(k-m-1, l-1) + (-1)^{m+1} \binom{m+1}{m+1} p(k-m-1, l) \right\} \\
&= (-1)^{m+1} \sum_{j=0}^{m+1} \binom{m+1}{j} (-1)^j p(k-m-1, l-m-1+j)
\end{aligned}$$

Here, the second equality follows from equation (37). Hence, $p(k,l) = (-1)^m \sum_{j=0}^{m} \binom{m}{j} (-1)^j p(k-m, l-m+j)$ by induction. If one chooses $m = k$, this formula is expressed as

$$p(k,l) = (-1)^k \sum_{j=0}^{k} \binom{k}{j} (-1)^j p(l-k+j). \tag{38}$$

By utilizing this expression, we can calculate the probability as follows:

$$
\begin{aligned}
&\mathrm{P}\left(exactly\ N-k\ of\ X_1,...,X_N\ are\ at\ most\ x\right)\\
=&\binom{N}{k}\mathrm{P}\left(X_1 \leq x,...,X_{N-k} \leq x, X_{N-k+1} > x,...,X_N > x\right)\\
=&\binom{N}{k}\mathrm{P}\left(X_1 > x,...,X_k > x, X_{k+1} \leq x,...,X_N \leq x\right)\\
=&\binom{N}{k}p(k,k+1) = \binom{N}{k}(-1)^k\sum_{j=0}^{k}\binom{k}{j}(-1)^j p(j+1).
\end{aligned}
\tag{39}
$$

The first and second equalities come from the exchangeable property, the third expression from the definition of $p(k,l)$, and the last equality from equation (38). □

Proof of Lemma 2.3

Proof. Since $C_{eq}(u) = C(u,...,u)$ and there is symmetry,

$$
\frac{dC_{eq}}{du} = \frac{d}{du}C(u,...,u) = \sum\frac{\partial}{\partial u}C(u,...,u) = Nc_{u_N}(u,...,u),
\tag{40}
$$

where $c_{u_N}(\cdot,...,\cdot)$ is the partial derivative of the copula with respect to the N thcoordinate. Then,

$$
\frac{dC^{\leftarrow}_{eq}(u)}{du} = \frac{1}{Nc_{u_N}(C^{\leftarrow}_{eq}(u),...,C^{\leftarrow}_{eq}(u))}.
\tag{41}
$$

From these formulas for C_{eq} and symmetry, one can derive

$$
\begin{aligned}
&\frac{d}{dx}C(1,...,1,C^{\leftarrow}_{eq}(F^{(N:N)}(x)),...,,C^{\leftarrow}_{eq}(F^{(N:N)}(x)))\\
=&\frac{N-j}{N}\frac{c_{u_N}(1,...,1,C^{\leftarrow}_{eq}(F^{(N:N)}(x)),...,,C^{\leftarrow}_{eq}(F^{(N:N)}(x)))}{c_{u_N}(C^{\leftarrow}_{eq}(F^{(N:N)}(x)),...,,C^{\leftarrow}_{eq}(F^{(N:N)}(x)))}.
\end{aligned}
\tag{42}
$$

□

Proof of Lemma 4.1

Proof. From Lemma 1 of Hubbard et al., 2012,

$$\frac{h(\varphi|\varphi)}{H(\varphi|\varphi)} = (N-1)h(\varphi)\frac{c_{u_1u_2}(H(\varphi), H(\varphi), ..., H(\varphi))}{c_{u_1}(H(\varphi), H(\varphi), ..., H(\varphi))}. \tag{43}$$

From the expression of $C(u_1, ..., u_N) = \boldsymbol{\Psi}\left(\Psi^{\leftarrow}(u_1), ..., \Psi^{\leftarrow}(u_N)\right)$,

$$c_{u_1}(u_1, ..., u_N) = \frac{\boldsymbol{\Psi}_1\left(\Psi^{\leftarrow}(u_1), ..., \Psi^{\leftarrow}(u_N)\right)}{\psi\left(\Psi^{\leftarrow}(u_1)\right)}, \tag{44}$$

$$c_{u_1u_2}(u_1, ..., u_N) = \frac{\boldsymbol{\Psi}_{12}\left(\Psi^{\leftarrow}(u_1), ..., \Psi^{\leftarrow}(u_N)\right)}{\psi\left(\Psi^{\leftarrow}(u_1)\right)\left(\Psi^{\leftarrow}(u_2)\right)}. \tag{45}$$

By the formula for the derivative of the inverse function and composite functions,

$$\begin{aligned}\frac{d}{d\varphi}\left[\boldsymbol{\Psi}_1\left(\Psi^{\leftarrow}(H(\varphi)), ..., \Psi^{\leftarrow}(H(\varphi))\right)\right]^2 =& (N-1)\boldsymbol{\Psi}_{12}\left(\Psi^{\leftarrow}(H(\varphi)), ..., \Psi^{\leftarrow}(H(\varphi))\right)\frac{h(\varphi)}{\psi(\Psi^{\leftarrow}(H(\varphi)))} \\ &+ \boldsymbol{\Psi}_{11}\left(\Psi^{\leftarrow}(H(\varphi)), ..., \Psi^{\leftarrow}(H(\varphi))\right)\frac{h(\varphi)}{\psi(\Psi^{\leftarrow}(H(\varphi)))}.\end{aligned} \tag{46}$$

Hence, $\frac{h(\varphi|\varphi)}{H(\varphi|\varphi)}$ is expressed as

$$\begin{aligned}\frac{h(\varphi|\varphi)}{H(\varphi|\varphi)} =& \frac{(N-1)h(\varphi)}{\psi\left(\Psi^{\leftarrow}(H(\varphi))\right)}\frac{\boldsymbol{\Psi}_{12}\left(\Psi^{\leftarrow}(H(\varphi)), ..., \Psi^{\leftarrow}(H(\varphi))\right)}{\boldsymbol{\Psi}_1\left(\Psi^{\leftarrow}(H(\varphi)), ..., \Psi^{\leftarrow}(H(\varphi))\right)} \\ =& \frac{d}{d\varphi}\log\left[\boldsymbol{\Psi}_1(\Psi^{\leftarrow}(H(\varphi)), ..., \Psi^{\leftarrow}(H(\varphi)))\right] + \frac{\boldsymbol{\Psi}_{11}\left(\Psi^{\leftarrow}(H(\varphi)), ..., \Psi^{\leftarrow}(H(\varphi))\right)}{\boldsymbol{\Psi}_1\left(\Psi^{\leftarrow}(H(\varphi)), ..., \Psi^{\leftarrow}(H(\varphi))\right)}\frac{d}{d\varphi}\Psi^{\leftarrow}(H(\varphi)).\end{aligned} \tag{47}$$

Therefore, if $\frac{\boldsymbol{\Psi}_{11}(u,...,u)}{\boldsymbol{\Psi}_1(u,...,u)} = \frac{d\tilde{\Psi}(u)}{du}$ for some function $\tilde{\Psi}$,

$$\begin{aligned}&\frac{d}{d\varphi}\log\left[\boldsymbol{\Psi}_1(\Psi^{\leftarrow}(H(\varphi)), ..., \Psi^{\leftarrow}(H(\varphi)))\right] + \frac{\boldsymbol{\Psi}_{11}\left(\Psi^{\leftarrow}(H(\varphi)), ..., \Psi^{\leftarrow}(H(\varphi))\right)}{\boldsymbol{\Psi}_1\left(\Psi^{\leftarrow}(H(\varphi)), ..., \Psi^{\leftarrow}(H(\varphi))\right)}\frac{d}{d\varphi}\Psi^{\leftarrow}(H(\varphi)) \\ =& \frac{d}{d\varphi}\left\{\log\left[\boldsymbol{\Psi}_1\left(\Psi^{\leftarrow}(H(\varphi)), ..., \Psi^{\leftarrow}(H(\varphi))\right)\exp\left(-\tilde{\Psi}(\Psi^{\leftarrow}(H(\varphi)))\right)\right]\right\}.\end{aligned} \tag{48}$$

Since we find the primitive function, we can also find the analytical expression for the distribution function:.

$$
\begin{aligned}
L(y|v) = & \exp\left(-\int_y^v \frac{h(\varphi|\varphi)}{H(\varphi|\varphi)}d\varphi\right) \\
= & \exp\left(\int_y^v \frac{d}{d\varphi}\left\{\log\left[\boldsymbol{\Psi}_1\left(\Psi^{\leftarrow}(H(\varphi)), ..., \Psi^{\leftarrow}(H(\varphi))\right)\exp\left(-\tilde{\Psi}(\Psi^{\leftarrow}(H(\varphi)))\right)\right]\right\}d\varphi\right) \\
= & \exp\left(\log\left[\frac{\boldsymbol{\Psi}_1\left(\Psi^{\leftarrow}(H(y)), ..., \Psi^{\leftarrow}(H(y))\right)\exp\left(-\tilde{\Psi}(\Psi^{\leftarrow}(H(y)))\right)}{\boldsymbol{\Psi}_1\left(\Psi^{\leftarrow}(H(v)), ..., \Psi^{\leftarrow}(H(v))\right)\exp\left(-\tilde{\Psi}(\Psi^{\leftarrow}(H(v)))\right)}\right]\right) \\
= & \left[\frac{\boldsymbol{\Psi}_1\left(\Psi^{\leftarrow}(H(y)), ..., \Psi^{\leftarrow}(H(y))\right)\exp\left(-\tilde{\Psi}(\Psi^{\leftarrow}(H(y)))\right)}{\boldsymbol{\Psi}_1\left(\Psi^{\leftarrow}(H(v)), ..., \Psi^{\leftarrow}(H(v))\right)\exp\left(-\tilde{\Psi}(\Psi^{\leftarrow}(H(v)))\right)}\right].
\end{aligned}
\tag{49}
$$

If $\tilde{\Psi} = \log\tilde{\tilde{\Psi}}$ for some function $\tilde{\tilde{\Psi}}$,

$$
L(y|v) = \left[\frac{\boldsymbol{\Psi}_1\left(\Psi^{\leftarrow}(H(y)), ..., \Psi^{\leftarrow}(H(y))\right)\tilde{\tilde{\Psi}}(\Psi^{\leftarrow}(H(v)))}{\boldsymbol{\Psi}_1\left(\Psi^{\leftarrow}(H(v)), ..., \Psi^{\leftarrow}(H(v))\right)\tilde{\tilde{\Psi}}(\Psi^{\leftarrow}(H(y)))}\right]. \tag{50}
$$

When the joint distribution follows the Aarchimedean family of copulas,

$$
\frac{\boldsymbol{\Psi}_{11}\left(u, ..., u\right)}{\boldsymbol{\Psi}_1\left(u, ..., u\right)} = \frac{d}{du}\log\left[-\frac{1}{N}\zeta'(\zeta^{-1}(Nu))\right]^{-1/N}. \tag{51}
$$

That is, $\tilde{\tilde{\Psi}} = \left[-\frac{1}{N}\zeta'(\zeta^{-1}(Nu))\right]^{-1/N}$, and

$$
\begin{aligned}
L(y|v) = & \left[\frac{\zeta'(\zeta^{-1}(N\zeta(H(v))))\zeta'(\zeta^{-1}(N\zeta(H(v))))^{-1/N}}{\zeta'(\zeta^{-1}(N\zeta(H(y))))\zeta'(\zeta^{-1}(N\zeta(H(y))))^{-1/N}}\right] \\
= & \left[\frac{\zeta'(\zeta^{-1}(N\zeta(H(v))))}{\zeta'(\zeta^{-1}(N\zeta(H(y))))}\right]^{\frac{N-1}{N}}.
\end{aligned}
\tag{52}
$$

$L(\cdot|x)$ is the distribution function (see,. e.g., Krishna, 2009). $\beta(v) = \int y dL(y|v)$. Hence, if $\tilde{y}_r(v)$ $(r = 1, ..., R)$ is generated from $L(\cdot|v)$, $\frac{1}{R}\sum_{r=1}^{R}\tilde{y}_r(v) \to \beta(v)$ almost surely as $R \to \infty$. □