Consistent Specification Test of the Quantile Autoregression*

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Abstract

This article proposes a specification test of a parametric Quantile Autoregression (QAR). Contrary to the existing procedures, we allow for the information set and/or the explanatory vector to include an unobservable random vector of factors characterising a much larger information set. We study the asymptotic distribution of the test statistics under fairly weak conditions and show that the inclusion of latent factors does not influence the asymptotic null distribution already derived for parametric quantile models with solely observable variables. We propose a bootstrap procedure for approximating the asymptotic critical values of the test. Finally, an application to the estimation of the distribution of UK inflation is suggested, which demonstrates that the proposed specification test may prove to be a suitable testing tool for policy makers, when deciding whether to include factors in a QAR estimation.

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1 Introduction

Quantile regression (QR) enables the analysis of a continuous range of conditional quantile functions, providing a more complete picture of the conditional dependence structure of the variables examined, rather than a single measure of conditional location. The QR approach has therefore emerged as a powerful complement to standard least squares regression, essentially a theory for conditional expectations models, in a variety of econometric applications including labour economics, demand analysis and finance (Koenker & Hallock, 2001). As the awareness of the importance of heterogeneity in the data increases, quantile regression has become more relevant (Koenker, 2017).

Parametric quantile regressions in particular, have been shown to be a useful and flexible modelling strategy in the case of economic and financial variables, especially in the context of forecasting. QR has been the primary tool in the estimation and forecasting of Value-at-Risk, a standard tool for measuring market risk (Engle & Manganelli, 2004), and studies of the wage and income distribution (Koenker & Xiao, 2002; Machado & Mata, 2005 among others). The presence, however, of time dependent data in economic and financial scenarios deems the Quantile Autoregression (QAR), as proposed by Koenker & Xiao (2006), a more suitable framework for relevant studies.

Nevertheless, inference on parametric quantile models critically depends on the validity of the parametric functional form that is specified for the particular quantiles under consideration. Therefore, in the case of quantile autoregression, any post-estimation inference is heavily dependent on the implicit assumption that the linear quantile specification is indeed correct. In accordance with the developments in the empirical field, tests for the correct specification of conditional quantiles functions over a continuum of quantile levels have been proposed by several authors, see e.g. Escanciano & Goh (2014), Escanciano & Velasco (2010), Conde-Amboage *et al.* (2015) and references therein.

Meanwhile, in empirical work the recent availability of large datasets, combined with advances in the field of statistics and econometrics, has generated interest in predictive models with many possible predictors. As a result, inference methods overcoming the curse of dimensionality have become increasingly popular. Factor analysis, an example of a dense modelling technique, recognises that all possible explanatory variables might have an impact on prediction, though such an impact might be of minimal size. On the other hand, sparse-modelling techniques like the LASSO, focus on selecting a small set of explanatory variables with the highest predictive power out of a much larger pool of regressors (Giannone *et al.*, 2018).

In the macroeconomics context particularly, policy makers have mainly focused in the use of factors as a way to summarise large amounts of information in a more parsimonious manner. Factors have been proven useful in overcoming the limited information bias, arising from the fact that the information set of decision makers is sufficiently larger than the information set captured by conventional empirical models, and the curse of dimensionality. Particularly for inflation, several authors suggest that (average) forecasts based on such factors seem to outperform univariate regressions (Stock & Watson, 2002; Bernanke *et al.*, 2005). Nevertheless, to the best of our knowledge, nobody has examined whether the aforementioned also holds in the case of conditional forecasts of the distribution of inflation through the QAR approach.

Therefore, in accordance with the developments in these two strands of the literature on QAR and factor models, we develop a consistent test for the correct specification of the linear QAR against the alternative of a Factor-Augmented QAR(FA-QAR), conditional on a given information set, with an empirical application to the distribution of inflation. Given that such factors are in practice unobservable what is used in the information set is their estimates and therefore in contrast to Escanciano & Velasco (2010), we have both parameter estimation error and factor estimation error present.

It is worth mentioning that there have been prior studies where the QR has been augmented with factors. Ando & Tsay (2011) have considered a quantile regression model with factor-augmented predictors, whose effect is allowed to vary across the different quantiles. Contrary to our work, their study models the quantile structure in a cross-sectional context. More recently, Ando & Bai (2017) introduced a new procedure for analysing the quantile co-movement of a large number of time series based on a large scale panel data model with factor structures. Though their model might at first look similar to our work, there is a substantial difference. In their study the latent factors are allowed to vary across the different quantiles and as such their model is a quantile factor model. Similarly, Chen *et al.* (2017) estimate scale-shifting factors and quantile dependent loadings, thus factors may shift characteristics (moments or quantiles) of the distribution of the set of directly observable measures, other than its mean and factor loadings are allowed to vary across the distributional characteristics of each variable. Our analysis however, solely involves mean-shifting factors used as additional predictors in a QAR and their effect is allowed to vary across the different quantiles of the variable of interest.

More precisely, suppose we observe a real-valued dependent variable y_t and the information vector $I_{t-1} = (Y'_{t-1}, \hat{F}'_{t-1}) \in \Re^d$, $d = (p+1) + k$, where $\hat{F}_{t-1} = (F_{1,t-1}, ..., F_{k,t-1}) \in \Re^k, k \in \aleph$, is the vector of estimated factors and $Y_{t-1} = (1, y_{t-1}, ..., y_{t-p}) \in \Re^{p+1}$, where A' denotes the transpose of A. We assume throughout the article that the time series process $\{(Y_t, \hat{F}'_{t-1})' : t = 0, \pm 1, \pm 2, ...\}$, defined on the probability space $(\Omega, \mathcal{A})$ is strictly stationary and ergodic. We can then define the τ^{th} conditional quantile of Y_t given I_{t-1} as the measurable function q_τ satisfying the conditional restriction

$$P(Y_t \leq q_\tau(I_{t-1}) \mid I_{t-1}) = \tau, \text{ almost surely (a.s).} \tag{1}$$

We use the Linear Quantile Autoregression (LQAR) model of order p as that is set out in Koenker & Xiao (2006) and in the primary stage of our testing procedure we do not augment the LQAR with the estimated factors that are in our information set. Our variable of interest can be characterised by the following equation

$$y_t = \theta_0(u_t) + \theta_1(u_t)y_{t-1} + ... + \theta_p(u_t)y_{t-p}, \tag{2}$$

where u_t is a sequence of i.i.d. standard uniform random variable, while $\theta_i(u_t)$ are unknown functions $[0,1] \to \mathbb{R}$. Provided that the right hand side of (2) is monotone increasing in u_t our specification takes the following form

$$\begin{aligned} m(I_{t-1}, \theta(\tau)) \equiv m(Y_{t-1}, \theta_0(\tau)) &= \theta_0(\tau) + \theta_1(\tau)y_{t-1} + ... + \theta_p(\tau)y_{t-p} \\ &= Y'_{t-1}\theta(\tau). \end{aligned} \tag{3}$$

Any inference therefore on the associated quantile process will heavily depend on the correct specification of this parametric quantile regression model. In this instance, we thus want to examine whether the estimated factors, which are included in the information set but have been neglected in the estimation, carry any relevant information for the quantile process of our dependent variable Y_t. Our test is therefore characterised by the infinite number of conditional moment restrictions

$$E[\mathbb{1}(y_t - Y'_{t-1}\theta(\tau) \leq 0) - \tau | I_{t-1}] = 0 \qquad \text{a.s. for some } \theta(\tau) \in \Theta. \tag{4}$$

The asymptotic theory is somehow complicated by the presence of the estimated and not directly observable factors in the information set. We solve this difficulty by employing the results of Bai (2003) who show that both the factor estimators and estimated factor loadings are consistent up to a normalisation, even in the presence of serial correlation and heteroscedasticity, for large N and T, as long as $\frac{\sqrt{T}}{N} \to 0$. We show that the asymptotic null distribution of the test statistic converges to the same asymptotic null distribution demonstrated in Escanciano & Velasco (2010) which depends on the specification under the null and the DGP. As a result, we propose to implement this test with the use of bootstrapping.

The remainder of the article is organised as follows. In Section 2 we outline the set-up and describe the testing procedure one should follow in order to examine whether a Quantile Autoregression should be augmented with factors. We define our test statistic and study the asymptotic properties of the suggested statistic. We show its convergence to a test statistic with solely observable variables, whose asymptotic distribution has already been established in the literature. Finally, section 3 provides an overview of the empirical application on the distribution of inflation, based on the use of a large-scale macroeconomic dataset. Information regarding proofs is referred to an appendix.

2 Test Statistic and Asymptotic Theory

We begin by outlining the factor model used in the sequel. Let

$$X_t = \Lambda_t F_t + e_t \tag{5}$$

where X_t is an $N \times 1$ vector of observable variables characterising the economy, Λ_t is an $N \times k$ matrix of factor loadings, F_t is a $k \times 1$ vector of the k latent common factors and e_t is an $N \times 1$ vector of idiosyncratic disturbances. The errors are allowed to be both serially and (weakly) cross sectionally correlated.

Factors are extracted via the principle components approach and the resulting principal components estimator of F is then $\hat{F} = \frac{X'\hat{\Lambda}}{N}$, where $\hat{\Lambda}$ is set equal to the eigenvectors of $X'X$ corresponding to its r largest eigenvalues (Stock & Watson, 2002). As N increases, the principle components estimator converges to the maximum likelihood estimator (Chamberlain & Rothschild, 1983) but given the computational simplicity of the former, it was deemed more appropriate.

We aim to test the null hypothesis of correct linear QAR specification:

$$H_0 : \quad E[\mathbb{1}(y_t - m(I_{t-1}, \theta(\tau)) \leq 0) - \tau | I_{t-1}] = 0 \qquad \text{a.s. for some } \theta(\tau) \in \Theta \tag{6}$$

against the alternative

$$H_A : \quad Pr\{E[\mathbb{1}(y_t - m(I_{t-1}, \theta(\tau)) \leq 0) - \tau | I_{t-1}] = 0\} < 1 \tag{7}$$

where $I_{t-1} = (Y'_{t-1}, \hat{F'}_{t-1}) \in \Re^d$ is the actual information set of a researcher, $d = (p+1) + k$, $Y_{t-1} = (1, y_{t-1}, ..., y_{t-p}) \in \Re^{p+1}$ is the vector of relevant lags for the dependent variable and a constant term and $\hat{F}_{t-1} \in \Re^k$ is the vector of estimated factors. Note that under H_0 (and a mild continuity condition), $m_{t-1}(\theta)$ is identified as the τ^{th} quantile of the conditional distribution of y_t given I_{t-1}, for all $\tau \in \mathscr{T}$

The null hypothesis demonstrates that if the specification is correct, the probability that the observed value of y_t falls below the estimated quantile should, on average, equal the nominal quantile level of interest (τ), with probability one. Therefore, if the null hypothesis holds, it is implied that the quantile of y_t is uncorrelated with any information in the information set that was not utilised in the estimation process.

Testing for the null hypothesis H_0 is a challenging problem, since it involves an infinite number of conditional moments parameterised by $\tau \in \mathscr{T}$, where $\mathscr{T}$ is a compact set comprising of the range of quantiles of interest ($\mathscr{T} \subset [0,1]$). Therefore we can characterise H_0 by the infinite number of unconditional moment restrictions:

$$E\{[\mathbb{1}(y_t - m(I_{t-1}, \theta(\tau)) \leq 0) - \tau] exp(b'\Phi(I_{t-1}))\} = 0 \tag{8}$$

where Φ is an arbitrary Borel Measurable bounded one-to-one mapping from $\Re^d$ to $\Re^d$ as in Bierens (1990) and b is a $d \times 1$ vector of weights. Conditioning on I_{t-1} is equivalent to conditioning on the bounded vector $\Phi(I_{t-1})$, for I_{t-1} and $\Phi(I_{t-1})$ generate the same Borel field.

Given therefore a sample $\{(y_t, \hat{I}'_{t-1})' : 1 \leq t \leq T\}$ and an estimated parameter value $\hat{\theta}(\tau)$ we consider the following **quantile empirical process**:

$$S_T(b, \tau, \theta_T) := T^{-\frac{1}{2}} \sum_{t=1}^{T} [\mathbb{1}(y_t - m(I_{t-1}, \theta(\tau)) \leq 0) - \tau] exp(b'_Y \Phi(Y_{t-1})) exp(b'_F \Phi(\hat{F}_{t-1})) \tag{9}$$

for a $\sqrt{T} - consistent$ estimator $\hat{\theta}_T(\tau)$ of $\theta(\tau)$. The most popular estimator for θ_0 is the Quantile Autoregression Estimstor (QARE), proposed by Koenker & Xiao (2006)). The QARE is defined as

$$\hat{\theta}_T = \arg \min_{\theta \in \mathbb{R}^{p+1}} \sum_{t=1}^{T} \rho_t(y_t - m(I_{t-1}, \theta(\tau))) \tag{10}$$

where $\rho_t(u) = |u|[\tau \mathbb{1}(u \geq 0) + (1-\tau)\mathbb{1}(u < 0)] = u(\tau - \mathbb{1}(u < 0))$ is the "tick" loss function.

The null hypothesis holds when the process $S_T(b, \tau, \theta_T)$ is close to zero for almost all $(b', \tau)' \in \Re^d \times \mathscr{T}$, and thus the test statistic is based on a distance from a standardised sample analogue of $E\{[\mathbb{1}(y_t - m(I_{t-1}, \theta(\tau)) \leq 0) - \tau] exp(b'_Y \Phi(Y_{t-1})) exp(b'_F \Phi(\hat{F}_{t-1}))\}$ to zero.

A joint desire to exploit an increasing number of moment restrictions for the sake of power and ensure that the test is asymptotically invariant to θ_T motivates a test statistic based on functionals of the aforementioned empirical process. Some popular norms we could consider are the following:

- **Cramer-von-Mises** $\Rightarrow$ $\mathrm{CvM_T} := \int_{\mathscr{T}} \int_{\Upsilon} |S_T(b,\tau,\theta_T)|^2 dZ(b) dW(\tau)$

- **Kolmogorov-Smirnov** $\Rightarrow$ $\mathrm{KS_T} := sup_{\tau\in\mathscr{T}} \int_{\Upsilon} |S_T(b,\tau,\theta_T)|^2 dZ(b)$

where Z and W are some integrating measures on Υ, a generic compact set of $\Re^d$ containing the origin, and $\mathscr{T}$ respectively. The test we propose therefore rejectsH_0 for "large" values of such functionals.

2.1 Assumptions

Let $\|A\| = [tr(A'A)^{\frac{1}{2}}]$ denote the norm of matrix A. Throughout, we let F_t be the $k\times 1$ vector of true factors and λ_i be the true loadings, with F and Λ being the corresponding matrices. The following assumptions are used in Bai (2003) to derive the limiting distributions of the estimated factors, factor loadings and common components. We rely on those same assumptions, to demonstrate that factor estimation error does not influence our test statistic.

Assumption A: Factors
$E\|F_t\|^4 \leq M < \infty$ and $T^{-1}\sum_{t=1}^{T} F_t F_t' \xrightarrow{\text{p}} \Sigma_F$ for some $k\times k$ positive definite matrix Σ_F.

Assumption B: Factor loadings
$E\|\lambda_i\| \leq \bar{\lambda} < \infty$ and $\|\frac{\Lambda\Lambda'}{N} - \Sigma_\Lambda\| \rightarrow 0$ for some $k\times k$ positive definite matrix Σ_Λ.

Assumption C: Time and cross-section dependence and heteroscedasticity
There exists a positive constant $M < \infty$ such that for all N and T,

1. $E(e_{it}) = 0, E|e_{it}|^8 < M$;

2. $E(e_s'e_t/N) = E(N^{-1}\sum_{i=1}^{N} e_{is}e_{it}) = \gamma_N(s,t), |\gamma_N(s,s)| \leq M$ for all s, and

   $$T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T} |\gamma_N(s,t)| \leq M;$$

3. $E(e_{it}e_{jt}) = c_{ij,t}$ with $|c_{ij,t}| \leq |c_{ij}|$ for some c_{ij} and for all t. In addition,

   $$N^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N} |c_{ij}| \leq M;$$

4. $E(e_{it}e_{js}) = c_{ij,ts}$ and $(NT)^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T} |c_{ij,ts}| \leq M$;

5. For every (t,s), $E|N^{-\frac{1}{2}}\sum_{i=1}^{N}[e_{is}e_{it}-E(e_{is}e_{it})]|^4 \leq M$.

Assumption D: Weak dependence between factors and idiosyncratic errors

$$E\big(\tfrac{1}{N}\sum_{i=1}^{N}\|\tfrac{1}{\sqrt{T}}\sum_{t=1}^{T}F_t e_{it}\|^2\big) \leq M.$$

Assumption E: Weak dependence
There exists $M<\infty$ such that for all T and N, and for every $t \leq T$ and every $i \leq N$,

1. $\sum_{s=1}^{T}|\gamma_N(s,t)| \leq M$

2. $\sum_{k=1}^{N}|c_{ki}| \leq M$

Assumption F: Moments and central limit theorem
There exists $M<\infty$ such that for all N, T

1. For each t,

$$E\|\tfrac{1}{\sqrt{NT}}\sum_{s=1}^{T}\sum_{k=1}^{N}F_t[e_{ks}e_{kt}-E(e_{ks}e_{kt})]\|^2 \leq M$$

2. The $k\times k$ matrix satisfies

$$E\|\tfrac{1}{\sqrt{NT}}\sum_{t=1}^{T}\sum_{k=1}^{N}F_t\lambda_k' e_{kt}\|^2 \leq M$$

3. For each t, as $N\rightarrow N$,

$$\tfrac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_i e_{it} \xrightarrow{\text{d}} N(0,\Gamma_t)$$

where $\Gamma_t = \lim_{N\rightarrow\infty}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_i\lambda_j E(e_{it}e_{jt})$

4. For each i, as $T\rightarrow\infty$,

$$\tfrac{1}{\sqrt{T}}\sum_{t=1}^{T}F_t e_{it} \xrightarrow{\text{d}} N(0,W_i),$$

where $W_i = \operatorname*{plim}_{T\rightarrow\infty}\frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}E[F_tF_s' e_{is}e_{it}]$.

Assumption A is more general than that of classical factor analysis, as it allows F_t to be dynamic such that $A(L)F_t = v_t$. However, this dynamic does not enter into X_{it} directly, so the relationship between X_{it} and F_t remains static. Assumption B ensures that each factor has a non-trivial contribution to the variance of X_t. Assumption C allows for limited time series and cross section dependence in the idiosyncratic components as well as heteroscedasticity in both dimensions. Assumption D does not require independence to hold and is implied by Assumptions A and C. Throughout this paper the number of factors (k) is assumed fixed as N and T grow. Assumptions A-D are sufficient for consistently estimating the number of factors (k), as well as the factors themselves and their corresponding loadings. In this paper we assume k is known, since the limiting distribution of the test statistic is not influenced when the number of factors is unknown and estimated, given that the limiting distribution of the factors themselves is not affected. Assumption E is a stronger version of assumptions C2 and C3 but is likely to hold for multiple empirical applications. Similarly, Assumption F is not stringent because sums in F1 and F2 involve zero mean random variables. The last two assumptions are central limit theorems, which are satisfied by several mixing processes.

To derive the asymptotic results of the quantile-marked empirical process we need to further consider the following assumptions and notation. The family $\mathcal{B}$, in which the parameter θ takes values is endowed with the sup norm, i.e., $\|\theta\|_{\mathcal{B}} = \sup_{\tau\in\Im} |\theta(\tau)|$. Let, for each $t \in \mathbb{Z}$, $\mathcal{F}_t = \sigma(I'_t, I'_{t-1}, \ldots)$ be the σ-field generated by the information set obtained up to time t. Define for each $t \in \mathbb{Z}$, the quantile "error" $\epsilon_t(\theta) \equiv \epsilon_t(\theta(\tau)) := Y_t - m(I_{t-1}, \theta(\tau))$. Define also the family of conditional distributions $F_b(y) := P(Y_t \leq y | I_{t-1} = b)$. Let f_b be the density function of the cumulative distribution function (cdf) F_b. In particular, $f_{I_{t-1}}(y)$ denoted the density of Y_t given I_{t-1}, evaluated at y. Let $N_{[\cdot]}(\delta, g, \|\cdot\|)$ be the δ-bracketing number of class of functions g with respect to a norm $\|\cdot\|$.[1]

Assumption G: Time series Model Checks

1. $\{(Y_t, F'_t)' : t = 0, \pm1, \pm2, \ldots\}$ is a strictly stationary and ergodic process. Under H_0, $\{\Psi_{\tau,t}(\theta), \mathcal{F}_t\}$ is a martingale difference sequence for all $\tau \in \Im$.

2. The parametric family $m(\cdot, \theta(\tau))$ is non-decreasing in τ a.s.

3. The family of distributions functions $\{F_b, b \in \Re^d\}$ has Lebesque densities $\{f_b, b \in \Re^d\}$ that are uniformly bounded away from zero for the quantiles of interest[2].

[1] In other words, the smallest number r such that there exists $f_1, \cdots, f_r$ and $\triangle_1, \cdots, \triangle_r$ such that $\max_{1\leq i\leq r} \| \triangle_i \| < \delta$ and for all $f \in g$, there exists an $1 \leq i \leq r$ such that $\|f - f_i\| < \triangle_i$

[2] In other words, $\inf_{b\in\Re^d,\tau\in\Im} |f_b(m(b, \theta(\tau)))| \geq C > 0$, satisfy $\sup_{b\in\Re^d,y\in\Im} |f_b(y)| \leq C$ and are equicontinuous: for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\sup_{b\in\Re^d,|y-z|\leq\delta} |f_b(y) - f_b(z)| \leq \varepsilon$

Assumption H: Compactness of the parameter space

The parametric space Θ is compact in $\Re^p$. The true parameter $\theta(\tau)$ belongs to the interior of Θ for each $\tau \in \Im$ and $\theta \in \mathcal{B}$. The class $\mathcal{B}$ satisfies

$$\int_0^\infty \left(log(N_{[\cdot]}(\delta^2, \}, \|\cdot\|_{\mathcal{B}}))\right)^{\frac{1}{2}} d\delta < \infty.$$

Assumption I: Estimator Consistency

The estimator θ_T satisfies that $P(\theta_t \in \mathcal{B}) \to 1$ as $T \to \infty$, and the following asymptotic expansion under H_0, uniformly in $\tau \in \Im$,

$$Q_T(\tau) = \sqrt{T}(\theta_T(\tau) - \theta(\tau))$$

$$= \frac{1}{\sqrt{T}} \sum_{t=1}^T l_\tau(Y_t, I_{t-1}, \theta(\tau)) + o_p(1)$$

where $\ell_\tau(\cdot)$ is such that $E[l_\tau(Y_1, I_0, \theta(\tau))] = 0$, $L_\tau(\theta(\tau)) = E[l_\tau(Y_1, I_0, \theta(\tau))l'_\tau(Y_1, I_0, \theta(\tau))]$ exists and is positive definite, and $E[l_\tau(Y_1, I_0, \theta(\tau))\Psi_\tau(Y_s - m(I_{s-1}, \theta(\tau))] = 0$ if $t \neq s$. Furthermore, as a process in $l^\infty(\Im)$, $Q_T(\tau)$ converges weakly to a Gaussian process $Q(\cdot)$ with zero mean and covariance function

$$K_Q(\tau_1, \tau_2) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^T \sum_{s=1}^T E[l_{\tau_1}(Y_1, I_0, \theta(\tau_1))l_{\tau_2}(Y_1, I_0, \theta(\tau_2))].$$

Assumption G1 is standard in time series model checks and is natural in the present context, while G3 is necessary for the asymptotic tightness of the process $S_T(b, \tau, \theta_T)$. Sufficient conditions for Assumption H can be found in Theorem 2.7.5 for monotone classes of functions which apply to the LQAR model in Vaart & Wellner (2000). Meanwhile, Assumption I has been established in the literature under a variety of conditions and different models and DGPs.

2.2 Asymptotic null distribution

In this subsection we establish the limit distribution of the quantile-marked empirical process $S_T(b, \tau, \theta_T)$ under the null hypothesis H_0.

We show that the null limit distribution of the test is identical to the limit distribution of an equivalent test where all of the variables in the information set are directly observable, as in Escanciano & Velasco (2010). The contribution of factor estimation error is therefore negligible when it comes to the asymptotic null distribution of the test statistic.

Theorem 1: Under the null hypothesis H_0 and Assumptions A-F,

$$
\begin{aligned}
S_T(b,\tau,\theta_T) &:= T^{-\frac{1}{2}}\sum_{t=1}^{T}[\mathbb{1}(y_t - m(I_{t-1},\theta(\tau)) \le 0) - \tau]exp(b_Y'\Phi(Y_{t-1}))exp(b_F'\Phi(\hat{F}_{t-1})) \\
&= T^{-\frac{1}{2}}\sum_{t=1}^{T}[\mathbb{1}(y_t - m(I_{t-1},\theta(\tau)) \le 0) - \tau]exp(b_Y'\Phi(Y_{t-1}))exp(b_F'\Phi(\hat{F}_{t-1})) \\
&\qquad \pm T^{-\frac{1}{2}}\sum_{t=1}^{T}[\mathbb{1}(y_t - m(I_{t-1},\theta(\tau)) \le 0) - \tau]exp(b_Y'\Phi(Y_{t-1}))exp(b_F'\Phi(F_{t-1}))] \\
&= T^{-\frac{1}{2}}\sum_{t=1}^{T}[\mathbb{1}(y_t - m(I_{t-1},\theta(\tau)) \le 0) - \tau]exp(b_Y'\Phi(Y_{t-1}))exp(b_F'\Phi(F_{t-1}))] + o_p(1)
\end{aligned}
$$

Under Assumption G1 and H_0, because $S_T(\nu)$ is a zero-mean-square-integrable martingale for each $\nu = (b', \tau)' \in \Pi$, under a suitable central limit theorem (CLT) for stationary ergodic martingale difference sequences, we have that the finite dimensional distributions of S_T converge to those of a multivariate normal distribution with a zero mean vector and a variance covariance matrix given by the covariance function

$$K_\infty(\nu_1,\nu_2) = (\tau_1 \wedge \tau_2 - \tau_1\tau_2)E[exp((b_1 - b_2)'\Phi(I_0))] \tag{11}$$

where, $\nu_1 = (b_1', \tau_1)'$ and $\nu_2 = (x_2', \tau 2)')$. With no mixing conditions required, the next theorem demonstrates that the finite-dimensional distributions of S_T weakly converge in the space $\ell^\infty(\Pi)$.

Theorem 2: Under the null hypothesis H_0, Theorem 1 and Assumption G,

$S_T(b,\tau,\theta) \Rightarrow S_\infty(b,\tau,\theta)$,

where R_∞ is a Gaussian process with zero mean and covariance function (11).

In practice θ_0 is also unknown and has to be estimated from a sample $\{(Y_t, I_{t-1}')' : 1 \le t \le T\}$ by an estimator θ_T. Therefore when we replace θ by θ_T in the quantile-marked empirical process we wish to see how this parameter estimation error affects the asymptotic properties of the process. Define the function
$G(b,\theta(\tau)) := E\big[g_{t-1}(\theta(\tau))f_{I_{t-1}}(m_{t-1},\theta)exp(b'\Phi(I_{t-1}))\big]$, $b \in \Upsilon$, $\tau \in \Im$.

Theorem 3: Under the null hypothesis H_0 and Assumptions G-H,
$\sup_{b\in\Upsilon,\tau\in\Im} |S_T(b,\tau,\theta_T) - S_T(b,\tau,\theta) + G'(b,\theta(\tau))T^{-\frac{1}{2}}\sum_{t=1}^{T} l_\tau(Y_t, I_{t-1},\theta(\tau))| = o_p(1)$.

Furthermore, under the assumptions of Theorem 3, we obtain the asymptotic null distribution of continuous functionals as CvM_T and KS_T.

Corollary 1: Under the assumptions of Theorem 3, for any continuous functional $\Gamma(\cdot)$ from $\ell^\infty(\Pi)$ to $\mathbb{R}$, given that $S_T(b,\tau,\theta_T) \Rightarrow S_\infty(b,\tau,\theta_T)$, where $S_\infty(\cdot) = S_\infty(\cdot) - G'(\cdot,\theta(\cdot))Q(\cdot)$,

$\Gamma(S_T(b,\tau,\theta_T)) \xrightarrow{\text{d}} \Gamma(S_\infty(b,\tau,\theta_T))$.

Details regarding the convergence of the statistic can be found in the Appendix.

2.3 Factor-Augmented Quantile Autoregression Alternative

Rejecting the null hypothesis can imply two different possibilities. The model can be either linearly misspecified or the estimated factors influence the quantile functions and should thus be included in the estimation process. The suggested next step is therefore to include such estimated factors in the estimation process of the quantile autoregression. The resulting model is what we call Factor-Augmented Quantile Autoregression and it is defined in the following manner:

$$y_t = \theta_0(u_t) + \theta_1(u_t)y_{t-1} + ... + \theta_p(u_t)y_{t-p} + \beta_1(u_t)\hat{F}_{1,t-1} + ... + \beta_k(u_t)\hat{F}_{k,t-1} \tag{12}$$

Provided that the right hand side of (2) is monotone increasing in u_t, it follows that the τ^{th} conditional quantile of y_t can be expressed as:

$$\begin{aligned} Q_{y_t}(\tau|y_{t-1},...,y_{t-p}) &= \theta_0(\tau) + \theta_1(\tau)y_{t-1} + ... + \theta_p(\tau)y_{t-p} + \beta_1(\tau)\hat{F}_{1,t-1} + ... + \beta_k(\tau)\hat{F}_{k,t-1} \\ &= Y'_{t-1}\theta(\tau) + \hat{F}'_{t-1}\beta(\tau) \end{aligned} \tag{13}$$

Once again factors in this case are still unobservable and thus replaced by their estimated counterparts. It is therefore evident that under this model factor estimation error would be influencing the test statistic in a duplex manner. In addition to its impact on the information set, factor estimation error will now also influence the quantile error (the difference between the estimated quantile and the true one). Nevertheless, once again keeping in mind that under conditions set in Bai (2003) factor estimation error is negligible, we are confident that the presence of estimated factors does not influence the asymptotic distribution under the null.

Therefore following the same testing procedure described before, one could eventually test whether the Factor-Augmented Quantile Autoregression is correctly specified. If the null hypothesis is rejected again, that is if the new moment condition still does not equal zero, then this would imply that the linear functional form of the quantile is the cause of misspecification. A non-parametric estimation would then be the logical step in an attempt to obtain a correctly specified quantile function.

A non-parametric approach would also be suitable in the case where the factors are such that the monotonicity assumption of the right hand side of (12) is violated. This is because even if the monotonicity conditions are not fulfilled at all t, the coefficient functions may still be identifiable and consistently estimated, but the conditional quantile regression estimate can be inconsistent. As a result, given that this estimate influences our test statistic, any inference might be contaminated from such inconsistency.

3 Empirical Application To The Inflation Distribution

The empirical application of the test involves the estimation of several quantiles of inflation, as a way to trace out its complete distribution. We consider the question of whether there has been a significant change in the shape or scale of the distribution of UK inflation, following the recent financial crisis, while considering that the location (i.e. the mean) has not experienced any dramatic changes. The reasoning behind our decision to focus on the UK inflation is twofold. On the one hand, research on the inflation dynamics of the UK is severely limited compared to that of other countries, like the US. On the other hand, we believe that the application to the UK inflation is of distinctive interest, given the extensive use of “fan charts” by the Bank of England (BoE) in its Inflation Reports, to describe its best prevision of future inflation to the general public ever since 1997.

The term “fan chart” was coined by the BoE itself. It is essentially a chart which joins a simple line chart for past observed data, by showing ranges for possible values of future data together with a line demonstrating the most likely outcome value. As uncertainty increases, with longer time horizons, these forecast ranges become larger thus creating a “fan shape”, hence the term. These "fan charts" therefore, explicitly aim to shift the discussion from the accuracy of the point forecasts to the assessment of uncertainties surrounding such forecasts. This is similarly the aim of quantile regression, to shift attention from the mean of the variable of interest, to higher moments and features of its distribution.

3.1 Data

As in Ellis *et al.* (2014) the dataset for the estimation of factors and inflation includes 350 data series containing data on inflation, real activity and indicators of money and key asset prices for the United Kingdom. The data is used to generate the k latent common factors that are included in the information set of the specification test. Data can be obtained from the Office of National Statistics (ONS), Bank of England and the Global Financial Database and is in its majority publicly available. The sample includes quarterly data spanning from the first quarter of 1989 up until the second quarter of 2017.

3.2 Critical Values Estimation

Under the null hypothesis, the quantile error $S_T(b, \tau, \theta)$ converges to a Gaussian process with zero mean and a given covariance structure. However, when the estimated parameter $\hat{\theta}_T$ is used in $S_T(b, \tau, \hat{\theta}_T)$ the parameter estimation error affects its asymptotic properties. Given that the asymptotic null distribution of $S_T(b, \tau, \hat{\theta}_T)$ will be dependent on the data generating process and thus is not nuisance parameter free, critical values for the test statistics cannot be tabulated for general cases.

Hence, we need to construct bootstrap critical values, following Corradi & Swanson (2006), which take into account the presence of parameter estimation error. We proceed therefore by drawing bl blocks (with replacement) of length le from the sample $B_t = (y_t, Y_{t-1}, \hat{F}_{t-1})$, where $T = le \times bl$, such that each block is equal to $B_{i+1}, ..., B_{i+le}$, for some i, with probability $\frac{1}{T-l+1}$. Having obtained the resampled series ${B_t}^* = ({y_t}^*, {Y_{t-1}}^*, \hat{F}^*_{t-1})$, which consists of bl blocks that are discrete i.i.d. uniform random variables, conditional on the sample, we estimate the bootstrap estimators, (i.e. $\hat{\theta}^*_T$), and consequently the bootstrap test statistics, ${CvM_T}^*$ or ${KS_T}^*$. Performing therefore B bootstrap replications, with B large, we compute the percentiles of the empirical distribution of the B bootstrapped statistics.

The null hypothesis H_0 is rejected if the test statistic based on the original sample (i.e. CvM_T or

KS_T) is greater than the $(1-\alpha)^{th}$ percentile of the empirical distribution, where α is the level of significance.

3.3 Preliminary Results

Preliminary results (shown in the Table 1) demonstrate that when estimating the different quantiles of inflation, the extreme quantiles, under the linear QAR model, are misspecified and thus need to be augmented with factors. The median on the other hand seems to not suffer from this misspecification. Table 2 also shows that when aggregating across multiple quantiles through the use of the continuous functional the model is overall misspecified. These preliminary results give us an indication that when a researcher wishes to study features of the distribution of inflation other than the mean, he or she must augment the model with factors extracted from variables characterising the economy, as additional predictors.

1000 bootstrap replications		**Critical Values**			
Quantile level	**Statistic**	$\alpha = 5\%$		$\alpha = 10\%$	
$\tau = 0.10$	4.540	-1.180	1.129	-0.962	0.694
$\tau = 0.20$	3.324	-0.922	1.255	-0.753	0.892
$\tau = 0.50$	-0.145	-0.670	1.394	-0.471	1.096
$\tau = 0.80$	-3.437	-0.806	1.578	-0.504	1.211
$\tau = 0.90$	-4.652	-0.602	1.745	-0.331	1.474

*p=5, k=2

Table 1: Results for extreme and median quantiles

1000 bootstrap replications		**Critical Values**			
Functional	**Statistic**	$\alpha = 5\%$		$\alpha = 10\%$	
KS	11.4364	-3.477	3.291	-2.473	2.609
CvM	4.063	-1.338	0.398	-1.0199	0.230

*p=5, k=2

Table 2: Results for continuous functionals

4 Appendix

Proof of Theorem 1. As mentioned in the main text, in practice the true factors F in our information set are unobservable and have to be replaced by their estimated counterparts $\hat{F}$. Therefore,once $\hat{F}_{t-1}$ are replaced in I_{t-1} resulting in $\hat{I}_{t-1}$, we demonstrate how this factor estimation error affects the asymptotic properties of $S_T(b,\tau,\theta)$. Theorem 1 claims that our statistic, which includes the estimated factors instead of the true ones, converges to the statistic of Escanciano & Velasco (2010) plus an additional element which converges with probability one to zero.

$$S_T(b,\tau,\theta) = T^{-\frac{1}{2}} \sum_{t=1}^{T} [\mathbb{1}(Y_t - Y_{t-1}'\boldsymbol{\theta(\tau)} \leq 0) - \tau] exp(b'\Phi(\hat{I}_{t-1}))$$

Bai (2003) developed an inferential theory for factor models where it was demonstrated that the estimated common components are asymptotically normal with a convergence rate equal to the minimum of the square roots of N and T.

Under the assumptions set out in their work, as $N,T \to \infty$,

(i) if $\frac{\sqrt{N}}{T} \to 0$, then for each t,

$$\sqrt{N}(\hat{F}_t - H'F_t) = V_{NT}^{-1}\Big(\frac{\hat{F}'F}{T}\Big)\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_i e_{it} + o_p(1)$$
$$\xrightarrow{d} N(0, V^{-1}Q\Gamma_t Q'V^{-1})$$

(ii) if $lim\ inf\ \frac{\sqrt{N}}{T} \geq c > 0$, then

$$T(\hat{F}_t - H'F_t) = O_p(1)$$

Their results were based on the following identity, set out in Bai & Ng (2002),

$$\hat{F}_t - H'F_t = V_{NT}^{-1}\big[\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\gamma_N(s,t) + \frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\zeta_{st} + \frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\eta_{st} + \frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\xi_{st}\big] \tag{14}$$

where

$$\gamma_N(s,t) = E(\frac{1}{N}\sum_{s=1}^{N} e_{is}e_{it}) = E\big(\frac{e_s'e_t}{N}\big)$$
$$\zeta_{st} = \frac{e_s'e_t}{N} - \gamma_N(s,t)$$
$$\eta_{st} = \frac{F_s'\Lambda'e_t}{N}$$
$$\xi_{st} = \frac{F_t'\Lambda'e_s}{N}$$

Therefore, using an intermediate value expansion around the true factors we obtain the following:

$$\begin{aligned}
S_T(b,\tau,\theta) &= T^{-\frac{1}{2}}\sum_{t=1}^{T}[\mathbb{1}(Y_t - \boldsymbol{Y'_{t-1}\theta(\tau)} \leq 0) - \tau]exp(b'\Phi(\hat{I}_{t-1})) \\
&= T^{-\frac{1}{2}}\sum_{t=1}^{T}[\mathbb{1}(\epsilon_t(\tau) \leq 0) - \tau]exp(b'_Y\Phi(Y_{t-1}))exp(b'_F\Phi(\hat{F}_{t-1})) \\
&= T^{-\frac{1}{2}}\sum_{t=1}^{T}\Psi_\tau(\epsilon_t)exp(b'_Y\Phi(Y_{t-1}))[exp(b'_F\Phi(HF_{t-1}) + \bigtriangledown_{F|\overline{HF}}(exp(b'_F\Phi(\hat{F}_{t-1})))[\hat{F}_{t-1} - HF_{t-1}]] \\
&= \underbrace{T^{-\frac{1}{2}}\sum_{t=1}^{T}\Psi_\tau(\epsilon_t)exp(b'_Y\Phi(Y_{t-1}))exp(b'_F\Phi(HF_{t-1}))}_{\text{Equivalent to the test statistic of Escanciano \& Velasco (2010)}^3} \quad + \\
&\qquad \underbrace{T^{-\frac{1}{2}}\sum_{t=1}^{T}\Psi_\tau(\epsilon_t)exp(b'_Y\Phi(Y_{t-1}))\bigtriangledown_{F|\overline{HF}}(exp(b'_F\Phi(\hat{F}_{t-1}))) * [\hat{F}_{t-1} - HF_{t-1}]}_{II}
\end{aligned}$$

Focusing then on the second term, using the identity of the forecast error as that is shown in Bai & Ng (2002) we get:

$$\begin{aligned}
II &= T^{-\frac{1}{2}}\sum_{t=1}^{T}\Psi_\tau(\epsilon_t)exp(b'_Y\Phi(Y_{t-1}))\bigtriangledown_{F|\overline{HF}}(exp(b'_F\Phi(\hat{F}_{t-1}))) * [\hat{F}_{t-1} - HF_{t-1}] \\
&= T^{-1}\sum_{t=1}^{T}\Psi_\tau(\epsilon_t)exp(b'_Y\Phi(Y_{t-1}))\bigtriangledown_{F|\overline{HF}}(exp(b'_F\Phi(\hat{F}_{t-1}))) * \sqrt{T} * \\
&\qquad \Big\{V_{NT}^{-1}[\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\gamma_N(s,t) + \frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\zeta_{st} + \frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\eta_{st} + \frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\xi_{\ st}]\Big\}
\end{aligned}$$

In the above term, e_{it} is determined by the relationship between the observables variables X and the factors F (equation 5) while ϵ_t is the residual of the quantile autoregression, thus the two errors are asymptotically independent between them.

[3] In Escanciano & Velasco (2010) the imaginary unit is used instead of the Borel measurable function, Φ, in order to bound the random variables but asymptotic results should still hold

Letting $J(t) = \Psi_\tau(\epsilon_t)exp(b'_Y\Phi(Y_{t-1}))\bigtriangledown_{F|\overline{HF}}(exp(b'_F\Phi(\hat{F}_{t-1})))$, by adding and subtracting $E\big(J(t)\big)$ we obtain the following:

$$II = \underbrace{T^{-\frac{1}{2}}\sum_{t=1}^{T}E\big(J(t)\big)\Big\{V_{NT}^{-1}\big[\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\gamma_N(s,t)+\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\zeta_{st}+\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\eta_{st}+\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\xi_{st}\big]\Big\}}_{A} \quad +$$

$$\underbrace{T^{-\frac{1}{2}}\sum_{t=1}^{T}[J(t)-E\big(J(t)\big)]\Big\{V_{NT}^{-1}\big[\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\gamma_N(s,t)+\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\zeta_{st}+\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\eta_{st}+\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\xi_{st}\big]\Big\}}_{B}$$

Therefore, the required proof involves demonstrating that the deterministic term A is converging to zero and the non-deterministic term B cannot be of a larger order than the former, which involves imposing some conditions on $\sup_t |J(t) - E[J(t)]|$.

To analyse the two terms, we need the following lemmas.
Lemma A.1: Under Assumptions A-D, as T, $N \to \infty$,

(*i*) $T^{-1}\hat{F}'(\frac{1}{TN}X'X)\hat{F} = V_{NT} \xrightarrow{\text{p}} V_{NT}$

(*ii*) $\frac{\hat{F}'F}{T}(\frac{\hat{Lambda}'Lambda}{N})\frac{F'\hat{F}}{T} \xrightarrow{\text{p}} V$
where V is the diagonal matrix consisting of the eigenvalues of $\Sigma_\Lambda\Sigma_F$.

Proof: This lemma is implicitly proved by Stock & Watson (1999).

Lemma A.2: Under assumptions A-D,

$$\delta^2_{NT}\Big(\frac{1}{T}\sum_{t=1}^{T}\|\hat{F}_t - H'F_t\|^2\Big) = O_p(1)$$

Proof: see Theorem 1 of Bai & Ng (2002).[4]

Also note that because V_{NT} converges to a positive definite matrix, by Lemma A.1, it follows that $\|V_{NT}\| = O_p(1)$. Furthermore, under assumptions A-B and Lemma A.3 with $\hat{F}'F/T = I$, it is implied that $\|H\| = O_p(1)$. Lastly note that $\sup_t |J(t) - E[J(t)]| = O_p(1)$, given the bounded structure of the variables involved and the nature of the residuals.

[4]Note that they have used $\|V_{NT}\hat{F}_t - V_{NT}H'F_t\|^2$.

Let us now consider the **first term of A**. By adding and subtracting terms,

$$T^{-\frac{1}{2}}\sum_{t=1}^{T}E\big(J(t)\big)V_{NT}^{-1}\big[\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\gamma_N(s,t)\big]$$

$$= T^{-\frac{1}{2}}\sum_{t=1}^{T}E\big(J(t)\big)V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^{T}H'F_s\gamma_N(s,t) + T^{-\frac{1}{2}}\sum_{t=1}^{T}E\big(J(t)\big)V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^{T}(\hat{F}_s - H'F_s)\gamma_N(s,t)$$

$$= H'E\big(J(t)\big)V_{NT}^{-1}T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}F_s\gamma_N(s,t) + E\big(J(t)\big)V_{NT}^{-1}T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}(\hat{F}_s - H'F_s)\gamma_N(s,t)$$

Now, $H'E\big(J(t)\big)V_{NT}^{-1}T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}F_s\gamma_N(s,t) = o_p(\frac{1}{T^{\frac{1}{2}}})$, since by assumption A and E1 $E|\sum_{s=1}^{T}F_s\gamma_N(s,t)| =$ $O_p(1)$ independent of t (thus a time average is also an $O_p(1)$) and under the null $E\big(J(t)\big) = 0$. For the second component of the first term, by Lemma A.2 and assumption E1 we obtain,:

$$E\big(J(t)\big)V_{NT}^{-1}T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}(\hat{F}_s - H'F_s)\gamma_N(s,t)$$

$$\leq E\big(J(t)\big)V_{NT}^{-1}T^{-\frac{1}{2}}\big(\frac{1}{T}\sum_{s=1}^{T}\parallel \hat{F}_s - H'F_s \parallel^2\big)^{\frac{1}{2}}\big(\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\mid \gamma_N(s,t)\mid^2\big)^{\frac{1}{2}}$$

$$\simeq E\big(J(t)\big)V_{NT}^{-1}T^{-\frac{1}{2}} * O_p(\frac{1}{\delta_{NT}}) * O_p(1)$$

$$\simeq o_p(1) * O_p(\frac{1}{\delta_{NT}\sqrt{T}})$$

$$\simeq o_p(\frac{1}{\delta_{NT}\sqrt{T}})$$

Therefore, the first term of A is an $o_p\big(\frac{1}{\sqrt{T}}\big)$.

Consider now the **second term of A**.

$$T^{-\frac{1}{2}}\sum_{t=1}^{T}E\big(J(t)\big)V_{NT}^{-1}\big[\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\zeta_{st}\big]$$

$$= T^{-\frac{1}{2}}\sum_{t=1}^{T}E\big(J(t)\big)V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^{T}H'F_s\zeta_{st} + T^{-\frac{1}{2}}\sum_{t=1}^{T}E\big(J(t)\big)V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^{T}(\hat{F}_s - H'F_s)\zeta_{st}$$

$$= H'E\big(J(t)\big)V_{NT}^{-1}T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}F_s\zeta_{st} + E\big(J(t)\big)V_{NT}^{-1}T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}(\hat{F}_s - H'F_s)\zeta_{st}$$

Now, given that $\frac{1}{\sqrt{NT}}\sum_{s=1}^{T}\sum_{k=i}^{N}F_s(e_{is}e_{it}-E(e_{is}e_{it}))=O_p(1)$ by Assumption F1,

$$
\begin{aligned}
H'E\big(J(t)\big)V_{NT}^{-1}T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}F_s\zeta_{st} &= H'E\big(J(t)\big)V_{NT}^{-1}T^{-\frac{1}{2}}\sum_{t=1}^{T}\frac{1}{NT}\sum_{s=1}^{T}\sum_{k=i}^{N}F_s(e_{is}e_{it}-E(e_{is}e_{it}))\\
&= H'E\big(J(t)\big)V_{NT}^{-1}\frac{1}{\sqrt{N}}*\frac{1}{T}\sum_{t=1}^{T}\frac{1}{\sqrt{NT}}\sum_{s=1}^{T}\sum_{k=i}^{N}F_s(e_{is}e_{it}-E(e_{is}e_{it}))\\
&\simeq o_p(1)*\frac{1}{\sqrt{N}}*O_p(1)\\
&\simeq o_p\big(\frac{1}{\sqrt{N}}\big)
\end{aligned}
$$

Meanwhile, under Lemma A.2 and assumption C5 the term $T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}(\hat{F}_s-H'F_s)\zeta_{st}$ is bounded by,

$$
\begin{aligned}
&T^{\frac{1}{2}}\big(\frac{1}{T}\sum_{s=1}^{T}\parallel\hat{F}_s-H'F_s\parallel^2\big)^{\frac{1}{2}}\big(\frac{1}{T}\sum_{s=1}^{T}\big(\frac{1}{T}\sum_{t=1}^{T}\zeta_{st}\big)^2\big)^{\frac{1}{2}}\\
&\qquad= T^{\frac{1}{2}}\big(\frac{1}{T}\sum_{s=1}^{T}\parallel\hat{F}_s-H'F_s\parallel^2\big)^{\frac{1}{2}}\big(\frac{1}{T}\sum_{s=1}^{T}\big(\frac{1}{T}\sum_{t=1}^{T}\frac{1}{N}\sum_{i=1}^{N}e_{is}e_{it}-E(e_{is}e_{it})\big)^2\big)^{\frac{1}{2}}\\
&\qquad\simeq\frac{\sqrt{T}}{\sqrt{N}}*O_p\big(\frac{1}{\delta_{NT}}\big)*O_p(1)\\
&\qquad\simeq O_p\big(\frac{\sqrt{T}}{\delta_{NT}\sqrt{N}}\big)
\end{aligned}
$$

Therefore the second term is an $o_p\big(\frac{\sqrt{T}}{\delta_{NT}\sqrt{N}}\big)$.

Consider now the **third term of A**. By adding and subtracting terms,

$$
\begin{aligned}
&T^{-\frac{1}{2}}\sum_{t=1}^{T}E\big(J(t)\big)V_{NT}^{-1}\big[\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\eta_{st}\big]\\
&\qquad= T^{-\frac{1}{2}}\sum_{t=1}^{T}E\big(J(t)\big)V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^{T}H'F_s\eta_{st}+T^{-\frac{1}{2}}\sum_{t=1}^{T}E\big(J(t)\big)V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^{T}(\hat{F}_s-H'F_s)\eta_{st}\\
&\qquad= H'E\big(J(t)\big)V_{NT}^{-1}T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}F_s\eta_{st}+E\big(J(t)\big)V_{NT}^{-1}T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}(\hat{F}_s-H'F_s)\eta_{st}
\end{aligned}
$$

Now, given that $\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \lambda_i e_{it} = o_p(1)$ by assumption F3,

$$
\begin{aligned}
H'E\big(J(t)\big)V_{NT}^{-1}T^{-\frac{3}{2}} \sum_{t=1}^{T}\sum_{s=1}^{T} F_s\eta_{st} &= H'E\big(J(t)\big)V_{NT}^{-1}\big(\frac{1}{T}\sum_{s=1}^{T} F_sF_s'\big)\big(\frac{1}{T}\sum_{t=1}^{T}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_i e_{it}\big)\frac{\sqrt{T}}{\sqrt{N}} \\
&\simeq \frac{\sqrt{T}}{\sqrt{N}} * o_p(1) * O_p(1) * O_p(1) \\
&\simeq o_p\big(\frac{\sqrt{T}}{\sqrt{N}}\big)
\end{aligned}
$$

Meanwhile, given Lemma A.2 and assumptions A and F3, the term $T^{-\frac{3}{2}} \sum_{t=1}^{T}\sum_{s=1}^{T}(\hat{F}_s - H'F_s)\eta_{st}$ is bounded by:

$$
\begin{aligned}
|T^{-\frac{3}{2}} \sum_{t=1}^{T}\sum_{s=1}^{T} \bar{J}(t)(\hat{F}_s - H'F_s)\eta_{st}| &\leq T^{-\frac{3}{2}} \sum_{t=1}^{T}\sum_{s=1}^{T} \|(\hat{F}_s - H'F_s)F_s'\|\|\frac{\Lambda e_t}{N}\| \\
&\leq \sqrt{T}(\frac{1}{T}\sum_{s=1}^{T}\|(\hat{F}_s - H'F_s)F_s'\|)(\frac{1}{T}\sum_{t=1}^{T}\|\frac{\Lambda e_t}{\sqrt{N}}\|)\frac{1}{\sqrt{N}} \\
&\leq \sqrt{T}(\frac{1}{T}\sum_{s=1}^{T}\|\hat{F}_s - H'F_s\|^2)^{\frac{1}{2}}(\frac{1}{T}\sum_{s=1}^{T}\|F_s\|^2)^{\frac{1}{2}}(\frac{1}{T}\sum_{t=1}^{T}\|\frac{\Lambda e_t}{\sqrt{N}}\|)\frac{1}{\sqrt{N}} \\
&\simeq \frac{\sqrt{T}}{\sqrt{N}} * O_p(\frac{1}{\delta_{NT}}) * O_p(1) * O_p(1) \\
&\simeq O_p(\frac{\sqrt{T}}{\delta_{NT}\sqrt{N}})
\end{aligned}
$$

Therefore the third term is an $o_p\big(\frac{\sqrt{T}}{\sqrt{N}}\big)$.

Lastly, consider the **fourth term**: By adding and subtracting terms,

$$
\begin{aligned}
&T^{-\frac{1}{2}} \sum_{t=1}^{T} E\big(J(t)\big)V_{NT}^{-1}[\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\xi_{st}] \\
&\quad = T^{-\frac{1}{2}} \sum_{t=1}^{T} E\big(J(t)\big)V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^{T} H'F_s\xi_{st} + T^{-\frac{1}{2}} \sum_{t=1}^{T} E\big(J(t)\big)V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^{T}(\hat{F}_s - H'F_s)\xi_{st} \\
&\quad = H'E\big(J(t)\big)V_{NT}^{-1}T^{-\frac{3}{2}} \sum_{t=1}^{T}\sum_{s=1}^{T} F_s\xi_{st} + E\big(J(t)\big)V_{NT}^{-1}T^{-\frac{3}{2}} \sum_{t=1}^{T}\sum_{s=1}^{T}(\hat{F}_s - H'F_s)\xi_{st}
\end{aligned}
$$

Now, given that $\frac{1}{\sqrt{TN}} \sum_{s=1}^{T} \sum_{i=1}^{N} F_s \lambda_i e_{is} F_t = O_p(1)$ by assumption F2,

$$\begin{aligned} H'E\big(J(t)\big)V_{NT}^{-1}T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}F_s\xi_{st} &= H'E\big(J(t)\big)V_{NT}^{-1}\frac{1}{T}\sum_{t=1}^{T}\big(\frac{1}{\sqrt{TN}}\sum_{s=1}^{T}\sum_{i=1}^{N}F_s\lambda_i e_{is}F_t\big)\frac{1}{\sqrt{N}} \\ &\simeq o_p(1) * o_p(1) * \frac{1}{\sqrt{N}} \\ &\simeq o_p(\frac{1}{\sqrt{N}}) \end{aligned}$$

Meanwhile, under lemma A.2 and assumptions A and F3, the term $T^{-\frac{3}{2}} \sum_{t=1}^{T} \sum_{s=1}^{T} (\hat{F}_s - H'F_s)\xi_{st}$ is bounded by:

$$\begin{aligned} &\frac{1}{\sqrt{N}}\big(\parallel \frac{1}{T}\sum_{s=1}^{T}(\hat{F}_s - H'F_s)\frac{\Lambda e_s}{\sqrt{N}} \parallel^2 \big)^{\frac{1}{2}}\big(\frac{1}{T}\sum_{t=1}^{T} \parallel F_t \parallel^2 \big)^{\frac{1}{2}} \\ &\leq \frac{1}{\sqrt{N}}\big\{\big[\big(\frac{1}{T}\sum_{s=1}^{T} \parallel \hat{F}_s - H'F_s \parallel^2 \big)^{\frac{1}{2}}\big(\frac{1}{T}\sum_{s=1}^{T} \parallel \frac{\Lambda e_s}{\sqrt{N}} \parallel^2 \big)^{\frac{1}{2}}\big]^2\big\}^{\frac{1}{2}}\big(\frac{1}{T}\sum_{t=1}^{T} \parallel F_t \parallel^2 \big)^{\frac{1}{2}} \\ &\simeq \frac{1}{\sqrt{N}} * O_p\big(\frac{1}{\delta_{NT}}\big) * O_p(1) * O_p(1) \\ &\simeq O_p\big(\frac{1}{\delta_{NT}\sqrt{N}}\big) \end{aligned}$$

Therefore the last term is an $o_p\big(\frac{1}{\sqrt{N}}\big)$.

Overall therefore,

$$\begin{aligned} II &= T^{-\frac{1}{2}}\sum_{t=1}^{T}\Psi_\tau(\epsilon_t)exp(b_Y'\Phi(Y_{t-1}))\bigtriangledown_{F|\overline{HF}}(exp(b_F'\Phi(\hat{F}_{t-1}))) * [\hat{F}_{t-1} - HF_{t-1}] \\ &\simeq o_p\big(\frac{1}{\sqrt{T}}\big) + o_p\big(\frac{\sqrt{T}}{\delta_{NT}\sqrt{N}}\big) + o_p\big(\frac{\sqrt{T}}{\sqrt{N}}\big) + o_p\big(\frac{1}{\sqrt{N}}\big) \end{aligned}$$

Therefore, either the first or third term drives the distribution, depending on $\max\{\sqrt{N}, \sqrt{T}\}$.

Consider now the **first term of B**. Let $\bar{J}(t) = J(t) - E\big(J(t)\big)$. Then by adding and subtracting terms,

$$
\begin{aligned}
T^{-\frac{1}{2}} \sum_{t=1}^{T} \big[J(t) - E\big(J(t)\big)\big] V_{NT}^{-1} \frac{1}{T} \sum_{s=1}^{T} \hat{F}_s \gamma_N(s,t) \\
&= T^{-\frac{1}{2}} \sum_{t=1}^{T} \bar{J}(t) V_{NT}^{-1} \frac{1}{T} \sum_{s=1}^{T} H' F_s \gamma_N(s,t) + T^{-\frac{1}{2}} \sum_{t=1}^{T} \bar{J}(t) V_{NT}^{-1} \frac{1}{T} \sum_{s=1}^{T} (\hat{F}_s - H' F_s) \gamma_N(s,t) \\
&= V_{NT}^{-1} T^{-\frac{3}{2}} \sum_{t=1}^{T} \sum_{s=1}^{T} \bar{J}(t) H' F_s \gamma_N(s,t) + V_{NT}^{-1} T^{-\frac{3}{2}} \sum_{t=1}^{T} \sum_{s=1}^{T} \bar{J}(t) (\hat{F}_s - H' F_s) \gamma_N(s,t)
\end{aligned}
$$

Now, disregarding the matrix H given that $\|H\| = O_p(1)$, by assumptions A and E1 we obtain the following,

$$
\begin{aligned}
T^{-\frac{3}{2}} \sum_{t=1}^{T} \sum_{s=1}^{T} \bar{J}(t) H' F_s \gamma_N(s,t) &\leq |T^{-\frac{3}{2}} \sum_{t=1}^{T} \sum_{s=1}^{T} \bar{J}(t) F_s \gamma_N(s,t)| \\
&\leq T^{-\frac{3}{2}} \sum_{t=1}^{T} \sum_{s=1}^{T} |\bar{J}(t)| \|F_s \gamma_N(s,t)\| \\
&\leq \sup_t |\bar{J}(t)| T^{-\frac{3}{2}} \sum_{t=1}^{T} \sum_{s=1}^{T} \|F_s\| |\gamma_N(s,t)| \\
&\leq \sup_t |\bar{J}(t)| \frac{1}{\sqrt{T}} \Big(\frac{1}{T} \sum_{s=1}^{T} \|F_s\|^2\Big)^{\frac{1}{2}} \Big(\frac{1}{T} \sum_{t=1}^{T} \big(\sum_{s=1}^{T} |\gamma_N(s,t)|\big)^2\Big)^{\frac{1}{2}} \\
&\simeq \sup_t |\bar{J}(t)| * \frac{1}{\sqrt{T}} * O_p(1) * O_p(1) \\
&\simeq \sup_t |\bar{J}(t)| * O_p\Big(\frac{1}{\sqrt{T}}\Big)
\end{aligned}
$$

For the second component, by Lemma A.2 and assumption E1 we have that,

$$
\begin{aligned}
T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\bar{J}(t)(\hat{F}_s-H'F_s)\gamma_N(s,t) &\leq |T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\bar{J}(t)(\hat{F}_s-H'F_s)\gamma_N(s,t)| \\
&\leq \sup_t|\bar{J}(t)|T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\|(\hat{F}_s-H'F_s)\||\gamma_N(s,t)| \\
&\leq \sup_t|\bar{J}(t)|\frac{1}{\sqrt{T}}(\sum_{s=1}^{T}\|\hat{F}_s-H'F_s\|^2)^{\frac{1}{2}}\big(\frac{1}{T}\sum_{t=1}^{T}(\sum_{s=1}^{T}|\gamma_N(s,t)|)^2\big)^{\frac{1}{2}} \\
&\simeq \sup_t|\bar{J}(t)|*\frac{1}{\sqrt{T}}*O_p(\frac{1}{\delta_{NT}})*O_p(1) \\
&\simeq \sup_t|\bar{J}(t)|*O_p(\frac{1}{\delta_{NT}\sqrt{T}})
\end{aligned}
$$

Therefore $\sup_t|\bar{J}(t)|O_p(\frac{1}{\sqrt{T}})=\sup_t|J(t)-E\big(J(t)\big)|O_p(\frac{1}{\sqrt{T}})$ drives the first term of B.

Consider now the **second term of B**. By adding and subtracting terms,

$$
\begin{aligned}
&T^{-\frac{1}{2}}\sum_{t=1}^{T}\big[J(t)-E\big(J(t)\big)\big]V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\zeta_{st} \\
&\qquad = T^{-\frac{1}{2}}\sum_{t=1}^{T}\bar{J}(t)V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^{T}H'F_s\zeta_{st}+T^{-\frac{1}{2}}\sum_{t=1}^{T}\bar{J}(t)V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^{T}(\hat{F}_s-H'F_s)\zeta_{st} \\
&\qquad = V_{NT}^{-1}T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\bar{J}(t)H'F_s\zeta_{st}+V_{NT}^{-1}T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\bar{J}(t)(\hat{F}_s-H'F_s)\zeta_{st}
\end{aligned}
$$

Now by assumptions A and C5 we obtain the following,

$$
\begin{aligned}
T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\bar{J}(t)H'F_s\zeta_{st} &\leq |T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\bar{J}(t)F_s\zeta_{st}| \\
&\leq \sup_t|\bar{J}(t)|T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\|F_s\|\|\big(\frac{e_s'e_t}{N}-\gamma_N(s,t)\big)\| \\
&\leq \sup_t|\bar{J}(t)|\sqrt{T}(\frac{1}{T}\sum_{s=1}^{T}\|F_s\|^2)^{\frac{1}{2}}\big[\frac{1}{T}\sum_{s=1}^{T}\big(\sum_{t=1}^{T}|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}e_{is}'e_{it}-E(e_{is}'e_{it})|\big)^2\big]^{\frac{1}{2}}\frac{1}{\sqrt{N}} \\
&\simeq \sup_t|\bar{J}(t)|*\frac{\sqrt{T}}{\sqrt{N}}*O_p(1)*O_p(1) \\
&\simeq \sup_t|\bar{J}(t)|*O_p(\frac{\sqrt{T}}{\sqrt{N}})
\end{aligned}
$$

Meanwhile, under Lemma A.2 and assumption C5, the second component is bounded by,

$$
\begin{aligned}
T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\bar{J}(t)(\hat{F}_s - H'F_s)\zeta_{st} &\leq |T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\bar{J}(t)(\hat{F}_s - H'F_s)\zeta_{st}| \\
&\leq \sup_t |\bar{J}(t)| T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\|(\hat{F}_s - H'F_s)\|\|\big(\frac{e_s'e_t}{N} - \gamma_N(s,t)\big)| \\
&\leq \sup_t |\bar{J}(t)|\sqrt{T}\big(\frac{1}{T}\sum_{s=1}^{T}\|\hat{F}_s - H'F_s\|^2\big)^{\frac{1}{2}}\Big[\frac{1}{T}\sum_{s=1}^{T}\Big(\frac{1}{T}\sum_{t=1}^{T}|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}e_{is}'e_{it} - E(e_{is}'e_{it})|\Big)^2\Big]^{\frac{1}{2}}\frac{1}{\sqrt{N}} \\
&\simeq \sup_t |\bar{J}(t)| * \frac{\sqrt{T}}{\sqrt{N}} * O_p\big(\frac{1}{\delta_{NT}}\big) * O_p(1) \\
&\simeq \sup_t |\bar{J}(t)| * O_p\big(\frac{\sqrt{T}}{\delta_{NT}\sqrt{N}}\big)
\end{aligned}
$$

Therefore $\sup_t |\bar{J}(t)| O_p(\frac{\sqrt{T}}{\sqrt{N}}) = \sup_t |J(t) - E\big(J(t)\big)| O_p(\frac{\sqrt{T}}{\sqrt{N}})$ drives the second term of B.

Consider now the **third term of B**. By adding and subtracting terms,

$$
\begin{aligned}
&T^{-\frac{1}{2}}\sum_{t=1}^{T}\big[J(t) - E\big(J(t)\big)\big]V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\eta_{st} \\
&\qquad = T^{-\frac{1}{2}}\sum_{t=1}^{T}\bar{J}(t)V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^{T}H'F_s\eta_{st} + T^{-\frac{1}{2}}\sum_{t=1}^{T}\bar{J}(t)V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^{T}(\hat{F}_s - H'F_s)\eta_{st} \\
&\qquad = V_{NT}^{-1}T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\bar{J}(t)H'F_s\eta_{st} + V_{NT}^{-1}T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\bar{J}(t)(\hat{F}_s - H'F_s)\eta_{st}
\end{aligned}
$$

By similar manipulations, under assumptions A and F3 we obtain the following,

$$
\begin{aligned}
T^{-\frac{3}{2}} \sum_{t=1}^{T} \sum_{s=1}^{T} \bar{J}(t) H' F_s \eta_{st} &\leq |T^{-\frac{3}{2}} \sum_{t=1}^{T} \sum_{s=1}^{T} \bar{J}(t) F_s \eta_{st}| \\
&\leq \sup_t |\bar{J}(t)| T^{-\frac{3}{2}} \sum_{t=1}^{T} \sum_{s=1}^{T} \| F_s \frac{F_s' \Lambda e_t}{N} \| \\
&\leq \sup_t |\bar{J}(t)| T^{-\frac{3}{2}} \sum_{t=1}^{T} \sum_{s=1}^{T} \| F_s F_s' \| \| \frac{\Lambda e_t}{N} \| \\
&\leq \sup_t |\bar{J}(t)| \sqrt{T} (\frac{1}{T} \sum_{s=1}^{T} \|F_s\|^2) \big(\frac{1}{T} \sum_{t=1}^{T} \| \frac{\Lambda e_t}{\sqrt{N}} \| \big) \frac{1}{\sqrt{N}} \\
&\simeq \sup_t |\bar{J}(t)| * \frac{\sqrt{T}}{\sqrt{N}} * O_p(1) * O_p(1) \\
&\simeq \sup_t |\bar{J}(t)| * O_p(\frac{\sqrt{T}}{\sqrt{N}})
\end{aligned}
$$

For the second component, Lemma A.2 and assumptions A and F3 imply that,

$$
\begin{aligned}
T^{-\frac{3}{2}} \sum_{t=1}^{T} \sum_{s=1}^{T} \bar{J}(t) (\hat{F}_s - H' F_s) \eta_{st} &\leq |T^{-\frac{3}{2}} \sum_{t=1}^{T} \sum_{s=1}^{T} \bar{J}(t) (\hat{F}_s - H' F_s) \eta_{st}| \\
&\leq \sup_t |\bar{J}(t)| T^{-\frac{3}{2}} \sum_{t=1}^{T} \sum_{s=1}^{T} \| (\hat{F}_s - H' F_s) F_s' \| \| \frac{\Lambda e_t}{N} \| \\
&\leq \sup_t |\bar{J}(t)| \sqrt{T} (\frac{1}{T} \sum_{s=1}^{T} \| (\hat{F}_s - H' F_s) F_s' \|) (\frac{1}{T} \sum_{t=1}^{T} \| \frac{\Lambda e_t}{\sqrt{N}} \|) \frac{1}{\sqrt{N}} \\
&\leq \sup_t |\bar{J}(t)| \sqrt{T} (\frac{1}{T} \sum_{s=1}^{T} \| \hat{F}_s - H' F_s \|^2)^{\frac{1}{2}} (\frac{1}{T} \sum_{s=1}^{T} \|F_s\|^2)^{\frac{1}{2}} (\frac{1}{T} \sum_{t=1}^{T} \| \frac{\Lambda e_t}{\sqrt{N}} \|) \frac{1}{\sqrt{N}} \\
&\simeq \sup_t |\bar{J}(t)| * \frac{\sqrt{T}}{\sqrt{N}} * O_p(\frac{1}{\delta_{NT}}) * O_p(1) * O_p(1) \\
&\simeq \sup_t |\bar{J}(t)| * O_p(\frac{\sqrt{T}}{\delta_{NT} \sqrt{N}})
\end{aligned}
$$

Therefore $\sup_t |\bar{J}(t)| O_p(\frac{\sqrt{T}}{\sqrt{N}}) = \sup_t |J(t) - E\big(J(t)\big)| O_p(\frac{\sqrt{T}}{\sqrt{N}})$ drives the third term of B.

Consider last the **fourth term of B**. By adding and subtracting terms,

$$
\begin{aligned}
&T^{-\frac{1}{2}}\sum_{t=1}^{T}\big[J(t)-E\big(J(t)\big)\big]V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\xi_{st} \\
&\qquad = T^{-\frac{1}{2}}\sum_{t=1}^{T}\bar{J}(t)V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^{T}H'F_s\xi_{st}+T^{-\frac{1}{2}}\sum_{t=1}^{T}\bar{J}(t)V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^{T}(\hat{F}_s-H'F_s)\xi_{st} \\
&\qquad = V_{NT}^{-1}T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\bar{J}(t)H'F_s\xi_{st}+V_{NT}^{-1}T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\bar{J}(t)(\hat{F}_s-H'F_s)\xi_{st}
\end{aligned}
$$

In a similar way, under assumptions A and F3, we obtain the following,

$$
\begin{aligned}
T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\bar{J}(t)H'F_s\eta_{st} &\leq |T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\bar{J}(t)H'F_s\xi_{st}| \\
&\leq \sup_t|\bar{J}(t)|T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\|H'F_s\frac{F_t'\Lambda'e_s}{N}\| \\
&\leq \sup_t|\bar{J}(t)|\sqrt{T}(\frac{1}{T}\sum_{s=1}^{T}\|F_s\frac{e_s\Lambda}{N}\|)\big(\frac{1}{T}\sum_{t=1}^{T}\|F_t\|\big) \\
&\leq \sup_t|\bar{J}(t)|\sqrt{T}(\frac{1}{T}\sum_{s=1}^{T}\|F_s\|^2)^{\frac{1}{2}}(\frac{1}{T}\sum_{s=1}^{T}\|\frac{e_s\Lambda}{\sqrt{N}}\|^2)^{\frac{1}{2}}\frac{1}{\sqrt{N}}\big(\frac{1}{T}\sum_{t=1}^{T}\|F_t\|\big) \\
&\simeq \sup_t|\bar{J}(t)|*\frac{\sqrt{T}}{\sqrt{N}}*O_p(1)*O_p(1)*O_p(1) \\
&\simeq \sup_t|\bar{J}(t)|*O_p(\frac{\sqrt{T}}{\sqrt{N}})
\end{aligned}
$$

For the second component, Lemma A.2 with assumptions A and F3 imply that:

$$
\begin{aligned}
T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\bar{J}(t)(\hat{F}_s-H'F_s)\eta_{st} &\leq |T^{-\frac{3}{2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\bar{J}(t)(\hat{F}_s-H'F_s)\xi_{st}| \\
&\leq \sup_t|\bar{J}(t)|\sqrt{T}(\frac{1}{T}\sum_{s=1}^{T}\|(\hat{F}_s-H'F_s)\frac{e_s\Lambda}{N}\|)(\frac{1}{T}\sum_{t=1}^{T}\|F_t\|) \\
&\leq \sup_t|\bar{J}(t)|\sqrt{T}(\frac{1}{T}\sum_{s=1}^{T}\|\hat{F}_s-H'F_s\|^2)^{\frac{1}{2}}(\frac{1}{T}\sum_{s=1}^{T}\|\frac{e_s\Lambda}{\sqrt{N}}\|^2)^{\frac{1}{2}}\frac{1}{\sqrt{N}}(\frac{1}{T}\sum_{t=1}^{T}\|F_t\|) \\
&\simeq \sup_t|\bar{J}(t)|*\frac{\sqrt{T}}{\sqrt{N}}*O_p(\frac{1}{\delta_{NT}})*O_p(1)*O_p(1) \\
&\simeq \sup_t|\bar{J}(t)|*O_p(\frac{\sqrt{T}}{\delta_{NT}\sqrt{N}})
\end{aligned}
$$

Therefore $\sup_t|\bar{J}(t)|O_p(\frac{\sqrt{T}}{\sqrt{N}})=\sup_t|J(t)-E\big(J(t)\big)|O_p(\frac{\sqrt{T}}{\sqrt{N}})$ drives the fourth term of B.

Overall therefore,

$$
\begin{aligned}
B &= T^{-\frac{1}{2}}\sum_{t=1}^{T}\big[J(t)-E\big(J(t)\big)\big]\Big\{V_{NT}^{-1}\big[\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\gamma_N(s,t)+\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\zeta_{st}+\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\eta_{st}+\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\xi_{st}\big] \\
&= \sup_t|J(t)-E[J(t)]|\big[O_p\big(\frac{1}{\sqrt{T}}\big)+O_p\big(\frac{\sqrt{T}}{\sqrt{N}}\big)+O_p\big(\frac{\sqrt{T}}{\sqrt{N}}\big)+O_p\big(\frac{\sqrt{T}}{\sqrt{N}}\big)\big] \\
&= \sup_t|J(t)-E[J(t)]|O_p\big(\frac{\sqrt{T}}{\sqrt{N}}\big)
\end{aligned}
$$

Therefore, given that all the elements of $J(t)-E[J(t)]$ are bounded by construction, then $\sup_t|J(t)-E[J(t)]|O_p\big(\frac{\sqrt{T}}{\sqrt{N}}\big)$ is also bounded. Hence $T^{-\frac{1}{2}}\sum_{t=1}^{T}[\mathbb{1}(Y_t-Y_{t-1}'\theta(\tau)\leq 0)-\tau]exp(b'\Phi(\hat{I}_{t-1}))-T^{-\frac{1}{2}}\sum_{t=1}^{T}[\mathbb{1}(Y_t-Y_{t-1}'\theta(\tau)\leq 0)-\tau]exp(b'\Phi(I_{t-1}))$ is bounded and thus the factor estimation error present in the test statistic is negligible.

Proof of Theorem 2. see proof of Theorem 1 in Escanciano & Velasco (2010).

Proof of Theorem 3. see proof of Theorem 2 in Escanciano & Velasco (2010).

References

Ando, Tomohiro, & Bai, Jushan. 2017. Quantile Co-Movement in Financial Markets; a Panel Quantile Model with Unobserved Heterogeneity. *SSRN Electronic Journal*.

Ando, Tomohiro, & Tsay, Ruey S. 2011. Quantile regression models with factor-augmented predictors and information criterion. *The Econometrics Journal*, **14**(1), 1–24.

Bai, Jushan. 2003. Inferential Theory for factor models of large dimensions. *Econometrica*, **71**(1), 135–171.

Bai, Jushan, & Ng, Serena. 2002. Determining the Number of Factors in Approximate Factor Models. *Econometrica*, **70**(1), 191–221.

Bernanke, Ben S., Boivin, Jean, & Eliasz, Piotr. 2005. Measuring the effects of monetary policy: a factor-augmented vector autoregressive (FAVAR) approach. *The Quarterly Journal of Economics*, **120**(1), 387–422.

Bierens, Herman J. 1990. A Consistent Conditional Moment Test of Functional Form. *Econometrica*, **58**(6), 1443–1458.

Chamberlain, Gary, & Rothschild, Michael. 1983. Arbitrage, Factor Structure, and Mean-Variance Analysis on Large Asset Markets. *Econometrica*, **51**(5), 1281–1304.

Chen, Liang, Dolado, Juan J., & Gonzalo, Jesús. 2017. Quantile factor models. *Unpublished Manuscript (available online)*.

Conde-Amboage, Mercedes, Sánchez-Sellero, César, & González-Manteiga, Wenceslao. 2015. A lack-of-fit test for quantile regression models with high-dimensional covariates. *Computational Statistics & Data Analysis*, **88**(Aug.), 128–138.

Corradi, Valentina, & Swanson, Norman R. 2006. Bootstrap conditional distribution tests in the presence of dynamic misspecification. *Journal of Econometrics*, **133**(2), 779–806.

Ellis, Colin, Mumtaz, Haroon, & Zabczyk, Pawel. 2014. What Lies Beneath? A Time-varying FAVAR Model for the UK Transmission Mechanism. *The Economic Journal*, **124**(576), 668–699.

Engle, Robert F, & Manganelli, Simone. 2004. CAViaR:Conditional autoregressive Value at Risk by regression quantiles. *Journal of Business & Economic Statistics*, **22**(4), 367–381.

Escanciano, J.C., & Goh, S.C. 2014. Specification analysis of linear quantile models. *Journal of Econometrics*, **178**(Jan.), 495–507.

Escanciano, Juan Carlos, & Velasco, Carlos. 2010. Specification tests of parametric dynamic conditional quantiles. *Journal of Econometrics*, **159**(1), 209–221.

Giannone, Domenico, Lenza, Michele, & Primiceri, Giorgio E. 2018. Economic Predictions with Big Data: The Illusion of Sparsity. *Federal Reserve Bank of New York*, **Staff reports 847**.

Koenker, Roger. 2017. Quantile Regression: 40 Years On. *Annual Review of Economics*, **9**(1), 155–176.

Koenker, Roger, & Hallock, Kevin F. 2001. Quantile regression. *Journal of Economic Perspectives*, **15**(4), 143–156.

Koenker, Roger, & Xiao, Zhijie. 2002. Inference on the quantile regression process. *Econometrica*, **70**(4), 1583–1612.

Koenker, Roger, & Xiao, Zhijie. 2006. Quantile autoregression. *Journal of the American Statistical Association*, **101**(475), 980–990.

Machado, José A. F., & Mata, José. 2005. Counterfactual decomposition of changes in wage distributions using quantile regression. *Journal of Applied Econometrics*, **20**(4), 445–465.

Stock, James H., & Watson, Mark W. 1999. Forecasting Inflation. *Journal of Monetary Economics*, **44**, 293–335.

Stock, James H, & Watson, Mark W. 2002. Forecasting using principal components from a large number of predictors. *Journal of the American Statistical Association*, **97**(460), 1167–1179.

Vaart, A. W. van der, & Wellner, Jon A. 2000. *Weak convergence and empirical processes: with applications to statistics*. New York: Springer. OCLC: 45749647.