A framework to analyze identification in DSGE models*

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Abstract

This paper offers a comprehensive framework to analyze global identification in DSGE models. Our approach relies on a formal identification condition that combines the links between the observationally equivalent state space representations with the inherent constraints imposed by the model solution on its deep parameters. We separate the identification problem into two stages, the more involved of which reduces to finding all roots of an appropriately defined system of polynomial equations. We show how this system can be solved using the concept of a Gröbner basis and recently developed analytical algorithms to obtain it. We illustrate the working of our framework with several examples, which also highlight its advantages in analyzing local identification.

JEL: C10, C51, C65, E32
Keywords: global identification, DSGE models, Gröbner basis

*We thank Frank Schorfheide for useful guidelines and comments to the early concept of this paper.
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1 Introduction

Parameter identification is one of the primary concerns of formal macroeconomic modelling. In the context of traditional simultaneous equations systems, the essence of the problem and its treatment was formalized already in the 1940s, mainly by various authors connected to the Cowles Commission for Research in Economics (see e.g. Koopmans, 1949). In recent decades, this class of purely backward-looking models have been gradually replaced, in both academic circles and policy making institutions, by the so-called dynamic stochastic general equilibrium (DSGE) models. In these mathematical constructs, the dynamics is driven mainly by unobserved stochastic processes, and model-consistent (and usually rational) expectations are the key building block. These features called for a modification of the existing approach to analyze and tackle identification.

One of the early contributions highlighting the problem in simple cases or developing simple diagnostic tools include Beyer and Farmer (2007), Fukac et al. (2007), Canova and Sala (2009) or Cochrane (2011). A more formal analysis soon followed, focusing first on local identification issues, and resulting in the rank conditions on an appropriately defined Jacobian matrix (Iskrev, 2010; Komunjer and Ng, 2011) or spectral density matrix (Qu and Tkachenko, 2012). Important progress has also been made towards resolving the problem of global identification. Qu and Tkachenko (2017) present a numerical routine that searches for observationally equivalent parameters by minimizing the Kullback-Leibler distance in a frequency domain. Kociecki and Kolasa (2018) develop an alternative algorithm, that relies on the conditions linking observationally equivalent state space representations from Komunjer and Ng (2011) and avoids solving the model for each candidate parameter.

This paper offers a more comprehensive framework to analyze local and global identification in dynamic linear systems with rational expectations, encompassing both determinate and indeterminate DSGE models. It tackles the identification problem by separating it into two levels. One is the relationship between the model's structural and semi-structural parameters, the latter defined as the coefficients showing up in the (linearized) model equilibrium conditions. This step is relatively straightforward as the connection between the two is given via analytical formulas. The second level concerns the relationship between the semi-structural parameters and the matrices defining the model's state-space representation. This step is more challenging as the latter cannot be obtained analytically, except in some some very simple special cases.

The essence of our approach consists of two insights. The first one reduces checking identification between the model's semi-structural parameters and its state-space representation to finding all roots of an appropriately defined system of polynomial equations. The second insight relies on applying the concept of the Gröbner basis to solve this system of equations. In short, and postponing the details for later, this last step boils down to transforming the original system of polynomials into an equivalent triangular system, which is done in a way resembling Gaussian elimination in linear algebra. Putting it other way, the method transforms the "tough" polynomial equations into equivalent "simple" polynomial equations, where equivalence means that the sets of solutions in both systems are the same.

The key advantage of our framework is that it explicitly derives and directly checks the global identification conditions. To the extent one can write the underlying problem using rational numbers, this gives the actual proof of global identification or lack thereof as the calculation of the Gröbner basis is exact. When implemented in the field of real numbers, the calculations can still be done with any degree of numerical precision, bringing the outcomes as close as possible to the formal proof. Our explicit approach is hence a major advantage over the existing methods to check global identification in DSGE models, that rely on searching over the parameter space (Kociecki and Kolasa, 2018; Qu and Tkachenko, 2017 or even precision refinement of the latter presented in Qu and Tkachenko (2018)), which in fact can never determine for sure that the model is globally identified. Another usefiul contribution of this paper is a generalization and unification of the conditions linking observationally equivalent state-space representations, derived by Komunjer and Ng (2011) separately for singular and non-singular cases.

Our framework offers also some additional insights into the local identification problem. In contrast to the existing and well-established approaches (Iskrev, 2010; Komunjer and Ng, 2011), a Gröbner basis explicitly produces the complete set of parameter vectors that are observationally equivalent to the one at which one checks identification. For example, it may indicate that some parameters "live" on the intesection of hyperplanes. Knowing that, it is straightforward to attain global identification by fixing some of these parameters, while leaving the remaining ones intact. This formal information concerning identifiability is absent in other approches and may be only gathered by some ad-hoc experiments.

Gröbner bases methods are a fast developing field in computational algebraic geometry. Despite their great potential, they are still very rarely used in economics, with only

few exceptions. Kubler and Schmedders (2010a) and Kubler and Schmedders (2010b) successfully applied these methods to determine an exact number of equilibria in several economic models and calculate them analytically. Datta (2010) exploited them to find Nash equilibria in games. Foerster et al. (2016) applies Gröbner bases to obtain higher-order approximations to the solutions of Markov-switching DSGE models.

Despite the existence of analytical algorithms to compute Gröbner bases that are proved to succeed after a finite number of iterations, and their efficient implementation in several software packages, it is well known that the these methods can be quite time and memory consuming for large systems of equations in practice. However, there are several features of our application that help alleviate this curse of dimensionality. One is that the system of polynomials generated by our identification conditions is of degree at most three and, for a typical DSGE model, is very sparse. Another important feature of our formulation of the problem, that allows to simplify it significantly, is that one of the roots is a point at which we check identification, which is known exactly. In fact, we show that our identification analysis can be applied not only to small DSGE models, but also to their richer versions represented by the popular Smets and Wouters (2007) setup. Lastly, real scientific applications usually "convey some structure" with them and it is well known that computation of Gröbner bases in real applications is usually simpler than it may appear by just counting the number of equations and their degrees.

The rest of this paper proceeds as follows. Section 2 presents the setup and establishes notation for a typical DSGE model and its state-space representation. Section 3 derives the conditions linking observationally equivalent state-space representations. Section 4 combines these links with the model deep parameters to establish the global identification conditions. Section 5 offers a brief introduction to the Gröbner basis concept that can be applied to checking the identification conditions. Several illustrative examples and applications to popular DSGE models from the literature are presented in Section 6. Section 7 concludes and discusses some possible further research directions. All proofs are relegated to the Appendix.

2 DSGE model

Let us write a linearized DSGE model as

$$\Gamma_0(\theta)\begin{bmatrix} s_t \\ p_t \end{bmatrix} = \Gamma_1(\theta)E_t\begin{bmatrix} s_{t+1} \\ p_{t+1} \end{bmatrix} + \Gamma_2(\theta)s_{t-1} + \Gamma_3(\theta)\varepsilon_t \tag{1}$$

where s_t is an $n \times 1$ vector of states, p_t is a $q \times 1$ vector of policy variables, matrices $\Gamma_0(\theta)$, $\Gamma_1(\theta)$, $\Gamma_2(\theta)$ and $\Gamma_3(\theta)$ are explicit functions of deep model parameters collected in an $m \times 1$ vector $\theta \in \Theta \subseteq \mathbb{R}^m$, $\varepsilon_t \sim i.i.d.\, N(0, \Sigma(\theta))$ is a $k \times 1$ vector of normally-distributed shocks, where $\Sigma(\theta)$ is a $k \times k$ symmetric positive definite matrix for every $\theta \in \Theta$. Vector ε_t can include both innovations to structural shocks, sunspot shocks and measurement errors, so that some columns in $\Gamma_3(\theta)$ might be zero.

A stable solution to (1) can be written in the following form

$$s_t = A(\theta)s_{t-1} + B(\theta)\varepsilon_t \tag{2}$$

$$p_t = F(\theta)s_{t-1} + G(\theta)\varepsilon_t \tag{3}$$

where $A(\theta)$ is an $n \times n$ matrix, $B(\theta)$ is an $n \times k$ matrix, $F(\theta)$ is a $q \times n$ matrix and $G(\theta)$ is a $q \times k$ matrix, all of which implicitly depend on deep model parameters θ. This is always the case if the model is determinate. Under indeterminacy, the solution has still this form as long as one allows for a sufficient number of sunspot shocks in ε_t, see Lubik and Schorfheide (2003). This becomes even more straightforward if, in the case of indeterminacy, one transforms the model as suggested by Farmer et al. (2015), i.e. redefines a sufficient number of errors in expectations as fundamentals. They show that this transformation is equivalent to the method proposed by Lubik and Schorfheide (2003).

Suppose the measurement equations can be written as

$$y_t = H(\theta) \begin{bmatrix} s_t \\ p_t \end{bmatrix} + J(\theta)\varepsilon_t \tag{4}$$

where y_t is an $r \times 1$ vector of observable variables, $H(\theta)$ is an $r \times (n+q)$ matrix and $J(\theta)$ is an $r \times k$ matrix, both of which may (explicitly) depend on θ.

Decomposing $H(\theta)$ into blocks corresponding to the state and policy variables $H(\theta) = [\ H^s(\theta) \quad H^p(\theta)\ ]$ and using equations (2) and (3) allows us to rewrite measurement equation (4) as

$$y_t = C(\theta)s_{t-1} + D(\theta)\varepsilon_t \tag{5}$$

where an $r \times n$ matrix $C(\theta)$ and an $r \times k$ matrix $D(\theta)$ are defined as

$$C(\theta) = H^s(\theta)A(\theta) + H^p(\theta)F(\theta) \tag{6}$$

$$D(\theta) = H^s(\theta)B(\theta) + H^p(\theta)G(\theta) + J(\theta) \tag{7}$$

Consequently, the law of motion for observable variables y_t has a state space form, given by transition equation (2) and measurement equation (5). For future reference, such a representation will be called the ABCD-representation.

3 Observationally equivalent state-space representations

One of the key insights from Komunjer and Ng (2011) is that the ABCD-representation of a DSGE model is not identified, and hence its elements cannot be treated as reduced-form parameters. In this section we generalize their results by developing a set of conditions linking the observationally equivalent ABCD-representations that encompass both singular and non-singular cases.[1] From now on, to save on notation, let us denote any matrix $X(\theta)$ that depends on θ simply as X. Similarly, when referring to other points in the deep parameter space $\bar{\theta}$, we will write $\bar{X}$ instead of $X(\bar{\theta})$.

To proceed, we need two assumptions. The first one concerns stability of the model solution.

Assumption 1. *(Stability) For every $\theta \in \Theta$ and for any $z \in \mathbb{C}$ (a set of complex numbers) $det(z\mathrm{I}_n - A) = 0$ implies $|z| < 1$.*

By Assumption 1, we can define the steady-state value $P := E(s_t s_t')$, which is a unique solution to the Lyapunov equation $P = APA' + B\Sigma B'$ implied by equation (2). Bearing in mind measurement equation (5), the autocovariance sequence $\Lambda_l = E(y_t y_{t-l}')$ is readily seen as $\Lambda_0 = CPC' + D\Sigma D'$ and $\Lambda_l = CA^{l-1}N$, for $l > 0$, where $N = APC' + B\Sigma D'$. Needless to say, we have $\Lambda_{-l} = \Lambda_l'$.

To state the second assumption, let us define $O = [C' \vdots A'C' \vdots A'^2C' \vdots \ldots \vdots A'^{n-1}C']'$ and $K = [N \vdots AN \vdots A^2N \vdots \ldots \vdots A^{n-1}N]$.

Assumption 2. *(Stochastic minimality) For every $\theta \in \Theta$, matrices O and K have full column and row rank, respectively, i.e. $rank(O) = rank(K) = n$.*

Assumption 2 (under this name) is well known in the linear system literature, see e.g. Lindquist and Picci (1996). It is exactly the same as in e.g. Komunjer and Zhu

[1] Non-singular models are the cases in which there are more shocks than observables ($k > r$) or when the system is square ($k = r$) but non-invertible.

(2017), who term it as autocovariance minimality. Its main purpose is to confine the analysis only to those ABCD-representations in which the dimension of the state vector is as small as possible. To this end, Assumption 2 ensures that the underlying infinite block Hankel matrix has the same rank n (see Appendix).

Assumption 2 differs from the assumptions made by Komunjer and Ng (2011) in how matrix K is defined. In their framework, N is replaced either by B (Assumption 5-S, applicable to the singular case) or the steady-state Kalman gain associated with the innovations representation of the original state-space system (Assumption 5-NS, for the non-singular case). Moreover, Komunjer and Ng (2011) additionally impose left-invertibility of the transfer function (Assumption 4-S for the singular case)[2] or full row rank of matrix D (Assumption 4-NS for the non-singular case). In our identification analysis, we do not need any of these additional assumptions. We also do not have to distinguish between singular and non-singular cases, which spares us reformulation of the original problem into its innovations representation in the latter case. In this sense, our framework can be seen both as some generalization (obtained under weaker conditions) and unification of that developed by Komunjer and Ng (2011).

As implied by our model formulation, we deal with a stationary Gaussian environment and ignore intercept in the measurement equation so that the unconditional mean of all observables is zero.[3] This allows us to define observational equivalence by using only second moments. More formally, let us define the spectral density of the ABCD-representation as $\Phi(z) = H(z)\Sigma H'(z^{-1})$, where $H(z) = D + C(z\mathrm{I}_n - A)^{-1}B$ is the transfer function and z^{-1} corresponds to its backward shift. Then we have definition

Definition 1. θ and $\bar{\theta}$ are observationally equivalent (written as $\theta \sim \bar{\theta}$) if $\Phi(z) = \bar{\Phi}(z)$ for all $z \in \mathbb{C}$.

What Definition 1 conveys is that two deep parameters sets are observationally equivalent if they result in the same autocovariances sequence, so that we can not distinguish them using second moments of observable variables. We are now ready to state the key theorem underlying this part of our framework.

[2]Under left-invertibility, finding all observationally equivalent ABCD-representations very much resembles the so-called deterministic realization problem, hence standard assumptions concerning observability and controllability are sufficient.

[3]If the measurement equation includes an intercept that depends on θ, additional identification conditions can be obtained from the first moments of observable variables. However, the mapping from unconditional means to deep model parameters in log-linearized DSGE models can be derived analytically and hence adding this information to identification analysis is straightforward, so we do not deal with this possibility here.

Theorem 1. *Let Assumptions 1 and 2 hold. Then $\theta \sim \bar{\theta}$ if and only if 1) $\bar{A} = TAT^{-1}$, 2) $\bar{C} = CT^{-1}$, 3) $AQA' - Q = T^{-1}\bar{B}\bar{\Sigma}\bar{B}'T'^{-1} - B\Sigma B'$, 4) $CQC' = \bar{D}\bar{\Sigma}\bar{D}' - D\Sigma D'$, 5) $AQC' = T^{-1}\bar{B}\bar{\Sigma}\bar{D}' - B\Sigma D'$, for some nonsingular matrix T and symmetric matrix Q. In addition, both T and Q are unique.*

Note that this theorem generalizes and unifies the key propositions 1-S and 1-NS in Komunjer and Ng (2011), who need to consider separately the singular and non-singular cases, and have to assume left-invertibility of the transfer function in the former case or the full row rank for D in the latter case.[4] Most importantly, Theorem 1 also allows us to treat the case $r < k$, which was not possible in a related identification framework developed by Kociecki and Kolasa (2018), and which arises naturally under indeterminacy as full characterization of the model solutions requires adding sunspot shocks (Lubik and Schorfheide, 2003).

Some specialization of Theorem 1 is possible if A is nonsingular (so that $\bar{A}$ is nonsingular too), which is very often the case in stochastically minimal state-space systems. Then, we can merge conditions 4) and 5) from this theorem, and we do not have to solve for Q since whatever it is, it will be just a unique (by Assumption 1) solution to the Lyapunov equation $AQA' - Q = T^{-1}\bar{B}\bar{\Sigma}\bar{B}'T'^{-1} - B\Sigma B'$. In this sense, Q plays no role any longer in establishing identification. Hence we get

Corollary 1. *Let Assumptions 1 and 2 hold. Assume that A is nonsingular. Then $\theta \sim \bar{\theta}$ if and only if 1) $\bar{A} = TAT^{-1}$, 2) $\bar{C} = CT^{-1}$, 3) $CA^{-1}T^{-1}\bar{B}\bar{\Sigma}\bar{D}' - CA^{-1}B\Sigma D' = \bar{D}\bar{\Sigma}\bar{D}' - D\Sigma D'$, for some nonsingular matrix T. In addition, T is unique.*

4 Global identification condition

The ABCD-representation is defined by matrices that, except for some very special cases, are only implicit functions of deep parameters θ. Therefore, to check identification we need to impose additional restrictions on the observationally equivalent ABCD-representations in Theorem 1 that would guarantee consistence with the original model. As in Kociecki and Kolasa (2018), these can be readily obtained by substituting the

[4] As a curiosity, it might be interesting to note that Theorem 1 accommodates the nonidentification of a simple MA(1) model, whereas the analogous proposition in Komunjer and Ng (2011) does not, exactly because left-invertibility is imposed. Indeed, two points $(A, B, C, D, \Sigma) = (0, b, 1, 1, \sigma^2)$ and $(\bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{\Sigma}) = (0, 1/b, 1, 1, b^2\sigma^2)$ are observationally equivalent from the spectral density, but left-invertibility assumption precludes the second point from the space of allowable points.

model solution (2)-(3) into model formulation (1), which we rewrite for convenience in a block form

$$\left[\begin{array}{cc} \Gamma_0^s & \Gamma_0^p \end{array}\right] \left[\begin{array}{c} s_t \\ p_t \end{array}\right] = \left[\begin{array}{cc} \Gamma_1^s & \Gamma_1^p \end{array}\right] E_t \left[\begin{array}{c} s_{t+1} \\ p_{t+1} \end{array}\right] + \Gamma_2 s_{t-1} + \Gamma_3 \varepsilon_t \tag{8}$$

Using $E_t \varepsilon_{t+1} = 0$ results in the following two matrix equation restrictions

$$\Gamma_0^s A + \Gamma_0^p F - \Gamma_1^s A^2 - \Gamma_1^p F A = \Gamma_2 \tag{9}$$

$$\Gamma_1^s AB + \Gamma_1^p FB - \Gamma_0^s B + \Gamma_3 = \Gamma_0^p G \tag{10}$$

A similar operation using the original measurement equation (4) results in two other matrix restrictions, that are already available as equations (6) and (7).

The full set of parameter vectors $\bar{\theta}$ that are observationally equivalent to θ are hence given by the following conditions

$$\bar{\Gamma}_0^s \bar{A} + \bar{\Gamma}_0^p \bar{F} - \bar{\Gamma}_1^s (\bar{A})^2 - \bar{\Gamma}_1^p \bar{F} \bar{A} = \bar{\Gamma}_2 \tag{11}$$

$$\bar{\Gamma}_1^s \bar{A} \bar{B} + \bar{\Gamma}_1^p \bar{F} \bar{B} - \bar{\Gamma}_0^s \bar{B} + \bar{\Gamma}_3 = \bar{\Gamma}_0^p \bar{G} \tag{12}$$

$$\bar{C} = \bar{H}^s \bar{A} + \bar{H}^p \bar{F} \tag{13}$$

$$\bar{D} = \bar{H}^s \bar{B} + \bar{H}^p \bar{G} + \bar{J} \tag{14}$$

$$\bar{A} = T A T^{-1} \tag{15}$$

$$\bar{C} = C T^{-1} \tag{16}$$

$$AQA' - Q = -B\Sigma B' + T^{-1} \bar{B} \bar{\Sigma} \bar{B}' (T^{-1})' \tag{17}$$

$$AQC' = T^{-1} \bar{B} \bar{\Sigma} \bar{D}' - B\Sigma D' \tag{18}$$

$$CQC' = \bar{D} \bar{\Sigma} \bar{D}' - D\Sigma D' \tag{19}$$

$$Q = Q' \tag{20}$$

In this system of equations, the unknowns are $\bar{\theta}$ (on which the following depend explicitly: $\bar{\Gamma}_0^s$, $\bar{\Gamma}_0^p$, $\bar{\Gamma}_1^s$, $\bar{\Gamma}_1^p$, $\bar{\Gamma}_2$, $\bar{\Gamma}_3$, $\bar{\Sigma}$, $\bar{H}^s$, $\bar{H}^p$, $\bar{J}$), as well as $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$, $\bar{F}$, $\bar{G}$, T, Q. All remaining matrices are functions of θ, and hence known while checking identification at this point. Therefore our final global identification condition can be stated as follows

Definition 2. *The model given by equations (1) and (4) is (globally) identified if and*

only if all (admissible) solutions to the system (11)-(20) are such that $\bar{\theta} = \theta$.

Note that being able to solve the system of equations above analytically, i.e. giving the full set of $\bar{\theta}$'s that satisfy it, essentially resolves the problem of identification in DSGE models. However, this is not easy as the equations are non-linear and their system is fairly large even for small-scale models. Naturally, one can try to solve it numerically as it is done in a less general framework by Kociecki and Kolasa (2018), but numerical methods can give only one solution rather than their full set. Our goal is to have an analytical solution, and to this end we will use some concepts developed in computational algebraic geometry.

Before we do it, let us first distinguish between two types of potential identification failure. The first one is related to the mapping between deep model parameters θ and the coefficients on the model variables that show up in the equations of the original model formulation (1) and (4). Let us denote the vector of such semi-structural parameters as α. Clearly, if the mapping from α to θ is not unique, the model is not identified. For example, this can easily happen if two structural parameters show up in just one semi-structural coefficient (say, slope of the Phillips curve). This type of identification failure is usually easy to detect and its nature is essentially the same as in standard identification analysis in econometrics, where one links the structural parameters to the reduced form parameters via a number of time-invariant restrictions.

The second type of identification failure is much more interesting, and the rest of our discussion in this and the next section focuses on this case. It is endemic to models having a solution in a state space form (which, as we know, is unidentified) and concerns the identifiability of α's, for a given normalization of the model equations (which of course does not affect identification).

To see the nature of the underlying problem a bit more clearly, let us eliminate some terms and reorganize to get

$$C = \bar{H}^s T A + \bar{H}^p \bar{\bar{F}} \tag{21}$$

$$\bar{D} = \bar{H}^s T \bar{\bar{B}} + \bar{H}^p \bar{G} + \bar{J} \tag{22}$$

$$\bar{\Gamma}_0^s T A + \bar{\Gamma}_0^p \bar{\bar{F}} - \bar{\Gamma}_1^s T A^2 - \bar{\Gamma}_1^p \bar{\bar{F}} A = \bar{\Gamma}_2 T \tag{23}$$

$$\bar{\Gamma}_1^s T A \bar{\bar{B}} + \bar{\Gamma}_1^p \bar{\bar{F}} \bar{\bar{B}} - \bar{\Gamma}_0^s T \bar{\bar{B}} + \bar{\Gamma}_3 = \bar{\Gamma}_0^p \bar{G} \tag{24}$$

$$AQA' - Q = -B\Sigma B' + \bar{\bar{B}} \bar{\Sigma} \bar{\bar{B}}' \tag{25}$$

$$AQC' = \bar{\bar{B}} \bar{\Sigma} \bar{D}' - B\Sigma D' \tag{26}$$

$$CQC' = \bar{D}\bar{\Sigma}\bar{D}' - D\Sigma D' \tag{27}$$

$$Q = Q' \tag{28}$$

where $\bar{\bar{F}} = \bar{F}T$ and $\bar{\bar{B}} = T^{-1}\bar{B}$.[5] We have thus turned our identification conditions into a system of polynomial equations. In this alternative formulation, the unknowns are: $\bar{\alpha}$ (on which the following depend explicitly and linearly: $\bar{\Gamma}_0^s$, $\bar{\Gamma}_0^p$, $\bar{\Gamma}_1^s$, $\bar{\Gamma}_1^p$, $\bar{\Gamma}_2$, $\bar{\Gamma}_3$, $\bar{\Sigma}$, $\bar{H}^s$, $\bar{H}^p$, $\bar{J}$), as well as matrices $\bar{\bar{B}}$, $\bar{D}$, $\bar{\bar{F}}$, $\bar{G}$, T, Q.

5 Implementation

As we have demonstrated in the previous section, the key step in solving the identification problem in a DSGE model boils down to solving a system of polynomial equations. In our implementation we draw on the concept of a Gröbner basis. Intuitively, calculating it is analogous to Gaussian elimination in systems of linear equations. There exist many algorithms that produce a Gröbner basis in finitely many steps and since the first algorithm proposed by Buchberger in the 1960s, enormous progress in computational efficiency has been made. Bellow we offer a brief introduction to the key concepts. For details, we refer the interested readers to widely suggested introductive textbooks on computational algebraic geometry (and Gröbner basis in particular) by Cox et al. (1997) and Cox et al. (2005). An excellent introduction in the context of finding all equilibria in economic models can be found in Kubler et al. (2014), and we can only imperfectly emulate their exposition.

Let $\mathbb{K}$ denote any field. For us the most important field will be $\mathbb{Q}$, i.e. that of rational numbers, however the field of real numbers $\mathbb{R}$, and complex numbers $\mathbb{C}$ will also be in use. In addition, let us denote as $\bar{\mathbb{K}}$ an algebraically closed field containing $\mathbb{K}$. Without going into deatils, one can think of $\bar{\mathbb{K}}$ as $\mathbb{C}$. The set of polynomials in variables $x_1, \ldots, x_l$ with coefficients in $\mathbb{K}$ will be denoted $\mathbb{K}(x_1, \ldots, x_l)$. Each polynomial equation is a finite sum of terms $cx_1^{d_1}x_2^{d_2}\cdots x_l^{d_l}$, where c is a coeffcient (in $\mathbb{K}$) and $x_1^{d_1}x_2^{d_2}\cdots x_l^{d_l}$ is called a monomial, where each d_i is a nonnegative integer. The degree of a monomial is $d_1 + \cdots + d_l$, and the degree of a polynomial equation is the maximum of the degrees of its all monomials.

Suppose we have a set of s polynomials $f_1, f_2, \ldots, f_s \in \mathbb{K}(x_1, \ldots, x_l)$. Then, a variety V is defined to be a set of all solutions to $f_1 = 0, f_2 = 0, \ldots, f_s = 0$, i.e. $V(f_1, \ldots, f_s) =$

[5]Note that, since T is nonsingular, there is a one-to-one relationship between these two newly defined matrices and $\bar{F}$ and $\bar{B}$.

$\{(a_1, \ldots, a_l) \in \bar{\mathbb{K}}^l | f_1 = 0, \ldots, f_s = 0\}$. Of course, the initial polynomials $f_1, \ldots, f_s$ only represent the variety. There are many other alternative sets, some of which could do a better job in the sense of the ease with which the underlying solutions can be read. In particular, this opens a way to a Gröbner basis. To this end, define an ideal generated by $f_1, \ldots, f_s$ as $I = \langle f_1, \ldots, f_s \rangle$, where $\langle f_1, \ldots, f_s \rangle = \{u_1 f_1 + \cdots + u_s f_s | u_i \in \mathbb{K}(x_1, \ldots, x_l), i = 1, \ldots, s\}$. The polynomials $f_1, \ldots, f_s$ are called generators of ideal I. The ideal is just a weighted sum of all initial polynomials in which the coefficients (weights) are polynomials themselves. What makes the ideal useful is that $V(f_1, \ldots, f_s) = V(I)$, i.e. the solution set of the intitial finite system of polynomials and that of an ideal generated by $f_1, \ldots, f_s$ (i.e. an infinite system) are the same. Evidently, any ideal can have different generators. As a matter of fact, if $\langle f_1, \ldots, f_s \rangle = \langle f'_1, \ldots, f'_{s'} \rangle$, then $V(f_1, \ldots, f_s) = V(I) = V(f'_1, \ldots, f'_{s'})$. Hence, the solutions to $f_1 = 0, \ldots, f_s = 0$ and to $f'_1 = 0, \ldots, f'_{s'} = 0$ are the same. In the essence, what defines the solution set is the ideal and not the initial polynomials. The main idea of a Gröbner basis is to find alternative generators that represent the ideal in a "better" way. For example, in the case of linear polynomials (equations), this "better" way is to find their row echelon form. Importantly, by the Hilbert basis theorem, each ideal must be generated by finite number of polynomials, hence the algorithmic methods to find the "better" representation of the variety may be safely applied.

Before we can define (and obtain) a Gröbner basis, we have to take a stand on the ordering of monomials since every algorithm to compute the basis must involve polynomial divisions. The ordering is just the rule that allows for a unique placement of terms in the polynomial. It turns out that the chosen ordering greatly influences the ultimate Gröbner basis and some orderings are particularly useful. For our purposes, the most important ordering is the so-called lexicographic ordering, i.e. monomial $x_1^{d_1} x_2^{d_2} \cdots x_l^{d_l} \succ$ (is greater than) $x_1^{e_1} x_2^{e_2} \cdots x_l^{e_l}$ if $d_1 = e_1, \ldots, d_m = e_m$ and $d_{m+1} > e_{m+1}$ (where possibly $m = 0$). For example, $x_1 x_2^2 x_3 \succ x_1 x_2^2 x_4^3$, since $d_3 = 1 > e_3 = 0$. Let us define $x^d := x_1^{d_1} x_2^{d_2} \cdots x_l^{d_l}$. If we choose a monomial ordering, each polynomial may be written uniquely as $f_i = c_d x^d + \cdots$. Then x^d is called the leading monomial and $c_d x^d$ is the leading term. We say that polynomials $g_1, \ldots, g_t \in I$ constitute the Gröbner basis for ideal I if the leading term of any (nonzero) polynomial in I is divisible by the leading term of one of $g_1, \ldots, g_t$.[6] Needless to say, $g_1, \ldots, g_t$ are generators for I, every (nonzero) I possesses a Gröbner basis and solutions to $g_1 = 0, \ldots, g_t = 0$ and to

[6] A nonzero term $c_d x^d$ is divisible by a nonzero term $c_e x^e$ if $d_i \geq e_i$ for all i.

the initial polynomials $f_1 = 0, \ldots, f_s = 0$ are the same. When there is only a finite number of solutions, the underlying ideal is called zero-dimensional, otherwise it will be non-zero-dimensional.

Using the lexicographic ordering, the resulting Gröbner basis represents an ideal particularly well since the polynomial system becomes "triangularized". The Gröbner basis contains a lot of information about the solutions set of the initial polynomial system. For example, the system does not have any solution iff the Gröbner basis contains only 1. Further, the fact whether an ideal is zero-dimensional or not is expicitly "coded" in the Gröbner basis and may be easily read off. The initial system of polynomials possesses a finite number of solutions (i.e. I is zero-dimensional) if and only if for every variable x_i there exists a polynomial in the Gröbner basis such that its leading monomial is equal to x_i^m, for some $m > 0$. Importantly, calculation of a Gröbner basis is analytical, i.e. numerical approximations are not involved at this stage. The latter may appear only when one wants to obtain explicitly those observationally equivalent points (provided that there is more than one solution). For example, when an ideal is zero-dimensional, in many cases numerical solving is usually confined to finding a solution to one polynomial equation in one variable, for which reliable numerical solvers exist. The rest of the solution given deterministically in a recursive fashion. When the ideal is not zero-dimensional, a numerical solution cannot be defined, but the solution set is still parameterized by the free variables.

Going back to our problem of identification in DSGE models, we showed that the key step in solving it amounts to finding all roots to a polynomial system of equations. In a typical case, the number of equations s exceeds the number of variables l, which is the so-called overdetermined case. For generic overdetermined polynomial systems, the solution set is empty, but of course in our formulation we do know that at least one solution exists. In order to deal effectively with overdetermined systems, we exploit the appraoches presented in Lazard (1992) or Moller (1993). The idea is to decompose the original ideal so that the solution set of the initial polynomials will be the disjoint (finite) union of solutions to some smaller systems of l equations in l variables. This leads to the so-called triangular decomposition. See Kubler et al. (2014), section 2.2.3, for some intuition.

There are several ways of making the underlying problem even simpler. First, without loss of generality, one can assume $\Sigma = \mathrm{I}_k$ and move the parameters describing the properties of stochastic shocks to Γ_3. Second, matrix H usually does not depend on the model parameters, i.e. $\bar{H} = H$. Third, in some important cases we do not need to

solve for Q (see Corollary 1). These all together imply that the system of polynomial equations becomes of the order at most two, which makes calculation of the Gröbner basis easier. One additional feature of our polynomial solution problem is that one of its roots, i.e. the one associated with the point at which we check identification, is known. We can effectively eliminate it by defining a quotient of the original ideal and an ideal generated by this known root, of which a Gröbner basis may be easier to calculate.

If the obtained Gröbner basis implies a unique solution (or there are no solution when we work with a quotient), we obtain a formal proof that the DSGE model is globally identified. If there exist alternative solutions, the last step is to solve for $\bar{\theta}$ associated with each of these solutions. As we have argued before, this last step is relatively straightforward as the link between structural and semi-structural parameters is analytical. If any of such obtained candidate deep parameter vectors is admissible (i.e. meets the restrictions defining Θ), we have a constructive proof that the model is not identified.

For all our calculations that follow, we use SINGULAR, a free and open source computer algebra system specialized in polynomial calculations, see at `www.singular.uni-kl.de`. It has implemented many routines for calculation of the Gröbner basis, which is useful as there is no single algorithm that beats in terms of computational efficiency all alternatives for all possible cases. Importantly, SINGULAR can be considered a repository of most state-of-the-art algorithms, with active community of users sharing their experience in approaching various problems. In the next section we demonstrate that applying the concept of a Gröbner basis is feasible for some widely used DSGE models. This observation per se probably constitutes a substantial contribution to the literature.

6 Examples

We demonstrae the working of our identification framework with several examples.

6.1 An-Schorfheide model

In this example, we exploit a small-scale DSGE model by An and Schorfheide (2007), AS henceforth, modified to allow for correlation between government spending and productivity as in Herbst and Schorfheide (2016). This is a very convenient case, allowing to demonstrate identification issues of various types, as done using a numerical

algorithm by Kociecki and Kolasa (2018).

6.1.1 Model summary

When written in log-linearized form, the model is given by the following equations

$$z_t = \rho_z z_{t-1} + \rho_{zg} g_{t-1} + \varepsilon_{z,t} \tag{29}$$

$$g_t = \rho_g g_{t-1} + \rho_{gz} z_{t-1} + \varepsilon_{g,t} \tag{30}$$

$$x_t = E_t x_{t+1} + g_t - E_t g_{t+1} - \frac{1}{\tau}(R_t - E_t \pi_{t+1} - E_t z_{t+1}) \tag{31}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa(x_t - g_t) \tag{32}$$

$$R_t = \rho_m R_{t-1} + (1-\rho_m)[\psi_1 \pi_t + \psi_2 (x_t - g_t)] + \varepsilon_{m,t} \tag{33}$$

There are three endogenous variables in the model: detrended output x_t, inflation π_t and the interest rate R_t. They are driven by two exogenous AR(1) processes for productivity growth z_t and government spending g_t, with innovations $\varepsilon_{z,t}$ and $\varepsilon_{g,t}$, respectively, and by an i.i.d. monetary policy shock $\varepsilon_{m,t}$. All of these i.i.d. innovations are assumed to be mutually uncorrelated and their standard deviations are σ_z, σ_g and σ_m, respectively. The 13-dimensional vector of deep parameters is hence $\theta = [\ \tau \quad \beta \quad \kappa \quad \psi_1 \quad \psi_2 \quad \rho_z \quad \rho_{zg} \quad \rho_g \quad \rho_{gz} \quad \rho_m \quad \sigma_z \quad \sigma_g \quad \sigma_m\ ]'$.

The model can be cast in form (1), with states $s_t = [\ z_t \quad g_t \quad R_t\ ]'$, policy variables $p_t = [\ x_t \quad \pi_t\ ]'$, shocks $\varepsilon_t = [\ \varepsilon_{z,t} \quad \varepsilon_{g,t} \quad \varepsilon_{m,t}\ ]'$ and matrices Γ_0, Γ_1, Γ_2, Γ_3 and Σ given by

$$\Gamma_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & \alpha_5 & 1 & 0 \\ 0 & \alpha_7 & 0 & -\alpha_7 & 1 \\ 0 & \alpha_{10} & 1 & -\alpha_{10} & -\alpha_9 \end{bmatrix} \qquad \Gamma_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \alpha_5 & -1 & 0 & 1 & \alpha_5 \\ 0 & 0 & 0 & 0 & \alpha_6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma_2 = \begin{bmatrix} \alpha_1 & \alpha_2 & 0 \\ \alpha_3 & \alpha_4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha_8 \end{bmatrix} \qquad \Gamma_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \alpha_3 & 0 & 0 \\ 0 & \alpha_6 & 0 \\ 0 & 0 & \alpha_{13} \end{bmatrix}$$

where the 13-dimensional vector of semi-structural parameters $\alpha = [\alpha_i]_{i=1}^{13}$ is related

to θ with the following restrictions: $\alpha_1 = \rho_z$, $\alpha_2 = \rho_{zg}$, $\alpha_3 = \rho_g$, $\alpha_4 = \rho_{gz}$, $\alpha_5 = \tau^{-1}$, $\alpha_6 = \beta$, $\alpha_7 = \kappa$, $\alpha_8 = \rho_m$, $\alpha_9 = (1-\rho_m)\psi_1$, $\alpha_{10} = (1-\rho_m)\psi_2$, $\alpha_{11} = \sigma_z^2$, $\alpha_{12} = \sigma_g^2$, $\alpha_{13} = \sigma_m^2$. It is easy to verify that, after imposing non-negativity on the standard deviations of shocks, there is a one-to-one relationship between α and θ, so that the first type of identification failure is not an issue here.[7] Note furthermore that, even though it is not really necessary in this case, while parametrizing the model with α, we imposed the obvious linear restrictions imposed by θ.

Our benchmark parametrization θ of the AS model is presented in Table 1. It is the same as in An and Schorfheide (2007), except for ρ_{zg} and ρ_{gz}, which are taken from Kociecki and Kolasa (2018).

Table 1: Benchmark parametrization of AS model

Parameter	Value	Parameter	Value
τ	2	ρ_{zg}	0.1
β	0.9975	ρ_{gz}	-0.075
κ	0.33	ρ_m	0.75
ψ_1	1.5	σ_z	0.3
ψ_2	0.125	σ_g	0.6
ρ_z	0.9	σ_m	0.2
ρ_g	0.95		

The vector of observable variables is $y_t = [\ R_t \quad x_t \quad \pi_t\ ]'$ and there are no measurement errors, which means that $H = [\ 0_{3\times 2} \quad \mathrm{I}_3\ ]$ and $J = [\ 0_{3\times 3}\ ]$ for all θ.

6.1.2 Global identification failure in a locally identified model

Let us start with the benchmark parametrization. Calculating the Gröbner basis results in the following set of solutions for $\bar{\alpha}$:[8]

[7] More precisely, it is an issue in the original An-Schorfheide model as κ is actually a semi-structural parameter, linked to the deep model parameters via $\kappa = \tau\frac{1-\nu}{\nu\pi^2\phi}$. Since ν, π and ϕ do not show up anywhere else in the model equations, including them in θ instead of κ trivially leads to (local) identification failure of the first type.

[8] To save space, in what follows we do not report the solutions for other "unknowns" in the system of identification conditions (21)-(28) as they are not of direct interest. We also show the numbers in decimal form and rounded to four decimal digits, even though they are in fact exact rational numbers.

$$
\begin{aligned}
\bar{\alpha}_1 &= 0.9155 - 0.0155u \\
\bar{\alpha}_2 &= 0.2415 - 0.1415u \\
\bar{\alpha}_3 &= 0.9345 + 0.0155u \\
\bar{\alpha}_4 &= 0.0209 - 0.0959u \\
\bar{\alpha}_5 &= 0.5 \\
\bar{\alpha}_6 &= 0.5351 + 0.4624u \\
\bar{\alpha}_7 &= 0.4912 - 0.1612u \\
\bar{\alpha}_8 &= 0.75 \\
\bar{\alpha}_9 &= 0.3283 + 0.0467u \\
\bar{\alpha}_{10} &= 0.0629 - 0.0317u \\
\bar{\alpha}_{11} &= 0.1279 - 0.0379u \\
\bar{\alpha}_{12} &= 0.3128 + 0.6728u \\
\bar{\alpha}_{13} &= 0.04 \\
0 &= u^2 - 1.8697u + 0.8697
\end{aligned}
$$

Hence, the model' semi-structural parameters are parametrized by u, which needs to be consistent with the last quadratic equation. The roots of this equation are: $u_1 = 1$ and $u_2 = 0.8697$. It is easy to verify that for $u = u_1$ we get our benchmark parameter vector α, while $u = u_2$ results in an observationally equivalent model parametrization that is exactly the same as obtained by Kociecki and Kolasa (2018) with a numerical algorithm. We have thus a formal and constructive proof that the AS model is not globally identified at θ, and that the identification failure concerns all semi-structural parameters but $\alpha_5, \alpha_8, \alpha_{13}$ (and hence all structural parameters except for τ, ρ_m and σ_m).

6.1.3 Local identification

Let us now rule out any spillovers between productivity and government spending shocks, i.e. fix $\rho_{zg} = \rho_{gz} = 0$, and hence also $\alpha_2 = \alpha_4 = 0$. Calculating the Gröb-

ner basis yields:

$$\begin{aligned}
\bar{\alpha}_1 &= 0.9 - 0.0078u \\
\bar{\alpha}_3 &= 0.95 \\
\bar{\alpha}_5 &= 0.5 - 0.0106u \\
\bar{\alpha}_6 &= 0.9975 \\
\bar{\alpha}_7 &= 0.33 \\
\bar{\alpha}_8 &= 0.75v \\
\bar{\alpha}_9 &= 3.1658 - 2.7908v \\
\bar{\alpha}_{10} &= -2.7682 + 2.7908v \\
\bar{\alpha}_{11} &= 0.09 + 8.1277u \\
\bar{\alpha}_{12} &= 0.36 \\
\bar{\alpha}_{13} &= 0.04v^2 \\
0 &= uv \\
0 &= u^2 - 49.3312u
\end{aligned}$$

Of the two roots of the last equation, only $u_1 = 0$ does not violate the restrictions on the deep model parameters (for example, the other root implies $\tau < 0$). If $u = 0$, the penultimate equation does not put any restrictions on v. Setting $v = 1$ results in the benchmark parameter vector, any other value of v meeting the restrictions on the deep parameters gives an alternative parameter vector that is observationally equivalent to the benchmark. The identification failure concerns exclusively α_8, α_9, α_{10} and α_{13}, and hence ρ_m, ψ_1, ψ_2 and σ_m, which is now proved in a constructive way.

One can think of this failure as a local one as it applies to any vicinity of $v = 1$. This outcome is consistent with the previous papers dealing with this version of the AS model (Qu and Tkachenko, 2012; Kociecki and Kolasa, 2018). Importantly, however, and in contrast to any of the existing approaches to analyze local identification (also Iskrev, 2010; Komunjer and Ng, 2011), our framework explicitly produces the whole set of parameter vectors that are observationally equivalent to the one at which we check identification. In this example, the set is one-dimensional in the semi-structural

parameters and reads in terms of the deep parameters as follows

$$\begin{aligned}
\rho_m &= 0.75v \\
\psi_1 &= \frac{3.1658 - 2.7908v}{1 - 0.75v} \\
\psi_2 &= \frac{-2.7682 + 2.7908v}{1 - 0.75v} \\
\sigma_m &= 0.2 \mid v \mid
\end{aligned}$$

where v is any real number that keeps the alternative model parametrization in the determinacy (and stability) region.

More generally, our method gives new insight into the concept of local identification, that seems to be new in the literature. In fact, what we demonstrated is that all observationally equivalent parameters in this example live on the intersection of some hyperplanes whose dimension is 1. It is not difficult to imagine other cases that possibly could emerge in other models, e.g. an intersection of hyperplanes of higher dimension or some polynomials in several variables (e.g. 2 polynomial equations of second degree in 3 variables).

6.1.4 Handling indeterminacy

In the previous two variants we have considered the parameter vectors that imply a unique stable solution. However, our framework can also handle indeterminate cases. To demonstrate it, let us consider the AS model parametrized as in Table 1, except that now $\psi_1 = 0.75$, i.e. half the benchmark value. It can be easily verified, e.g. by checking the Blanchard-Kahn conditions, that there are infinitely many stable equilibria under such parametrization. As shown by Lubik and Schorfheide (2003), the full set of these equilibria are still given by equations (2) and (3), except that the vector of shocks ε_t must include a sufficient number of sunspots. Moreover, expectations of forward-looking variables become new states and hence need to be included in vector s_t. As demonstrated by Farmer et al. (2015), an equivalent characterization of indeterminate equilibria is to redefine a subset of expectational errors as new fundamentals. This is what we do here.

The order of indeterminacy in the considered model is one, so we need to pick one expectational error. Without loss of generality, let us pick the one associated with the

output gap x_t. Then, the AS model can be written as

$$z_t = \rho_z z_{t-1} + \rho_{zg} g_{t-1} + \varepsilon_{z,t} \tag{34}$$

$$g_t = \rho_g g_{t-1} + \rho_{gz} z_{t-1} + \varepsilon_{g,t} \tag{35}$$

$$x_t = \tilde{x}_t + g_t - E_t g_{t+1} - \frac{1}{\tau}(R_t - E_t \pi_{t+1} - E_t z_{t+1}) \tag{36}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa(x_t - g_t) \tag{37}$$

$$R_t = \rho_m R_{t-1} + (1 - \rho_m)[\psi_1 \pi_t + \psi_2 (x_t - g_t)] + \varepsilon_{m,t} \tag{38}$$

$$x_t - \tilde{x}_{t-1} = \rho_{sz} \varepsilon_{z,t} + \rho_{sg} \varepsilon_{g,t} + \rho_{sm} \varepsilon_{m,t} + \varepsilon_{s,t} \tag{39}$$

where $\tilde{x}_t = E_t x_{t+1}$, $\varepsilon_{s,t}$ is an i.i.d. sunspot shock with standard deviation σ_s and, as evident from equation (39), we allow for possible correlation between expectational errors and other structural shocks.

As an illustration, we check identification of this model at $\rho_{sz} = \rho_{sg} = \rho_{sm} = \sigma_s = 0.1$. Calculating the Gröbner basis implies a unique solution to our identification restrictions $\bar{\theta} = \theta$, and hence we have proved that the AS model is globally identified at this indeterminate parameterization θ. We arrive at the same conclusion also if we fix $\rho_{zg} = \rho_{gz} = 0$, thus confirming the outcome obtained by Qu and Tkachenko (2017) with a numerical algorithm that searches over the parameter space.

6.2 Other examples

We have also successfully applied our identification framework to the widely cited model of Smets and Wouters (2007). We fix the five parameters as in the orignal application, and also the growth rate of neutral technology it can be easily identified from the first moments. When checking identification at θ corresponding to the reported posterior mean, we find that only $\bar{\theta} = \theta$ meets our formal conditions given by Definition 2. We can hence conclude that the model is globally identified at this point.

7 Conclusions

In this paper we have developed a comprehensive framework to analyze local and global identification in DSGE models or, more generally, dynamic linear systems with rational expectations. Its main advantage is analytical flavor, which effectively allows to prove

identification or lack thereof. The essence of our approach is application of a Gröbner basis to solve analytically for all roots of a system of polynomial equations, which make up a formal identification condition that we derive.

Calculation of the Gröbner bases is known to be computationally involved for large systems, but we show that it can be still successfully applied to small and medium-sized DSGE models. It is worth stressing that this is not the only possible way to make use of our identification condition. One potentially attractive avenue to explore is application of all-solution homotopy methods, recently brought to the attention of economists by Kubler et al. (2014). While numerical in its nature, it may be a useful complement to the Gröbner basis due to its computational advantage, arising from the use of floating point arithmetic and parallelizability.

Appendix

A.1 Proof of Theorem 1

Recalling the notation introduced in the main text, let us define the infinite block Hankel matrix as

$$\mathcal{H} = \begin{bmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 & \cdots \\ \Lambda_2 & \Lambda_3 & \Lambda_4 & \cdots \\ \Lambda_3 & \Lambda_4 & \Lambda_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ \vdots \end{bmatrix} \begin{bmatrix} N & AN & A^2N & A^3N & \cdots \end{bmatrix}$$

Assuming stochastic minimality and using Sylvester's rank inequality, it may be easily shown that $rank(\mathcal{H}) = n$. Suppose that two sets of deep parameters $\bar{\theta} \neq \theta$ genarate the same autocovariances. Looking at the Hankel matrix, this implies $OAK = \bar{O}\bar{A}\bar{K}$. By Assumption 2 it follows that $\bar{A} = (\bar{O}'\bar{O})^{-1}\bar{O}'OAK\bar{K}'(\bar{K}\bar{K}')^{-1}$. Let us denote $T = (\bar{O}'\bar{O})^{-1}\bar{O}'O$ (which is nonsingular by Assumption 2). Since we also have $OK = \bar{O}\bar{K}$, we get $T^{-1} = K\bar{K}'(\bar{K}\bar{K}')^{-1}$, hence $\bar{A} = TAT^{-1}$. Looking at the first block row of the Hankel matrix, we have $CK = \bar{C}\bar{K} \Rightarrow \bar{C} = CT^{-1}$. Uniqueness of T follows from equality $\bar{O} = OT^{-1}$. Suppose that there was other T^* matrix. Then $\bar{O} = OT^{-1} = O(T^*)^{-1} \Leftrightarrow O(T^{-1} - (T^*)^{-1}) = 0$. Since O has full column rank, it follows $T = T^*$. Now suppose that $\bar{\theta} \neq \theta$ results in the same spectral density $\Phi(z) = \bar{\Phi}(z)$ for all $z \in \mathbb{C}$ (i.e. $\bar{\theta} \sim \theta$), that is

$$[C(z\mathrm{I}_n - A)^{-1}{:}\mathrm{I}_r] \begin{bmatrix} B\Sigma B' & B\Sigma D' \\ D\Sigma B' & D\Sigma D' \end{bmatrix} [C(z^{-1}\mathrm{I}_n - A)^{-1}{:}\mathrm{I}_r]' =$$

$$[\bar{C}(z\mathrm{I}_n - \bar{A})^{-1}{:}\mathrm{I}_r] \begin{bmatrix} \bar{B}\bar{\Sigma}\bar{B}' & \bar{B}\bar{\Sigma}\bar{D}' \\ \bar{D}\bar{\Sigma}\bar{B}' & \bar{D}\bar{\Sigma}\bar{D}' \end{bmatrix} [\bar{C}(z^{-1}\mathrm{I}_n - \bar{A})^{-1}{:}\mathrm{I}_r]'$$

Since we know $\bar{A} = TAT^{-1}$ and $\bar{C} = CT^{-1}$, hence

$$[C(z\mathrm{I}_n - A)^{-1}{:}\mathrm{I}_r] \begin{bmatrix} B\Sigma B' & B\Sigma D' \\ D\Sigma B' & D\Sigma D' \end{bmatrix} [C(z^{-1}\mathrm{I}_n - A)^{-1}{:}\mathrm{I}_r]'$$

$$= [CT^{-1}(z\mathrm{I}_n - TAT^{-1})^{-1}\vdots\mathrm{I}_r] \begin{bmatrix} \bar{B}\bar{\Sigma}\bar{B}' & \bar{B}\bar{\Sigma}\bar{D}' \\ \bar{D}\bar{\Sigma}\bar{B}' & \bar{D}\bar{\Sigma}\bar{D}' \end{bmatrix} [CT^{-1}(z^{-1}\mathrm{I}_n - TAT^{-1})^{-1}\vdots\mathrm{I}_r]'$$

$$= [C(z\mathrm{I}_n - A)^{-1}\vdots\mathrm{I}_r] \begin{bmatrix} T^{-1} & 0 \\ 0 & \mathrm{I}_r \end{bmatrix} \begin{bmatrix} \bar{B}\bar{\Sigma}\bar{B}' & \bar{B}\bar{\Sigma}\bar{D}' \\ \bar{D}\bar{\Sigma}\bar{B}' & \bar{D}\bar{\Sigma}\bar{D}' \end{bmatrix} \begin{bmatrix} T'^{-1} & 0 \\ 0 & \mathrm{I}_r \end{bmatrix} [C(z^{-1}\mathrm{I}_n - A)^{-1}\vdots\mathrm{I}_r]'$$

$$= [C(z\mathrm{I}_n - A)^{-1}\vdots\mathrm{I}_r] \begin{bmatrix} T^{-1}\bar{B}\bar{\Sigma}\bar{B}'T'^{-1} & T^{-1}\bar{B}\bar{\Sigma}\bar{D}' \\ \bar{D}\bar{\Sigma}\bar{B}'T'^{-1} & \bar{D}\bar{\Sigma}\bar{D}' \end{bmatrix} [C(z^{-1}\mathrm{I}_n - A)^{-1}\vdots\mathrm{I}_r]'$$

From the last expression, it follows that state-space representations (A, B, C, D, Σ) and $(A, T^{-1}\bar{B}, C, \bar{D}, \bar{\Sigma})$ also result in the same spectral density. Define Lyapunov equations connected with the first representation as $P = APA' + B\Sigma B'$, and the second one as $\bar{P} = A\bar{P}A' + T^{-1}\bar{B}\bar{\Sigma}\bar{B}'T'^{-1}$. Since A is stable, both P and $\bar{P}$ are unique. To proceed further, we need to use a well known lemma. Let X be any symmetric $n \times n$ matrix X, then

$$[C(z\mathrm{I}_n - A)^{-1}\vdots\mathrm{I}_r] \begin{bmatrix} X - AXA' & -AXC' \\ -CXA' & -CXC' \end{bmatrix} [C(z^{-1}\mathrm{I}_n - A)^{-1}\vdots\mathrm{I}_r]' = 0$$

which can be easily proved using the so-called Kalman-Yakubovich trick, i.e. $X - AXA' = (z\mathrm{I}_n - A)X(z^{-1}\mathrm{I}_n - A') + (z\mathrm{I}_n - A)XA' + AX(z^{-1}\mathrm{I}_n - A')$.

Let us use this lemma and substract from the left hand side

$$[C(z\mathrm{I}_n - A)^{-1}\vdots\mathrm{I}_r] \begin{bmatrix} P - APA' & -APC' \\ -CPA' & -CPC' \end{bmatrix} [C(z^{-1}\mathrm{I}_n - A)^{-1}\vdots\mathrm{I}_r]' (\equiv 0)$$

and from the right hand side expression

$$[C(z\mathrm{I}_n - A)^{-1}\vdots\mathrm{I}_r] \begin{bmatrix} \bar{P} - A\bar{P}A' & -A\bar{P}C' \\ -C\bar{P}A' & -C\bar{P}C' \end{bmatrix} [C(z^{-1}\mathrm{I}_n - A)^{-1}\vdots\mathrm{I}_r]' (\equiv 0)$$

Keeping in mind Lyapunov equations we get

$$[C(z\mathrm{I}_n - A)^{-1}\vdots\mathrm{I}_r] \begin{bmatrix} 0 & B\Sigma D' + APC' \\ D\Sigma B' + CPA' & D\Sigma D' + CPC' \end{bmatrix} [C(z^{-1}\mathrm{I}_n - A)^{-1}\vdots\mathrm{I}_r]' =$$

$$[C(z\mathrm{I}_n - A)^{-1}\vdots\mathrm{I}_r] \begin{bmatrix} 0 & T^{-1}\bar{B}\bar{\Sigma}\bar{D}' + A\bar{P}C' \\ \bar{D}\bar{\Sigma}\bar{B}'T'^{-1} + C\bar{P}A' & \bar{D}\bar{\Sigma}\bar{D}' + C\bar{P}C' \end{bmatrix} [C(z^{-1}\mathrm{I}_n - A)^{-1}\vdots\mathrm{I}_r]'$$

or using notation $S = B\Sigma D' - T^{-1}\bar{B}\bar{\Sigma}\bar{D}' + A(P-\bar{P})C'$, $R = D\Sigma D' - \bar{D}\bar{\Sigma}\bar{D}' + C(P-\bar{P})C'$

$$[C(z\mathrm{I}_n - A)^{-1}\vdots\mathrm{I}_r] \begin{bmatrix} 0 & S \\ S' & R \end{bmatrix} [C(z^{-1}\mathrm{I}_n - A)^{-1}\vdots\mathrm{I}_r]' = 0$$

Since $(z\mathrm{I}_n - A)^{-1} = z^{-1}\mathrm{I}_n + z^{-2}A + z^{-3}A^2 + \cdots$ and $(z^{-1}\mathrm{I}_n - A')^{-1} = z\mathrm{I}_n + z^2A' + z^3A'^2 + \cdots$, multiplying all terms we get

$$S'(z\mathrm{I}_n + z^2A' + z^3A'^2 + \cdots)C' + C(z^{-1}\mathrm{I}_n + z^{-2}A + z^{-3}A^2 + \cdots)S + R = 0$$

Any polynomial is identically (i.e. for all $z \in \mathbb{C}$) equal to zero iff its all coefffcients are zeros. Using this fact we get $R = D\Sigma D' - \bar{D}\bar{\Sigma}\bar{D}' + C(P - \bar{P})C' = 0$ (coefficient of $z = 0$). Moreover, all coefficients of all z^i must be equal to 0. Stacking (part of) these restrictions together we have

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} S = 0$$

By Assumption 2 it yields $S = B\Sigma D' - T^{-1}\bar{B}\bar{\Sigma}\bar{D}' + A(P - \bar{P})C' = 0$. Lastly, combining two Lyapunov equations into one equation we have $A(P - \bar{P})A' - (P - \bar{P}) = T^{-1}\bar{B}\bar{\Sigma}\bar{B}'T'^{-1} - B\Sigma B'$. Setting $Q = P - \bar{P}$, we arrive at the theorem. Uniqueness of Q follows by uniqueness of P and $\bar{P}$.

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