Simple Nonparametric Bounds on the Joint Distribution of Welfare and Endogenous Choice(s) in Discrete Random Utility Models

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Abstract

This paper studies the nonparametric identification of the joint distribution of money metric utility and endogenous choice(s) in discrete choice random utility models with unrestricted unobserved heterogeneity. Identifying this joint distribution allows researchers to study the welfare consequences of exogenous price changes for groups that are endogenous to the model. This is an important consideration in many current policy debates and provides answers to questions like 'What is the welfare impact of a refundable tax credit on the unemployed?' and 'How do congestion taxes affect the welfare of drivers?'. When panel data are available, I find the joint distribution of the compensating or equivalent variation and the endogenous pre-reform choice to be point identified from the observed uncompensated Marshallian transition probabilities. When only (repeated) cross-sectional data are available, however, I show that this object can still be set identified by exploiting stochastic revealed preference conditions which impose restrictions on the now unobserved transition probabilities. The bounds of this identified set can be easily computed by solving two constrained linear optimization programs along a one-dimensional grid. My simulations suggest that these bounds are in general sufficiently informative for meaningful applied welfare analysis. [Empirical application to be added.]

Keywords: compensating variation, equivalent variation, nonparametric identification, unrestricted heterogeneity, discrete choice models, random utility models, applied welfare analysis

JEL codes: C14, C25, C35, D11, D12, D63

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1 Introduction

Discrete choice random utility models (DC-RUMs) have a long tradition in both theoretical and applied microeconometric research. Since the pioneering work of McFadden (1974), these models have been applied to a wide range of problems in transportation, education, health care, industrial organisation, marketing, labour, and public finance. This success can be explained by DC-RUMs' ability to model consumer demand among a discrete set of alternatives in a flexible way, allowing for the presence of unobserved heterogeneity. Some models within this class — like the multinomial logit model — also yield convenient closed-form choice probabilities, which reduces the computational burden in both estimation and simulation considerably (for a comprehensive overview, see Train, 2003).[1]

The structural modelling of individual preferences in DC-RUMs renders this class of models especially suitable for applied welfare analysis. In such an analysis, researchers try to quantify the impact of exogenous price and quality changes on consumer welfare. However, as choices in DC-RUMs are (partially) random from the point of view of the econometrist, also money metrics like the compensating variation (CV) or equivalent variation (EV) become stochastic in nature, which complicates the analysis considerably (e.g., see Small and Rosen, 1981).

Until only recently, no closed-form expressions for the marginal distribution of the CV and EV existed, such that researchers had to resort to approximations, except for the most simple of DC-RUMs.[2] For the expected value of the CV, for example, the proposed approximations are either biased (Morey et al., 1993), rather uninformative (Herriges and Kling, 1999), or computationally burdensome (McFadden, 1999). This changed due to Dagsvik and Karlström (2005), who derive exact expressions for the distribution of the CV in additive DC-RUMs (with random effects), based on compensated Hicksian choice probabilities. They provide simple formulas for models where unobserved heterogeneity is GEV distributed. Alternatively, de Palma and Kilani (2011) advance a direct approach for the class of additve DC-RUMs, in which they express this distribution in terms of uncompensated Marshallian transition probabilities. Bhattacharya (2015, 2018) shows that the distribution of the CV (or EV) can be written as a simple functional of uncompensated Marshallian choice probabilities, even when unobserved heterogeneity is essentially unrestricted, and therefore possibly nonadditive.[3] His results readily imply that this distribution is nonparametrically identified from (repeated) cross-sectional data.

The joint distribution of money metric utility and endogenous choice(s), however, has received much less attention in the literature.[4] Knowledge on this joint distribution allows researchers to study the welfare consequences of exogenous reforms for groups that are endogenous to the model. This is an important consideration in many current policy debates and provides answers to questions like 'What is the welfare impact of a refundable tax credit on the unemployed?' and 'How do congestion taxes affect the welfare of drivers?'.

The main contributions of this paper are threefold. Firstly, I prove that the joint distribution of money metric

[1]Especially popular DC-RUM specifications in empirical research are the binary logit, the binary probit, the multinomial logit, and the mixed multinomial logit model.

[2]Small and Rosen (1981) and McFadden (1999) show that the *logsum* formula can be used to calculate the exact expected CV, given that individuals have constant marginal utility of income and given that unobserved heterogeneity is addtive and generalized extreme value (GEV) distributed.

[3]This result only holds for price changes. When quality changes occur, Bhattacharya (2018) demonstrates that functionals of the choice probabilities only set identify the distribution of the CV (or EV).

[4]Dagsvik and Karlström (2005) and de Palma and Kilani (2011) derive theoretical results for DC-RUMs, but only consider models with additive unobserved heterogeneity. In addition, they do not study whether this object can be nonparametically identified.

utility and pre- or post-reform endogenous choice(s) are point identified from Marshallian transition probabilities even in the presence of unrestricted unobserved heterogeneity. This result holds for the large class of DC-RUMs in which agents' utility functions are continuous and strictly increasing in the numeraire. An immediate implication is that the joint distribution can be recovered nonparametrically from panel data, given that sufficient cross-price variation is available. Secondly, I demonstrate how the restrictions stemming from stochastic revealed preference inequalities allows one to derive bounds on Marshallian transition probabilities when only (repeated) cross-sectional data is available. These bounds are simple functionals of Marshallian choice probabilities, which are nonparametric identified in a cross-section. Thirdly, in combining both these results, I show that the joint distribution of money metric utility and endogenous choice(s) can be set identified in a cross-section. The bounds on this distribution are easy to compute by means of two constrained linear optimization programs along a one-dimensional grid. From these distributional bounds, bounds on averages and quantiles of welfare are readily available. My simulations suggest that these bounds are in general sufficiently informative for meaningful applied welfare analysis. [Empirical application to be added.]

This paper is related to various other strands of literature. Over the last decades, several semiparametric methods have been developed to relax functional form assumptions on either the deterministic utility function or the distribution of unobserved heterogeneity in additive DC-RUMs (e.g., for early results see Manski, 1975; Matzkin, 1991; and Klein and Spady, 1993). Other contributions introduce entirely nonparametric methods — which refrain from imposing functional form restrictions on both these components — for this class of additive models based on shape restrictions (e.g., see Matzkin, 1993) or the availability of a large-support *special regressor* (e.g., see Lewbel, 2000). In a more recent paper, Briesch et al. (2010) use such a regressor to identify the distribution of nonadditive unrestrictive unobserved heterogeneity in an additive DC-RUM; Fox et al. (2012) nonparametrically identify the distribution of random coefficients in the canonical conditional logit model. The approach I follow in this paper deviates from this literature in that my objective is not to separately recover deterministic preferences and the distribution of unobserved heterogeneity, but instead to identify a higher level concept which is a function of both model primitives.

Another stream of research focusses on the nonparametric identification of counterfactual choices and welfare under unobserved heterogeneity in models where demand is continuous. Most results exploit the smoothness restrictions on the underlying individual demand functions to arrive at Slutsky-like conditons for the observed average or quantile demands (e.g., see Blomquist et al., 2015; Dette et al., 2016; Hausman and Newey, 2016; Blundell, Horowitz and Parey, 2017; and Hoderlein and Vanhems, 2018). An alternative approach is offered by Blundell, Kristensen and Matzkin (2017), who combine invertability conditions and an exclusion restriction to bound individual-level counterfactuals when unobserved heterogeneity is multivariate.

There is also a growing interest in incorporating nonparametric demand estimation into a revealed preference (RP) framework. Blundell et al. (2003, 2008) show how semiparametric Engel curves can improve the counterfactual predictions delivered by the strong axiom of RP, imposing unobserved heterogeneity to be additive and scalar-valued. Also using the strong axiom of RP, Blundell et al. (2014) improve the bounds on counterfactual quantile demand functions in a two-goods setting where individual demand is invertible in unobserved heterogeneity. In a multi-goods setting, Cosaert and Demuynck (2018) operationalize the weak axiom of stochastic RP by means of nonparametric density estimation and attain set identification for quantiles of counterfactual demand and money metrics, allowing for unrestricted unobserved heterogeneity.

Finally, there exists a small literature that studies transition probabilities in parametric DC-RUMs. Duncan and Weeks (2000) propose a simulation procedure to approximate transition probabilities numerically through a calibration approach. Bonin and Schneider (2006) derive closed-form expressions for the canonical conditional logit model. For the class of additive DC-RUMs, de Palma and Kilani (2011) show that transition probabilities can be derived by one-dimensional integration over partial derivatives of the Marshallian choice probabilities. Delle Site and Salucci (2013) provide results for additive models where there is imperfect correlation in unobserved heterogeneity before and after a reform. Similarily, Dagsvik (2002) derives transition probabilities when unobserved heterogeneity evolves according to a specific intertemporal stochastic process.

The remainder of this paper is organized as follows. Section 2 lays out the conceptual framework. Section 3 presents my main results on the identification of the joint distribution of money metric utility and endogenous choice(s). Section 4 details the nonparametric estimation strategy and shows how to deal with some forms of endogeneity. Section 5 contains simulation results for the canonical conditional logit model. Section 6 illustrates the results in an empirical application on the choice of fishing-mode. Finally, Section 7 concludes the paper.

2 Conceptual framework

My conceptual framework is similar to that of Bhattacharya (2015, 2018) and allows for unrestricted unobserved heterogeneity in DC-RUMs. As this set-up does not impose any restrictions on *observed* individual characteristics, the entire identificational analysis could be thought of as being conditional on these covariates.

2.1 Set-up

Suppose that there exists a probability space $(\Omega, \mathcal{F}, \Pr_\omega[\cdot])$, which defines preference types $\omega \in \Omega$ and allows to make probabilitistic statements over (countable) unions of these types. Every preference type can be though of as a different individual, who has his idiosyncratic preferences over a finite set of alternatives $\mathcal{C}$.[5] These idiosyncratic preferences are assumed to be representable by a utility function $U(y - p_c, c, \omega) : \mathbb{R} \times \mathcal{C} \times \Omega \to \mathbb{R}$, in which y denotes exogenous income and p_c the price of chosing alternative c. The only economically substantial restriction I impose on this function is that for every alternative $c \in \mathcal{C}$ and every type $\omega \in \Omega$, utility is continuous and strictly increasing in its first argument — the numeraire.[6] For notational clarity, I will compress the utility function as $U_c^\omega(y - p_c)$ in the sequel.

In addition, I assume that observed choice behaviour is generated by a Random Utility Model (for a detailed technical overview, see McFadden, 2005). It implies that an individual chooses a given alternative $i \in \mathcal{C}$ if and only if this alternative yields the highest utility among the elements in his choice set,

$$U_i^\omega(y - p_i) = \max_{c \in \mathcal{C}} \{U_c^\omega(y - p_c)\}.$$

In keeping the analysis tractable, I further impose that the set of types who are indifferent between two or more

[5]Note that when the set of alternatives is finite, the number of logically distinct preference types is also finite (e.g., see Manski, 2007).

[6]For the sake of technical rigour, I also require $U(\cdot, \omega)$ to be a $\mathcal{F}$-measurable function.

alternatives in their choice set has probability measure zero. This allows me to safely neglect the pathological cases where demand is multi-valued.

Finally, I also assume that the distribution of unrestricted unobserved heterogeneity $F_\omega(\omega)$ is independent of prices $\mathbf{p}$ (and also $\mathbf{q}$ and $\mathbf{r}$ in the sequel) and income y: i.e. $F_{\omega|\mathbf{p},y}(\omega \mid \mathbf{p}, y) = F_\omega(\omega)$ This is a strong but standard assumption in the literature on nonparametric identification of demand and welfare under unrestricted unobserved heterogeneity. Indeed, to the best of my knowledge there do not exist theoretical results that allow the budget or choice set to be endogenous in general ways.

For further reference, I restate the main assumptions maintained below.

Assumption 1. There exists a probability space $(\Omega, \mathcal{F}, \Pr_\omega[\cdot])$ that defines preference types $\omega \in \Omega$.

Assumption 2. Individual preferences can be represented by a utility function $U(y-p_c, c, \omega) : \mathbb{R}\times\mathcal{C}\times\Omega \to \mathbb{R}$ that is continuous and strictly increasing in its first argument for every alternative $c \in \mathcal{C}$ and preference type $\omega \in \Omega$. In addition, $U(\cdot, \omega)$ is a $\mathcal{F}$-measurable function.

Assumption 3. Observed choice behaviour is generated by a Random Utility Model. The set of types who are indifferent between two or more alternatives in their choice set has probability measure zero.

Assumption 4. The distribution of unobserved heterogeneity $F_\omega(\omega)$ is independent of prices $\mathbf{p}$ and income y: i.e. $F_{\omega|\mathbf{p},y}(\omega \mid \mathbf{p}, y) = F_\omega(\omega)$.

2.2 Choice and transition probabilities

As mentioned before, the choices induced by the DC-RUM framework are stochastic from the point of view of the researcher. When this random variation is averaged out, one attains uncompensated Marshallian *choice* probabilities,

$$\begin{aligned} P_i(\mathbf{p}, y) &= \Pr_\omega\left[\left\{\omega : U_i^\omega(y-p_i) = \max_{c\in\mathcal{C}}\{U_c^\omega(y-p_c)\}\right\}\right] \\ &= \int_w \mathbb{1}\left[U_i^w(y-p_i) = \max_{c\in\mathcal{C}}\{U_c^w(y-p_c)\}\right] dF_\omega(w), \end{aligned}$$

which describe choice frequencies conditional on prices $\mathbf{p}$ and income y.[7] If cross-sectional data contains enough exogenous price and income variation, one can readily provide semi- or nonparametrically estimates for these objects.

Note from the second equality, however, that these probabilities are functions of both preferences and the distribution of unobserved heterogeneity. As such, they are not sufficiently informative to separately identify the two model primitives. Fortunately, knowledge on both primitives will prove to be unnecessary to attain (set) identification for the joint distribution of money metrics and endogenous choice(s).

Another concept induced by the DC-RUM framework are uncompensated Marshallian *transition* probabilities,

$$\begin{aligned} P_{i\to j}(\mathbf{p}, \mathbf{q}, y) &= \Pr_\omega\left[\left\{\omega : U_i^\omega(y-p_i) = \max_{c\in\mathcal{C}}\{U_c^\omega(y-p_c)\}\right\} \cap \left\{\omega : U_j^\omega(y-q_j) = \max_{c\in\mathcal{C}}\{U_c^\omega(y-q_c)\}\right\}\right] \\ &= \int_w \mathbb{1}\left[U_i^w(y-p_i) = \max_{c\in\mathcal{C}}\{U_c^w(y-p_c)\}\right] \mathbb{1}\left[U_j^w(y-q_j) = \max_{c\in\mathcal{C}}\{U_c^w(y-q_c)\}\right] dF_\omega(w), \end{aligned}$$

which describe transition frequencies conditional on pre-reform prices $\mathbf{p}$, post-reform prices $\mathbf{q}$, and income y. Of course, if pre-reform equal post-reform prices, their are no transitions between different choices. In principle, these objects can be semi- or nonparametrically estimated from panel data with at least two periods.

[7] This concept is also known as the *average structural function* (e.g., see Blundell and Powell, 2004).

Again, it is important to stress that I assume unobserved heterogeneity to be unaffected by a reform. The (perfect) correlation between pre- and post-reform preference types therefore implies that transition probabilities are not simply equal to the product of their marginals: i.e. $P_{i\to j}(\mathbf{p},\mathbf{q},y) \neq P_i(\mathbf{p},y)P_j(\mathbf{q},y)$.

2.3 Money metric utility

Money metrics induce a particular normalization to individual preferences, such that interpersonal welfare comparisons can be conducted. The compensating variation (CV) refers to the amount of additional money an individual needs to receive after a reform to be equally well off as before this reform. In DC-RUMs this concept is implicitly defined by

$$\max_{c\in\mathcal{C}}\{U_c^\omega(y-p_c)\} = \max_{c\in\mathcal{C}}\{U_c^\omega(y-q_c+cv)\}.$$

Another closely related concept to measure welfare changes is the the equivalent variation (EV). It refers to the amount of money an individual is willing to forgo before a reform to be equally well off as after this reform,

$$\max_{c\in\mathcal{C}}\{U_c^\omega(y-p_c-ev)\} = \max_{c\in\mathcal{C}}\{U_c^\omega(y-q_c)\}.$$

Bhattacharya (2018) shows that under the assumptions maintained above, both welfare concepts exist and are unique for every possible preference type. As their explicit solutions $cv^\omega(\mathbf{p},\mathbf{q},y)$ and $ev^\omega(\mathbf{p},\mathbf{q},y)$ depend on these unobserved types, they are stochastic from the point of view of the econometrist. The main goal of this paper is to nonparametrically identify the joint distribution of the CV or EV and *pre-reform* choices,

$$P^{cv}_{i\to\cdot}(z;\ \mathbf{p},\mathbf{q},y) = \Pr_\omega\left[\left\{\omega : cv^\omega(\mathbf{p},\mathbf{q},y)\le z\right\}\cap\left\{\omega : U_i^\omega(y-p_i) = \max_{c\in\mathcal{C}}\{U_c^\omega(y-p_c)\}\right\}\right],$$
$$P^{ev}_{i\to\cdot}(z;\ \mathbf{p},\mathbf{q},y) = \Pr_\omega\left[\left\{\omega : ev^\omega(\mathbf{p},\mathbf{q},y)\le z\right\}\cap\left\{\omega : U_i^\omega(y-p_i) = \max_{c\in\mathcal{C}}\{U_c^\omega(y-p_c)\}\right\}\right],$$

and similarily the joint distribution of the CV or EV and *post-reform* choices,

$$P^{cv}_{\cdot\to j}(z;\ \mathbf{p},\mathbf{q},y) = \Pr_\omega\left[\left\{\omega : cv^\omega(\mathbf{p},\mathbf{q},y)\le z\right\}\cap\left\{\omega : U_j^\omega(y-q_j) = \max_{c\in\mathcal{C}}\{U_c^\omega(y-q_c)\}\right\}\right],$$
$$P^{ev}_{\cdot\to j}(z;\ \mathbf{p},\mathbf{q},y) = \Pr_\omega\left[\left\{\omega : ev^\omega(\mathbf{p},\mathbf{q},y)\le z\right\}\cap\left\{\omega : U_j^\omega(y-q_j) = \max_{c\in\mathcal{C}}\{U_c^\omega(y-q_c)\}\right\}\right].$$

In Section 3.2, which contains the additional results, I also briefly discuss the nonparametric identification of the joint distribution of the CV or EV, pre-reform choices, and post-reform choices,

$$P^{cv}_{i\to j}(z;\ \mathbf{p},\mathbf{q},y) = \Pr_\omega\Bigg[\left\{\omega : cv^\omega(\mathbf{p},\mathbf{q},y)\le z\right\}\cap\left\{\omega : U_i^\omega(y-p_i) = \max_{c\in\mathcal{C}}\{U_c^\omega(y-p_c)\}\right\}$$
$$\cap\left\{\omega : U_j^\omega(y-q_j) = \max_{c\in\mathcal{C}}\{U_c^\omega(y-q_c)\}\right\}\Bigg],$$
$$P^{ev}_{i\to j}(z;\ \mathbf{p},\mathbf{q},y) = \Pr_\omega\Bigg[\left\{\omega : ev^\omega(\mathbf{p},\mathbf{q},y)\le z\right\}\cap\left\{\omega : U_i^\omega(y-p_i) = \max_{c\in\mathcal{C}}\{U_c^\omega(y-p_c)\}\right\}$$
$$\cap\left\{\omega : U_j^\omega(y-q_j) = \max_{c\in\mathcal{C}}\{U_c^\omega(y-q_c)\}\right\}\Bigg].$$

3 Identification

3.1 Main results

In this section, I present the identificational results for the joint distribution of money metric utility and endogenous pre-reform *or* post-reform choices. I will assume everywhere — without loss of generality — that the alternatives contained in the choice set $\mathcal{C}$ are labeled according to increasing price differentials, i.e. $q_1 - p_1 \leq \ldots \leq q_C - p_C$, and that $q_t - p_t \leq z < q_{t+1} - p_{t+1}$. All proofs are in the Appendix.

My first result is an immediate extension of Theorem 1 in Bhattacharya (2018) and shows that the joint distribution is point identified from uncompensated Marshallian transition probabilities, even in the presence of unrestricted unobserved heterogeneity. It therefore generalizes the analysis of de Palma and Kilani (2011), who prove this for the class of additive DC-RUMs.

Result 1. Suppose Assumptions 1 – 4 hold. Then the joint distribution of the compensating or equivalent variation and the ex-ante choice, i.e. $\{P^{\text{cv}}_{i\to\cdot};\ i \in \mathcal{C}\}$ and $\{P^{\text{ev}}_{i\to\cdot};\ i \in \mathcal{C}\}$ respectively, are point identified from $\{P_{i\to j};\ i, j \in \mathcal{C}\}$. In addition, also the joint distribution of the compensating or equivalent variation and the ex-post choice, i.e. $\{P^{\text{cv}}_{\cdot\to j};\ j \in \mathcal{C}\}$ and $\{P^{\text{ev}}_{\cdot\to j};\ j \in \mathcal{C}\}$ respectively, are point identified from $\{P_{i\to j}\}$. Point estimates for $\{P^{\text{cv}}_{i\to\cdot}\}$, $\{P^{\text{ev}}_{i\to\cdot}\}$, $\{P^{\text{cv}}_{\cdot\to j}\}$, and $\{P^{\text{ev}}_{\cdot\to j}\}$ can be constructed from functions of the (estimated) transition probabilities that are evaluated at counterfactual price vectors:

$$
\begin{aligned}
P^{\text{cv}}_{i\to\cdot}(z;\ \mathbf{p}, \mathbf{q}, y) &= \sum_{k=1}^{t} P_{i\to k}(\mathbf{p};\ q_1 - z, \ldots, q_t - z, p_{t+1}, \ldots, p_C;\ y), \\
P^{\text{ev}}_{i\to\cdot}(z;\ \mathbf{p}, \mathbf{q}, y) &= \sum_{k=1}^{t} P_{i\to k}(\mathbf{p};\ q_1, \ldots, q_t, p_{t+1} + z, \ldots, p_C + z;\ y), \\
P^{\text{cv}}_{\cdot\to j}(z;\ \mathbf{p}, \mathbf{q}, y) &= \sum_{k=1}^{t} P_{k\to j}(q_1 - z, \ldots, q_t - z, p_{t+1}, \ldots, p_C;\ \mathbf{q};\ y), \\
P^{\text{ev}}_{\cdot\to j}(z;\ \mathbf{p}, \mathbf{q}, y) &= \sum_{k=1}^{t} P_{k\to j}(q_1, \ldots, q_t, p_{t+1} + z, \ldots, p_C + z;\ \mathbf{q};\ y).
\end{aligned}
$$

As mentioned before, Marshallian transition probabilities are nonparametrically identified from panel data with sufficient exogenous price and income variation. Result 1 therefore implies that the joint distribution of money metric utility and pre- or post-reform choices is also nonparametrically identified in such a setting. One simply has to evaluate the (estimated) transition probabilities at counterfactual price vectors.

This result can be slightly adjusted to cover the case in which one of the modelled alternatives is an outside good. For an outside good, there is no price variation present in the data such that choice probabilities can not be evaluated at some values of z within its support $\mathcal{Z}$. This problem can be circumvented by exploiting the individual intertemporal variation in exogenous income y, as is presented in Corollary 1.

Corollary 1. Suppose Assumptions 1 – 4 hold. Suppose that there exists an outside good $c_0 \in \mathcal{C}$ that does not exhibit price variation. Then $\{P^{\text{cv}}_{i\to\cdot}\}$, $\{P^{\text{ev}}_{i\to\cdot}\}$, $\{P^{\text{cv}}_{\cdot\to j}\}$, and $\{P^{\text{ev}}_{\cdot\to j}\}$ are still identified from individual variation in the exogenous income y:

$$P^{cv}_{i\to\cdot}(z;\mathbf{p},\mathbf{q},y)=\begin{cases}\sum_{k=1}^{t}P_{i\to k}(\mathbf{p};\ q_1-z,...,q_t-z,p_{t+1},...,p_C;\ y;\ y), & \text{if } z<0,\\ \sum_{k=1}^{t}P_{i\to k}(\mathbf{p};\ q_1,...,q_t,p_{t+1}+z,...,p_C+z;\ y;\ y+z), & \text{if } z\geq 0,\end{cases}$$

$$P^{ev}_{i\to\cdot}(z;\mathbf{p},\mathbf{q},y)=\begin{cases}\sum_{k=1}^{t}P_{i\to k}(\mathbf{p};\ q_1-z,...,q_t-z,p_{t+1},...,p_C;\ y;\ y-z), & \text{if } z<0,\\ \sum_{k=1}^{t}P_{i\to k}(\mathbf{p};\ q_1,...,q_t,p_{t+1}+z,...,p_C+z;\ y;\ y), & \text{if } z\geq 0,\end{cases}$$

$$P^{cv}_{\cdot\to j}(z;\mathbf{p},\mathbf{q},y)=\begin{cases}\sum_{k=1}^{t}P_{k\to j}(q_1-z,...,q_t-z,p_{t+1},...,p_C;\ \mathbf{q};\ y;\ y), & \text{if } z<0,\\ \sum_{k=1}^{t}P_{k\to j}(q_1,...,q_t,p_{t+1}+z,...,p_C+z;\ \mathbf{q};\ y+z;\ y), & \text{if } z\geq 0,\end{cases}$$

$$P^{ev}_{\cdot\to j}(z;\mathbf{p},\mathbf{q},y)=\begin{cases}\sum_{k=1}^{t}P_{k\to j}(q_1-z,...,q_t-z,p_{t+1},...,p_C;\ \mathbf{q};\ y-z;\ y), & \text{if } z<0,\\ \sum_{k=1}^{t}P_{k\to j}(q_1,...,q_t,p_{t+1}+z,...,p_C+z;\ \mathbf{q};\ y;\ y), & \text{if } z\geq 0.\end{cases}$$

In many empirical applications, however, researchers only have access to (repeated) cross-sectional data. This type of data nonparametrically identifies the uncompensated Marshallian *choice* probabilities, but not the associated *transition* probabilities. Using a stochastic revealed preference argument, I show that one can derive simple bounds on the transition probabilities when prices are changed exogenously from $\mathbf{p}$ to $\mathbf{r}$.

Lemma 1. Suppose Assumptions 1 – 4 hold. Define $\underline{m}(c)=\min\{p_c,r_c\}$ and $\overline{m}(c)=\max\{p_c,r_c\}$. Then the transition probabilities $\{P_{i\to j}(\mathbf{p},\mathbf{r},y);\ i,j\in\mathcal{C}\}$ are set identified from the choice probabilities $\{P_i(\mathbf{p},y);\ i\in\mathcal{C}\}$ with bounds

$$P^{L}_{i\to i}(\mathbf{p},\mathbf{r},y)=P_i(\underline{m}(1),...,\overline{m}(i),...,\underline{m}(C);\ y)$$
$$P^{U}_{i\to i}(\mathbf{p},\mathbf{r},y)=\min\left\{P_i(\mathbf{p};\ y),\ P_i(\mathbf{r};\ y)\right\},$$

for $P_{i\to i}(\mathbf{p},\mathbf{r},y)$, and

$$P^{L}_{i\to j}(\mathbf{p},\mathbf{r},y)=0$$
$$P^{U}_{i\to j}(\mathbf{p},\mathbf{r},y)=\min\left\{P_i(\mathbf{p};\ y),\ P_j(\mathbf{r};\ y)\right\},$$

for $\{P_{i\to j}(\mathbf{p},\mathbf{r},y)\}$. The transition probabilities $\{P_{i\to j}(\mathbf{p},\mathbf{r},y)\}$ are point identified — and equal zero — whenever $p_i>r_i$ and $p_j<r_j$.

Similary as before, also this result can be slightly adjusted to cover the case in which one of the modelled alternatives is an outside good. Corollary 2 shows how the bounds on the transition probabilities are calculated in this particular instance.

Corollary 2. Suppose Assumptions 1 – 4 hold. Suppose that there exists an outside good $c_0\in\mathcal{C}$ that does not exhibit price variation. Then the transition probabilities $\{P_{i\to j}(\mathbf{p},\mathbf{r},y);\ i,j\in\mathcal{C}\}$ are still set identified from cross-sectional variation in the exogenous income y:

$$P^{L}_{i\to i}(\mathbf{p},\mathbf{r},y)=\begin{cases}P_i(\underline{m}(1),...,\overline{m}(i),...,\underline{m}(C);\ y+(p_{c_0}-\underline{m}(c_0))) & \text{if } c_0\neq i\\ P_i(\underline{m}(1),...,\overline{m}(i),...,\underline{m}(C);\ y+(p_{c_0}-\overline{m}(c_0))) & \text{if } c_0=i\end{cases}$$

$$P^{U}_{i\to i}(\mathbf{p},\mathbf{r},y)=\min\left\{P_i(\mathbf{p};\ y),\ P_j(r_1+(p_{c_0}-r_{c_0}),...,p_{c_0},...,r_C+(p_{c_0}-r_{c_0});\ y+(p_{c_0}-r_{c_0}))\right\},$$

for $P_{i\to i}(\mathbf{p},\mathbf{r},y)$, and

$$P^L_{i\to j}(\mathbf{p},\mathbf{r},y) = 0$$
$$P^U_{i\to j}(\mathbf{p},\mathbf{r},y) = \min\left\{P_i(\mathbf{p};\ y),\ P_j(r_1+(p_{c_0}-r_{c_0}),...,p_{c_0},...,r_C+(p_{c_0}-r_{c_0});\ y+(p_{c_0}-r_{c_0}))\right\},$$

for $\{P_{i\to j}(\mathbf{p},\mathbf{r},y)\}$.

Combining Result 1 with Lemma 1 immediately leads to Result 2, which presents simple bounds on the joint distribution of money metric utility and pre- or post-reform choices. To recover both the lower and upper bound, one has to solve two constrained linear optimization problems along a one-dimensional grid. This grid should cover the entire support of z, i.e. $\mathcal{Z}$.

Result 2. Suppose Assumptions 1 – 4 hold. Then the joint distribution of the compensating or equivalent variation and the ex-ante choice, i.e. $\{P^{cv}_{i\to\cdot};\ i\in\mathcal{C}\}$ or $\{P^{cv}_{i\to\cdot};\ i\in\mathcal{C}\}$ respectively, are set identified from the choice probabilities $\{P_i;\ i\in\mathcal{C}\}$. The lower and upper bounds of these convex sets, i.e. $\{LB^{ev/cv}_{i\to\cdot};\ i\in\mathcal{C}\}$ and $\{UB^{ev/cv}_{i\to\cdot};\ i\in\mathcal{C}\}$ respectively, are defined by means of two constrained linear optimization programs,

$$LB^{cv/ev}_{i\to\cdot}(z;\ \mathbf{p},\mathbf{q},y) = \min_{\{\underline{x}_c(z);\ c\in\mathcal{C}\}} \sum_{k=1}^{t} \underline{x}_k(z),$$
$$\text{subject to} \begin{cases} P^L_{i\to k}(\mathbf{p},\mathbf{q}',y) \le \underline{x}_k(z) \le P^U_{i\to k}(\mathbf{p},\mathbf{q}',y), & \forall k\in\mathcal{C}, \\ \sum_{k=1}^{C} \underline{x}_k(z) = P_i(\mathbf{p};\ y), \end{cases}$$
$$UB^{cv/ev}_{i\to\cdot}(z;\ \mathbf{p},\mathbf{q},y) = \max_{\{\overline{x}_c(z);\ c\in\mathcal{C}\}} \sum_{k=1}^{t} \overline{x}_k(z),$$
$$\text{subject to} \begin{cases} P^L_{i\to k}(\mathbf{p},\mathbf{q}',y) \le \overline{x}_k(z) \le P^U_{i\to k}(\mathbf{p},\mathbf{q}',y), & \forall k\in\mathcal{C}, \\ \sum_{k=1}^{C} \overline{x}_k(z) = P_i(\mathbf{p};\ y), \end{cases}$$

in which the price vector $\mathbf{q}'$ equals $(q_1-z,...,q_t-z,p_{t+1},...,p_C)$ for the compensating and $(q_1,...,q_t,p_{t+1}+z,...,p_C+z)$ for the equivalent variation.

As both the lower and upper bound are cumulative distribution functions — conditional on the endogenous pre-reform or post-reform choice —, they should be monotonically increasing on their domain $\mathcal{Z}$. As this is not guaranteed by the optimization programs in Result 2, I will impose this additional restriction in the estimation procedure (see Section 4).

3.2 Additional results

In this section, I present additional results for the joint distributions of money metric utility and endogenous pre-reform *and* post-reform choices. Again using Theorem 1 from Bhattacharya (2018), I find that this distribution is identified from *double* transition probabilities. In principle, these double transition probabilities can be estimated from panel data with at least three periods.[8]

[8]Note, however, that estimating *multiple* transition probabilities is increasingly subject to the curse of dimensionality.

Result 3. Suppose Assumptions 1 – 4 hold. Then the joint distribution of the compensating or equivalent variation and the ex-ante and ex-post choice choices, i.e. $\{P^{\text{cv}}_{i\to j};\ i,j\in\mathcal{C}\}$ and $\{P^{\text{ev}}_{i\to j};\ i,j\in\mathcal{C}\}$ respectively, are point identified from observing double transition probabilities, $\{P_{i\to j\to k};\ i,j,k\in\mathcal{C}\}$. Point estimates for $\{P^{cv}_{i\to j}\}$ and $\{P^{ev}_{i\to j}\}$ can be constructed from functions of the (estimated) double transition probabilities that are evaluated at counterfactual price vectors:

$$P^{cv}_{i\to j}(z;\ \mathbf{p},\mathbf{q},y) = \sum_{k=1}^{t} P_{i\to j\to k}(\mathbf{p};\ \mathbf{q};\ q_1 - z,\dots,q_t - z, p_{t+1},\dots,p_C;\ y),$$
$$P^{ev}_{i\to j}(z;\ \mathbf{p},\mathbf{q},y) = \sum_{k=1}^{t} P_{i\to j\to k}(\mathbf{p};\ \mathbf{q};\ q_1,\dots,q_t, p_{t+1}+z,\dots,p_C+z;\ y).$$

Again, revealed preference inequalities can be used to restrict the double transition probabilities.

Lemma 2. Suppose Assumptions 1 – 4 hold. Define $\underline{m}_{x,y}(c)=\min\{x_c,y_c\}$ and $\overline{m}_{x,y}(c)=\max\{x_c,y_c\}$. Then the double transition probabilities $\{P_{i\to j\to k}(\mathbf{p},\mathbf{q},\mathbf{r},y);\ i,j,k\in\mathcal{C}\}$ are set identified from the transition choice probabilities $\{P_{i\to j}(\mathbf{p},\mathbf{q},y);\ i,j\in\mathcal{C}\}$ with bounds

$$P^{L}_{i\to i\to i}(\mathbf{p},\mathbf{q},\mathbf{r},y) = \max\left\{\begin{array}{l} P_{i\to i}\left(\underline{m}_{p,r}(1),\dots,\overline{m}_{p,r}(i),\dots,\underline{m}_{p,r}(C);\ \underline{m}_{q,r}(1),\dots,\overline{m}_{q,r}(i),\dots,\underline{m}_{q,r}(C);\ y\right), \\ P_{i\to i}\left(\underline{m}_{p,q}(1),\dots,\overline{m}_{p,q}(i),\dots,\underline{m}_{p,q}(C);\ \underline{m}_{p,r}(1),\dots,\overline{m}_{p,r}(i),\dots,\underline{m}_{p,r}(C);\ y\right), \\ P_{i\to i}\left(\underline{m}_{p,q}(1),\dots,\overline{m}_{p,q}(i),\dots,\underline{m}_{p,q}(C);\ \underline{m}_{q,r}(1),\dots,\overline{m}_{q,r}(i),\dots,\underline{m}_{q,r}(C);\ y\right) \end{array}\right\},$$
$$P^{U}_{i\to i\to i}(\mathbf{p},\mathbf{q},\mathbf{r},y) = \min\left\{P_{i\to i}(\mathbf{p},\mathbf{q},y),\ P_{i\to i}(\mathbf{p},\mathbf{r},y),\ P_{i\to i}(\mathbf{q},\mathbf{r},y),\ P_i(\mathbf{p},y),\ P_i(\mathbf{q},y),\ P_i(\mathbf{r},y)\right\},$$

for $P_{i\to i\to i}(\mathbf{p},\mathbf{q},\mathbf{r},y)$, and

$$\begin{aligned} P^{L}_{i\to j\to k}(\mathbf{p},\mathbf{q},\mathbf{r},y) &= 0 \\ P^{U}_{i\to j\to k}(\mathbf{p},\mathbf{q},\mathbf{r},y) &= \min\{P_i(\mathbf{p},y),P_j(\mathbf{q},y),P_k(\mathbf{r},y)\} \end{aligned} \tag{1}$$

for $\{P_{i\to j\to k}(\mathbf{p},\mathbf{q},\mathbf{r},y)\}$. The double choice probabilities $\{P_{i\to j\to k}\}$ are point identified — and equal zero — whenever [To be added].[9]

Similary as before, combining Result 3 with Lemma 2 immediately leads to Result 4.

Result 4. Suppose Assumptions 1, 2 and 3 hold. Then the joint distribution of the compensating or equivalent variation and the ex-ante and ex-post choice choices, i.e. $\{P^{\text{cv}}_{i\to j};\ i,j\in\mathcal{C}\}$ and $\{P^{\text{ev}}_{i\to j};\ i,j\in\mathcal{C}\}$ respectively, are set identified from the transition probabilities $\{P_{i\to j};\ i,j\in\mathcal{C}\}$.

3.3 Average of money metrics conditional on endogenous choices

An other important object of interest is the cumulative distribution of money metric utility conditional on endogenous choice(s), which allows researchers to assess the distribution of welfare gains or losses for groups that are endogenous

[9] Note that $P_{i\to i}(\mathbf{r},\mathbf{r},y)$ equals $P_i(\mathbf{r},y)$ by definition.

to the model. This object is easily derived from the joint distributions:

$$P^{cv/ev}_{|i\to\cdot}(z;\ \mathbf{p},\mathbf{q},y) = \frac{P^{cv/ev}_{i\to\cdot}(z;\ \mathbf{p},\mathbf{q},y)}{P_i(\mathbf{p},y)}$$

$$P^{cv/ev}_{|\cdot\to j}(z;\ \mathbf{p},\mathbf{q},y) = \frac{P^{cv/ev}_{\cdot\to j}(z;\ \mathbf{p},\mathbf{q},y)}{P_j(\mathbf{q},y)}.$$

Bounds on average conditional welfare can be calculated as expressed in Proposition 1.

Proposition 1. Suppose Assumptions 1, 2, and 3 hold. Then average welfare for the conditional distribution of the compensating and equivalent variation, i.e. $\{\mathbb{E}(cv \mid i \to \cdot);\ i \in \mathcal{C}\}$ and $\{\mathbb{E}(ev \mid i \to \cdot);\ i \in \mathcal{C}\}$ respectively, are bound from below by the mean for the conditional upper bounds $\{UB^{cv/ev}_{|i\to\cdot};\ i \in \mathcal{C}\}$ and from above by the mean for the conditional lower bound $\{LB^{cv/ev}_{|i\to\cdot};\ i \in \mathcal{C}\}$. The bounds for average welfare can be calculated directly from the conditional CDFs as follows:

$$\left(\int_0^\infty (1 - UB^{cv/ev}_{|i\to\cdot}(u;\ \cdot))du - \int_{-\infty}^0 UB^{cv/ev}_{|i\to\cdot}(u;\ \cdot)du\right)$$
$$\leq \mathbb{E}(cv/ev \mid i \to \cdot) \leq$$
$$\left(\int_0^\infty (1 - LB^{cv/ev}_{|i\to\cdot}(u;\ \cdot))du - \int_{-\infty}^0 LB^{cv/ev}_{|i\to\cdot}(u;\ \cdot)du\right).$$

4 Estimation

4.1 Nonparametric estimation

[To be added]

4.2 Endogeneity

[To be added]

5 Simulation

In this section, I assess the performance of the bounds developed above in a simulation exercise. Due to space constraints, I limit the scope of this exercise to the distribution of the CV and EV conditional on the pre-reform choices (see Section 3.3).

5.1 Set-up

Simulation 1 The first simulation centers around the conditional logit model, which has been applied extensively in empirical research. This model belongs to the class of additive DC-RUMs, such that the utility function can be further decomposed as

$$U^\omega_c(y - p_c) = u_c(y - p_c) + \varepsilon_c(\omega),$$

where $\varepsilon_c(\cdot)$ is i.i.d. across alternatives and individuals and is drawn from an extreme value type I distribution.[10] Furthermore, I set $u_c(y - p_c) \equiv a_c(1 + \log(y - p_c))$, which introduces nonlinear income effects.

[10]The CDF of the EV I distribution is $F_\varepsilon = \exp(-\exp(-\varepsilon))$.

A well-known result by McFadden (1974) shows that this type of model has especially tractable uncompensated choice probabilities:

$$P_i(\mathbf{p}, y) = \frac{\exp(u_i(y - p_i))}{\sum_{c=1}^{C} \exp(u_c(y - p_c))}.$$

I use these choice probabilities to derive restrictions on the uncompensated transition probabilities as described in Lemma 1. To compare my bounds with the actual solution, I also derive the closed-form expression for these transition probabilities from Proposition 1 in de Palma and Kilani (2011),

$$P_{i\to i} = \frac{\exp(u_i(y - p_i))}{n(i)},$$

$$P_{i\to j} = \begin{cases} \sum_{r=i}^{j-1} \left(\frac{\exp(u_i(y-p_i))}{n(r+1)} - \frac{\exp(u_i(y-p_i))}{n(r)} \right) \frac{\exp(u_j(y-q_j))}{\sum_{k>r} \exp(u_k(y-q_k))}, & \text{if } j > i, \\ 0, & \text{if } j < i, \end{cases}$$

in which $n(c) = \sum_{k \le c} \exp(u_k(y - p_k)) + \exp(u_c(y - p_c) - u_c(y - q_c)) \sum_{k>c} \exp(u_k(y - q_k))$.

In particular, I simulate a price reform for a DC-RUM with 4 alternatives (labeled 1 to 4). Table 1 contains the vectors with preference shifters ($\mathbf{a}$) and the pre-reform ($\mathbf{p}$) and post-reform ($\mathbf{q}$) prices.

Table 1: Simulation set-up

	Alternative (c)			
	1	2	3	4
preference shifter ($\mathbf{a}$)	3	2.5	3.5	4
ex-ante price ($\mathbf{p}$)	2	1.75	3	5
ex-post price ($\mathbf{q}$)	1	1.25	3.75	6.5

[Additional simulations to be added]

5.2 Results

Figures 1 – 2 and Table 2 contain the main simulation results for the conditional distribution of the CV and EV. Overall, the results suggest that the bounds developed in this paper are sufficiently informative for meaningful applied welfare analysis.

6 Empirical illustration

[To be added]

7 Conclusion

[To be added]

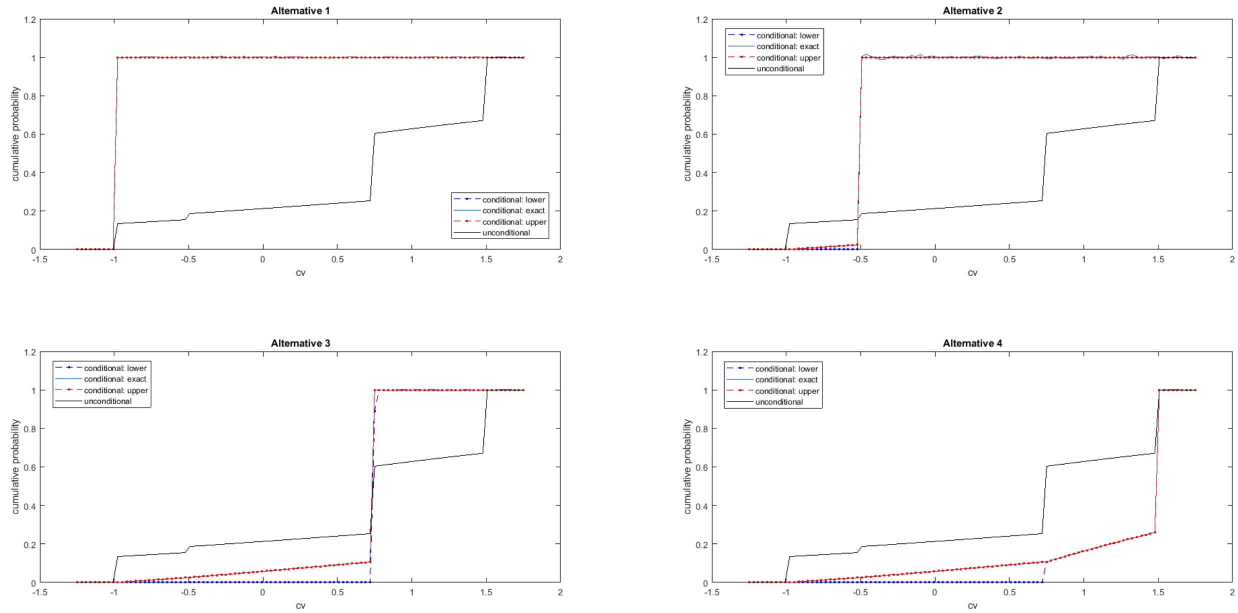

Figure 1: CDF of the compensating variation conditional on the endogenous choice

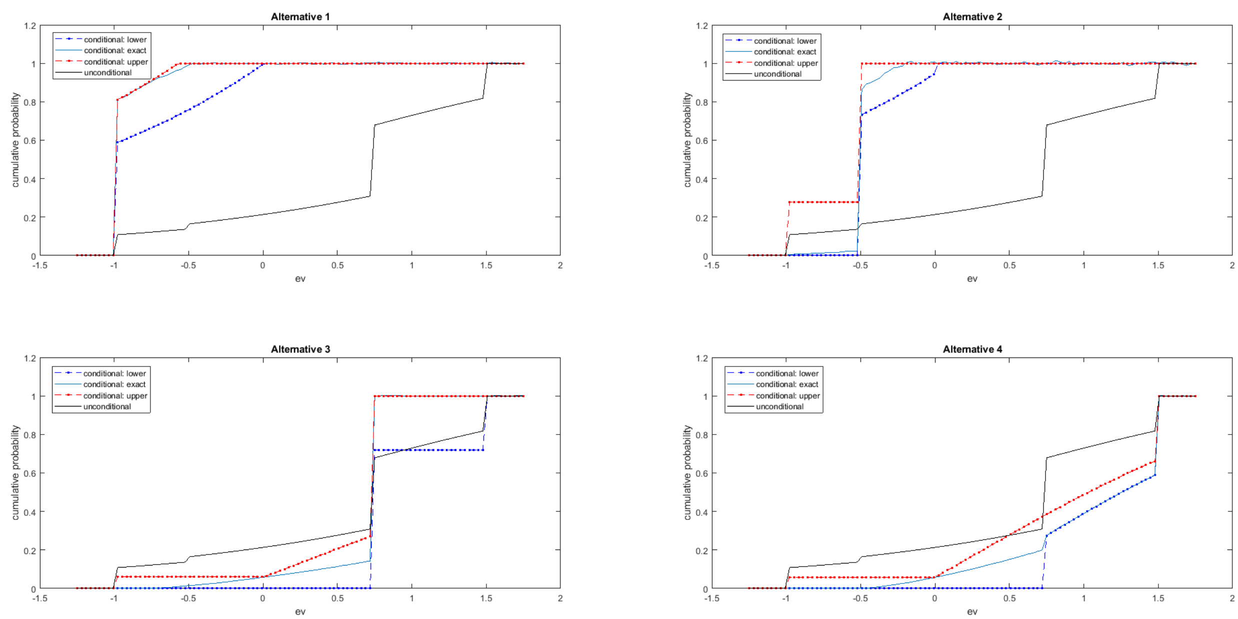

Figure 2: CDF of the equivalent variation conditional on the endogenous choice

Table 2: Main simulation results

	Chosen baseline alternative							
	1		2		3		4	
	R	NB	R	NB	R	NB	R	NB
Compensating variation								
Mean	−0.98	−0.98	−0.51	−0.51 / −0.50	0.63	0.63 / 0.72	1.24	1.24 / 1.33
Q25	−1.01	−1.01	−0.52	−0.52	0.72	0.72	1.42	1.42
Q50	−1.01	−1.01	−0.52	−0.52	0.72	0.72	1.48	1.48
Q75	−1.01	−1.01	−0.52	−0.52	0.72	0.72	1.48	1.48
Equivalent variation								
Mean	−0.94	−0.94 / −0.76	−0.49	−0.64 / −0.42	0.62	0.53 / 0.92	1.04	0.85 / 1.14
Q25	−1.01	−1.01	−0.52	−0.52 / −1.01	0.72	0.63 / 0.72	0.72	0.42 / 0.72
Q50	−1.01	−1.01	−0.52	−0.52	0.72	0.72	1.23	1.02 / 1.23
Q75	−1.01	−1.01 / −0.52	−0.52	−0.52 / −0.46	0.72	0.72 / 1.48	1.48	1.48

R: real values NB: nonparametric bounds

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A Appendix

Proof of Result 1. We first consider the compensating variation. Define $d_c \equiv q_c - p_c$ as the price difference for all $c \in \mathcal{C}$ and assume w.l.o.g. that $d_1 \leq ... \leq d_C$ and that $d_t \leq z < d_{t+1}$. From Theorem 1 in Bhattacharya (2018), it follows the subset of Ω for which the compensating variation is smaller than z can be expressed as

$$\{\omega : cv \leq z\} \iff \cup_{k=1}^{t}\Big\{\omega : U_k(y - q_k + z) = \max\big\{U_1(y - q_1 + z), ..., U_t(y - q_t + z), U_{t+1}(y - p_{t+1}), ..., U_C(y - p_C)\big\}\Big\},$$

which we abbreviate as $\cup_{k=1}^{t}\{\omega : \mathcal{W}_k^{cv}\}$ in the sequel.

The *joint distribution of the compensating variation and the ex-ante choice*, i.e. $P_{i\to\cdot}^{\text{cv}}$, is defined as $\Pr[(\cup_{k=1}^{t}\mathcal{W}_k^{cv}) \cap \mathcal{A}]$, where $\mathcal{A}$ denotes the subset of Ω for which the observed ex-ante alternative i has the highest utility,

$$\mathcal{A} = \Big\{\omega : U_i^{\omega}(y - p_i) = \max\big\{U_1^{\omega}(y - p_1), ..., U_C^{\omega}(y - p_C)\big\}\Big\}.$$

By the distributive property of sets, this joint distribution can be rewritten as

$$\begin{aligned} P_{i\to\cdot}^{cv}(\cdot) &= \Pr_{\omega}[(\cup_{k=1}^{t}\mathcal{W}_k^{cv}) \cap \mathcal{A}] \\ &= \Pr_{\omega}[\cup_{k=1}^{t}(\mathcal{W}_k^{cv} \cap \mathcal{A})] \\ &= \sum_{k=1}^{t}\Pr_{\omega}[\mathcal{W}_k^{cv} \cap \mathcal{A}]. \end{aligned}$$

It then follows immediately that this object is point identified from $P_{i\to j}$ by evaluating this transition probability function t times at a counterfactual price vector:

$$P_{i\to\cdot}^{cv}(z;\ \mathbf{p}, \mathbf{q}, y) = \sum_{k=1}^{t} P_{i\to k}(\mathbf{p};\ q_1 - z, ..., q_t - z, p_{t+1}, ..., p_C;\ y).$$

The derivation for the *joint distribution of the compensating variation and the ex-post choice* is equivalent, as one only needs to replace the set $\mathcal{A}$ by

$$\mathcal{P} = \Big\{\omega : U_j^{\omega}(y - q_j) = \max\big\{U_1^{\omega}(y - q_1), ..., U_C^{\omega}(y - q_C)\big\}\Big\},$$

which is the subset of Ω for which the observed ex-post alternative j has the highest utility. The object of interest is retrieved by

$$P_{\cdot\to j}^{cv}(z;\ \mathbf{p}, \mathbf{q}, y) = \sum_{k=1}^{t} P_{k\to j}(q_1 - z, ..., q_t - z, p_{t+1}, ..., p_C;\ \mathbf{q};\ y).$$

We now consider the equivalent variation. Again from Theorem 1 in Bhattacharya (2018), it follows that the subset of Ω for which the equivalent variation is smaller than z can be expressed as

$$\{\omega : ev \leq z\} \iff \cup_{k=1}^{t}\Big\{\omega : U_k(y - q_k) = \max\big\{U_1(y - q_1), ..., U_t(y - q_t), U_{t+1}(y - p_{t+1} - z), ..., U_C(y - p_C - z)\big\}\Big\},$$

which we abbreviate as $\cup_{k=1}^{t}\{\omega : \mathcal{W}_k^{ev}\}$. Using the similar arguments as above, the *joint distribution of the equivalent* *variation and the ex-ante or ex-post choice*, i.e. $P^{ev}_{i\to\cdot}$ or $P^{ev}_{\cdot\to j}$ respectively, can be rewritten as follows:

$$P^{ev}_{i\to\cdot}(\cdot) = \sum_{k=1}^{t} \Pr_{\omega}[\mathcal{W}^{ev}_k \cap \mathcal{A}],$$
$$P^{ev}_{\cdot\to j}(\cdot) = \sum_{k=1}^{t} \Pr_{\omega}[\mathcal{W}^{ev}_k \cap \mathcal{P}].$$

Finally, in this case, the objects of interest become

$$P^{ev}_{i\to\cdot}(z;\ \mathbf{p},\mathbf{q},y) = \sum_{k=1}^{t} P_{i\to k}(\mathbf{p};\ q_1,\ldots,q_t,p_{t+1}+z,\ldots,p_C+z;\ y),$$
$$P^{ev}_{\to j}(z;\ \mathbf{p},\mathbf{q},y) = \sum_{k=1}^{t} P_{k\to j}(q_1,\ldots,q_t,p_{t+1}+z,\ldots,p_C+z;\ \mathbf{q};\ y).$$

□

Proof of Corollary 1. We first consider the compensating variation. Note that when there exists an outside good $c_o \in \mathcal{C}$ such that $d_{c_o} = 0$, one cannot evaluate the expressions in Result 1 for values $z \geq 0$ because there is no price variation present in the data for this particular good. The issue can be circumvented by substituting the exogenous ex-post income with $y' \equiv y + z$ such that

$$P^{cv}_{i\to\cdot}(z;\ \mathbf{p},\mathbf{q},y) = \begin{cases} \sum_{k=1}^{t} P_{i\to k}(\mathbf{p};\ q_1-z,\ldots,q_t-z,p_{t+1},\ldots,p_C;\ y;\ y), & \text{if } z<0,\\ \sum_{k=1}^{t} P_{i\to k}(\mathbf{p};\ q_1,\ldots,q_t,p_{t+1}+z,\ldots,p_C+z;\ y;\ y'), & \text{if } z\geq 0,\end{cases}$$

which exploits individual level income variation. As defined in the main text, $P_{i\to j}(\mathbf{p},\mathbf{q},y_p,y_q)$ denotes the transition probability when both individual price and income variation is present in the data.

Similarly, by substituting the ex-ante income we attain that

$$P^{cv}_{\cdot\to j}(z;\ \mathbf{p},\mathbf{q},y) = \begin{cases} \sum_{k=1}^{t} P_{k\to j}(q_1-z,\ldots,q_t-z,p_{t+1},\ldots,p_C;\ \mathbf{q};\ y;\ y), & \text{if } z<0,\\ \sum_{k=1}^{t} P_{k\to j}(q_1,\ldots,q_t,p_{t+1}+z,\ldots,p_C+z;\ \mathbf{q};\ y';\ y), & \text{if } z\geq 0,\end{cases}$$

which also does not require income variation in c_0.

We now consider the equivalent variation. In this case, one cannot evaluate the expressions in Result 1 for values $z < 0$ because there is no price variation present in the data for the outside good. By substituting the exogenous ex-post or ex-ante income with $y'' \equiv y - z$, we find that

$$P^{ev}_{i\to\cdot}(z;\ \mathbf{p},\mathbf{q},y) = \begin{cases} \sum_{k=1}^{t} P_{i\to k}(\mathbf{p};\ q_1-z,\ldots,q_t-z,p_{t+1},\ldots,p_C;\ y;\ y''), & \text{if } z<0,\\ \sum_{k=1}^{t} P_{i\to k}(\mathbf{p};\ q_1,\ldots,q_t,p_{t+1}+z,\ldots,p_C+z;\ y;\ y), & \text{if } z\geq 0,\end{cases}$$

$$P^{ev}_{\cdot\to j}(z;\ \mathbf{p},\mathbf{q},y) = \begin{cases} \sum_{k=1}^{t} P_{k\to j}(q_1-z,\ldots,q_t-z,p_{t+1},\ldots,p_C;\ \mathbf{q};\ y'';\ y), & \text{if } z<0,\\ \sum_{k=1}^{t} P_{k\to j}(q_1,\ldots,q_t,p_{t+1}+z,\ldots,p_C+z;\ \mathbf{q};\ y;\ y), & \text{if } z\geq 0,\end{cases}$$

respectively.

□

Proof of Lemma 1. We first consider the case with no transitions. The transition probability $P_{i\to i}(\mathbf{p},\mathbf{r},y)$ can be further decomposed as the intersection of multiple inequalities,

$$\begin{aligned} P_{i\to i}(\mathbf{p},\mathbf{r},y) &= \Pr_{\omega}[\{\omega : U_i^{\omega}(y-p_i) = \max\{U_1(y-p_1),...,U_C(y-p_C)\}\} \\ &\qquad \cap \{\omega : U_i^{\omega}(y-r_i) = \max\{U_1(y-r_1),...,U_C(y-r_C)\}\}] \\ &= \Pr_{\omega}\left[(\cap_{c\neq i}\{\omega : U_i^{\omega}(y-p_i) > U_c^{\omega}(y-p_c)\}) \cap (\cap_{c\neq i}\{\omega : U_i^{\omega}(y-r_i) > U_c^{\omega}(y-r_c)\})\right] \\ &= \Pr_{\omega}\left[\cap_{c\neq i}\underbrace{(\{\omega : U_i^{\omega}(y-p_i) > U_c^{\omega}(y-p_c)\} \cap \{\omega : U_i^{\omega}(y-r_i) > U_c^{\omega}(y-r_c)\})}_{\mathcal{X}_c^{ii}}\right]. \end{aligned}$$

We distinguish four different subcases for the series of events $\{\mathcal{X}_c^{ii}, c \neq i \in \mathcal{C}\}$, for which we derive subsets $\{\mathcal{L}_c^{ii}, c \neq i \in \mathcal{C}\}$ and supersets $\{\mathcal{U}_c^{ii}, c \neq i \in \mathcal{C}\}$:[11]

- $p_i > r_i$ and $p_c > r_c$

  In this case, both good i and c become less expensive. By monotonicity we have that $U_i^{\omega}(y-p_i) < U_i^{\omega}(y-r_i)$ and $U_c^{\omega}(y-p_c) < U_c^{\omega}(y-r_c)$. One can then easily show that

$$\{\omega : U_i^{\omega}(y-p_i) > U_c^{\omega}(y-r_c)\} \subseteq \mathcal{X}_c^{ii} \subseteq \begin{cases} \{\omega : U_i^{\omega}(y-p_i) > U_c^{\omega}(y-p_c)\}, \\ \{\omega : U_i^{\omega}(y-r_i) > U_c^{\omega}(y-r_c)\}. \end{cases}$$

- $p_i > r_i$ and $p_c < r_c$

  In this case, good i becomes less and good c more expensive. By monotonicity we have that $U_i^{\omega}(y-p_i) < U_i^{\omega}(y-r_i)$ and $U_c^{\omega}(y-p_c) > U_c^{\omega}(y-r_c)$. One can then easily show that

$$\mathcal{X}_c^{ii} = \{\omega : U_i^{\omega}(y-p_i) > U_c^{\omega}(y-p_c)\} \subseteq \{\omega : U_i^{\omega}(y-r_i) > U_c^{\omega}(y-r_c)\}.$$

- $p_i < r_i$ and $p_c > r_c$

  In this case, good i becomes more and good c less expensive. By monotonicity we have that $U_i^{\omega}(y-p_i) > U_i^{\omega}(y-r_i)$ and $U_c^{\omega}(y-p_c) < U_c^{\omega}(y-r_c)$. One can then easily show that

$$\mathcal{X}_c^{ii} = \{\omega : U_i^{\omega}(y-r_i) > U_c^{\omega}(y-r_c)\} \subseteq \{\omega : U_i^{\omega}(y-p_i) > U_c^{\omega}(y-p_c)\}.$$

- $p_i < r_i$ and $p_c < r_c$

  In this case, both good i and c become more expensive. By monotonicity we have that $U_i^{\omega}(y-p_i) > U_i^{\omega}(y-r_i)$ and $U_c^{\omega}(y-p_c) > U_c^{\omega}(y-r_c)$. One can then easily show that

$$\{\omega : U_i^{\omega}(y-r_i) > U_c^{\omega}(y-p_c)\} \subseteq \mathcal{X}_c^{ii} \subseteq \begin{cases} \{\omega : U_i^{\omega}(y-p_i) > U_c^{\omega}(y-p_c)\}, \\ \{\omega : U_i^{\omega}(y-r_i) > U_c^{\omega}(y-r_c)\}. \end{cases}$$

Putting these results together, and exploiting that $\left(\cap_{c\neq i}\mathcal{L}_c^{ii}\right) \subseteq \left(\cap_{c\neq i}\mathcal{X}_c^{ii}\right) \subseteq \left(\cap_{c\neq i}\mathcal{U}_c^{ii}\right)$ always holds by elementary set theory, we immediately find that[12]

$$\cap_{c\neq i}\{\omega : U_i^{\omega}(y-\overline{m}(i)) > U_c^{\omega}(y-\underline{m}(c))\} \subseteq \left(\cap_{c\neq i}\mathcal{X}_c^{ii}\right) \subseteq \begin{cases} \cap_{c\neq i}\{\omega : U_i^{\omega}(y-p_i) > U_c^{\omega}(y-p_c)\}, \\ \cap_{c\neq i}\{\omega : U_i^{\omega}(y-r_i) > U_c^{\omega}(y-r_c)\}, \end{cases}$$

[11] These subsets and supersets are allowed to coincide. When this occurs, one attains point instead of set identification.

[12] Note that in Lemma 1 we defined $\underline{m}(c) = \min\{p_c, r_c\}$ and $\overline{m}(c) = \max\{p_c, r_c\}$.

such that bounds on the transition probabilities are given by

$$\Pr_{\omega}\left[\cap_{c\neq i}\{\omega : U_i^{\omega}(y-\overline{m}(i)) > U_c^{\omega}(y-\underline{m}(c))\}\right]$$

$$\leq P_{i\to i}(\mathbf{p},\mathbf{r},y) \leq$$

$$\min\left\{\Pr_{\omega}\left[\cap_{c\neq i}\{\omega : U_i^{\omega}(y-p_i) > U_c^{\omega}(y-p_c)\}\right],\ \Pr_{\omega}\left[\cap_{c\neq i}\{\omega : U_i^{\omega}(y-r_i) > U_c^{\omega}(y-r_c)\}\right]\right\}.$$

Note that both the lower and upper bound can easily be calculated from the observed choice probabilities $\{P_i(\mathbf{p};\ y), i \in \mathcal{C}\}$ evaluated at counterfactual price vectors.

We now consider the case <u>with transitions</u>. Again, the transition probabilities $\{P_{i\to j}(\mathbf{p},\mathbf{r},y);\ i,j \in \mathcal{C}\}$ can be further decomposed as the intersection of multiple inequalities,

$$\begin{aligned} P_{i\to j}(\mathbf{p},\mathbf{r},y) &= \left(\cap_{c\neq i}\{\omega : U_i^{\omega}(y-p_i) > U_c^{\omega}(y-p_c)\}\right) \cap \left(\cap_{c\neq j}\{\omega : U_j^{\omega}(y-r_j) > U_c^{\omega}(y-r_c)\}\right) \\ &= \underbrace{\left(\{\omega : U_i^{\omega}(y-p_i) > U_j^{\omega}(y-p_j)\} \cap \{\omega : U_j^{\omega}(y-r_j) > U_i^{\omega}(y-r_i)\}\right)}_{\mathcal{X}^{ij}} \\ &\quad \cap_{c\neq i,j} \underbrace{\left(\{\omega : U_i^{\omega}(y-p_i) > U_c^{\omega}(y-p_c)\} \cap \{\omega : U_j^{\omega}(y-r_j) > U_c^{\omega}(y-r_c)\}\right)}_{\mathcal{X}_c^{ij}} \end{aligned}$$

We first discuss the event $\mathcal{X}^{ij}$. Again, we distinguish four different subcases, for which we derive a subset $\mathcal{L}^{ij}$ and a superset $\mathcal{U}^{ij}$:

- $p_i > r_i$ and $p_j > r_j$

  In this case, both good i and j become less expensive. By monotonicity we have that $U_i^{\omega}(y-p_i) < U_i^{\omega}(y-r_i)$ and $U_j^{\omega}(y-p_j) < U_j^{\omega}(y-r_j)$. One can then only derive the trivial bounds

$$\emptyset \subseteq \mathcal{X}^{ij} \subseteq \begin{cases} \{\omega : U_i^{\omega}(y-p_i) > U_j^{\omega}(y-p_j)\}, \\ \{\omega : U_j^{\omega}(y-r_j) > U_i^{\omega}(y-r_i)\}. \end{cases}$$

- $p_i > r_i$ and $p_j < r_j$

  In this case, good i becomes less and good j more expensive. By monotonicity we have that $U_i^{\omega}(y-p_i) < U_i^{\omega}(y-r_i)$ and $U_j^{\omega}(y-p_j) > U_j^{\omega}(y-r_j)$. One can then easily show that this event cannot occur, such that

$$\mathcal{X}^{ij} = \emptyset.$$

- $p_i < r_i$ and $p_j > r_j$

  In this case, good i becomes more and good j less expensive. By monotonicity we have that $U_i^{\omega}(y-p_i) > U_i^{\omega}(y-r_i)$ and $U_j^{\omega}(y-p_j) < U_j^{\omega}(y-r_j)$. One can then only derive the trivial bounds

$$\emptyset \subseteq \mathcal{X}^{ij} \subseteq \begin{cases} \{\omega : U_i^{\omega}(y-p_i) > U_j^{\omega}(y-p_j)\}, \\ \{\omega : U_j^{\omega}(y-r_j) > U_i^{\omega}(y-r_i)\}. \end{cases}$$

- $p_i < r_i$ and $p_j < r_j$

  In this case, both good i and j become more expensive. By monotonicity we have that $U_i^{\omega}(y-p_i) > U_i^{\omega}(y-r_i)$ and $U_j^{\omega}(y-p_j) > U_j^{\omega}(y-r_j)$. One can then only derive the trivial bounds

$$\emptyset \subseteq \mathcal{X}^{ij} \subseteq \begin{cases} \{\omega : U_i^{\omega}(y-p_i) > U_j^{\omega}(y-p_j)\}, \\ \{\omega : U_j^{\omega}(y-r_j) > U_i^{\omega}(y-r_i)\}. \end{cases}$$

Now consider the events $\{\mathcal{X}_c^{ij};\ c \neq i,j \in \mathcal{C}\}$. Again, revealed preferences do not seem to offer lower bounds $\{\mathcal{L}_c^{ij};\ c \neq i,j \in \mathcal{C}\}$ for these events. Upper bounds $\{\mathcal{U}_c^{ij};\ c \neq i,j \in \mathcal{C}\}$, however, can always be derived for any intersection. It therefore holds that

$$\emptyset \subseteq \mathcal{X}_c^{ij} \subseteq \begin{cases} \{\omega : U_i^\omega(y-p_i) > U_c^\omega(y-p_c)\}, \\ \{\omega : U_j^\omega(y-r_j) > U_c^\omega(y-r_c)\}. \end{cases}$$

for every $c \neq i,j \in \mathcal{C}$. Putting these results together, and exploiting that $\left(\mathcal{L}^{ij} \cap_{c\neq i,j} \mathcal{L}_c^{ij}\right) \subseteq \left(\mathcal{X}^{ij} \cap_{c\neq i,j} \mathcal{X}_c^{ij}\right) \subseteq \left(\mathcal{U}^{ij} \cap_{c\neq i,j} \mathcal{U}_c^{ij}\right)$, we immediately find that

$$\emptyset \subseteq \left(\mathcal{X}^{ij} \cap_{c\neq i,j} \mathcal{X}_c^{ij}\right) \subseteq \begin{cases} \cap_{c\neq i}\{\omega : U_i^\omega(y-p_i) > U_c^\omega(y-p_c)\}, \\ \cap_{c\neq j}\{\omega : U_j^\omega(y-r_j) > U_c^\omega(y-r_c)\}, \end{cases}$$

such that bounds on the transition probabilities are given by

$$\begin{gathered} 0 \\ \leq P_{i\to j}(\mathbf{p},\mathbf{r},y) \leq \\ \min\left\{\Pr_\omega\left[\cap_{c\neq i}\{\omega : U_i^\omega(y-p_i) > U_c^\omega(y-p_c)\}\right],\ \Pr_\omega\left[\cap_{c\neq j}\{\omega : U_j^\omega(y-r_j) > U_c^\omega(y-r_c)\}\right]\right\}. \end{gathered}$$

□

Proof of Corollary 2. The proof is similar to that of Corollary 1.

□

Proof of Result 2. The optimization program follows immediately from combining Result 1 and Lemma 1.

Since the constraints jointly require that possible solutions $\{a_c(z);\ c \in \mathcal{C}\}$ belong to a convex subspace in $\mathbb{R}^C$, it follows from the linearity of the objective function that its image $\sum_{k=1}^t a_k(z)$ is also convex. Therefore, all values between the lower and upper bound are feasible for a given value of z.

□

Proof of Result 3. This follows readily from the proof of Result 1 by noting that

$$\begin{aligned} P_{i\to j}^{cv/ev}(\cdot) &= \Pr_\omega[(\cup_{k=1}^t \mathcal{W}_k^{cv/ev}) \cap \mathcal{A} \cap \mathcal{P}] \\ &= \Pr_\omega[\cup_{k=1}^t (\mathcal{W}_k^{cv/ev} \cap \mathcal{A} \cap \mathcal{P})] \\ &= \sum_{k=1}^t \Pr_\omega[\mathcal{W}_k^{cv/ev} \cap \mathcal{A} \cap \mathcal{P}], \end{aligned}$$

which is point identified from the set of double transition probabilities $\{P_{i\to j\to k};\ i,j,k \in \mathcal{C}\}$, evaluated at a counterfactual price vector. In particular we have that

$$P_{i\to j}^{cv}(z;\ \mathbf{p},\mathbf{q},y) = \sum_{k=1}^t P_{i\to j\to k}(\mathbf{p};\ \mathbf{q};\ q_1 - z, ..., q_t - z, p_{t+1}, ..., p_C;\ y),$$

for the compensating variation, and

$$P_{i\to j}^{ev}(z;\ \mathbf{p},\mathbf{q},y) = \sum_{k=1}^t P_{i\to j\to k}(\mathbf{p};\ \mathbf{q};\ q_1, ..., q_t, p_{t+1} + z, ..., p_C + z;\ y)$$

for the equivalent variation.

□

Proof of Lemma 2. We first consider the case with no transitions. The double transition probability, $P_{i\to i\to i}$ can be further decomposed as the intersection of multiple inequalities,

$$\begin{aligned}
P_{i\to i\to i} &= \Pr_{\omega}[(\cap_{c\neq i}\{\omega : U_i^{\omega}(y-p_i) > U_c^{\omega}(y-p_c)\}) \cap (\cap_{c\neq i}\{\omega : U_i^{\omega}(y-q_i) > U_c^{\omega}(y-q_c)\}) \\
&\qquad \cap (\cap_{c\neq i}\{\omega : U_i^{\omega}(y-r_i) > U_c^{\omega}(y-r_c)\})] \\
&= \Pr_{\omega}[((\cap_{c\neq i}\{\omega : U_i^{\omega}(y-p_i) > U_c^{\omega}(y-p_c)\}) \cap (\cap_{c\neq i}\{\omega : U_i^{\omega}(y-r_i) > U_c^{\omega}(y-r_c)\})) \\
&\qquad \cap ((\cap_{c\neq i}\{\omega : U_i^{\omega}(y-q_i) > U_c^{\omega}(y-q_c)\}) \cap (\cap_{c\neq i}\{\omega : U_i^{\omega}(y-r_i) > U_c^{\omega}(y-r_c)\}))] \\
&= \Pr_{\omega}[\left(\cap_{c\neq i}\mathcal{X}_c^{ii}(p,r)\right) \cap \left(\cap_{c\neq i}\mathcal{X}_c^{ii}(q,r)\right)],
\end{aligned}$$

in which the last equality follows from Lemma 1. By using elementary set theory, it holds that

$$\left(\cap_{c\neq i}\mathcal{L}_c^{ii}(p)\right) \cap \left(\cap_{c\neq i}\mathcal{L}_c^{ii}(q)\right) \subseteq \underbrace{\left(\cap_{c\neq i}\mathcal{X}_c^{ii}(p)\right) \cap \left(\cap_{c\neq i}\mathcal{X}_c^{ii}(q)\right)}_{\mathcal{X}_c^{iii}} \subseteq \left(\cap_{c\neq i}\mathcal{U}_c^{ii}(p)\right) \cap \left(\cap_{c\neq i}\mathcal{U}_c^{ii}(q)\right),$$

such that we attain[13]

$$\mathcal{X}_c^{iii} \supseteq \left(\cap_{c\neq i}\{\omega : U_i^{\omega}(y-\overline{m}_p(i)) > U_c^{\omega}(y-\underline{m}_p(c))\}\right) \cap \left(\cap_{c\neq i}\{\omega : U_i^{\omega}(y-\overline{m}_q(i)) > U_c^{\omega}(y-\underline{m}_q(c))\}\right),$$

for the subset, and

$$\mathcal{X}_c^{iii} \subseteq \begin{cases}
(\cap_{c\neq i}\{\omega : U_i^{\omega}(y-p_i) > U_c^{\omega}(y-p_c)\}) \cap (\cap_{c\neq i}\{\omega : U_i^{\omega}(y-q_i) > U_c^{\omega}(y-q_c)\}), \\
(\cap_{c\neq i}\{\omega : U_i^{\omega}(y-p_i) > U_c^{\omega}(y-p_c)\}) \cap (\cap_{c\neq i}\{\omega : U_i^{\omega}(y-r_i) > U_c^{\omega}(y-r_c)\}), \\
(\cap_{c\neq i}\{\omega : U_i^{\omega}(y-q_i) > U_c^{\omega}(y-q_c)\}) \cap (\cap_{c\neq i}\{\omega : U_i^{\omega}(y-r_i) > U_c^{\omega}(y-r_c)\}), \\
(\cap_{c\neq i}\{\omega : U_i^{\omega}(y-r_i) > U_c^{\omega}(y-r_c)\}) \cap (\cap_{c\neq i}\{\omega : U_i^{\omega}(y-r_i) > U_c^{\omega}(y-r_c)\}),
\end{cases}$$

for the superset. By symmetry, we can also decompose the events $\{\mathcal{X}_c^{iii}\}$ as the intersection of $\{\mathcal{X}_c^{ii}(q,p) \cap \mathcal{X}_c^{ii}(r,p)\}$ or $\{\mathcal{X}_c^{ii}(p,q) \cap \mathcal{X}_c^{ii}(r,p)\}$, which yields additional subsets,

$$\mathcal{X}_c^{iii} \supseteq \begin{cases}
\left(\cap_{c\neq i}\{\omega : U_i^{\omega}(y-\overline{m}_{p,r}(i)) > U_c^{\omega}(y-\underline{m}_{p,r}(c))\}\right) \cap \left(\cap_{c\neq i}\{\omega : U_i^{\omega}(y-\overline{m}_{q,r}(i)) > U_c^{\omega}(y-\underline{m}_{q,r}(c))\}\right), \\
\left(\cap_{c\neq i}\{\omega : U_i^{\omega}(y-\overline{m}_{p,q}(i)) > U_c^{\omega}(y-\underline{m}_{p,q}(c))\}\right) \cap \left(\cap_{c\neq i}\{\omega : U_i^{\omega}(y-\overline{m}_{p,r}(i)) > U_c^{\omega}(y-\underline{m}_{p,r}(c))\}\right), \\
\left(\cap_{c\neq i}\{\omega : U_i^{\omega}(y-\overline{m}_{p,q}(i)) > U_c^{\omega}(y-\underline{m}_{p,q}(c))\}\right) \cap \left(\cap_{c\neq i}\{\omega : U_i^{\omega}(y-\overline{m}_{q,r}(i)) > U_c^{\omega}(y-\underline{m}_{q,r}(c))\}\right),
\end{cases}$$

and supersets,

$$\mathcal{X}_c^{iii} \subseteq \begin{cases}
(\cap_{c\neq i}\{\omega : U_i^{\omega}(y-p_i) > U_c^{\omega}(y-p_c)\}) \cap (\cap_{c\neq i}\{\omega : U_i^{\omega}(y-q_i) > U_c^{\omega}(y-q_c)\}), \\
(\cap_{c\neq i}\{\omega : U_i^{\omega}(y-p_i) > U_c^{\omega}(y-p_c)\}) \cap (\cap_{c\neq i}\{\omega : U_i^{\omega}(y-r_i) > U_c^{\omega}(y-r_c)\}), \\
(\cap_{c\neq i}\{\omega : U_i^{\omega}(y-q_i) > U_c^{\omega}(y-q_c)\}) \cap (\cap_{c\neq i}\{\omega : U_i^{\omega}(y-r_i) > U_c^{\omega}(y-r_c)\}), \\
(\cap_{c\neq i}\{\omega : U_i^{\omega}(y-p_i) > U_c^{\omega}(y-p_c)\}) \cap (\cap_{c\neq i}\{\omega : U_i^{\omega}(y-p_i) > U_c^{\omega}(y-p_c)\}), \\
(\cap_{c\neq i}\{\omega : U_i^{\omega}(y-q_i) > U_c^{\omega}(y-q_c)\}) \cap (\cap_{c\neq i}\{\omega : U_i^{\omega}(y-q_i) > U_c^{\omega}(y-q_c)\}), \\
(\cap_{c\neq i}\{\omega : U_i^{\omega}(y-r_i) > U_c^{\omega}(y-r_c)\}) \cap (\cap_{c\neq i}\{\omega : U_i^{\omega}(y-r_i) > U_c^{\omega}(y-r_c)\}),
\end{cases}$$

[13]Note that in Lemma 2 we defined $\underline{m}_x(c) = \min\{x_c, r_c\}$ and $\overline{m}_x(c) = \max\{x_c, r_c\}$.

From this, we immediately get the following probability bounds

$$\max\left\{\begin{array}{l}\Pr_\omega[\big(\cap_{c\neq i}\{\omega: U_i^\omega(y-\overline{m}_{p,r}(i)) > U_c^\omega(y-\underline{m}_{p,r}(c))\}\big)\cap\big(\cap_{c\neq i}\{\omega: U_i^\omega(y-\overline{m}_{q,r}(i)) > U_c^\omega(y-\underline{m}_{q,r}(c))\}\big)],\\ \Pr_\omega[\big(\cap_{c\neq i}\{\omega: U_i^\omega(y-\overline{m}_{p,q}(i)) > U_c^\omega(y-\underline{m}_{p,q}(c))\}\big)\cap\big(\cap_{c\neq i}\{\omega: U_i^\omega(y-\overline{m}_{p,r}(i)) > U_c^\omega(y-\underline{m}_{p,r}(c))\}\big)],\\ \Pr_\omega[\big(\cap_{c\neq i}\{\omega: U_i^\omega(y-\overline{m}_{p,q}(i)) > U_c^\omega(y-\underline{m}_{p,q}(c))\}\big)\cap\big(\cap_{c\neq i}\{\omega: U_i^\omega(y-\overline{m}_{q,r}(i)) > U_c^\omega(y-\underline{m}_{q,r}(c))\}\big)]\end{array}\right\}$$

$$\leq P_{i\to i\to i}(\mathbf{p},\mathbf{q},\mathbf{r},y)\leq$$

$$\min\left\{\begin{array}{l}\Pr_\omega[(\cap_{c\neq i}\{\omega: U_i^\omega(y-p_i) > U_c^\omega(y-p_c)\})\cap(\cap_{c\neq i}\{\omega: U_i^\omega(y-q_i) > U_c^\omega(y-q_c)\})],\\ \Pr_\omega[(\cap_{c\neq i}\{\omega: U_i^\omega(y-p_i) > U_c^\omega(y-p_c)\})\cap(\cap_{c\neq i}\{\omega: U_i^\omega(y-r_i) > U_c^\omega(y-r_c)\})],\\ \Pr_\omega[(\cap_{c\neq i}\{\omega: U_i^\omega(y-q_i) > U_c^\omega(y-q_c)\})\cap(\cap_{c\neq i}\{\omega: U_i^\omega(y-r_i) > U_c^\omega(y-r_c)\})],\\ \Pr_\omega[(\cap_{c\neq i}\{\omega: U_i^\omega(y-p_i) > U_c^\omega(y-p_c)\})\cap(\cap_{c\neq i}\{\omega: U_i^\omega(y-p_i) > U_c^\omega(y-p_c)\})],\\ \Pr_\omega[(\cap_{c\neq i}\{\omega: U_i^\omega(y-q_i) > U_c^\omega(y-q_c)\})\cap(\cap_{c\neq i}\{\omega: U_i^\omega(y-q_i) > U_c^\omega(y-q_c)\})],\\ \Pr_\omega[(\cap_{c\neq i}\{\omega: U_i^\omega(y-r_i) > U_c^\omega(y-r_c)\})\cap(\cap_{c\neq i}\{\omega: U_i^\omega(y-r_i) > U_c^\omega(y-r_c)\})]\end{array}\right\},$$

which are themselves point identified from the transition probability $P_{i\to i}(\cdot)$, evaluated at alternative price vectors.

We now consider the case <u>with transitions</u>. Again, the double transition probabilities $\{P_{i\to j\to k}(\mathbf{p},\mathbf{q},\mathbf{r},y);\ i,j,k\in\mathcal{C}\}$ can be further decomposed as the intersection of multiple inequalities,

$$\begin{aligned}P_{i\to j\to i} &= \Pr_\omega[(\cap_{c\neq i}\{\omega: U_i^\omega(y-p_i) > U_c^\omega(y-p_c)\})\cap\big(\cap_{c\neq j}\{\omega: U_j^\omega(y-q_j) > U_c^\omega(y-q_c)\}\big)\\ &\qquad\cap(\cap_{c\neq i}\{\omega: U_i^\omega(y-r_i) > U_c^\omega(y-r_c)\})]\\ &= \Pr[\underbrace{(\{\omega: U_i^\omega(y-p_i) > U_j^\omega(y-p_j)\}\cap\{\omega: U_j^\omega(y-q_j) > U_i^\omega(y-q_i)\}\cap\{\omega: U_i^\omega(y-r_i) > U_j^\omega(y-r_j)\})}_{\mathcal{X}^{iji}}\\ &\qquad\cap\underbrace{((\cap_{c\neq i,j}\{\omega: U_i^\omega(y-p_i) > U_c^\omega(y-p_c)\})\cap(\cap_{c\neq i,j}\{\omega: U_j^\omega(y-q_j) > U_c^\omega(y-q_c)\}))}_{\mathcal{X}_c^{ij}(p,q)}\\ &\qquad\cap\underbrace{((\cap_{c\neq i,j}\{\omega: U_i^\omega(y-r_i) > U_c^\omega(y-r_c)\})\cap(\cap_{c\neq i,j}\{\omega: U_j^\omega(y-q_j) > U_c^\omega(y-q_c)\}))}_{\mathcal{X}_c^{ij}(r,q)}]\end{aligned}$$

[To be added]

Now consider the events for $\{\mathcal{X}_c^{ijk};\ c\neq i,j,k\in\mathcal{C}\}$. Again, revealed preferences do not seem to offer lower bounds $\{\mathcal{L}_c^{ijk};\ c\neq i,j,k\in\mathcal{C}\}$ for these events. Upper bounds $\{\mathcal{U}_c^{ijk};\ c\neq i,j,k\in\mathcal{C}\}$, however, can always be derived for any intersection. It therefore holds that

$$\emptyset\subseteq\mathcal{X}_c^{ijk}\subseteq\left\{\begin{array}{l}\{\omega: U_i^\omega(y-p_i) > U_c^\omega(y-p_c)\},\\ \{\omega: U_j^\omega(y-q_j) > U_c^\omega(y-q_c)\},\\ \{\omega: U_k^\omega(y-r_k) > U_c^\omega(y-r_c)\},\end{array}\right.$$

for every $c\neq i,j,k\in\mathcal{C}$. [To be added]

□

Proof of Result 4. This follows immediately from combining Result 3 and Lemma 2.

□

Proof of Proposition 1. A well-known implication of Fubini's theorem is that the mean of any random variable X can be directly derived from its cumulative density function $F_X(\cdot)$, i.e.

$$\mathbb{E}(X) = \int_0^{\infty} (1 - F_X(u))du - \int_{-\infty}^{0} F_X(u)du.$$

This result allows us to calculate average welfare for any feasible conditional distribution $FP^{cv/ev}_{|i \to \cdot}(z;\ \cdot)$ of the compensating and equivalent variation. Note, however, that from the definition of the lower and upper bounds, it always holds that $LB^{cv/ev}_{|i \to \cdot}(z;\ \cdot) \leq FP^{cv/ev}_{|i \to \cdot}(z;\ \cdot) \leq UB^{cv/ev}_{|i \to \cdot}(z;\ \cdot)$ for every $z \in \mathcal{Z}$, such that $\int LB^{cv/ev}_{|i \to \cdot}(u;\ \cdot)du \leq \int FB^{cv/ev}_{|i \to \cdot}(u;\ \cdot)du \leq \int UB^{cv/ev}_{|i \to \cdot}(u;\ \cdot)du$. It then follows immediately that average welfare is bounded from below and above as stated in Proposition 1.

□