Risk-Sensitive Linear Approximations*

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Abstract

I construct risk-sensitive approximations of policy functions of DSGE models around the stochastic steady state and ergodic mean that are linear in the state variables. The method requires only the solution of linear equations using standard perturbation output to construct the approximation and is uniformly more accurate than standard linear approximations. In an application to real business cycles with recursive utility and growth risk, the approximation successfully estimates risk aversion using the Kalman filter, where a standard linear approximation provides no information and alternative methods require computationally intensive procedures such as particle filters. At the posterior mode, the model's market price of risk is brought in line with the postwar US Sharpe ratio without compromising the fit of the macroeconomy.

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1 Introduction

What are the effects of risk? There is a consensus that standard linear approximations around the deterministic steady state are insufficient to address this question satisfactorily in a variety of settings, such as for conditional asset pricing, under recursive utility, for welfare comparisons, etc., where precautionary motives play a significant role in economic decision making.[1] At fault is the certainty equivalence of standard linear approximations around the deterministic steady state that makes them invariant to the higher moments of the distribution of exogenous shocks (i.e., to risk).

I reconcile the linear framework with risk by constructing approximations of the policy functions of DSGE models that are linear in states but that account for risk in the points and slopes used to construct the linear approximation; I call these risk-sensitive linear approximations.[2] I construct two different such approximate solutions, one around the stochastic steady state and one around the ergodic mean. The method can be used profitably in estimation. Due to the linearity in states and under the assumption of normally distributed shocks, the Kalman filter is operational for the risk-sensitive linear approximation. I find the risk-sensitive linear approximation using the Kalman filter is equally successful as standard perturbation particle filter[3] estimation—both with the state space and nonlinear moving average policy function representations[4]—in identifying parameters outside the reach of standard linear approximations. The advantage, then, is that the risk-sensitive linear approximation, by employing the Kalman filter, is several orders of magnitude faster and is not subject to the sampling variation that the particle filter faces when identification is weak.

This method is only valuable if it can be fruitfully applied when conventional (log-linear) meth-

[1]Kim and Kim (2003) provide an insightful example, where blind application of linear approximations leads to the spurious results that autarky is preferred by risk averse agents over risk sharing.

[2]I construct only risk-sensitive linear approximations and not second, third, or higher order risk-sensitive approximations. The method uses some derivatives calculated at the deterministic steady state from a given order and sacrifices others to maintain linearity in states while rigorously accounting for the nonlinearities associated with risk. It is less clear what the gain over a standard perturbation would be for higher order risk-sensitive approximations.

[3]See Fernández-Villaverde and Rubio-Ramírez (2007) and Fernández-Villaverde, Guerrón-Quintana, Rubio-Ramírez, and Uribe (2011) for details on particle filtering in DSGE models as well as applications to risk.

[4]See Jin and Judd (2002), Schmitt-Grohé and Uribe (2004), Kim, Kim, Schaumburg, and Sims (2008), Lombardo (2010) and Lan and Meyer-Gohde (2013c)

ods are inadequate. Thus, I apply the method to a real business cycle model with risk sensitivity, using recursive preferences (Epstein and Zin 1989, Weil 1990), when long run risk (Bansal and Yaron 2004) is present. The risk-sensitive linear approximations match the stochastic steady state, ergodic mean, and impulse responses reported in previous nonlinear studies.[5] On the basis of Euler equation errors, I find the risk-sensitive linear approximation uniformly improves the accuracy of the standard linear approximation. As risk aversion is increased, allowing risk to play a greater role in the utility maximization problem, the risk-sensitive linear approximation demonstrates accuracy in the vicinity of the stochastic steady state and ergodic mean that is comparable to second and third order perturbations. I find that US post war data on consumption, output, and excess retruns leads the likelihood function to favor higher levels of risk aversion with the posterior mode at about 30. The market price of risk of the risk-sensitive posterior mode is about 0.2, in line with the sample spread of the NYSE weighted portfolio over 3 month T-Bills. The likelihood function, however, is rather flat in the dimension of risk aversion and fully nonlinear approximations which employ the particle filter suffer from sampling variation that impedes reliable inference and are four orders of magnitude slower; for standard linear approximations, the likelihood function is entirely flat in the dimension of risk aversion and the posterior is identical to the prior.

The method I propose is not the first to explicitly incorporate a risk adjustment,[6] but it is different in two important respects. First, it works solely with the derivatives obtained from a standard perturbation and known moments of the exogenous process—no reevaluation of derivatives or recalculation of policy rules are required to construct the approximation. The resulting equations are

[5] The stochastic steady state derived here is identical to that of Lan and Meyer-Gohde (2013b), the ergodic mean identical to that of Lan and Meyer-Gohde (2013a) and Andreasen, Fernández-Villaverde, and Rubio-Ramírez (2013), and the impulse responses to those that would result from the method of Lan and Meyer-Gohde (2013c)

[6] Kim and Kim (2003) as well as Collard and Juillard (2001b) and Collard and Juillard (2001a) are early DSGE bias reduction or risk correction techniques. Coeurdacier, Rey, and Winant (2011) uses a second order approximation to the equilibrium conditions to solve for the stochastic steady state in a portfolio problem, de Groot (2013) extends this to general settings as a matrix quadratic problem. Juillard (2011) and Kliem and Uhlig (2013) use iterative techniques, solving for implied stochastic steady states given an approximated solution and then recalculating the approximation at the new implied stochastic steady state. Evers (2012) solves for the stochastic steady state implied by a risk perturbation of the equilibrium conditions and then solves for a perturbation in the states of these perturbed equilibrium conditions.

linear in the unknown coefficients of my approximation, entirely avoiding fixed point or other recursive algorithms with unknown convergence properties. Second, I can construct the approximation around the ergodic mean as well as around the stochastic steady state—competing methods can provide only the latter. Both of these two features are accomplished by working implicitly with the unknown policy function instead of the model equilibrium conditions.

The remainder of the paper is organized as follows. In section 2, I lay out the model class and assumptions underlying the local, risk corrected procedure behind risk-sensitive linear approximations before I derive the approximations in section 3. A real business cycle model with recursive preferences and long run risk serves as the laboratory for studying and illustrating my solution method in section 4. I analyze two calibrated versions of the real business cycle model in terms of accuracy in section 5. In section 6, I assess the likelihood properties of the risk-sensitive linear approximation relative to particle filters and standard linearizations and estimate risk aversion and long run risk using post war US data. Section 7 concludes.

2 DSGE Model: Assumptions and Local Approximation

2.1 Generic Model Description with Parameterized Risk

I will analyze a family of discrete-time rational expectations models given by

$$0 = E_t[f(y_{t+1}, y_t, y_{t-1}, \sigma\varepsilon_t)] \tag{1}$$

$f : \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_e} \to \mathbb{R}^{n_y}$ is a n_y-dimensional vector-valued function collecting the equilibrium conditions that describe the model; $y_t \in \mathbb{R}^{n_y}$ is the vector of n_y endogenous and exogenous variables;[7] and $\varepsilon_t \in \mathbb{R}^{n_e}$ the vector of n_e exogenous shocks, where n_y and n_e are positive integers ($n_y, n_e \in \mathbb{N}$).

The auxiliary parameter $\sigma \in \mathbb{R}$ scales the risk in the model. The stochastic model (1) corresponds to $\sigma = 1$ and the deterministic version to $\sigma = 0$. Indexing solutions by σ yields

$$y_t = g(y_{t-1}, \sigma\varepsilon_t, \sigma), \quad y : \mathbb{R}^{n_y} \times \mathbb{R}^{n_e} \times \mathbb{R} \to \mathbb{R}^{n_y} \tag{2}$$

[7] Nonlinearity or serial correlation in exogenous processes can be captured by including the processes themselves in the vector y_t and including functions in f that specify the nonlinearity or correlation pattern.

That σ scales risk can be seen by expressing the expectations operator explicitly

$$0 = \int_{\Omega} f\left(y_{t+1}, y_t, y_{t-1}, \sigma\varepsilon_t\right) \phi\left(\varepsilon_{t+1}\right) d\varepsilon_{t+1} = \int_{\Omega} f\left(y_{t+1}, y_t, y_{t-1}, \tilde{\varepsilon}_t\right) \phi\left(\frac{\tilde{\varepsilon}_{t+1}}{\sigma}\right) d\tilde{\varepsilon}_{t+1} \tag{3}$$

where Ω is the support and ϕ the probability density function of ε_{t+1}, and $\tilde{\varepsilon}_t \doteq \sigma\varepsilon_t$. Letting σ go to zero collapses the entire distribution $\phi(\cdot)$, and this can be rewritten as

$$0 = \lim_{\sigma\to 0} \int_{\Omega} f\left(y_{t+1}, y_t, y_{t-1}, \tilde{\varepsilon}_t\right) \phi\left(\frac{\tilde{\varepsilon}_{t+1}}{\sigma}\right) d\tilde{\varepsilon}_{t+1} = f\left(y_{t+1}, y_t, y_{t-1}, 0\right) \tag{4}$$

the deterministic counterpart of (1).

2.2 Local Approximations and Points of Expansion

The deterministic steady state $\bar{y}$, defined subsequently in (5), is recovered by solving the for a fixed point of (4), the deterministic version of (1) in the absence of risk and shocks, $0 = f\left(\bar{y}, \bar{y}, \bar{y}, 0\right)$.

Definition 2.1. *Deterministic Steady State*

Let $\bar{y}^{det} \in \mathbb{R}^{n_y}$ define a fixed point of (2) given by

$$\bar{y}^{det} = g(\bar{y}^{det}, 0, 0) \tag{5}$$

i.e., a fixed point of (2) in the absence of both risk ($\sigma = 0$) and shocks ($\varepsilon_t = 0$).

I make the following assumptions on the functions f and g and the moments of ε_t

Assumption 2.2.

1. *Local Analyticity: the functions f in (1) and g in (2) are locally analytic around the deterministic steady state ($y_{t-1} = \bar{y}$, $\varepsilon_t = 0$, $\sigma = 0$) with a domain of convergence that contains the stochastic steady state and ergodic mean.*

2. *Local Stability: the eigenvalues of g_y evaluated at the deterministic steady state ($y_{t-1} = \bar{y}$, $\varepsilon_t = 0$, $\sigma = 0$) are all inside the unit circle.*

3. *Exogenous Moments: the elements of ε_t are i.i.d. with $E\left[\varepsilon_t\right] = 0$ and $E\left[\varepsilon_t^{\otimes[m]}\right]$ finite $\forall m \leq M$.*[8]

The first assumption ensures that the functions involved are smooth at least in the region of interest and that the true policy function has an infinite order Taylor series representation that remains

[8] $\varepsilon_t^{\otimes[m]}$ is the m-fold Kronecker product of ε_t with itself: $\varepsilon_t \otimes \varepsilon_t \cdots \otimes \varepsilon_t$, m times.

valid around the stochastic steady state and ergodic mean (both to be defined shortly) as well as the deterministic steady state. The second that the solution is locally stable at the deterministic steady state. The third that that the exogenous process is defined at least out to the order of approximation.[9]

Standard DSGE perturbation constructs a Taylor series expansion of the locally analytic policy function, (2), up to some, say M-th, order around the deterministic steady state, given by[10]

$$y_t \approx \sum_{j=0}^{M} \frac{1}{j!}\left[\sum_{i=0}^{M-j} \frac{1}{i!} g_{z^j\sigma^i}\sigma^i\right](z_t - \overline{z})^{\otimes[j]} \tag{6}$$

where

$$g_{z^j\sigma^i} \doteq \mathscr{D}^{j+i}_{z^j_{t-1}\sigma^i}\{y(\sigma, z_t)\} \in \mathbb{R}^{n_y \times n_z^j}, \text{ with } n_z = n_y + n_\varepsilon \tag{7}$$

is the partial derivative of the vector function y with respect to the state vector $z_t \doteq \begin{bmatrix} y'_{t-1} & \sigma\varepsilon'_t \end{bmatrix}'$ j times and the perturbation parameter σ i times evaluated at the deterministic steady state.[11]

The deterministic steady state is the fixed point for the deterministic, $\sigma = 0$, model but not the stochastic, $\sigma = 1$, model. This manifests itself in the Taylor series (6) as the constant terms $\sum_{i=1}^{M} \frac{1}{i!} g_{\sigma^i}\sigma^i$ that move the Taylor series away from the deterministic steady state. The deterministic steady state is neither a fixed point of the stochastic model nor of the Taylor series approximation of the stochastic, $\sigma = 1$, model for $M \geq 2$, when σ corrects for the second moment of risk.

The stochastic steady state incorporates risk and is the steady state of the stochastic model,

Definition 2.3. *Stochastic Steady State*

Let $\overline{y}^{stoch} \in \mathbb{R}^{n_y}$ *define a fixed point of (2) given by*

$$\overline{y}^{stoch} = g(\overline{y}^{stoch}, 0, 1) \tag{8}$$

i.e., a fixed point of (2) in the presence of risk ($\sigma = 1$) *but in the absence of shocks* ($\varepsilon_t = 0$)*.*

That is, the fixed point in the state space in the absence of shocks, but while expecting future

[9] Jin and Judd (2002) would additionally require bounded support for the exogenous process. Kim, Kim, Schaumburg, and Sims (2008) offer skepticism regarding the necessity of a boundedness assumption.

[10] See Lan and Meyer-Gohde (2014) for a derivation of this multivariate Taylor series approximation.

[11] The notation is outlined in detail in appendix A.1. As the perturbation parameter also scales ε_t, I should say this is the partial derivative with respect to the third argument but I choose not to do so as not to overload the notation. The complete, direct and indirect, derivative of y_t with respect to σ, $\mathscr{D}_\sigma\{y_t\}$, in my notation is given by $\mathscr{D}_\sigma\{y_t\} = y_\varepsilon\varepsilon_t + y_\sigma$

shocks with a known probability distribution.[12] Solving for an approximation of the stochastic steady state is not trivial using the state space formulation of the policy function, see Coeurdacier, Rey, and Winant (2011), Juillard (2011), and de Groot (2013). The difficulty arises as $\bar{y}^{stoch}$ is defined only implicitly in (8) and the method of solving $0 = f(\bar{y}, \bar{y}, \bar{y}, 0)$ to recover the deterministic steady state is not available as the presence of risk requires the integral over the probability distribution of future shocks embodied by the expectations operator be maintained in (1).

Alternatively, assumption 2.2 validates the nonlinear moving average representation of the policy function that results upon inverting or recursively substitution the state space policy function (2).[13] Approximated out to some, say M-th, order around the deterministic steady state as[14]

$$y_t \approx \sum_{m=0}^{M} \frac{1}{m!} \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_m=0}^{\infty} \left[\sum_{n=0}^{M-m} \frac{1}{n!} y_{\sigma^n i_1 i_2 \cdots i_m} \right] (\varepsilon_{t-i_1} \otimes \varepsilon_{t-i_2} \otimes \cdots \otimes \varepsilon_{t-i_m}) \tag{9}$$

where $y_{\sigma^n i_1 \cdots i_m} \sigma^n$ is the derivative of y_t with respect to the m'th fold Kronecker products of exogenous innovations i_1, i_2, ... and i_m periods ago and with respect to the perturbation parameter, σ, n times. The stochastic steady state now follows by letting the history of shocks be equal to zero at all dates (i.e., letting y_t converge to its fixed point), but letting $\sigma = 1$ to correct for risk to M'th order, yielding

$$\bar{y}^{stoch} \approx \sum_{n=0}^{M} \frac{1}{n!} y_{\sigma^n} \tag{10}$$

as an approximation of the stochastic steady state.

The ergodic mean of y_t, a potentially useful point of expansion for likelihood estimation, is

Definition 2.4. *Ergodic Mean*

Let $\bar{y}^{mean} \in \mathbb{R}^{n_y}$ *be a vector such that*

$$\bar{y}^{mean} \doteq E[y_t] = E[g(y_{t-1}, \varepsilon_t, 1)] \tag{11}$$

being the unconditional expectation of (2) in the presence of uncertainty ($\sigma = 1$) *and shocks* (ε_t).

Again, the definition in (11) is not directly useful, as calculating the mean requires integration

[12] I.e., the rest point obtained by simulating the model with all realizations of the shock vector, ε_t, set to zero despite the model having been solved under the assumption of a non-degenerate distribution.

[13] Details can be found in appendix A.2.

[14] See Lan and Meyer-Gohde (2013b) for the mapping between the partial derivatives $g_{z^j \sigma^i}$ and $y_{\sigma^n i_1 i_2 \cdots i_m}$.

over the endogenous variables through the unknown policy function. The nonlinear moving average validated by assumption 2.2, however, approximates the ergodic mean as

$$\overline{y}^{mean} \doteq E\left[y_t\right] \approx \sum_{m=0}^{M} \frac{1}{m!} \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_m=0}^{\infty} \left[\sum_{n=0}^{M-m} \frac{1}{n!} y_{\sigma^n i_1 i_2 \cdots i_m}\right] E\left[\varepsilon_{t-i_1} \otimes \varepsilon_{t-i_2} \otimes \cdots \otimes \varepsilon_{t-i_m}\right] \tag{12}$$

Due to the analyticity assumed in a domain containing both the ergodic mean and the stochastic steady state, the policy function is analytic at these points as well. If the function is analytic at a point, it is infinitely differentiable at the point as well. Hence, it is certainly once differentiable there as well. Thus, the stochastic steady state and the ergodic mean and the first derivative of the policy function at these points can be recovered using derivative information at the deterministic steady state and the moments of the exogenous shocks of the model. Now I shall proceed to do exactly that and assemble the points and slopes into linear approximations.

3 Risk-Sensitive Linear Approximations

Define a risk-sensitive linear approximation as an approximation that is linear in the states, y_{t-1} and ε_t, but adjusted to arbitrary order for risk, i.e., for the desired moments of the distribution of the exogenous shocks ε_t in (1). The adjustment for risk is accomplished by expanding the policy function nonlinearly in σ, the scaling index. Expanding to first order in σ corrects for the first moment of ε_t,[15] to second order in σ corrects for the second moment of ε_t, and so forth.

I will write such a σ dependent or risk-sensitive linear approximation as

$$y_t \approx \overline{y}(\sigma) + \overline{y_y}(\sigma)\left(y_{t-1} - \overline{y}(\sigma)\right) + \overline{y_\varepsilon}(\sigma)\varepsilon_t \tag{13}$$

where $\overline{y}(\sigma)$ is a σ dependent or risk-sensitive point for y_t and $\overline{y_y}(\sigma)$ and $\overline{y_\varepsilon}(\sigma)$ are the σ dependent or risk-sensitive first derivatives of y_t at this risk-sensitive point. I shall consider two risk-sensitive points, the stochastic steady state and the ergodic mean.

From standard DSGE perturbation, I have derivatives at the deterministic steady state.[16] I will

[15] As the shock is assumed mean zero, this correction does not alter the policy functions compared with their deterministic counterparts. See Schmitt-Grohé and Uribe (2004) and Lan and Meyer-Gohde (2014).

[16] As will become evident in the construction of the approximation, I will need derivatives from a $M+1$-th order perturbation to construct the risk-sensitive linear approximation corrected out to M-th order for risk.

now show that the first derivatives, $\overline{y_y}(\sigma)$ and $\overline{y_\varepsilon}(\sigma)$, will depend on the risk-sensitive point, $\overline{y}(\sigma)$, and that my two choices for the risk-sensitive point, the stochastic steady state and the ergodic mean, along with these first derivatives at these points can be obtained from the derivatives at the deterministic steady state along with the moments of ε_t.

I summarize the procedure for constructing a risk-sensitive linear approximation in the following

Risk-Sensitive Linear Approximation

1. Express the model as a set of equilibrium conditions in the form of (1)
2. Perform a standard perturbation around the deterministic steady state to the order $m+1$, where m is desired order of approximation in risk
3. Use the resulting derivatives at the deterministic steady state (7) to construct approximations in risk of
   (a) a point—$\overline{y}^{stoch}$ from (8) or $\overline{y}^{mean}$ from (11). Define this as $\overline{y}(\sigma)$.
   (b) derivatives at this point. Define these as $\overline{y_y}(\sigma)$ and $\overline{y_\varepsilon}(\sigma)$.
4. Assemble the risk-sensitive linear approximation of (13) using these point and slopes.

and turn now to the derivation of approximations in risk of the point and slopes used in the risk-sensitive linear approximation of (13).

3.1 Risk-Sensitive Points of Approximation

3.1.1 Stochastic Steady State

The stochastic and deterministic steady states can be embedded in a σ-dependent steady state

$$\overline{y}^{stoch}(\sigma) = g(\overline{y}^{stoch}(\sigma), 0, \sigma) \tag{14}$$

Here, $\sigma = 1$ gives the stochastic and $\sigma = 0$ the deterministic steady state. A Taylor expansion of $\overline{y}^{stoch}(\sigma)$ around the $\sigma = 0$ deterministic steady state can be written as

$$\overline{y}^{stoch}(\sigma) = \sum_{i=0}^{\infty} \frac{1}{i!} \overline{y}^{stoch}_{\sigma^i}(0)\,\sigma^i \tag{15}$$

Using the Taylor expansion in σ, the stochastic steady state can be approximated by solving sets of linear equations with inhomogeneous constants collecting lower order terms and standard DSGE perturbation output, as I summarize in the following

Proposition 3.1. *σ Approximation of the Stochastic Steady State*

Let assumption 2.2 hold, the stochastic steady state in (8) can be approximated in σ using only derivatives from standard perturbations—$g_{z^j\sigma^i}$ in (6).

Proof. See appendix A.4. □

To capture the effect of the first two moments of the exogenous processes (i.e., that of the variance of the mean zero growth shocks in the model of section 4) on the stochastic steady state, a second order approximation in σ evaluated at $\sigma = 1$ is needed.

(16) $$\overline{y}^{stoch}(\sigma) = \overline{y} + \overline{y}^{stoch}_{\sigma}(0) + \frac{1}{2}\overline{y}^{stoch}_{\sigma^2}(0) + O(\sigma^3)$$

This second order in σ approximation of the stochastic steady state[17] is in terms of the derivatives of the standard perturbation output by

Corollary 3.2. *Second Order σ Approximation of the Stochastic Steady State*

The stochastic steady state in (8) can be approximated to second order in σ as

(17) $$\overline{y}^{stoch} \approx \overline{y} + \frac{1}{2}\left(I_{n_y} - g_y\right)^{-1} g_{\sigma^2}$$

Proof. See appendix A.5. □

3.1.2 Ergodic Mean

The ergodic mean and the deterministic steady state can be embedded in a σ-dependent point

(18) $$\overline{y}^{mean}(\sigma) \doteq E\left[g(y_{t-1}, \sigma\varepsilon_t, \sigma)\right]$$

Here, $\sigma = 1$ gives the ergodic mean and $\sigma = 0$ the deterministic steady state. Due to the singularity induced by $\sigma = 0$, which turns off the stochastics in the model, the steady state coincides with the mean in this deterministic setting, which I exploit to extrapolate from the deterministic steady state to the ergodic mean. A Taylor expansion of $\overline{y}(\sigma)$ around the $\sigma = 0$ deterministic steady state is

(19) $$\overline{y}^{mean}(\sigma) = \sum_{i=0}^{\infty}\frac{1}{i!}\overline{y}^{mean}_{\sigma^i}(0)\,\sigma^i$$

[17] Lan and Meyer-Gohde (2013b) report the same value using a second order nonlinear moving average.

Using the Taylor expansion in σ, the ergodic mean can be approximated by solving sets of linear equations with inhomogeneous constants collecting lower order terms, standard DSGE perturbation output, and the given moments of the exogenous driving force, as I summarize in the following

Proposition 3.3. *σ Approximation of the Ergodic Mean*

Let assumption 2.2 hold, the ergodic mean in (11) can be approximated in σ using derivatives from standard perturbations—$g_{z^j\sigma^i}$ in (6)—and the given moments of ε_t.

Proof. See appendix A.6. □

Discarding terms of order higher than two in (19) and evaluating at $\sigma = 1$ gives an approximation of the ergodic mean that captures the effects of the first two moments of the exogenous processes (i.e., that of the variance of the mean zero growth shocks in the model of section 4)

$$\bar{y}^{mean}(\sigma) = \bar{y} + \bar{y}^{mean}_{\sigma}(0) + \frac{1}{2}\bar{y}^{mean}_{\sigma^2}(0) + O(\sigma^3) \tag{20}$$

In terms of the derivatives of the standard perturbation output and moments of ε_t, the ergodic mean is give out to second order[18] by

Corollary 3.4. *Second Order σ Approximation of the Ergodic Mean*

The ergodic mean in (11) can be approximated to second order in σ as

$$\bar{y}^{mean} \approx \bar{y} + \frac{1}{2}\left(I_{n_y} - g_y\right)^{-1}\left(g_{\sigma^2} + \left(g_{\varepsilon^2} + \left(I_{n_y^2} - g_y^{\otimes[2]}\right)^{-1} g_{\varepsilon}^{\otimes[2]}\right) E\left[\varepsilon_t^{\otimes[2]}\right]\right) \tag{21}$$

Proof. See appendix A.7. □

3.2 Risk-Sensitive First Derivatives

Given a risk-sensitive point from above, I only need the first derivatives with respect to states and shocks in order to complete the construction of the risk-sensitive linear approximation in (13).[19]

[18] This value is identical to those reported in Lan and Meyer-Gohde (2013a) and in Andreasen, Fernández-Villaverde, and Rubio-Ramírez (2013) using second order nonlinear moving average and pruned state space approximations.

[19] The risk-sensitive derivatives will, as is to be expected, depend on where in the state space they are being evaluated; that is on which of the risk-sensitive points, $\bar{y}^{\text{mean}}(\sigma)$ or $\bar{y}^{\text{mean}}(\sigma)$, from above is chosen. The derivation, however, is completely symmetric in both cases and thus I refer to the risk-sensitive point simply as $\bar{y}(\sigma)$ leaving the modeler to fill in the desired superscript.

The first derivatives with respect to states and shocks around a risk-sensitive point are given by

Definition 3.5. *First Derivatives at a σ Adjusted Point*

The derivatives of y_t with respect to y_{t-1} and ε_t at a risk-sensitive point, $\bar{y}(\sigma)$ are

$$\overline{y_y}(\sigma) \doteq g_y(\bar{y}(\sigma), 0, \sigma), \quad \overline{y_\varepsilon}(\sigma) \doteq g_\varepsilon(\bar{y}(\sigma), 0, \sigma) \tag{22}$$

The first derivatives are σ dependent functions, both directly—with σ being the third argument of the function—and indirectly—as the first argument of the function is risk-sensitive point of approximation, $\bar{y}(\sigma)$. Here, $\sigma = 1$ gives the first derivatives at the risk-sensitive point of approximation and $\sigma = 0$ at the deterministic steady state. Taylor expansions of $\overline{y_y}(\sigma)$ and $\overline{y_\varepsilon}(\sigma)$ around the $\sigma = 0$ deterministic steady state can be written as

$$\overline{y_y}(\sigma) = \sum_{i=0}^{\infty} \frac{1}{i!} \bar{y}_{y\sigma^i}(0)\, \sigma^i, \quad \overline{y_\varepsilon}(\sigma) = \sum_{i=0}^{\infty} \frac{1}{i!} \bar{y}_{\varepsilon\sigma^i}(0)\, \sigma^i \tag{23}$$

As was the case with the two risk-sensitive points considered above, the first derivatives at these points depend only on standard output from perturbation algorithms: derivatives of the policy function at the deterministic steady state and the moments (through the derivatives of the risk-sensitive ergodic mean) of the exogenous shocks, ε_t, as I summarize in the following

Proposition 3.6. *σ Approximation of the First Derivatives*

The first derivatives in (22) can be approximated in σ using derivatives from standard perturbations— $g_{z^j\sigma^i}$ in (6)—and the derivatives in σ from the chosen risk-sensitive point of approximation.

Proof. See appendix A.8. □

Approximating out to second order in σ and evaluating at $\sigma = 1$ gives the following

$$\overline{y_y}(\sigma) = \overline{y_y} + \bar{y}_{y\sigma}(0) + \frac{1}{2}\bar{y}_{y\sigma^2}(0) + O(\sigma^3), \quad \overline{y_\varepsilon}(\sigma) = \overline{y_\varepsilon} + \bar{y}_{\varepsilon\sigma}(0) + \frac{1}{2}\bar{y}_{\varepsilon\sigma^2}(0) + O(\sigma^3) \tag{24}$$

In terms of the derivatives of the standard perturbation and of the chosen risk-sensitive point of approximation, the first derivatives can be written to second order in σ as

Corollary 3.7. *Second Order σ Approximation of the First Derivatives*

The first derivatives in (22) can be approximated to second order in σ as

$$\overline{y_y}(1) \approx g_y + \frac{1}{2}\left(g_{y^2}\left(\overline{y}_{\sigma^2}(0) \otimes I_{n_y}\right) + g_{\sigma^2 y}\right), \quad \overline{y_\varepsilon}(1) \approx g_\varepsilon + \frac{1}{2}\left(g_{y\varepsilon}\left(\overline{y}_{\sigma^2}(0) \otimes I_{n_e}\right) + g_{\sigma^2\varepsilon}\right) \tag{25}$$

Proof. See appendix A.9. □

These risk-sensitive first derivatives are identical to the first derivatives of the Taylor series evaluated at the risk-sensitive points to a given order in σ, see appendix A.3. They differ from those used in Andreasen, Fernández-Villaverde, and Rubio-Ramírez (2013), whose pruning procedure gives risk-sensitive derivatives evaluated at the deterministic steady state. Thus, my procedure provides the correct risk-sensitive point of interest and derivatives at this point up to the chosen order in σ.

With the risk-sensitive first derivatives in hand, the risk-sensitive linear approximation in (13) can be constructed by choosing either the approximation of the stochastic steady state or of the ergodic mean and calculating the associated first derivatives.

4 Long Run Risk and the Real Business Cycle

Two features of an otherwise canonical real business cycle model in the spirit of Kydland and Prescott (1982) will serve to emphasize the role of risk: recursive—or risk-sensitive—preferences and long run risk. I follow Epstein and Zin (1989), Weil (1990), and others by replacing the continuation value of household utility with a power certainty equivalent to introduce risk sensitivity and to separate risk aversion and the inverse elasticity of intertemporal substitution. I confront agents with long run real risk in the form of stochastic trends in productivity,[20] which adds risk to the growth of consumption, making the stochastic driving force of the model welfare-relevant in the sense of Lucas (1987). My choice of model is very similar to the model used in the numerical study of Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012), where I have replaced their stochastic

[20] See, e.g., Bansal and Yaron (2004) in an endowment and Rudebusch and Swanson (2012) in a production model.

volatility with long run risk.[21]

The social planner maximizes the following expected discounted lifetime utility sum[22]

$$U_t = \max_{C_t, L_t} \left[(1-\beta) \left(C_t^{\nu} (1-L_t)^{1-\nu} \right)^{\frac{1-\gamma}{\theta}} + \beta \left(E_t \left[U_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}} \tag{26}$$

where C_t is consumption, L_t labor, $\beta \in (0,1)$ the discount factor, ν a labor supply parameter, γ risk aversion,[23] and $\theta = (1-\gamma)/(1-1/\psi)$—where ψ is the elasticity of intertemporal substitution (IES)—separates the (inverse) IES and risk aversion. The resource constraint is

$$C_t + K_t = K_{t-1}^{\xi} \left(e^{Z_t} L_t \right)^{1-\xi} + (1-\delta) K_{t-1} \tag{27}$$

with K_t being capital, ξ its output elasticity and δ its depreciation rate, and Z_t productivity given by

$$a_t \doteq Z_t - Z_{t-1} = \overline{a} + \overline{\sigma} \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0,1) \tag{28}$$

with $\overline{\sigma}$ the standard deviation of a_t and $\overline{a}$ the drift of the random walk.

The first order conditions are the intratemporal condition

$$\frac{1-\nu}{\nu} \frac{C_t}{1-L_t} = (1-\xi) e^{(1-\xi)Z_t} K_{t-1}^{\xi} L_t^{-\xi} \tag{29}$$

and the intertemporal condition

$$1 = E_t \left[M_{t+1} \left(\xi K_t^{\xi-1} \left(e^{Z_{t+1}} L_{t+1} \right)^{1-\xi} + 1 - \delta \right) \right] \tag{30}$$

where the pricing kernel is given by

$$M_{t+1} \doteq \frac{\partial U_t / \partial C_{t+1}}{\partial U_t / \partial C_t} = \beta \frac{C_t}{C_{t+1}} \frac{\left(C_{t+1}^{\nu} (1-L_{t+1})^{1-\nu} \right)^{\frac{1-\gamma}{\theta}}}{\left(C_t^{\nu} (1-L_t)^{1-\nu} \right)^{\frac{1-\gamma}{\theta}}} \left(\frac{U_{t+1}^{1-\gamma}}{E_t \left[U_{t+1} \right]^{1-\gamma}} \right)^{1-\frac{1}{\theta}} \tag{31}$$

The presence of U_{t+1} here necessitates the inclusion of the value function evaluated at the optimum

$$U_t = \left[(1-\beta) \left(C_t^{\nu} (1-L_t)^{1-\nu} \right)^{\frac{1-\gamma}{\theta}} + \beta \left(E_t \left[U_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{\gamma}} \right]^{\frac{\theta}{1-\gamma}} \tag{32}$$

along with the first order conditions, the resource constraint (27), and the exogenous driving force

[21] As I will be examining linear approximations, I would only be able to capture the precautionary effects or average effects of stochastic volatility and would miss the time varying effects of changes in conditional heteroskedasticity. One could conceivably move the approximation towards the conditionally normal one used in Justiniano and Primiceri (2008) or linear in endogenous states one of Benigno, Benigno, and Nisticò (2013), to retain some of the advantages of linearity, but this is beyond the scope of this paper.

[22] See Epstein and Zin (1989) and Weil (1990), as well as Backus, Routledge, and Zin (2005) for a review.

[23] In the presence of an adjustable labor margin, the standard measure of risk aversion does not directly apply, see Swanson (2012a). Swanson (2012b) presents measures of risk aversion under recursive preferences in the presence of a labor margin. I maintain the misnomer of referring to γ as risk aversion for expositional ease.

(28) to characterize an equilibrium. With the stochastic trend in the model, I detrend all variables (apart from L_t and M_t) with $x_t \doteq X_t/e^{Z_t}$.[24] I reexpress all macroeconomic variables through a log transformation $\hat{x}_t = \ln(x_t)$ so that deviations in these variables from any given value can be interpreted as percentage deviations and a linear approximation of $\hat{x}_t$ gives a log linear approximation.[25]

I will include two conditional asset pricing variables: the expected risk premium

$$erp_t = E_t\left[r^k_{t+1} - r^f_t\right] \tag{33}$$

where the risk-free rate, r^f_t, is given by $r^f_t \doteq \frac{1}{E_t[M_{t+1}]}$ and the return on capital, r^k_t, is given by $r^f_t \doteq \xi K^{\xi-1}_{t-1}\left(e^{Z_t}L_t\right)^{1-\xi} + 1 - \delta$; and the (squared)[26] conditional market price of risk

$$cmpr_t = \frac{E_t\left[(M_{t+1} - E_t\left[M_{t+1}\right])^2\right]}{E_t\left[M_{t+1}\right]^2} \tag{34}$$

Finally, I will include excess returns

$$rp_t = r^k_t - r^f_{t-1} \tag{35}$$

as an observable counterpart to the expected risk premium version above.

5 Accuracy of the Risk-Sensitive Linear Approximation

5.1 Calibration

In table 1, I report the parameter values common to both calibrations I consider. The calibration for these parameters largely follows Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012) and reflects standard observations on the post war US economy: ξ is set to match the labor share of national income; β to reflect an annual interest rate of about 3.5 %; the value of ν induces work to occupy roughly one third of the time endowment; and δ aligns the model in the deterministic steady state to the investment output ratio. The value of $\overline{a}$ is set to match the average growth rate of output.

[Table 1 about here.]

[24] The value function is detrended slightly differently, $u_t \doteq U_t/e^{\nu Z_t}$. See appendix A.10 for the detrended model.

[25] See, e.g., Uhlig (1999).

[26] This is necessary as the usual conditional market price of risk—conditional standard deviation over conditional mean—is not differentiable at the deterministic steady state, violating the analyticity assumption on f in 2.2.

In table 2, I report the baseline calibration in the first three columns. Here, I set risk aversion, γ, to 5, following the baseline parameterization of Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012). The elasticity of intertemporal substitution (IES) and the standard deviation of technology growth shocks are set to match the standard deviations of log consumption and output growth for the third order nonlinear moving average perturbation solution of the model.[27] This value for the IES lies in the range of 0.5 to 1.5 examined in Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012), reflecting conservative bounds on the parameter advocated in the literature.

[Table 2 about here.]

The extreme calibration can be found in the last three columns of table 2. Risk aversion, γ, is equal to 40, following Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012). The IES and the standard deviation of technology growth shocks are again set to match the standard deviations of log consumption and output growth for the third order in perturbation solution of the model. When calibrating to the two macro data sets, the values for the IES and the standard deviation of technology growth shocks remain virtually unchanged, reflecting the well known result—see, e.g., Tallarini (2000)—that macro series are driven primarily by the IES and not risk aversion.

5.2 Euler Equation Errors

I follow Judd and Guu (1997) and Judd (1998) and use unit-free Euler equation residuals to measure the accuracy of the the risk-sensitive linear approximation at the stochastic steady state.[28]

The Euler equation error expressed as a fraction of time t consumption is given by

$$\frac{E_t\left[\beta\left(c_{t+1}e^{a_{t+1}}\right)^{\nu\frac{1-\gamma}{\theta}-1}\left(\frac{1-L_{t+1}}{1-L_t}\right)^{\frac{(1-\nu)(1-\gamma)}{\theta}}\left(\frac{\left(v_{t+1}e^{\nu a_{t+1}}\right)^{1-\gamma}}{E_t\left[\left(v_{t+1}e^{\nu a_{t+1}}\right)^{1-\gamma}\right]}\right)^{1-\frac{1}{\theta}}\left(\alpha k_t^{\alpha-1}\left(e^{a_{t+1}}L_{t+1}\right)^{1-\alpha}+1-\delta\right)\right]^{\frac{1}{\nu\frac{1-\gamma}{\theta}-1}}}{c_t}-1 \tag{36}$$

[27] Lan and Meyer-Gohde (2013a) provide closed forms for the theoretical moments of the nonlinear moving average.

[28] The results are virtually unchanged at the ergodic mean. See also Aruoba, Fernández-Villaverde, and Rubio-Ramírez (2006) and Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012) for comparative studies of solution methods that use Euler equation residuals.

Here, the value of, say, $1E-2$ implies a $1 mistake for each $100 spent and the value of $1E-3$ implies a $1 mistake for each $1000 and so forth. In figure 1, I plot the Euler equation errors with the current shock set to zero and examine how this error depends on the endogenous state, k_{t-1}.

[Figure 1 about here.]

In figure 1a, the Euler equation errors for the baseline calibration can be found. In general, higher order perturbations improve the accuracy of the standard perturbation approximation. The risk-sensitive linear approximation uniformly improves on the linear approximation, while lagging behind the second and third order perturbation.[29] In the vicinity of the steady states,[30] this improvement is more than one order of magnitude. In sum, the risk-sensitive linear approximation, while still linear in states and shocks, is uniformly more accurate than the standard linear approximation.

The Euler equation errors for the extreme calibration are depicted in figure 1b. As risk becomes more important through the increase in risk aversion in the extreme calibration, so does the relative performance of the risk-sensitive linear approximation. Now the risk-sensitive linear approximation is two orders of magnitude more accurate than the standard linear approximation over a broad vicinity encompassing the steady states.[31] Furthermore, the risk-sensitive linear approximation is roughly comparable to higher order approximations despite its linearity in states and shocks.

Thus, for small shock realizations and values of the state close to the stochastic steady state, I conclude that the risk-sensitive linear approximation outperforms the standard linear approximation and performs favorably compared with higher order perturbations.

[29] Again as mentioned in the introduction, although one could adjust higher order perturbations for a yet higher order of risk, it is not clear what the gain over a standard perturbation would be for such a higher order risk-sensitive approximation.

[30] Under the baseline calibration, the stochastic and deterministic steady states are nearly the same.

[31] The stochastic and deterministic steady states can be distinguished visually: the minimum of the risk-sensitive linear approximation is the stochastic and that of the standard linear approximation the deterministic steady state.

5.3 Impulse Response Analysis

A consequence of the linearity in states and shocks of the approximation is that impulse responses are standard. Whereas nonlinear methods must take a stance regarding the specific assumptions regarding a generalized impulse response,[32] the risk-sensitive linear approximation and its standard linear approximation require no such discussion. Of course, the risk-sensitive linear approximation as any linear approximation will accordingly miss the nonlinear effects of asymmetry or large shocks that a nonlinear approximation can pick up. The model under study here, however, is nearly log-linear in the state dimension with nonlinearities primarily in the risk dimension that the risk-sensitive method is designed to capture.

[Figure 2 about here.]

In figure 2, the impulses of selected macroeconomic and financial variables with respect to a one standard deviation shock to the growth rate of technology are plotted. Figure 2a contains the impulses of consumption and capital to a growth rate shock. Impulse responses from the risk-sensitive linear (here the stochastic steady state version), standard linear, and third order nonlinear moving average approximations are indistinguishable up to numerical rounding. Consumption and capital, both detrended, fall in response to the shock as the capital stock and consumption catch up to the accelerated growth path. Figure 2b contains the impulses of the expected risk premium and the conditional market price of risk. The standard linear approximation fails to capture the movement in these conditional asset pricing variables, while the risk-sensitive linear approximation matches the impulses generated by the full nonlinear third order moving average approximation of Lan and Meyer-Gohde (2013c).

[32] See Lan and Meyer-Gohde (2013c), Andreasen, Fernández-Villaverde, and Rubio-Ramírez (2013), and Borovička and Hansen (2014).

6 Estimation using Risk-Sensitive Linear Approximations

I begin by exploring the properties of the likelihood function for the risk-sensitive linear approximation in a Monte Carlo experiment and then turn to the Bayesian estimation of risk aversion and the standard deviation of technology growth rate shocks using US post war data.

6.1 Monte Carlo Study of Estimation Properties

Here, I study the ability of the risk-sensitive linear approximation to estimate deep parameters and compare the efficiency with which it is able to do such with perturbation-based particle filters that enjoy relative solution efficiency advantages over alternative nonlinear methods—see, e.g., Aruoba, Fernández-Villaverde, and Rubio-Ramírez (2006).[33]

For the estimation exercise, I begin by generating two $10,000$ period series of the logarithm of detrended output, one for each calibration in section 5.1, using a third order nonlinear moving average. I then estimate the parameters for risk aversion, γ, and the standard deviation of growth shocks, $\overline{\sigma}$ one at a time, holding all other parameters constant, using different solution methods. The methods I will compare are the risk-sensitive linear method, conventional linearization, third order state space perturbation, and the third order nonlinear moving average. To address the robustness of the results I repeat the exercise with data on output growth and then on excess returns.

The risk-sensitive linear method maintains linearity in states and shocks, which, given the assumed normality of growth rate shocks, enables the use of the Kalman filter. Here, I choose the ergodic mean of section 3.1.2 as the risk-sensitive point so that the mean of the risk-sensitive linear approximation coincides with the approximation, in σ, of the ergodic mean of the true nonlinear model. The standard linearization is also estimated with the Kalman filter. The standard approach for the two nonlinear perturbations, nonlinear moving average and standard state space, is to estimate using a sequential importance sampler with resampling, i.e., the particle filter, see Fernández-

[33] Fernández-Villaverde and Rubio-Ramírez (2007) point out that perturbation is neither required for nor necessarily the preferred method for taking every model to the particle filter.

Villaverde and Rubio-Ramírez (2007), which simulates the distributions of the unobservable states. Unfortunately, the particle filter can be very demanding computationally, precluding its use currently in many policy relevant models, especially at orders of approximation higher than two.[34] I set the number of particles in the filter to be $40,000$ and add measurement noise accounting for 1% of the variance of y_t to operationalize a version of the particle filter following, e.g., Bidder and Smith (2012).

[Figure 3 about here.]

In figure 3 the likelihood function—normalized relative to the maximum likelihood value for each method—of risk aversion, γ, and the standard deviation of technology growth shocks, $\overline{\sigma}$, are plotted for the baseline calibration. The standard linear approximation is a certainty equivalent approximation and changes in risk aversion, figure 3a, have no effect on the approximation: the likelihood function is entirely flat in this dimension. The risk-sensitive linear approximation, however, is not certainty equivalent and correctly estimates the level of risk aversion in figure 3a. Both of the particle filter based policy functions correctly estimate the degree of risk aversion, but as can be seen in figure 3a, there is clearly sampling variation and the number of particles would clearly need to be increased past $40,000$ to operationalize a numerical maximization routine. As the scale of the y axis in figure 3a indicates, risk aversion of this small degree is only weakly identified, placing high demands on the particle filters; the risk-sensitive linear approximation, however, has no difficulties with this weak identification. All four of the methods correctly estimate the standard deviation of growth shocks, as can be seen in figure 3b. The likelihood cuts for both of the particle filter estimated perturbations coincide and the risk-sensitive and standard linear approximations display slightly more dispersion than the perturbation methods.

[34] Fernández-Villaverde, Guerrón-Quintana, Kuester, and Rubio-Ramírez (2011) is the exception, successfully applying the particle filter to a third-order model of time-varying policy risk. van Binsbergen, Fernández-Villaverde, Koijen, and Rubio-Ramírez (2012) and Born and Pfeifer (2014) highlight the challenges of the procedure with the demands of the particle filter leading the former to model inflation exogenously and to focus on the estimation exercise itself and the latter to abandon the likelihood perspective altogether when estimating their structural model.

[Figure 4 about here.]

The likelihood cuts—expressed relative to the maximum log likelihood value for each method—of risk aversion, γ, and the standard deviation of technology growth shocks, $\overline{\sigma}$, are plotted in figure 4 for the extreme calibration. Again, the standard linear approximation is a certainty equivalent approximation and changes in risk aversion have no effect on the approximation as can be garnered from the entirely flat likelihood function figure 4a. Once again, the risk-sensitive linear approximation advocated in previous sections, however, is not certainty equivalent and correctly estimates the level of risk aversion, albeit with slightly more dispersion relative to the particle based filters. Both of the particle filter based policy functions correctly estimate the level of risk aversion and nearly coincide in figure 4a. Note that sampling variation in the particle filters is not noticeable in figure 4a, as risk aversion is clearly more strongly identified as can be garnered from the scale of the y-axis. Turning to the standard deviation of growth shocks in figure 4b, the standard linear approximation clearly fails to correctly estimate this parameter. As the standard linearization does not capture risk aversion, it attributes the increase in risk sensitivity to an increase in risk itself. The risk-sensitive linear approximation and the two particle filter based perturbations correctly estimate the standard deviation and posit the same likelihood contours.

Figures 5 and 6 display likelihood cuts under the baseline calibration using output growth—$\log(Y_t) - \log(Y_{t-1})$—as the observable. As figure 5a indicates, this series is unable to reveal the degree of risk aversion. Risk enters this model primarily as a constant and first differencing removes the constant correction for risk in the policy functions, eliminating the role for risk in the observable, as all approximations reflect with their flat likelihood surfaces.

[Figure 5 about here.]

The story is different with data generated by excess returns (35). This measure reveals significant information on the level of risk aversion, as indicated by the curvature of the likelihood function in

6a. With the effect of increased risk sensitivity incorporated, all measures but the standard linear approximation agree upon a relative reduction of the source of constant risk, the standard deviation of growth shocks—see figure 6b.

[Figure 6 about here.]

In table 3 the different computation costs, measured in terms of computation time per likelihood evaluation.[35] As can be seen, the risk-sensitive linear was negligibly slower than the standard linear with the additional costs coming from the need to calculate the third order perturbation that delivers the higher order derivatives used to correct the linear terms for risk. Compared to the perturbation solutions that use the particle filter, the difference is striking. The risk-sensitive linear method of the previous sections is four orders of magnitude faster than the particle filter based methods. This despite their similar performance in estimating the parameters and, as the presence of sampling variation implies, the choice of the number of particles appears to have been conservative.

[Table 3 about here.]

The Monte Carlo exercise provides strong evidence in favor of the risk-sensitive linear approximation for use in estimation. It is meets or exceeds—if the potential sampling variability with particle filters is taken into account—higher order perturbation methods in identifying nonlinear parameters, like risk aversion, that standard linear approximations cannot identify while simultaneously maintaining the computational efficiency provided by the linear in state and shock framework.

6.2 US Post-War Estimation of Risk and Risk Sensitivity

I now turn to the estimation of risk and risk sensitivity using post war US data. While estimating, I take a Bayesian perspective following standard practice in the DSGE literature.[36] Taking the results

[35]Comparisons computed on an Intel Xeon E5-2690 with 16 cores at 2.90 GHz on Matlab R2013b. Approximately 61% of the processor resources were used by the particle filter at any given point in time during the calculations.

[36]See Smets and Wouters (2003) and Smets and Wouters (2007) for prominent and Del Negro, Schorfheide, Smets, and Wouters (2007) and An and Schorfheide (2007) for instructive examples of Bayesian estimation of DSGE models.

of the previous section into account, I shall include excess returns along with consumption and output growth in the data set.[37] I find that the risk-sensitive linear approximation introduced here calls for more risk and risk aversion, as is to be expected with the inclusion of excess returns in the data set. The particle filter based methods suffer from sampling variation close to the posterior mode, which makes estimating the mode infeasible.

[Table 4 about here.]

Table 4 contains the priors of the standard deviation of growth shocks and risk aversion. Both priors are relatively loose, with the prior on risk aversion centered roughly in between the two values of the calibrated model. The standard deviation of the growth shock has its prior mean and mode below the calibrated values but assigns substantial probability mass to the region around that value. Table 4 contains point estimates from the posterior from the risk-sensitive and conventional linear approximations. The risk-sensitive linear approximation favors more risk aversion and more risk than the standard linear approximation, whose estimate of risk aversion is entirely prior driven with prior and posterior modes coinciding and the likelihood function entirely flat along this dimension.

[Figure 7 about here.]

Figure 7 depicts the posterior as well as the likelihood using the risk-sensitive linear approximation. The likelihood function, figure 7b, indicates that the data is informative in both dimensions. While the likelihood and posterior, figure 7a, both favor a similar value for the standard deviation of growth rate shocks, $\overline{\sigma}$, they differ substantially over the parameter controlling risk aversion, γ. As discussed also by, e.g., Tallarini (2000), production models with recursive utility can match the slope of the market line (or market price of risk) but require exorbitant levels of risk aversion to come close to the average risk premium, see table 5. The posterior tempers this tendency, yielding a modest increase in risk aversion relative over the prior.

[37] See appendix A.11 for details on the data series.

[Figure 8 about here.]

In figure 7, the posterior and likelihood using the standard linear approximation can be found. As was to be expected from the results of the preceding sections, the likelihood is flat along the dimension of the parameter controlling risk aversion. In other words, the precautionary component of the risk premium in the data is entirely ignored and risk aversion is completely prior driven.

The posteriors and likelihoods for the nonlinear moving average perturbation can be found in figure 9.[38] As was the case for two of the four sets of synthetic data from the calibrated models, sampling variation in the particle filter is visible here with the post war US data set with the dimension in the risk aversion parameter, γ, being most obviously impacted.[39] This is not surprising, as the likelihood surface for the risk-sensitive linear approximation indicates that this dimension of the likelihood function is nearly flat, especially for values of the standard deviation of growth shocks, $\overline{\sigma}$, close to the mode. Nonetheless, for larger values of $\overline{\sigma}$, a clear upward slope for larger values of γ emerges, consistent with the model requiring more risk aversion to increase the risk premium.

[Figure 9 about here.]

Table 5 gives the asset pricing variable moments.[40] As discussed above, the model does not match the magnitude of empirical excess returns. The risk-sensitive linear approximation is, however, able to bring the market price of risk from the pricing kernel ($\text{std}(m_t)/E[m_t]$) and the Sharpe ratio from the excess return on risky capital ($E[r_t^k - r^f]/\text{std}(r_t^k - r^f)$) close to the empirical market price of risk as measured by the NYSE value weighted portfolio over the secondary market rate for three month Treasury bill.[41] As the standard linear approximation does not generate a risk premium

[38]The results for the standard perturbation were essentially the same and have been omitted for brevity.

[39]Here I increased the number of particles to 100,000, which reduced but did not eliminate sampling variation.

[40]The macroeconomic variables remain essentially unchanged as on the risk aversion has been changed substantially and it is known, see Tallarini (2000) for example, that macroeconomic variables are virtually invariant to the level of risk aversion, holding the intertemporal elasticity of substitution constant. Tables with empirical as well as the posterior model based business cycle measures have thusly been relegated to appendix A.12.

[41]A description of the post war US data used for the empirical values can be found in appendix A.11.

at all, its Sharpe ratio is zero, and the standard linear approximation produces a market price of risk that is half the size as generated by the risk-sensitive linear approximation.

[Table 5 about here.]

Informing the estimation with excess returns along with consumption and output growth leads the posterior with the risk-sensitive linear approximation to favor a higher level of risk aversion than under the prior. The conventional linear approximation is invariant to the level of risk aversion and so the likelihood function is unable to inform the posterior. As the likelihood function is rather flat in the dimension of risk aversion, full nonlinear estimation is infeasible as the particle filter suffers from a sampling variation large enough to mask the curvature in the likelihood function. Under the posterior mode estimates, the model's predictions of the market price of risk and the Sharpe ratio are brought closer to the observe sample market price of risk.

7 Conclusion

I have derived and analyzed a risk-sensitive linear approximation of the policy function for DSGE models. The method solves linear equations in standard perturbation output, requiring neither fixed point nor other recursive methods, by operating implicitly with the unknown policy function instead of the equilibrium conditions of the model. This direct approach along with the minimal costs associated with standard perturbation methods allow me provide a certainty non equivalent method suitable for the estimation and analysis of policy relevant DSGE policy under risk without needed to turn to the particle filter or alternate algorithms with unknown convergence properties that correct for risk. Finally, the method presented is able to provide a risk corrected linear approximation around the ergodic mean as well as the stochastic steady state.

In the real business cycle model with long run risk and recursive preferences, I find that the risk-sensitive linear approximation is a uniform improvement over the standard linear approximation and, as risk becomes more important in the the model, the accuracy of the algorithm becomes comparable to second and third perturbations. The method is able to model the responses of conditional asset pricing variables to shocks, which are beyond the reach of standard linear approximations. Finally, in a estimation exercise, I show that the risk-sensitive linear approximation estimated using the Kalman filter correctly identifies risk and risk aversion along with the particle filter estimations of standard perturbation and nonlinear moving average approximation. Thus, the risk-sensitive linear approximation combines the efficiency in estimation (with the Kalman filter here being four orders of magnitude faster than the particle filter) of linear formulations with the information from nonlinear approximations needed to identify parameters such as the degree of risk aversion that are beyond the reach of standard linear approximations. Indeed, in the application to post war US data, the likelihood function is entirely flat in the dimension of risk aversion for a standard linear approximation and sufficiently flat for third order perturbations using the particle filter that sampling variation precludes reliable inference. The risk-sensitive linear approximation, however, yields a posterior with higher risk aversion than in the prior and a market price of risk and Sharpe ratio in line with the data.

The method here could be extended using an accuracy improving change of variables following Fernández-Villaverde and Rubio-Ramírez (2006) or to conditional linear approximations from Justiniano and Primiceri (2008) and Benigno, Benigno, and Nisticò (2013). Finally, the method developed here could be applied to policy relevant models that require capturing risk, e.g., to match financial market date, but whose size precludes the application of alternative nonlinear methods, e.g., the computational costs of the particle filter are too burdensome.

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A Appendix

A.1 Matrix Derivatives

The partial derivative of the vector function y with respect to the state vector z_t using the method of Lan and Meyer-Gohde (2013c) that differentiates conformably with the Kronecker product is[42]

$$g_{z^j\sigma^i} \doteq \mathscr{D}^{j+i}_{z^j_{t-1}\sigma^i}\{y(\sigma, z_t)\} \doteq \left(\left[\frac{\partial}{\partial z_{1,t-1}} \quad \cdots \quad \frac{\partial}{\partial z_{n_z,t-1}}\right]^{\otimes[j]} \otimes \left(\frac{\partial}{\partial \sigma}\right)^{\otimes[i]}\right) \otimes y_t \tag{A-1}$$

$$= \left(\left[\frac{\partial}{\partial z_{1,t-1}} \quad \cdots \quad \frac{\partial}{\partial z_{n_z,t-1}}\right]^{\otimes[j]} \left(\frac{\partial}{\partial \sigma}\right)^{i}\right) \otimes y_t \tag{A-2}$$

where the second line follows as σ is a scalar. The terms $\left[\sum_{i=0}^{M-j} \frac{1}{i!} y_{z^j\sigma^i}\sigma^i\right]$ in (6) collect all the coefficients associated with the j'th fold Kronecker product of the state vector, $(z_t - \overline{z})$. Higher orders of σ in $g_{z^j\sigma^i}$ correct the Taylor series coefficients for uncertainty by successively opening the coefficients to higher moments in the distribution of future shocks.[43]

A.2 Assumptions, Stochastic Steady State, and Ergodic Mean

From the assumption of local analyticity for g, y_t has the Taylor series representation

$$y_t = \sum_{j=0}^{\infty} \frac{1}{j!}\left[\sum_{i=0}^{\infty} \frac{1}{i!} g_{z^j\sigma^i}\sigma^i\right]\left[y'_{t-1} - \overline{y} \quad \sigma\varepsilon'_t\right]^{\prime\otimes[j]} \tag{A-3}$$

Thus, increasing the approximation order in the standard perturbation brings the approximation closer to the true policy function in the sense that an infinite order perturbation will recover the true policy function. Local analyticity of f and g and the existence of finite moments validates the standard DSGE perturbation practice of successively differentiating f to produce equations that the coefficients, $g_{z^j\sigma^i}$, in (A-3) solve.[44]

The assumption of local stability in addition to that of local analyticity for g ensures that the

[42] Details of the associated calculus that generalizes familiar chain and product rules as well as Taylor approximations to multidimensional settings can be found in Lan and Meyer-Gohde (2013c) and Lan and Meyer-Gohde (2014).

[43] A similar interpretation can be found in Judd and Mertens (2012) for univariate expansions and in Lan and Meyer-Gohde (2013c) for expansions in infinite sequences of innovations.

[44] Additionally, Lan and Meyer-Gohde (2014) prove that assumptions 2.2 are sufficient to guarantee the solvability of DSGE perturbations; that is, that successively differentiating f delivers equations that can be uniquely solved to deliver the coefficients, $g_{z^j\sigma^i}$, in (A-3).

policy function can be inverted, at least locally, as

$$y_t = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{i_1=0}^{\infty} \cdots \sum_{i_m=0}^{\infty} \left[\sum_{n=0}^{\infty} \frac{1}{n!} y_{\sigma^n i_1 \cdots i_m} \sigma^n \right] (\sigma \varepsilon_{t-i_1} \otimes \cdots \otimes \sigma \varepsilon_{t-i_m}) \tag{A-4}$$

delivering a infinite nonlinear moving average or Volterra series representation.[45]

The assumption that analyticity holds over a domain larger than the deterministic steady state ensures that the Taylor series representation in (A-3) remains valid beyond the immediate vicinity of the deterministic steady state. This assumption is crucial. Analyticity in σ from zero to one connects the deterministic and stochastic models, enabling our use of the derivatives of g at $\sigma = 0$ to approximate the stochastic, $\sigma = 1$, model. In this stochastic model, both the fixed point—the stochastic steady state, (8)—and average value—the ergodic mean, (11)—of the policy function are generically different from the deterministic steady state. As such, virtually any policy experiment or simulation will leave the vicinity of the deterministic steady state. Hence, for standard perturbations to be applicable in settings useful for analysis, they must maintain their validity in a region of the state space that contains the deterministic steady state, stochastic steady state, and ergodic mean, as well as in the perturbation parameter, σ, over the deterministic and stochastic models.

In this case, both the stochastic steady state

$$\overline{y}^{\text{stoch}} = \sum_{j=0}^{\infty} \frac{1}{j!} \left[\sum_{i=0}^{\infty} \frac{1}{i!} g_{z^j \sigma^i} \right] \left[\overline{y}^{\text{stoch}\prime} - \overline{y} \quad 0' \right]'^{\otimes[j]} \tag{A-5}$$

and the ergodic mean

$$E[y_t] = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{i_1=0}^{\infty} \cdots \sum_{i_m=0}^{\infty} \left[\sum_{n=0}^{\infty} \frac{1}{n!} y_{\sigma^n i_1 \cdots i_m} \right] E\left[\varepsilon_{t-i_1} \otimes \cdots \otimes \varepsilon_{t-i_m} \right] \tag{A-6}$$

are recoverable from the implicit function theorem, requiring only derivatives of g evaluated at the deterministic steady state as enter the Taylor series (A-3) and the moments of ε_t.

A.3 Relation to Perturbation First Derivatives

The first derivative of the Taylor series in (6) with respect to the state vector z_t is

$$\mathscr{D}_{z_t} y_t \approx \sum_{j=1}^{M} \frac{1}{(j-1)!} \left[\sum_{i=0}^{M-j} \frac{1}{i!} g_{z^j \sigma^i} \sigma^i \right] \left[(z_t - \overline{z})^{\otimes[j-1]} \otimes I_{n_z} \right]$$

[45] See Lan and Meyer-Gohde (2013c) and Sandberg (1983).

(A-7)
$$\approx \sum_{j=0}^{M-1} \frac{1}{j!} \left[\sum_{i=0}^{M-j-1} \frac{1}{i!} g_{z^{j+1}\sigma^i} \sigma^i \right] \left[(z_t - \bar{z})^{\otimes[j]} \otimes I_{n_z} \right]$$

Evaluated at the deterministic steady state, $z_t = \bar{z}$, the foregoing collapses to

(A-8)
$$\mathscr{D}_{z_t} y_t \approx \sum_{i=0}^{M-1} \frac{1}{i!} g_{z\sigma^i} \sigma^i$$

For $M = 3$, the first derivative from a third order perturbation approximation, setting σ to one and recalling that terms first order in σ are zero,[46] is

(A-9)
$$\mathscr{D}_{z_t} y_t \approx g_z + \frac{1}{2} g_{z\sigma^2}$$

$g_{z\sigma^i}$ is the third order time varying risk correction in the pruning algorithm of Andreasen, Fernández-Villaverde, and Rubio-Ramírez (2013) and matched perturbation of Lombardo (2010).[47]

Of interest here are the derivatives at the risk-sensitive points from section 2. Recall, (15) and (19), that the risk-sensitive points can be expressed as Taylor series in σ; i.e.,

(A-10)
$$\bar{y}(\sigma) = \sum_{i=0}^{\infty} \frac{1}{i!} \bar{y}_{\sigma^i}(0) \sigma^i$$

setting $y_t = \bar{y}(\sigma)$ and $\varepsilon_t = 0$ in (A-7) yields

(A-11)
$$\mathscr{D}_{z_t} y_t \approx \sum_{j=0}^{M-1} \frac{1}{j!} \left[\sum_{i=0}^{M-j-1} \frac{1}{i!} g_{z^{j+1}\sigma^i} \sigma^i \right] \left(\begin{bmatrix} \bar{y}(\sigma) - \bar{y} \\ 0 \end{bmatrix}^{\otimes[j]} \otimes I_{n_z} \right)$$

For a second order in σ approximation of a point from section 2, this expression becomes

$$\mathscr{D}_{z_t} y_t \approx \sum_{j=0}^{M-1} \frac{1}{j!} \left[\sum_{i=0}^{M-j-1} \frac{1}{i!} g_{z^{j+1}\sigma^i} \sigma^i \right] \left(\begin{bmatrix} \frac{1}{2} \bar{y}_{\sigma^2}(0) \sigma^2 \\ 0 \end{bmatrix}^{\otimes[j]} \otimes I_{n_z} \right)$$

(A-12)
$$\approx \sum_{j=0}^{M-1} \frac{1}{j!} \left[\sum_{i=0}^{M-j-1} \frac{1}{i!} g_{z^{j+1}\sigma^i} \right] \left(\begin{bmatrix} \frac{1}{2} \bar{y}_{\sigma^2}(0) \\ 0 \end{bmatrix}^{\otimes[j]} \otimes I_{n_z} \right) \sigma^{i+2j}$$

Discarding terms in σ of order higher than two in order to obtain a second order in σ approximation of the matrix of first derivatives at the risk-sensitive point of interest gives

(A-13)
$$\mathscr{D}_{z_t} y_t \approx g_z + \frac{1}{2} \left[g_{z\sigma^2} + g_{z^2} \left(\begin{bmatrix} \bar{y}_{\sigma^2} \\ 0 \end{bmatrix} \otimes I_{n_z} \right) \right]$$

or in terms of derivatives with respect to y_{t-1} and ε_t separately

(A-14)
$$\mathscr{D}_{y_{t-1}} y_t \approx g_y + \frac{1}{2} \left[g_{y\sigma^2} + g_{y^2} \left(\bar{y}_{\sigma^2} \otimes I_{n_y} \right) \right], \quad \mathscr{D}_{\varepsilon_t} y_t \approx g_\varepsilon + \frac{1}{2} \left[g_{\varepsilon\sigma^2} + g_{y\varepsilon} \left(\bar{y}_{\sigma^2} \otimes I_{n_e} \right) \right]$$

which are identical to the results presented in section 3.2 for my risk-sensitive linear approximation.

[46] See Jin and Judd (2002), Schmitt-Grohé and Uribe (2004), and Lan and Meyer-Gohde (2014).

[47] See Lan and Meyer-Gohde (2013b) for a detailed comparison of these and other pruning algorithms.

A.4 Proof of Proposition 3.1

Successive differentiation of (14) yields equations recursively linear in $\bar{y}_{\sigma^i}$ taking as given lower order terms of the form $\bar{y}_{\sigma^i}$ and derivatives of g with respect to y_{t-1} and σ. For solvability, following the implicit function theorem, the matrix g_y, the first derivative of the policy function at the deterministic steady state with respect to endogenous variables, must have all eigenvalues inside the unit circle; this holds under local saddle stability of (1).

A.5 Proof of Corollary 3.2

For a second-order (in σ) approximation of the stochastic steady state, differentiate $\bar{y}(\sigma) = g(\bar{y}(\sigma), 0, \sigma)$ at $\sigma = 0$ once for

$$\bar{y}'(0) = g_y \bar{y}'(0) + g_\sigma = \left(I - g_y\right)^{-1} g_\sigma = 0 \tag{A-15}$$

and twice for

$$\bar{y}''(0) = g_{y^2} \bar{y}'(0)^{\otimes[2]} + 2 g_{y\sigma} \bar{y}_\sigma + g_y \bar{y}''(0) + g_{\sigma^2} = \left(I - g_y\right)^{-1} g_{\sigma^2} \tag{A-16}$$

Thus, up to second order in σ, the stochastic steady state is

$$\bar{y}^{\text{stoch}} \approx \bar{y} + \frac{1}{2}\left(I - g_y\right)^{-1} g_{\sigma^2} \tag{A-17}$$

as claimed in corollary 3.2.

A.6 Proof of Proposition 3.3

Successive differentiation of (2) with respect to σ evaluated at the deterministic steady state gives recursive equations $\mathscr{D}_{\sigma^i} y_t$ that depend on lower order derivatives of $\mathscr{D}_{\sigma^i} y_t$,[48] derivatives of the g function evaluated at the deterministic steady state, and the exogenous vector ε_t. Successive differentiation of (18) yields equations recursively linear in $\bar{y}_{\sigma^i}$ taking as given lower order terms of the form $\bar{y}_{\sigma^i}$, derivatives of the g function evaluated at the deterministic steady state, and expectations

[48] $\mathscr{D}_{\sigma^i} y_t$ denotes the i'th order derivative of y_t with respect to σ. The alternative notation, y_{σ^i}, refers to the i'th derivative of y_t with respect to its third argument, i.e., the "direct" derivative with respect to σ, neglecting derivatives involving σ that enter through the term $\sigma\varepsilon_t$ that are included in the notation $\mathscr{D}_{\sigma^i} y_t$.

of terms involving $\mathscr{D}_{\sigma^i} y_t$ and ε_t. For solvability both of the expectations of $\mathscr{D}_{\sigma^i} y_t$ and of derivatives of (18), following the implicit function theorem, the matrix g_y, the first derivative of the policy function at the deterministic steady state with respect to endogenous variables, must have all eigenvalues inside the unit circle; this holds under local saddle stability of (1). Under this condition and if the moments of ε_t exists and are finite, the terms involving expectations and the derivatives of (18), $\bar{y}_{\sigma^i}$, can be solved uniquely from the given moments of ε_t and derivative information of the g function evaluated at the deterministic steady state.

A.7 Proof of Corollary 3.4

For a second-order (in σ) approximation of the ergodic mean, differentiate $\bar{y}(\sigma) = E\left[g(y_{t-1}, \sigma\varepsilon_t, \sigma)\right]$ at $y_{t-1} = \bar{y}(0)$ and $\sigma = 0$ once for

(A-18) $$\bar{y}'(0) = E\left[g_y \mathscr{D}_\sigma\{y_{t-1}\} + g_\varepsilon \varepsilon_t + g_\sigma\right] = \left(I - g_y\right)^{-1} g_\sigma = 0$$

and twice for

(A-19) $$\bar{y}''(0) = E\left[g_y \mathscr{D}_{\sigma^2}\{y_{t-1}\} + g_{y^2} \mathscr{D}_\sigma\{y_{t-1}\}^{\otimes[2]} + 2 g_{y\varepsilon} \varepsilon_t \otimes \mathscr{D}_\sigma\{y_{t-1}\}\right.$$

(A-20) $$\left. + 2 g_{y\sigma} \mathscr{D}_\sigma\{y_{t-1}\} + 2 g_{\varepsilon\sigma} \varepsilon_t + g_{\varepsilon^2} \varepsilon_t^{\otimes[2]} + g_{\sigma^2}\right]$$

(A-21) $$= \left(I - g_y\right)^{-1} \left(g_{y^2} E\left[\mathscr{D}_\sigma\{y_{t-1}\}^{\otimes[2]}\right] + g_{\varepsilon^2} E\left[\varepsilon_t^{\otimes[2]}\right] + g_{\sigma^2}\right)$$

(A-22) $$= \left(I_{n_y} - g_y\right)^{-1} \left(g_{\sigma^2} + \left(g_{\varepsilon^2} + \left(I_{n_y^2} - g_y^{\otimes[2]}\right)^{-1} g_\varepsilon^{\otimes[2]}\right) E\left[\varepsilon_t^{\otimes[2]}\right]\right)$$

where the last line follows from $E\left[\mathscr{D}_\sigma\{y_t\}^{\otimes[2]}\right] = E\left[\left(g_y \mathscr{D}_\sigma\{y_{t-1}\} + g_\varepsilon \varepsilon_t + g_\sigma\right)^{\otimes[2]}\right] = g_y^{\otimes[2]} E\left[\mathscr{D}_\sigma\{y_{t-1}\}^{\otimes[2]}\right] + g_\varepsilon^{\otimes[2]} E\left[\varepsilon_t^{\otimes[2]}\right]$ Thus, up to second order in σ, the ($\sigma = 1$) ergodic mean is

(A-23) $$\bar{y}^{\text{mean}} \approx \bar{y} + \frac{1}{2} \left(I_{n_y} - g_y\right)^{-1} \left(g_{\sigma^2} + \left(g_{\varepsilon^2} + \left(I_{n_y^2} - g_y^{\otimes[2]}\right)^{-1} g_\varepsilon^{\otimes[2]}\right) E\left[\varepsilon_t^{\otimes[2]}\right]\right)$$

as claimed in corollary 3.4.

A.8 Proof of Proposition 3.6

Successive differentiation of (22) yields $\bar{y}_{y\sigma^i}$ and $\bar{y}_{\varepsilon\sigma^i}$ as functions of derivatives of g with respect to y_{t-1} and σ as well as derivatives of the chosen risk-sensitive point of approximation $y(\sigma)$.

A.9 Proof of Corollary 3.7

For a second-order (in σ) approximation of $\overline{y_y}(\sigma)$, differentiate $\overline{y_y}(\sigma) = g_y(\overline{y}(\sigma), 0, \sigma)$ once for

$$\overline{y_y}'(0) = g_{y^2}\overline{y}'(0) \otimes I_{n_y} + g_{\sigma y} = 0 \tag{A-24}$$

and twice for

$$\overline{y_y}''(0) = g_{y^2}\left(\overline{y}''(0) \otimes I_{n_y}\right) + g_{\sigma^2 y} \tag{A-25}$$

Thus, up to second order in σ, the ($\sigma = 1$) derivative in y is

$$\overline{y_y}(1) \approx g_y + \frac{1}{2}\left(g_{y^2}\left(\overline{y}''(0) \otimes I_{n_y}\right) + g_{\sigma^2 y}\right) \tag{A-26}$$

as was claimed in corollary 3.7. Analogous derivations follow for $\overline{y_\varepsilon}(\sigma) \approx \overline{y_\varepsilon}(0) + \overline{y_\varepsilon}'(0)\sigma + \frac{1}{2}\overline{y_\varepsilon}''(0)\sigma^2$.

A.10 Detrended Model

Detrending with $x_t \doteq X_t/e^{Z_t}$ ($u_t \doteq U_t/e^{\nu Z_t}$) gives

$$u_t = \left[(1-\beta)\left(c_t^{\nu}(1-L_t)^{1-\nu}\right)^{\frac{1-\gamma}{\theta}} + \beta\left(E_t\left[(u_{t+1}e^{\nu a_t})^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}\right]^{\frac{\theta}{1-\gamma}} \tag{A-27}$$

$$c_t + k_t = e^{-\xi a_t}k_{t-1}^{\xi}L_t^{1-\xi} + (1-\delta)\,e^{-a_t}k_{t-1} \tag{A-28}$$

$$1 = E_t\left[M_{t+1}\left(\xi e^{(1-\xi)a_{t+1}}k_t^{\xi-1}L_{t+1}^{1-\xi} + 1 - \delta\right)\right] \tag{A-29}$$

$$M_{t+1} = \beta\left(\frac{c_{t+1}}{c_t}e^{a_{t+1}}\right)^{\nu\frac{1-\gamma}{\theta}-1}\left(\frac{1-L_{t+1}}{1-L_t}\right)^{\frac{(1-\nu)(1-\gamma)}{\theta}}\left(\frac{(u_{t+1}e^{\nu a_{t+1}})^{1-\gamma}}{E_t\left[(u_{t+1}e^{\nu a_{t+1}})^{1-\gamma}\right]}\right)^{1-\frac{1}{\theta}} \tag{A-30}$$

$$\frac{1-\nu}{\nu}\frac{c_t}{1-L_t} = (1-\xi)\,e^{-\xi a_t}k_{t-1}^{\xi}L_t^{-\xi} \tag{A-31}$$

A.11 Data

All series are quarterly and were retrieved from the Federal Reserve Economic Data (FRED) database of the Federal Reserve Bank of St. Louis except for the risky return.

Investment is the sum of the National Income and Product Accounts (NIPA) measures of Personal Consumption Expenditures on Durable Goods, Private Nonresidential Fixed Investment, and Private Residential Fixed Investment; Consumption the sum of the NIPA measures of Personal Consumption Expenditures on Nondurable Goods and Services; Output is Gross Domestic Product (GDP) expressed at an annual rate; Hours are measured by Hours Worked by Full-Time and Part-Time Employees, interpolated to a quarterly series by the growth rate of Civilian Noninstitutional Population series. Investment, Consumption, and Output are expressed in real per capita terms by deflating by the Civilian Noninstitutional Population series and the chain-type GDP deflator. The risky return is the return on the NYSE value weighted portfolio from the CRSP dataset and the risk-free return is secondary market rate for the three month Treasury bill. Both returns have been deflated by the implicit deflator of the Personal Consumption Expenditures Nondurables and Services series.

A.12 Business Cycle Tables

Table 6 summarizes the first two moments of output, consumption, investment, and hours.

[Table 6 about here.]

Table 7 summarizes the first two moments of output, consumption, investment, and hours from the model of section 4 evaluated at the posterior mode with the risk-sensitive linear approximation.

[Table 7 about here.]

Table 1: Common Calibration

Parameter	$\overline{a}$	δ	ν	β	ξ
Value	0.46%	0.0196	0.357	0.991	0.3

Table 2: Baseline and Extreme Calibration

	Baseline			Extreme		
Parameter	γ	ψ	$\overline{\sigma}$	γ	ψ	$\overline{\sigma}$
Value	5	1.008	1.12625%	40	1.0085	1.1269%

Table 3: Computational Costs: Monte Carlo Estimation

Method	Linear	Risk-Sensitive Linear	3rd Order Pert.	3rd Order Pert. (pruned)
Evaluation Time	0.44	0.47	430	690

in seconds, per likelihood evaluation

Table 4: Priors and Posteriors

	γ	$\overline{\sigma}$
	Priors	
Type	Shifted Gamma	Inverse Gamma
Mean	20	0.22%
Mode	14.737	0.11%
Standard Deviation	10	0.6%
Domain	$(1, \infty)$	$(0, \infty)$
	Posteriors	
Risk-Sensitive Linear Mode	29.296	1.0032%
Standard Linear Mode	14.737	0.9911%

Table 5: Asset Return Properties

	Empirical		Risk-Sensitive Linear		Standard Linear	
Return	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
r^k	2.14	8.25	0.5003	0.0801	0.5502	0.0758
r^f	0.26	0.62	0.4980	0.0767	0.5502	0.0726
$r^k - r^f$	1.88	8.25	0.0023	0.0212	0.000	0.0217
Market Price of Risk	0.2283		0.2004		0.1049	
Sharpe Ratio			0.1072		0.0000	

All returns are measured as real quarterly percentage returns.
See appendix A.11 for details on the series.
The model based numbers were derived using the posterior mode from the risk-sensitive linear model, see table 4.

Table 6: U.S. Business Cycle Data, 1948:2-2013:2

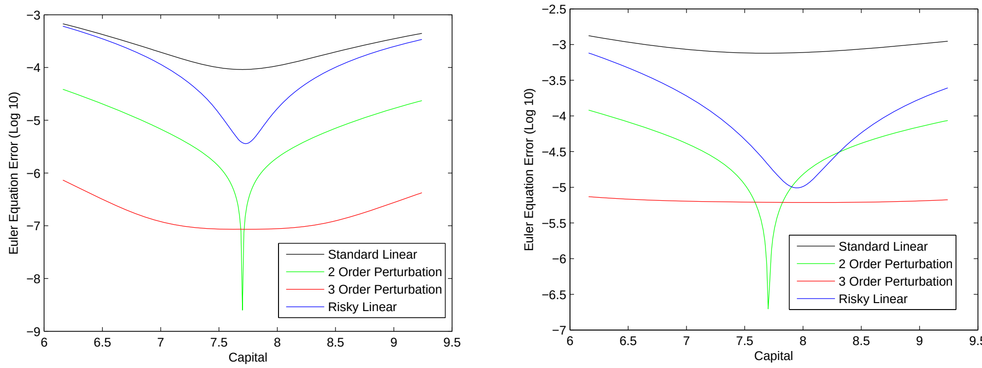
Variable	Mean	Std. Dev.	Relative Std. Dev.	Autocorrelations 1	2	3	Cross Corr. w$\Delta \ln Y_t$
$\Delta \ln Y_t$	0.458	0.988	1.000	0.381	0.266	0.046	1.000
$\Delta \ln C_t$	0.497	0.565	0.572	0.257	0.205	0.074	0.531
$\Delta \ln I_t$	0.420	2.527	2.558	0.335	0.249	0.043	0.662
$\Delta \ln N_t$	0.328	1.188	1.202	-0.020	-0.010	-0.008	0.388
$\ln N_t$	119.993	2.786	2.820	0.999	0.998	0.997	-0.141
$\ln C_t - \ln Y_t$	—	5.956	6.029	0.990	0.979	0.965	-0.173
$\ln I_t - \ln Y_t$	—	7.328	7.418	0.962	0.911	0.843	0.129

See appendix A.11 for details on the series.

Table 7: Risk-Sensitive Linear Approximation Business Cycle Properties, Posterior Mode

Variable	Mean	Std. Dev.	Relative Std. Dev.	Autocorrelations 1	2	3	Cross Corr. w$\Delta \ln Y_t$
$\Delta \ln Y_t$	0.46	0.863	1.000	0.0083	0.0080	0.0078	1.000
$\Delta \ln C_t$	0.46	0.515	0.596	0.0665	0.0641	0.0619	0.992
$\Delta \ln I_t$	0.46	1.712	1.983	-0.0156	-0.0150	-0.0145	0.996
$\Delta \ln N_t$	0	0.231	0.268	-0.0237	-0.0229	-0.0221	0.984
$\ln N_t$	-1.035	0.864	1.001	0.9643	0.9303	0.8980	0.308
$\ln C_t - \ln Y_t$	—	1.341	1.553	0.9643	0.9303	0.8980	-0.308
$\ln I_t - \ln Y_t$	—	3.203	3.710	0.9643	0.9303	0.8980	0.308

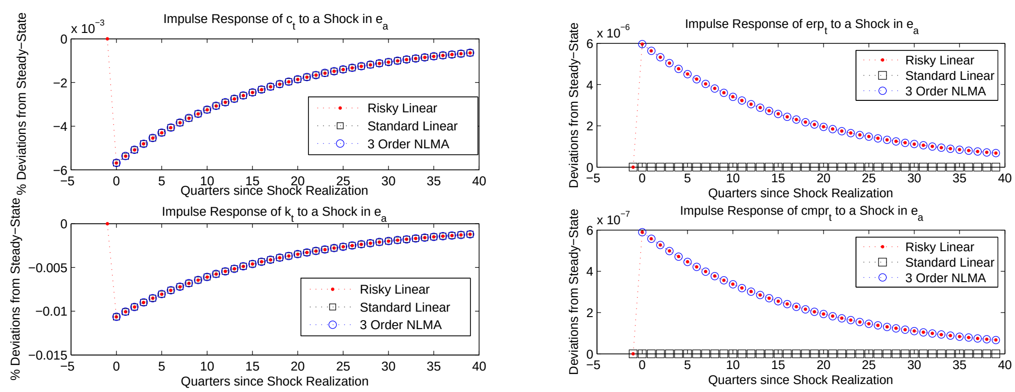
Compare with the empirical moments in table 6.



(a) Baseline Calibration

(b) Extreme Calibration

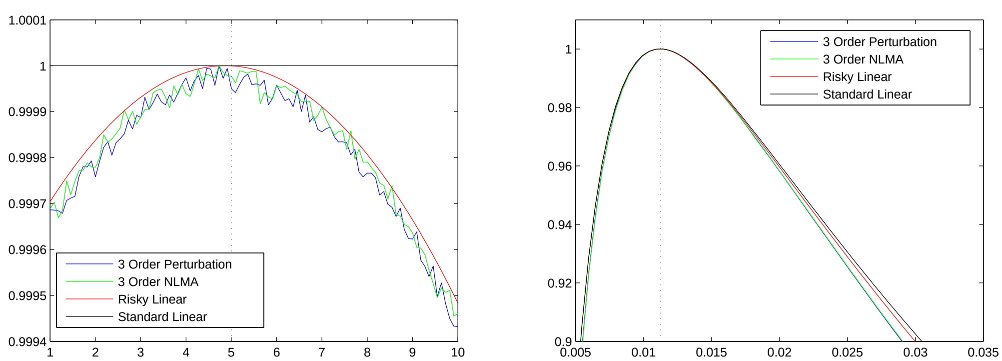
Figure 1: Euler Equation Errors



(a) Macroeconomic Variables

(b) Asset Pricing Variables

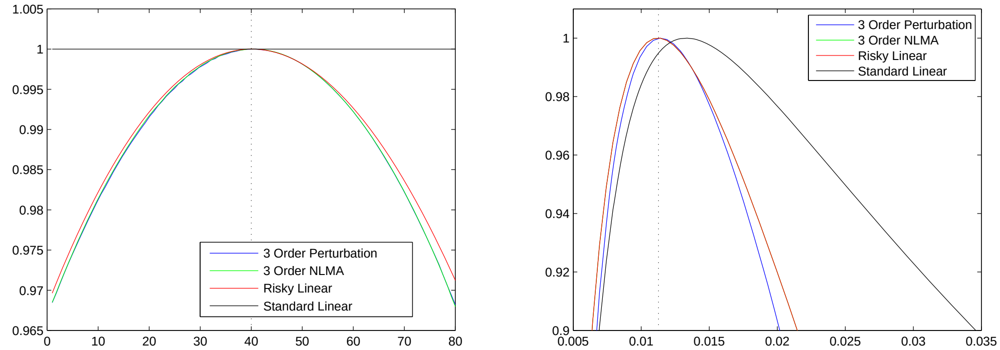
Figure 2: Impulse Response Functions



(a) Risk Aversion (γ)

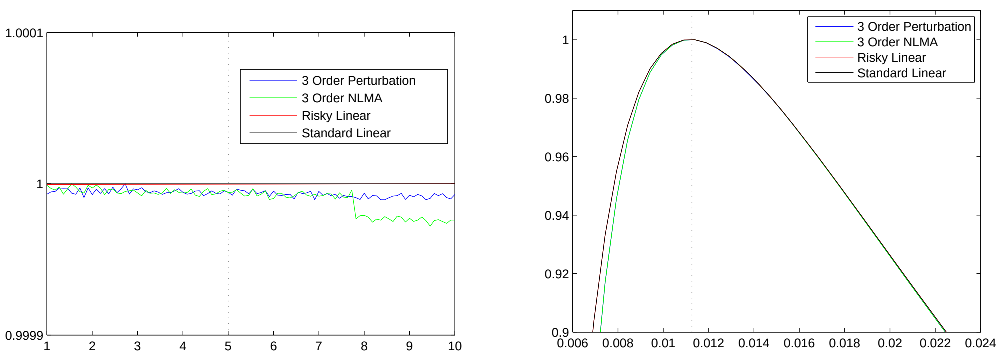
(b) Growth Shock Std. Dev. ($\overline{\sigma}$)

Figure 3: Likelihood Cuts: Baseline Calibration



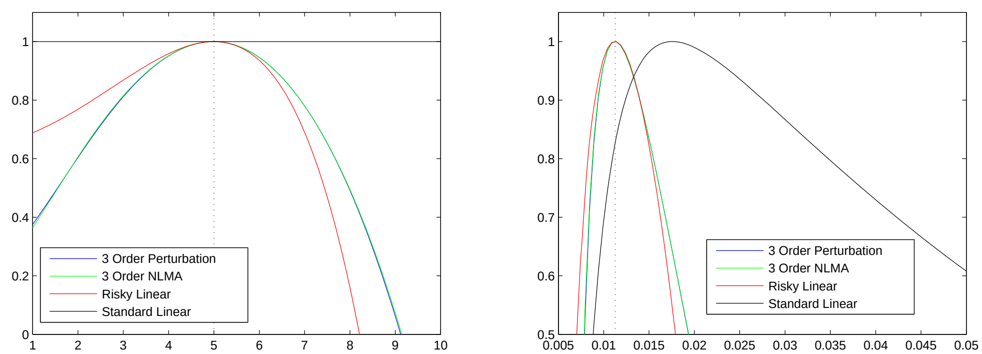
(a) Risk Aversion (γ) (b) Growth Shock Std. Dev. ($\overline{\sigma}$)

Figure 4: Likelihood Cuts: Extreme Calibration



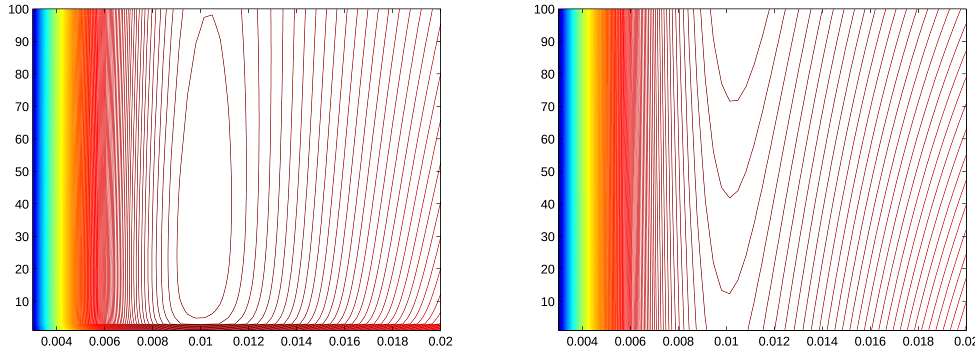
(a) Risk Aversion (γ) (b) Growth Shock Std. Dev. ($\overline{\sigma}$)

Figure 5: Likelihood Cuts: Baseline Calibration, Data on ΔY_t



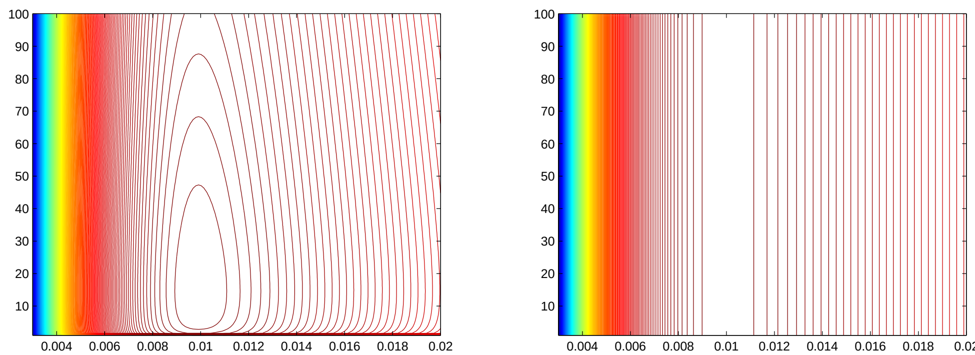
(a) Risk Aversion (γ) (b) Growth Shock Std. Dev. ($\overline{\sigma}$)

Figure 6: Likelihood Cuts: Baseline Calibration, Data on rp_t



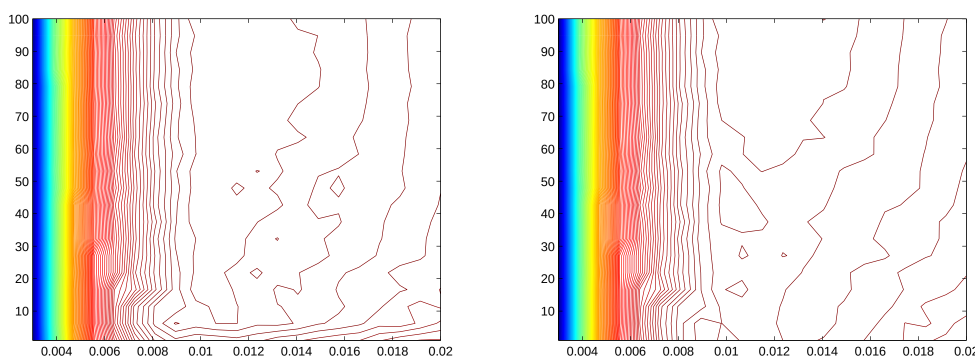
(a) Posterior (b) Likelihood

Figure 7: Risk-Sensitive Linear Estimation Results; x-axis: $\overline{\sigma}$; y-axis: γ



(a) Posterior (b) Likelihood

Figure 8: Standard Linear Estimation Results; x-axis: $\overline{\sigma}$; y-axis: γ



(a) Posterior (b) Likelihood

Figure 9: Third Order Nonlinear Moving Average Estimation Results; x-axis: $\overline{\sigma}$; y-axis: γ