Monitoring Parameter Changes in Models with a Trend

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Abstract

In this paper, we develop a monitoring procedure for detecting structural changes in models with a trend. The procedure is based on the cumulative sum (CUSUM) of the ordinary least squares residuals and a proper boundary function is designed to control the size. We derive the asymptotic distribution of the detecting statistic under the null hypothesis, while proving the consistency of the procedure under the alternative. In addition, we derive the asymptotic distribution of the delay time for the CUSUM procedure as well as the fluctuation procedure proposed by Qi *et al*. (2016). Then, we compare these two monitoring procedures in a small simulation study and the results indicate that although neither procedure is uniformly superior to the other, the CUSUM test is more suitable for an early break. An empirical example is provided to support the theoretical analyses.

JEL classification: C12, C22

Key words: sequential test, CUSUM test, fluctuation test, structural change

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1. Introduction

Structural change has long been an important issue in the statistics and econometrics literature. Much research effort has been devoted to testing for parameter instability and estimating the change points given a fixed size dataset. Such an approach in a given sample is called a retrospective or a posteriori test. However, a retrospective test cannot be applied to detect parameter changes every time new data become available because multiple testing with a given critical value will result in an uncontrollable empirical size of the test. On the contrary, to sequentially detect structural changes, a sequential or priori test is designed such that the model can be estimated from historical and new data and an online decision then made.

The first contribution to continuously monitoring parameter changes in the econometrics literature was by Chu *et al.* (1996). They introduced a monitoring scheme by setting a training period of size m in which the parameters are known to be stable as a reference for comparison with new data and argued that the key feature of the sequential tests is to construct a nondecreasing boundary function such that the tests can maintain a proper size. This approach has been developed in many directions. Leisch *et al.* (2000) extended the fluctuation test of Chu *et al.* (1996) based on moving estimates, with the boundary function having a slower growth rate to improve the sensitivity to a late break in a monitoring period. The MOSUM (moving sum) procedure was further investigated by Horváth *et al.* (2008), who indicated that prior information on the moment structure of innovations is required to choose a suitable boundary function. Horváth *et al.* (2004) discussed two classes of the residual-based cumulative sum (CUSUM) monitoring procedure with an infinite monitoring horizon and introduced an appropriate boundary function with the parameter $\gamma \in [0, 1/2)$ to deal with different timings of changes. Since the speed of detection is a crucial measure, Aue and Horváth (2004) derived the limit distribution of the stopping time for a changing mean model, which is asymptotically normal, while Aue *et al.* (2009) extended a local-level model to a linear regression model. They found that γ close to $1/2$ implies a shorter detection delay for an early break. Horváth *et al.* (2007), Aue, Horváth, Kokoszka, and Steinebach (2008), and Aue and Kühn (2008) further investigated the behaviors of the delay time in the case of $\gamma = 1/2$. Following the work of Aue and Horváth (2004), Fremdt (2014, 2015) derived the asymptotic distribution of Page's sequential CUSUM procedure and compared the asymptotic normality of the stopping time with that of the ordinary CUSUM version under a weaker condition on the change. Furthermore, the monitoring procedure for sequentially detecting parameter instability has been investigated extensively in various models. For example, Carsoule and Franses (2003) and Lee *et al.* (2009) developed sequential tests in autoregressive models, while Na *et al.* (2011) applied the monitoring procedure to detect changes for autocorrelation function, parameter instability in GARCH models, and distributional changes. Xia *et al.* (2011) and Kurozumi (2017) considered a monitoring scheme for linear models with endogenous regressors.

All the aforementioned sequential tests focus on models with nontrending regressors. However, as noted by Perron (1989) and others, macroeconomic time series are sometimes better characterized by trend-stationary series with possible change(s) in deterministics. Such evidence with an upward or downward trend has also been found in the fields of tourism, marketing, and environmental studies. For models with trending regressors, Chu and White (1992), Kuan (1998), and Aue, Horváth, Hušková, and Kokoszka (2008) among others proposed tests of parameter instability based on a given historical sample, while Qi *et al.* (2016) extended the generalized fluctuation test to monitor structural changes in polynomial regressions.

In this study, we develop a CUSUM-type monitoring scheme based on ordinary least squares (OLS) residuals to detect parameter instability in a model with a trend. A new boundary function is introduced to maintain a proper size. We derive the limit distribution of the CUSUM detecting statistic under the null hypothesis, while proving that the test is consistent under the alternative. Moreover, we investigate the asymptotic distribution of the delay time for the CUSUM (OLS-based) test as well as the fluctuation one proposed by Qi *et al.* (2016) in a model with an early change. We find that the delay time of the CUSUM test grows at a slower rate than that of the fluctuation test, which implies that the latter requires a longer time to detect an early change than the former. We also compare these two types of monitoring procedures in a small simulation study and apply them to macroeconomic time series. The results confirm that the performance of the tests strongly depends on the timing of changes. The CUSUM test is good at detecting an early change soon after the training period and has a shorter detection time than the fluctuation test, while the fluctuation test is suitable for a late break.

The remainder of the paper is as follows. In Section 2, we introduce the model and our assumptions. The asymptotic properties are investigated in Section 3. We compare the CUSUM and fluctuation tests in finite samples via Monte Carlo simulations in Section 4. Section 5 provides an empirical example and concluding remarks are given in Section 6. The mathematical proofs are relegated to the Appendix.

2. Model and Assumptions

We consider the following model:

$$y_t = x_t'\beta_t + \epsilon_t \qquad (t = 1, 2, \cdots, m, m+1, \cdots), \tag{1}$$

where $x_t = [1, t/m]'$ is a regressor including a constant term and a trend, ϵ_t is an unobservable stochastic disturbance, and $\beta_t = [\beta_{0t}, \beta_{1t}]'$ is a vector of the coefficients. Our asymptotic results do not change if the regressor is replaced by $x_t = [1, t]$.

The "noncontamination assumption", as noted by Chu *et al.* (1996), is particularly important, and we suppose that there is no change in the training period of size m, that is,

$$\beta_t = \beta_0, \quad t = 1, 2, \cdots, m.$$

The historical data are set as a reference to compare with the new data.

We are interested in testing the null hypothesis that β_t is stable and allows for a one-time change in the parameters under the alternative. Thus, we consider the testing problem given by

$$H_0: \ \beta_t = \beta_0, \quad t = m+1, m+2, \cdots$$

against the alternative hypothesis

$$H_1: \ \text{There is } k^* \geq 1 \text{ such that } \ \beta_t = \beta_0, \quad t = m+1, m+2, \cdots, m+k^*-1,$$

$$\text{but} \quad \beta_t = \beta_0 + \Delta, \quad t = m+k^*, m+k^*+1, \cdots. \quad \text{with} \quad \Delta = [\Delta_1, \Delta_2]'.$$

We reject the null hypothesis if the detecting statistic (detector) $\Gamma(m,k)$ exceeds a boundary function $g(m,k)$ for some $k \geq 1$. The detector and boundary function must be designed such that

$$\lim_{m\to\infty} P(\tau_m < \infty) = \alpha, \quad \text{under} \quad H_0, \tag{2}$$

$$\lim_{m\to\infty} P(\tau_m < \infty) = 1, \quad \text{under} \quad H_1, \tag{3}$$

where the stopping time τ_m is defined by

$$\tau_m := \begin{cases} \inf\{k \geq 1 : |\Gamma(m,k)| \geq g(m,k)\}, \\ \infty, \quad \text{if } |\Gamma(m,k)| < g(m,k) \text{ for all } k = 1, 2, \cdots . \end{cases} \tag{4}$$

Condition (2) ensures that the probability of a false alarm is given by α, while Condition (3) means that we reject the hypothesis of no change with a probability approaching one under the alternative.

To investigate the asymptotic properties of the monitoring test, we impose the following assumption.

Assumption 1 *For every m, there are two independent sequences of Wiener processes $\{W_{1,m}(t), t \geq 0\}$, $\{W_{2,m}(t), t \geq 0\}$ and a constant $\sigma > 0$ such that, for some $\nu > 2$,*

$$\sup_{1\leq k<\infty} \frac{1}{k^{1/\nu}} \left| \sum_{t=m+1}^{m+k} \epsilon_t - \sigma W_{1,m}(k) \right| = O_p(1), \tag{5}$$

$$\sum_{t=1}^{m} \epsilon_t - \sigma W_{2,m}(m) = o_p(m^{1/\nu}). \tag{6}$$

The sequence ϵ_t satisfying Conditions (5) and (6) includes not only an *i.i.d.* sequence but also a dependent sequence with some regularity conditions, as discussed by Aue *et al*. (2004). From Conditions (5) and (6), we can derive the following approximation.

$$\sup_{1\leq m<\infty} \frac{1}{m^{1/\nu}} \left| \sum_{t=1}^{m} \frac{t}{m} \epsilon_t - \frac{\sigma}{m} \int_0^m x dW_{2,m}(x) \right| = O_p(1). \tag{7}$$

See, for example, Aue, Horváth, Hušková, and Kokoszka (2008).

3. Monitoring Procedure for a Change in the Trend

3.1 CUSUM-based monitoring procedure

The CUSUM procedure is based on the future residuals of the model given by

$$\hat{\epsilon}_t := y_t - x_t'\hat{\beta}_m, \tag{8}$$

where $\hat{\beta}_m$ is an OLS estimator from the historical data given by

$$\hat{\beta}_m := \left(\sum_{t=1}^{m} x_t x_t'\right)^{-1} \sum_{t=1}^{m} x_t y_t.$$

The CUSUM detector is defined by

$$\Gamma(m,k) := \frac{1}{\hat{\sigma}_m}\tilde{\Gamma}(m,k) \qquad \text{where} \qquad \tilde{\Gamma}(m,k) := \sum_{t=m+1}^{m+k} \hat{\epsilon}_t,$$

for $k = 1, 2, \ldots$ where $\hat{\sigma}_m^2$ is the consistent estimator of σ^2 obtained from the training period.

Letting $k/m = \lambda$, (20) in the proof of Theorem 1(i) provides the asymptotic distribution of the detector as follows:

$$\begin{aligned} \frac{1}{\hat{\sigma}_m\sqrt{m}} \sum_{t=m+1}^{m+k} \hat{\epsilon}_t &\Rightarrow (\lambda+1)W_1\left(\frac{\lambda}{\lambda+1}\right) + \sqrt{3}\lambda(\lambda+1)W_2(1) \\ &=: G(\lambda), \end{aligned} \tag{9}$$

where $W_1(\cdot)$ and $W_2(\cdot)$ are independent Wiener processes. This process has zero mean and a growing variance

$$3\lambda^2(\lambda+1)^2 + \lambda(\lambda+1), \tag{10}$$

which implies that a constant boundary cannot be used for the monitoring procedure because the detecting statistic will eventually exceed a constant boundary and the null hypothesis will be rejected with a probability approaching one even if the parameter is stable.

As noted by Chu *et al*. (1996), the growing variance of the asymptotic distribution of the detector induces an increasing monitoring boundary. The boundary function cannot grow at a too slow rate because the monitoring procedure will have a high probability of type one error, while a boundary function with a too fast growth rate will result in the low power of the test. The distribution of $G(\lambda)$ shows that the first term is dominated by the second one as λ increases, which determines the growth rate of the limiting process, and this enables us to find a suitable form of the boundary function. Based on the boundary function proposed by Horváth *et al*. (2004), we also allow for a flexible adjustment of the test to deal with an early break in the monitoring period in terms of the parameter $\gamma \in (0, 1/2)$. Thus, we design the boundary function such that it grows at the rate $\sqrt{3}\lambda^\gamma(\lambda+1)^{2-\gamma}$. This means that the probability of the excess over the boundary can be controlled to maintain a proper size. Then, we propose the boundary function given by

$$g(m,k) := c\sqrt{3m}\left(\frac{k}{m}\right)^{\gamma}\left(1+\frac{k}{m}\right)^{2-\gamma}, \quad \text{for some} \quad 0<\gamma<\frac{1}{2}.$$

This boundary function over $\sqrt{m}$ grows approximately at the rate $\lambda^\gamma(1+\lambda)^{2-\gamma}$ to ensure that $P\{|G(\lambda)| \geq c\sqrt{3}\lambda^\gamma(1+\lambda)^{2-\gamma}\}$ equals α for some c.

We next derive the limiting properties of the procedure in Theorem 1.

Theorem 1 *Suppose that Assumption 1 holds.*

(i) Under the null hypothesis, we have

$$\lim_{m\to\infty} P\left(\sup_{1\leq k<\infty}|\Gamma(m,k)| \leq g(m,k)\right) = P\left(\sup_{0\leq t\leq 1}\left|\frac{1-t}{\sqrt{3}t^\gamma}W_1(t)+t^{1-\gamma}W_2(1)\right| \leq c\right),$$

where $\{W_1(t), 0\leq t<\infty\}$ *and* $\{W_2(t), 0\leq t<\infty\}$ *are independent Wiener processes.*

(ii) Suppose that $\Delta_2 \neq 0$. *Then, under the alternative, we have*

$$\sup_{1\leq k<\infty}\frac{|\Gamma(m,k)|}{g(m,k)} \to \infty, \quad \textit{as} \quad m\to\infty.$$

In Theorem 1(i), the critical value $c = c(\alpha)$ determined by the significance level α can be obtained from the asymptotic distribution of the detector. In practice, the monitoring period of the procedure cannot go to infinity; instead, a researcher determines how long he/she would like to monitor a change (the length of the monitoring horizon). Therefore, we suppose that k is given by κm for some $\kappa > 0$. This means that we start testing at time $m+1$ and stop at time $m+\kappa m$. Then, the critical values are obtained from

$$\lim_{m\to\infty} P(\sup_{1\leq k<\kappa m}|\Gamma(m,k)| \leq g(m,k)) = P\left(\sup_{0\leq t\leq \frac{\kappa}{\kappa+1}}\left|\frac{1-t}{\sqrt{3}t^\gamma}W_1(t)+t^{1-\gamma}W_2(1)\right| \leq c(\alpha)\right) = \alpha, \tag{11}$$

which depends on the various selections of κ and γ in the boundary function. We choose $\kappa = 1, 2, \ldots, 8$, and $\gamma = 0.05, 0.15, \ldots, 0.45$, and approximate Brownian motions using 1,000 independent normal random variables with 1,000,000 replications to obtain the critical values in Table 1.

Theorem 1(i) shows that the asymptotic distribution is composed of two independent Wiener processes. In the case of $t \to 0$, because of the law of the iterated logarithm, the condition $\gamma < 1/2$ ensures that the term $|W_1(t)|/t^{\gamma}$ converges to zero and consequently the asymptotic distribution tends to zero. As $t \to 1$, the first component associated with $W_1(t)$ converges to zero and thus the distribution of the procedure is determined by the second Wiener process. As a result, the proposed boundary function enables the CUSUM monitoring procedure to maintain a nondegenerate and finite limit. Theorem 1(ii) implies that the CUSUM monitoring test is consistent and that (3) is satisfied. We can see that the diverging order of the detecting statistic crucially depends on the term $\sup_{1 \le k < \infty} |\sum_{t=m+k^*}^{m+k} x_t' \Delta|/g(m, k)$ under the alternative from the proof of Theorem 1(ii). This term is guaranteed to diverge to infinity as far as $\Delta_2 \neq 0$.

3.2 Asymptotic distributions of the stopping times

The monitoring procedure generally rejects the null hypothesis of no change possibly with a delay after the break. Since a shorter detection delay implies a more reliable conclusion and lower cost, the speed of detection is an important measure for the sequential test. Thus, we expect that the procedure should reject the null hypothesis as soon as possible under the alternative. In this section, we derive the limiting distribution of the stopping time based on the CUSUM monitoring test as well as that based on the test presented by Qi *et al.* (2016), who also proposed a monitoring test for a change in a trend. We thus investigate the theoretical difference in their limiting behaviors.

To investigate the asymptotic property of the stopping time based on the CUSUM detector, we make the following assumption related to k^* and Δ.

Assumption 2 *(a) There exists a $\theta > 0$ such that $k^* = O(m^{\theta})$ for some $0 \le \theta < \frac{1-2\gamma}{4(1-\gamma)}$.*
(b) Let $\delta := d'\Delta = \Delta_1 + \Delta_2$ where $d := [1, 1]'$. There are positive constants C_1 and C_2 such

that,

$$C_1 \le |\delta| \le C_2.$$

Assumption 2(a) implies that the order of the change-point k^* is related to the historical sample size m. We focus on the same case as Aue *et al.* (2009) that a break occurs shortly after the end of the training period.[2] Assumption 2(b) assumes that the magnitude of the change is bounded and excludes by a technical reason the case where a change in the trend coefficient is in the opposite direction to a change in a constant with the same magnitude ($\Delta_2 = -\Delta_1$).

Under this assumption, we derive the asymptotic distribution of the stopping time based on the CUSUM detector.

Theorem 2 *Suppose that Assumptions 1 and 2 hold. Then, we have*

$$\lim_{m\to\infty} P(\tau_m \le a_m + b_m z) = \Phi(z),$$

where $\Phi(\cdot)$ *is the cumulative distribution function of a standard normal distribution,*

$$a_m := \left\{ c_m^{1-\gamma} - \frac{1}{c_m^{\gamma}\delta} \sum_{t=m+k^*}^{m+c_m} (x_t - d)'\Delta \right\}^{\frac{1}{1-\gamma}},$$

$$b_m := \frac{\sqrt{c_m}\sigma}{(1-\gamma)|\delta|}, \quad \text{and} \quad c_m := \left(\frac{\sqrt{3}c\sigma m^{1/2-\gamma}}{|\delta|} \right)^{\frac{1}{1-\gamma}}.$$

We next derive the asymptotic distribution of the delay time based on the maximal-type fluctuation procedure of Qi *et al.* (2016). The detector of Qi *et al.* (2016) is defined by $\Gamma_m^{FL}(k,\ell) := \hat{\sigma}_m^{-1}\tilde{\Gamma}_m^{FL}(k,\ell)$, where $\ell \ne k$ is proportional to k as supposed in Assumption 3(a) and

$$\tilde{\Gamma}_m^{FL}(k,\ell) := \sum_{t=1}^{m+k} y_t - \frac{m+k}{m}\sum_{t=1}^{m} y_t - \frac{k(m+k)}{\ell(m+\ell)} \left(\sum_{t=1}^{m+\ell} y_t - \frac{m+\ell}{m}\sum_{t=1}^{m} y_t \right),$$

and the boundary function is given by

$$g^{FL}(m,k) := c^{FL}\sqrt{m}\left(\frac{m+k}{m}\right)^2.$$

[2]The order of k^* can be slightly relaxed to $O(m^{3/8})$ for the fluctuation test, as is seen in the proof of Lemma 3.1(i) and (53).

Then, the corresponding stopping time is defined as

$$\tau_m^{FL} := \inf\left\{k \geq 1 : |\Gamma^{FL}(k,\ell)| \geq g^{FL}(m,k)\right\}.$$

The critical value $c^{FL} = c^{FL}(\alpha)$ is determined by the asymptotic distribution of the detector under the null hypothesis. See Qi *et al.* (2016) for more details.[3]

To derive the limiting distribution of the stopping time based on the fluctuation test, we need to make an additional assumption.

Assumption 3 *(a) Let $\ell = [\frac{k+1}{\eta}]$ with $\eta > 1$, where the bracket means the integer part of the term.*
(b) Suppose that $\Delta_1 + \Delta_2/2 \neq 0$.

Assumption 3(a) defines the relationship between the parameters k and ℓ, where k is required to be greater than ℓ. Qi *et al.* (2016) used $\ell = k/2$ in their simulations but it can be relaxed as in (a). Assumption 3(b) is a necessary technical condition to ensure that the limit results in Lemmas 3.2 and 3.6 can be derived.

The asymptotic distribution of the stopping time based on the fluctuation test is given in the following theorem.

Theorem 3 *Suppose that Assumptions 1, 2(a), and 3 hold. Then, we have*

$$\lim_{m\to\infty} P\left(\tau_m^{FL} \leq a_m^{FL} + b_m^{FL} z\right) = \Phi(z),$$

where

$$a_m^{FL} := \left\{\frac{c^{FL}\sigma m^{3/2}}{\left|\left(\frac{1}{\eta}-1\right)\left(\delta - \frac{\Delta_2}{2}\right)\right|}\right\}^{1/2} \quad \textit{and} \quad b_m^{FL} := \frac{\sqrt{\eta-1}m\sigma}{2\sqrt{a_m^{FL}}\left|\left(\frac{1}{\eta}-1\right)\left(\delta - \frac{\Delta_2}{2}\right)\right|}.$$

Theorems 2 and 3 show that the limit distributions of the stopping times are normal. The sequences a_m, b_m, a_m^{FL}, and b_m^{FL} are used to standardize the variables to obtain the limiting distributions. We can show that $\tau_m/a_m \xrightarrow{p} 1$ and $\tau_m^{FL}/a_m^{FL} \xrightarrow{p} 1$. Since both a_m and a_m^{FL}

[3]Qi *el al.* (2016) also considered the range-type test. However, its finite sample property is inferior to the maximum-type test considered in this article according to their simulations and thus we focus on the latter test.

diverge to infinity, both the stopping times also go to infinity. Of importance is the difference in the diverging rates. In the case of the CUSUM detector, a_m is of the order $m^{(1/2-\gamma)/(1-\gamma)}$, which takes the value among $(m^0, m^{1/2})$ depending on γ, whereas a_m^{FL} is of the order $m^{3/4}$. This implies that the stopping time based on the fluctuation test grows at a faster rate than that based on the CUSUM detector. Thus, we expect that the delay time based on the CUSUM procedure tends to be shorter than that based on the fluctuation one. In other words, the monitoring test based on the CUSUM detector has a theoretical advantage over that based on the fluctuation test as far as the break occurs early in the monitoring period. This is confirmed by the Monte Carlo simulations in Section 4.

4. Finite Sample Properties

In this section, we investigate the finite sample performance of the tests considered in the previous section. The data-generating process we consider is given by

$$y_t = x_t'(\beta + \delta 1_{\{t \geq m+k^*\}}) + \epsilon_t, \quad \epsilon_t = \rho\epsilon_{t-1} + e_t, \quad t = 1, \ldots, m, m+1, \ldots, m + \kappa m.$$

where $x_t = [1, t/m]'$, $\beta = [1, 1]'$, and $\{e_t\} \sim i.i.d.N(0, (1-\rho)^2)$, meaning that the long-run variance of ϵ_t is 1. The settings for δ and k^* are explained later. In finite samples, we consider that the monitoring period stops at $2m$ ($\kappa = 1$), while the training period m is 50, 100, and 250. The parameter γ in the boundary function is set to 0.15 and 0.45. We allow for serial correlation in the errors and the coefficient ρ is 0.4 and 0.8. To obtain a consistent estimate $\hat{\sigma}_m^2$ of the variance of the errors based on the historical data, we use the prewhitened kernel estimator proposed by Andrews and Monahan (1992), which is defined by

$$\hat{\sigma}_m^2 \coloneqq (1-\hat{\rho})^{-1}\hat{\Omega}(1-\hat{\rho})^{-1},$$

where $\hat{\Omega}$ is a standard kernel heteroskedasticity and autocorrelation consistent estimator given by

$$\hat{\Omega} \coloneqq \frac{m}{m-2}\left\{\hat{\Gamma}_0 + \sum_{j=1}^{m-1} k\left(\frac{j}{\hat{S}_m}\right)(\hat{\Gamma}_j + \hat{\Gamma}_j')\right\}, \quad \text{with} \quad \hat{\Gamma}_j \coloneqq \frac{1}{m}\sum_{t=j+1}^{m} \hat{e}_t\hat{e}_{t-j}.$$

The coefficient estimate $\hat{\rho}$ and residuals $\hat{e}_t$ are obtained by regressing $\hat{\epsilon}_t$ on $\hat{\epsilon}_{t-1}$, where the OLS residuals $\hat{\epsilon}_t$ are calculated from regressing y_t on x_t. In this simulation, we use the quadratic spectral kernel as $k(\cdot)$, which is defined by

$$k(x) \coloneqq \frac{25}{12\pi^2 x^2}\left(\frac{\sin(6\pi x/5)}{6\pi x/5} - \cos(6\pi x/5)\right),$$

while the bandwidth $\hat{S}_m$ is selected based on Andrews (1991) given by

$$\hat{S}_m \coloneqq 1.3221(\hat{\alpha}(2)m)^{1/5} \quad \text{where} \quad \hat{\alpha}(2) \coloneqq \frac{4\hat{\rho}^2\hat{\sigma}_e^4}{(1-\hat{\rho})^8} \Bigg/ \frac{\hat{\sigma}_e^4}{(1-\hat{\rho})^4},$$

and $\hat{\sigma}_e^2$ is the estimated variance of the residuals $\hat{e}_t$. The significance level is set to 0.05, the number of replications is 5,000, and all computations are conducted using the GAUSS matrix language.

Table 2 summarizes the empirical sizes of the monitoring procedures. The sizes of both the CUSUM and fluctuation tests in the cases of $m = 100, \rho = 0.4$ and $m = 250, \rho = 0.4$ are controlled well, whereas for the other cases, the sizes are relatively distorted, especially when $\rho = 0.8$.[4]

For a comparison of the power performance, the change in the coefficient is specified by $\delta = bd$, where the magnitude b is set to $0, 0.5, 1.0, 1.5, 2.0$ and $d = [-1, -1]'$ or $[1, 1]'$. Tables 3 and 4 report the powers of the monitoring tests corresponding to an early break ($m + k^* = m + 1$) and a late break ($m + k^* = 1.8m$), respectively. The results imply that neither the CUSUM test nor the fluctuation test dominates the other in small samples. The CUSUM tests are more powerful than the fluctuation tests when the break occurs soon after the historical data. For a late break in the monitoring period, the powers of the fluctuation tests are higher than those of the CUSUM tests. Additionally, all the monitoring tests are more powerful for a larger magnitude of change. Table 4 indicates that the power of all the tests is particularly low for a small shift ($b = 0.5$) because there are fewer observations after a late break.

We further investigate the effect of the time of the change on the power performance. The break date is controlled by $m + k^* = 1.1m, 1.2m, ..., 1.9m$ ($k^* = 0.1m, 0.2m, ..., 0.9m$); Table 5 reports the results. The CUSUM-based monitoring procedure performs better under early-change settings and the earlier the change occurs, the better the tests perform. On the

[4] We also conducted simulations for the range-type test by Qi *et al.* (2016), but the maximum-type test outperforms the rage-type one in many cases and thus we report only the former result.

contrary, the power of the fluctuation tests increases first and then decreases as k^* changes from $0.1m$ to $0.9m$.

Next, we compare the delay times of the two procedures since the detection speed is regarded as an important indicator of the performance of the monitoring tests. We set $d = [-1, -1]'$ such that Assumptions 2(b) and 3(b) can be satisfied and consider the breaks occurring at $m + k^* = m + 1$ and $1.8m$. Tables 6 and 7 summarize the minimum value, quartiles, and maximum value of the delay time. The CUSUM procedure with $\gamma = 0.45$ has a shorter detection delay than the procedure with $\gamma = 0.15$. If a change occurs rapidly after the end of the training period, the CUSUM version rejects the null hypothesis earlier than the fluctuation version and the minimum value, quartiles, and maximum value of the delay time for the CUSUM tests are much smaller. This is consistent with the theoretical result that the stopping time based on the CUSUM version grows at a slower rate than for the fluctuation version. For a late break in the monitoring period, there is a slight difference between two procedures and the delay time of the fluctuation test is shorter in some cases.

In summary, the CUSUM procedure is good at detecting a change occurring soon after the training period, whereas the fluctuation test can detect a late break in the monitoring period better. Neither version is uniformly superior to the other according to the comparison of their power. However, we find that the CUSUM procedure can reject the null hypothesis earlier than the fluctuation test in many cases from the performance of the delay time.

5. Empirical Example

In this section, we apply the monitoring tests to sequentially detect parameter instability in macroeconomic time series. The following simple linear trend model is considered:

$$y_t = \alpha + \beta t + \epsilon_t,$$

where y_t is the logarithm of real GDP measured in the domestic currency. Three countries, namely Denmark, Japan, and New Zealand, are selected and quarterly data are taken from the International Financial Statistics database. The sample periods are different for each country and Figure 1 describes the logarithm of the real GDP series of the three countries.

We first apply the historical test proposed by Perron and Yabu (2009) to detect structural changes in the whole sample and find that the null hypothesis of no change in the parameters is rejected. Then, we estimate the break date by minimizing the sum of the squared residuals and test for parameter constancy in the period before the estimated break. Because GDP series can have a unit root, we also investigate the presence of a unit root. The results in Table 8 indicate that the parameters are stable and that the unit root hypothesis can be rejected for the three series, which implies that we can set the period before the estimated break as the training period.

We next investigate whether the two procedures can successfully detect the parameter changes in the three GDP series and compare their speed of detection. We set different training periods corresponding to the different timings of the break in the monitoring period and the results are summarized in Table 9. In the case of Japan, the end points of the training period are set to 1998Q4, 2006Q2, and 2007Q4, which correspond to the late, moderate, and early breaks in the monitoring period. We can see that all the tests reject the null hypothesis of no change in the parameters, except the fluctuation test when the break occurs early. Moreover, the maximal fluctuation test has a much longer detection delay than the CUSUM procedure in the case of the late break, while for a moderate change, the maximal fluctuation test performs better than the others. For Denmark, both procedures can detect an early change and the CUSUM method rejects the null hypothesis of no change much earlier than the fluctuation method, which is consistent with our theoretical analysis that the CUSUM test is expected to have a shorter detection delay than the fluctuation one if the change occurs early in the monitoring period. We also find evidence of the better performance of the CUSUM-based tests for an early change in the case of New Zealand, while the fluctuation test is good at detecting a relatively late break in the monitoring period, as shown in the simulations.

6. Conclusion

In this study, we applied the CUSUM test based on OLS residuals to sequentially detect structural change in models with a trend. The asymptotic property of the CUSUM monitoring procedure was investigated and the results indicated that it can successfully reject the null hypothesis of no change. We further derived the asymptotic distributions of the stopping times based on the CUSUM and fluctuation procedures and found that the delay time based on the CUSUM procedure is shorter than that based on the fluctuation one in the case of an early break. This tendency is confirmed in finite samples, although the fluctuation test works better in some cases.

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Appendix

In this appendix, we replace $\hat{\sigma}_m^2$ with σ^2 because it is consistent under both the null and alternative hypotheses. Let

$$C_m \coloneqq \sum_{t=1}^{m} x_t', \quad C \coloneqq \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}.$$

Then, we have

$$\left\| \frac{1}{m} C_m - C \right\| = O\left(\frac{1}{m}\right), \tag{12}$$

$$\left\| \left(\frac{1}{m} C_m\right)^{-1} - C^{-1} \right\| = O\left(\frac{1}{m}\right). \tag{13}$$

We rewrite $\tilde{\Gamma}(m,k)$ as follows:

$$\sum_{t=m+1}^{m+k} \hat{\epsilon}_t = \sum_{t=m+1}^{m+k} \epsilon_t - \sum_{t=m+1}^{m+k} x_t' \left(\sum_{t=1}^{m} x_t x_t'\right)^{-1} \sum_{t=1}^{m} x_t \epsilon_t. \tag{14}$$

Lemma 1.1. Under Assumption 1,

$$\sup_{1 \le k < \infty} \frac{\left| \sum\limits_{t=m+1}^{m+k} x_t' C_m^{-1} \sum\limits_{t=1}^{m} x_t \epsilon_t - \frac{1}{m} \sum\limits_{t=m+1}^{m+k} x_t' C^{-1} \sum\limits_{t=1}^{m} x_t \epsilon_t \right|}{h(m,k)} = o_p(1), \quad \text{as} \quad m \to \infty,$$

where $h(m,k) \coloneqq g(m,k)/c$ with $c = c(\alpha)$ determined by the given significance level α.

Proof. Relations (6) and (7) imply that

$$\left\| \sum_{t=1}^{m} x_t \epsilon_t \right\| = O_p(\sqrt{m}), \quad \text{as} \quad m \to \infty. \tag{15}$$

Putting together (12), (13), and (15), we have

$$\begin{aligned}
&\sup_{1 \le k < \infty} \frac{\left| \sum\limits_{t=m+1}^{m+k} x_t' C_m^{-1} \sum\limits_{t=1}^{m} x_t \epsilon_t - \frac{1}{m} \sum\limits_{t=m+1}^{m+k} x_t' C^{-1} \sum\limits_{t=1}^{m} x_t \epsilon_t \right|}{h(m,k)} \\
\le\; & \sup_{1 \le k < \infty} \frac{\left\| \frac{1}{m} \sum\limits_{t=m+1}^{m+k} x_t' \right\| \left\| (\frac{1}{m} C_m)^{-1} - C^{-1} \right\| \left\| \sum\limits_{t=1}^{m} x_t \epsilon_t \right\|}{h(m,k)} \\
=\; & \sup_{1 \le k < \infty} \frac{\left\| \left[\frac{k}{m}, \frac{k}{m} + \frac{k^2}{2m^2} + \frac{k}{2m^2} \right] \right\|}{h(m,k)} O\left(\frac{1}{m}\right) O_p(\sqrt{m}).
\end{aligned}$$

Since $k/m + k^2/(2m^2)$ is the dominating term of $\sum_{t=m+1}^{m+k} x_t/m$, and

$$\sup_{1\le k<\infty} \frac{\left|\frac{k}{m}+\frac{k^2}{2m^2}\right|}{\left(\frac{k}{m}\right)^{\gamma}\left(1+\frac{k}{m}\right)^{2-\gamma}} \le \sup_{1\le k<\infty}\left(\frac{k}{m}\right)^{1-\gamma}\left(1+\frac{k}{m}\right)^{\gamma-1} = O(1),$$

then the proof is complete.■

Lemma 1.2. Under Assumption 1,

$$\sup_{1\le k<\infty} \frac{\left|\sum_{t=m+1}^{m+k}\epsilon_t - \frac{1}{m}\sum_{t=m+1}^{m+k} x_t' C^{-1}\sum_{t=1}^{m} x_t\epsilon_t - \tilde{G}(m,k)\right|}{h(m,k)} = o_p(1), \tag{16}$$

where

$$\tilde{G}(m,k) := \sigma W_{1,m}(k) + \frac{k}{m}\tilde{G}_1(m) + \left(\frac{k^2}{m^2}+\frac{k}{m^2}\right)\tilde{G}_2(m),$$
$$\tilde{G}_1(m) := 2\sigma W_{2,m}(m) - 6\sigma\int_0^m \frac{x}{m}dW_{2,m}(x),$$
$$\tilde{G}_2(m) := 3\sigma W_{2,m}(m) - 6\sigma\int_0^m \frac{x}{m}dW_{2,m}(x).$$

Proof. The left-hand side of (16) becomes

$$\begin{aligned}
&\sup_{1\le k<\infty} \frac{\left|\sum_{t=m+1}^{m+k}\epsilon_t - \frac{1}{m}\sum_{t=m+1}^{m+k} x_t' C^{-1}\sum_{t=1}^{m} x_t\epsilon_t - \tilde{G}(m,k)\right|}{h(m,k)}\\
\le\ & \sup_{1\le k<\infty}\frac{\left|\sum_{t=m+1}^{m+k}\epsilon_t - \sigma W_{1,m}(k)\right|}{h(m,k)} + \sup_{1\le k<\infty}\frac{\frac{k}{m}\left|2\sum_{t=1}^{m}\epsilon_t - 6\sum_{t=1}^{m}\frac{t}{m}\epsilon_t - \tilde{G}_1(m)\right|}{h(m,k)}\\
&+ \sup_{1\le k<\infty}\frac{\left(\frac{k^2}{m^2}+\frac{k}{m^2}\right)\left|3\sum_{t=1}^{m}\epsilon_t - 6\sum_{t=1}^{m}\frac{t}{m}\epsilon_t - \tilde{G}_2(m)\right|}{h(m,k)}\\
=:\ & A_1 + A_2 + A_3.
\end{aligned}$$

Under Assumption 1, $1/\nu < 1/2$, and $\gamma < 1/2$, we can see that $A_1 = o_p(1)$ because

$$\sup_{1\le k\le m}\frac{\left|\sum_{t=m+1}^{m+k}\epsilon_t - \sigma W_{1,m}(k)\right|}{h(m,k)} = \sup_{1\le k\le m}\frac{O_p(k^{1/\nu})m^{\gamma-1/2}}{k^{\gamma}(1+k/m)^{2-\gamma}} \le \left\{\begin{array}{ll} O_p(1)m^{\gamma-1/2} & \text{if } 1/\nu<\gamma \\ O_p(1)m^{1/\nu-1/2} & \text{if } 1/\nu\ge\gamma \end{array}\right\} = o_p(1),$$

and in the case of $m < k < \infty$,

$$\sup_{m<k<\infty} \frac{O_p(k^{1/\nu})m^{\gamma-1/2}}{k^\gamma(1+k/m)^{2-\gamma}}$$
$$< \left\{ \begin{array}{ll} O_p(1)m^{1/\nu-1/2} & \text{if } 1/\nu < \gamma \\ O_p(1) \sup\limits_{m<k<\infty} k^{1/\nu-\gamma}m^{\gamma-1/2}(\frac{m}{k})^{2-\gamma} < O_p(1)m^{1/\nu-1/2} & \text{if } 1/\nu \geq \gamma \end{array} \right\} = o_p(1).$$

For A_2, we have

$$\begin{aligned} A_2 &\leq \sup_{1\leq k<\infty} \frac{\frac{k}{m}2\left|\sum\limits_{t=1}^m \epsilon_t - \sigma W_{2,m}(m)\right|}{h(m,k)} + \sup_{1\leq k<\infty} \frac{\frac{k}{m}6\left|\sum\limits_{t=1}^m \frac{t}{m}\epsilon_t - \sigma\int_0^m \frac{x}{m}dW_{2,m}(x)\right|}{h(m,k)} \\ &= \sup_{1\leq k<\infty} \frac{k/m}{\sqrt{3}(k/m)^\gamma(1+k/m)^{2-\gamma}} O_p(m^{1/\nu-1/2}). \end{aligned}$$

Since

$$\sup_{1\leq k\leq m} \frac{k/m}{(k/m)^\gamma(1+k/m)^{2-\gamma}} = \sup_{1\leq k\leq m} \frac{(k/m)^{1-\gamma}}{(1+k/m)^{2-\gamma}} = O(1),$$
$$\sup_{m<k<\infty} \frac{k/m}{(k/m)^\gamma(1+k/m)^{2-\gamma}} < \sup_{m<k<\infty} \left(\frac{k}{m}\right)^{1-\gamma}\left(\frac{m}{k}\right)^{2-\gamma} = O(1),$$

we obtain $A_2 = o_p(1)$. Similarly, for the term A_3, we can see that

$$A_3 = O\left(\sup_{1\leq k<\infty} \frac{\frac{k^2}{m^2} + \frac{k}{m^2}}{\sqrt{3}(k/m)^\gamma(1+k/m)^{2-\gamma}}\right) O_p(m^{1/\nu-1/2}) = o_p(1).$$

Thus, the proof is complete.■

Proof of Theorem 1:(i) Since the distribution of $\{W_{1,m}(t), W_{2,m}(t), 0 \leq t < \infty\}$ does not depend on m, we omit the subscript m in the following. We first establish that

$$\begin{aligned} &\frac{1}{\sigma}\sup_{1\leq k<\infty} \frac{|\tilde{G}(m,k)|}{h(m,k)} \\ &\stackrel{\text{D}}{=} \sup_{1\leq k<\infty} \frac{\left| \begin{array}{r} W_1(k) + \frac{k}{m}(2W_2(m) - 6\int_0^m \frac{x}{m}dW_2(x)) \\ +\frac{k^2}{m^2}(3W_2(m) - 6\int_0^m \frac{x}{m}dW_2(x)) \end{array} \right| \Big/ \sqrt{m}}{\sqrt{3}(k/m)^\gamma(1+k/m)^{2-\gamma}} \quad (17) \\ &\stackrel{\text{D}}{=} \sup_{1\leq k<\infty} \frac{|(\frac{k}{m}+1)W_1(\frac{k}{k+m}) + \sqrt{3}\frac{k}{m}(\frac{k}{m}+1)W_2(1)|}{\sqrt{3}(k/m)^\gamma(1+k/m)^{2-\gamma}}. \quad (18) \end{aligned}$$

The first equality in the distribution holds because

$$\sup_{1\leq k<\infty} \left| \frac{k}{m^2}\left(3\frac{1}{\sqrt{m}}W_2(m) - 6\frac{1}{\sqrt{m}}\int_0^m \frac{x}{m}dW_2(x)\right)\right| \Big/ \left\{\sqrt{3}(k/m)^\gamma(1+k/m)^{2-\gamma}\right\} = o_p(1),$$

which can be shown by noting that the process $(3W_2(m) - 6\int_0^m \frac{x}{m} dW_2(x))/\sqrt{m}$ has zero expectation and a finite variance independent of m, which implies that $\left|(3W_2(m) - 6\int_0^m \frac{x}{m} dW_2(x))/\sqrt{m}\right| = O_p(1)$, and

$$\sup_{1 \le k < \infty} \frac{|k/m^2|}{\sqrt{3}(k/m)^\gamma (1+k/m)^{2-\gamma}} = o(1).$$

For the second equality in distribution given by (18), let $k/m = \lambda$, and the numerator of (17) can be written as

$$G(\lambda) := W_1(\lambda) + \lambda\left(2\frac{1}{\sqrt{m}}W_2(m) - 6\frac{1}{\sqrt{m}}\int_0^m \frac{x}{m}dW_2(x)\right) + \lambda^2\left(3\frac{1}{\sqrt{m}}W_2(m) - 6\frac{1}{\sqrt{m}}\int_0^m \frac{x}{m}dW_2(x)\right).$$

This is a Gaussian process with zero mean and covariance function given by

$$E[G(s)G(t)] = s(t+1) + 3st(s+1)(t+1) \quad \text{for} \quad s < t, \tag{19}$$

which can be decomposed into the two independent processes $X_1(\lambda)$ and $X_2(\lambda)$ with the covariance functions $s(t+1)$ and $3st(s+1)(t+1)$, respectively. Since the covariance function of $X_1(\lambda)$ can be written as $E[X_1(s)X_1(t)] = u(s)v(t)$ for $s < t$, where $u(s) := s$ and $v(t) := t+1$, we can use the technique of Doob (1949) to transform $X_1(\lambda)$ into a Brownian motion. Let $a(\lambda) := u(\lambda)/v(\lambda) = \lambda/(\lambda+1)$, which is continuous and monotonically increasing with inverse $b(\lambda) := \lambda/(1-\lambda)$. Then, $X_1(b(\lambda))/v(b(\lambda))$ is a standard Brownian motion because $E[X_1(b(\lambda))/v(b(\lambda))] = 0$ and the covariance function is $\min(s,t)$. This implies that $X_1(\lambda) \overset{\mathrm{D}}{=} v(\lambda)W(a(\lambda))$. And we can also see that $X_2(\lambda) \overset{\mathrm{D}}{=} \sqrt{3}\lambda(\lambda+1)W(1)$. Hence, we transform a Gaussian process $G(\lambda)$ into a functional of the two independent Brownian motions $W_1(\cdot)$ and $W_2(\cdot)$ as follows:

$$G(\lambda) \overset{\mathrm{D}}{=} (\lambda+1)W_1\left(\frac{\lambda}{\lambda+1}\right) + \sqrt{3}\lambda(\lambda+1)W_2(1), \tag{20}$$

and we thus obtain the second equality in distribution in (18).

We next derive the limiting distribution of (18). The continuity of $\{(t+1)W_1(\frac{t}{t+1}) + \sqrt{3}t(t+1)W_2(1)\}/\{\sqrt{3}t^\gamma(1+t)^{2-\gamma}\}$ on $[0,T]$ for a given $T > 0$ yields that,

$$\sup_{1 \le k \le mT} \frac{\left|(\frac{k}{m}+1)W_1(\frac{k}{k+m}) + \sqrt{3}\frac{k}{m}(\frac{k}{m}+1)W_2(1)\right|}{\sqrt{3}(k/m)^\gamma(1+k/m)^{2-\gamma}} \to \sup_{0 < t \le T} \frac{\left|(t+1)W_1(\frac{t}{t+1}) + \sqrt{3}t(t+1)W_2(1)\right|}{\sqrt{3}t^\gamma(t+1)^{2-\gamma}} \quad \text{a.s.}$$

For $k \ge mT$, we have

$$\sup_{mT \le k < \infty} \frac{\left|(\frac{k}{m}+1)W_1(\frac{k}{k+m})\right|}{\sqrt{3}(k/m)^\gamma(1+k/m)^{2-\gamma}} \le \sup_{T \le t < \infty} \frac{\left|(t+1)W_1(\frac{t}{t+1})\right|}{\sqrt{3}t^\gamma(t+1)^{2-\gamma}} = \sup_{T \le t < \infty} \frac{\left|W_1(\frac{t}{t+1})\right|}{\sqrt{3}t^\gamma(t+1)^{1-\gamma}}, \tag{21}$$

and from the law of the iterated logarithm, we have for any $\delta > 0$,

$$\lim_{T\to\infty} P\left(\sup_{T\le t<\infty} \frac{\left|W_1(\frac{t}{t+1})\right|}{\sqrt{3}t^\gamma (t+1)^{1-\gamma}} > \delta \right) = 0. \tag{22}$$

We also have

$$\sup_{mT\le k<\infty} \left| \frac{\sqrt{3}\frac{k}{m}(\frac{k}{m}+1)}{\sqrt{3}(k/m)^\gamma (1+k/m)^{2-\gamma}} - 1 \right| \le \sup_{T\le t<\infty} \left| \frac{\sqrt{3}t(t+1)}{\sqrt{3}t^\gamma (1+t)^{2-\gamma}} - 1 \right| \to 0, \quad \text{as} \quad T\to\infty. \tag{23}$$

Putting together (21), (22), and (23), we can see that as $m\to\infty$ and $T\to\infty$,

$$\left| \sup_{mT\le k<\infty} \frac{\left|(\frac{k}{m}+1)W_1(\frac{k}{k+m}) + \sqrt{3}\frac{k}{m}(\frac{k}{m}+1)W_2(1)\right|}{\sqrt{3}(k/m)^\gamma (1+k/m)^{2-\gamma}} - \sup_{T\le t<\infty} \frac{\left|(t+1)W_1(\frac{t}{t+1}) + \sqrt{3}t(t+1)W_2(1)\right|}{\sqrt{3}t^\gamma (t+1)^{2-\gamma}} \right| = o_p(1).$$

Hence,

$$\sup_{1\le k<\infty} \frac{\left|G(\frac{k}{m})\right|}{h(m,k)} \xrightarrow{d} \sup_{0\le t<\infty} \frac{\left|(t+1)W_1(\frac{t}{t+1}) + \sqrt{3}t(t+1)W_2(1)\right|}{\sqrt{3}t^\gamma (t+1)^{2-\gamma}}. \tag{24}$$

From the scalar transformation, we have

$$\left\{ \frac{(t+1)W_1(\frac{t}{t+1}) + \sqrt{3}t(t+1)W_2(1)}{\sqrt{3}t^\gamma (t+1)^{2-\gamma}}, \quad 0\le t<\infty \right\} \overset{\mathrm{D}}{=} \left\{ \frac{1-t}{\sqrt{3}t^\gamma} W_1(t) + t^{1-\gamma} W_2(1), \quad 0\le t\le 1 \right\}.$$

Therefore, we obtain

$$\sup_{1\le k<\infty} \frac{\left|G(\frac{k}{m})\right|}{h(m,k)} \xrightarrow{d} \sup_{0\le t\le 1} \left| \frac{1-t}{\sqrt{3}t^\gamma} W_1(t) + t^{1-\gamma} W_2(1) \right|. \tag{25}$$

Theorem 1(i) is obtained from Lemmas 1.1 and 1.2, (14), and (25).■

(ii) Let $\tilde{k} > k^*$ and $\Delta = [\Delta_1, \Delta_2]'$. Then, the detector is expressed as

$$\sum_{t=m+1}^{m+\tilde{k}} \hat{\epsilon}_t = \sum_{t=m+1}^{m+\tilde{k}} \epsilon_t - \sum_{t=m+1}^{m+\tilde{k}} x_t'(\hat{\beta}_m - \beta_0) + \sum_{t=m+k^*}^{m+\tilde{k}} x_t'\Delta.$$

From Theorem 1, we have

$$\sup_{1\le \tilde{k}<\infty} \left| \sum_{t=m+1}^{m+\tilde{k}} \epsilon_t - \sum_{t=m+1}^{m+\tilde{k}} x_t'(\hat{\beta}_m - \beta_0) \right| \Big/ h(m,\tilde{k}) = O_p(1).$$

We then focus on the last term and will show that

$$\left| \sum_{t=m+k^*}^{m+\tilde{k}} x_t'\Delta \right| \Big/ h(m,\tilde{k}) = \left| \Delta_1 + \Delta_2 + \frac{\tilde{k}+k^*}{2m}\Delta_2 \right| (\tilde{k} - k^* + 1)/h(m,\tilde{k}) \to \infty. \tag{26}$$

Suppose that $k^* = O(m^\theta)$ with $0 \le \theta < 1$ and $\Delta_1 + \Delta_2 \neq 0$. Let $\tilde{k} = m^{\tilde{\theta}} + k^* - 1$ with $\tilde{\theta}$ satisfying $\max(\theta,\ (1-2\gamma)/\{2(1-\gamma)\}) < \tilde{\theta} < 1$. Then, we have

$$\left| \Delta_1 + \Delta_2 + \frac{\tilde{k} + k^*}{2m} \Delta_2 \right| \to |\Delta_1 + \Delta_2| > 0. \tag{27}$$

$$\frac{\tilde{k} - k^* + 1}{m^{1/2}(\tilde{k}/m)^\gamma (1 + \tilde{k}/m)^{2-\gamma}} = \frac{O(m^{\tilde{\theta}(1-\gamma)})}{O(m^{1/2-\gamma})} = O(m^{\tilde{\theta}(1-\gamma)-(1/2-\gamma)}) \to \infty. \tag{28}$$

Thus, (27) and (28) imply (26).

When $\Delta_1 + \Delta_2 = 0$, let $\tilde{k} = m + k^* - 1$ ($\tilde{\theta} = 1$). Since $\Delta_2 \neq 0$, we have

$$\left| \Delta_1 + \Delta_2 + \frac{\tilde{k} + k^*}{2m} \Delta_2 \right| \to \frac{|\Delta_2|}{2} > 0.$$

Since (28) holds with $\tilde{\theta} = 1$, we have (26).

Similarly, we can prove (26) for $\theta \ge 1$ and thus the test is consistent.■

The proof of Theorem 2 is based on the framework of Aue *et al*. (2009) through a series of propositions and lemmas. The basic idea is to find a sequence $N = N(m, x)$ such that

$$P\{\tau_m \ge N\} = P\left\{ \max_{1 \le k \le N} \frac{|\Gamma(m,k)|}{g(m,k)} \le 1 \right\} \to \Phi(x), \quad \text{for all real} \quad x.$$

Now, we define N as

$$\begin{aligned}
N^{1-\gamma} \quad &:= \quad \frac{\sqrt{3} c \sigma m^{1/2-\gamma}}{|\delta|} - \frac{1}{c_m^\gamma \delta} \sum_{t=m+k^*}^{m+c_m} (x_t - d)'\Delta - \sigma x \left(\frac{(\sqrt{3} c\sigma)^{1/2-\gamma} m^{(1/2-\gamma)^2}}{|\delta|^{3/2-2\gamma}} \right)^{\frac{1}{1-\gamma}} \\
&= \quad c_m^{1-\gamma} - \frac{1}{c_m^\gamma \delta} \sum_{t=m+k^*}^{m+c_m} (x_t - d)'\Delta - \frac{\sigma x}{|\delta|} c_m^{1/2-\gamma} \\
&= \quad a_m^{1-\gamma} - x \frac{(1-\gamma) b_m}{c_m^\gamma}.
\end{aligned} \tag{29}$$

The following proof assumes that δ is positive and the same result can be derived under the condition $\delta < 0$. We first derive the order of the maximum of the partial sums of $x_t - d$ in Propositions 1 and 2, where $d := [1, 1]'$. Note that d is not the mean of regressor x_t.

Proposition 1. For $j = 1$ and 2, as $m \to \infty$,

$$\max_{k^* \le k \le c_m} \left| \sum_{t=m+k^*}^{m+k} (x_{jt} - d_j) \right| = o(1),$$

$$\max_{1\le k\le k^*}\frac{1}{k}\left|\sum_{t=m+1}^{m+k}(x_{jt}-d_j)\right| = o(1).$$

Proof. The result for $j = 1$ is obvious because the first element of x_t is unity. In the case of $j = 2$,

$$\max_{k^*\le k\le c_m}\left|\sum_{t=m+k^*}^{m+k}\left(\frac{t}{m}-1\right)\right| = \max_{k^*\le k\le c_m}\left|\frac{(k+k^*)(k-k^*+1)}{2m}\right| \le \frac{(c_m+k^*)(c_m-k^*+1)}{2m} = o(1),$$

since $k^{*2}/m = o(1)$ and $c_m^2/m = o(1)$ from Lemmas 2.1(i) and (iii).

We can also see that

$$\max_{1\le k\le k^*}\frac{1}{k}\left|\sum_{t=m+1}^{m+k}\left(\frac{t}{m}-1\right)\right| = \max_{1\le k\le k^*}\left|\frac{k+1}{2m}\right| \le \frac{k^*+1}{2m} = o(1),$$

and we finish the proof.■

Proposition 2. Under Assumption 2, as $m\to\infty$,

$$\frac{N}{c_m}\to 1 \qquad \text{and} \qquad \frac{a_m}{c_m}\to 1, \tag{30}$$

$$\max_{k^*\le k\le N}\frac{1}{k^\gamma}\left|\sum_{t=m+k^*}^{m+k}(x_t-d)'\Delta\right| = O\left(\frac{N^{2-\gamma}}{m}\right), \tag{31}$$

$$\max_{k^*\le k\le N}\frac{1}{k^\gamma}\left\|\sum_{t=m+k^*}^{m+k}(x_t-d)\right\| = O\left(\frac{N^{2-\gamma}}{m}\right). \tag{32}$$

Proof. From the definition of N, we find that

$$\left(\frac{N}{c_m}\right)^{1-\gamma} = 1 - \frac{1}{c_m\delta}\sum_{t=m+k^*}^{m+c_m}(x_t-d)'\Delta - \frac{\sigma x}{\sqrt{c_m}|\delta|}.$$

Lemma 2.1 implies that the second term is $o_p(1)$ because

$$\frac{1}{c_m\delta}\sum_{t=m+k^*}^{m+c_m}(x_t-d)'\Delta = \frac{(c_m^2-k^{*2}+c_m+k^*)\Delta_2}{2mc_m\delta} = o(1),$$

which implies $(a_m/c_m)^{1-\gamma}\to 1$, while the third term also goes to 0 according to the definition of c_m. Thus, we have $N/c_m\to 1$.

We next observe that

$$\begin{aligned} \max_{k^*\le k\le N}\frac{1}{k^\gamma}\left|\sum_{t=m+k^*}^{m+k}(x_t-d)'\Delta\right| &< \max_{k^*\le k\le N}\frac{(k^2+k+k^*)|\Delta_2|}{2mk^\gamma} \\ &\le \left(\frac{N^{2-\gamma}}{2m}+\frac{N^{1-\gamma}}{2m}+\frac{k^{*(1-\gamma)}}{2m}\right)|\Delta_2| \qquad (33)\\ &= O\left(\frac{N^{2-\gamma}}{m}\right), \end{aligned}$$

since for the third term of (33), according to Assumption 2(a) and (30), we have

$$O\left(\frac{k^{*(1-\gamma)}}{m}\right)=O\left(\frac{k^{*(1-\gamma)}}{N^{2-\gamma}}\frac{N^{2-\gamma}}{m}\right)=O\left(m^{\theta(1-\gamma)-\frac{(1/2-\gamma)(2-\gamma)}{1-\gamma}}\right)O\left(\frac{N^{2-\gamma}}{m}\right)=o\left(\frac{N^{2-\gamma}}{m}\right).$$

Similarly, we can derive (32) and we omit the proof.■

Lemma 2.1. Under Assumption 2, as $m\to\infty$,

(i) $\dfrac{k^{*2}}{m}\to 0$.

(ii) $\dfrac{N}{m}\to 0$.

(iii) $\dfrac{c_m^2}{m}\to 0$.

(iv) $\dfrac{k^*}{\sqrt{c_m}}\to 0$ and $\dfrac{k^*}{\sqrt{N}}\to 0$.

Proof. (i) This is obvious by noting that $k^*=O(m^\theta)$ with $0\le\theta<1/2$.

(ii) It is proven that

$$\left(\frac{c_m}{m}\right)^{1-\gamma}=\frac{\sqrt{3}c\sigma}{|\delta|\sqrt{m}}=o(1).$$

Applying (30), we obtain $N/m=o(1)$.

(iii) From the definition of c_m, if $\gamma\neq 0$ holds, we have

$$\left(\frac{c_m^2}{m}\right)^{1-\gamma}=\frac{3c^2\sigma^2}{\delta^2m^\gamma}=o(1).$$

(iv) It can be verified by the assumption of k^* and the definitions of c_m that

$$\frac{k^*}{\sqrt{c_m}}=O\left(m^{\theta-(1/2-\gamma)/(2(1-\gamma))}\right)=o(1),$$

and c_m can be replaced by N from (30). Thus, the proof is complete.■

Lemma 2.2. Under Assumption 2, for all real x,

$$\lim_{m\to\infty}\left(\frac{N}{m}\right)^{\gamma-1/2}\left(c\sigma-\left|\frac{1}{\sqrt{3m}(N/m)^{\gamma}}S_m(k^*,N)\right|\right)=\frac{\sigma x}{\sqrt{3}}, \tag{34}$$

where $S_m(k^*,a) \coloneqq \sum_{t=m+k^*}^{m+a} x_t'\Delta$.

Proof. It can be verified that

$$\begin{aligned}&\left(\frac{N}{m}\right)^{\gamma-1/2}\frac{1}{\sqrt{3m}(N/m)^{\gamma}}S_m(k^*,N)\\ &= \left(\frac{N}{m}\right)^{\gamma-1/2}\left(\frac{N\delta}{\sqrt{3m}(N/m)^{\gamma}}+\frac{1}{\sqrt{3m}(N/m)^{\gamma}}\sum_{t=m+k^*}^{m+N}(x_t-d)'\Delta\right)+o(1).\end{aligned} \tag{35}$$

Since the first term in the parentheses can be shown to dominate the second one, we can see for a large m that $|S_m(k^*,N)| = S_m(k^*,N)$ when $\delta > 0$. Then, the left-hand side of (34) can be rewritten by inserting (35) as

$$\left(\frac{N}{m}\right)^{\gamma-1/2}\left(c\sigma-\frac{N\delta}{\sqrt{3m}(N/m)^{\gamma}}-\frac{1}{\sqrt{3m}(N/m)^{\gamma}}\sum_{t=m+k^*}^{m+N}(x_t-d)'\Delta\right)+o(1). \tag{36}$$

From the definition of N, we find that

$$\frac{N\delta}{\sqrt{3m}(N/m)^{\gamma}}=c\sigma-\frac{1}{\sqrt{3m}(c_m/m)^{\gamma}}\sum_{t=m+k^*}^{m+c_m}(x_t-d)'\Delta-\frac{\sigma x}{\sqrt{3}}\left(\frac{m}{c_m}\right)^{\gamma-1/2}. \tag{37}$$

If we prove that

$$N^{\gamma-1/2}(R(c_m,c_m)-R(N,N))=o(1),\quad \text{where}\quad R(y,z)\coloneqq\frac{1}{y^{\gamma}}\sum_{t=m+k^*}^{m+z}(x_t-d)'\Delta,$$

the lemma is proven by using (36), (37), and the fact that $(N/m)^{\gamma-1/2}(m/c_m)^{\gamma-1/2}\to 1$. We start by transforming

$$N^{\gamma-1/2}|R(c_m,c_m)-R(N,N)|\le N^{\gamma-1/2}|R(c_m,c_m)-R(N,c_m)|+N^{\gamma-1/2}|R(N,c_m)-R(N,N)|. \tag{38}$$

Applying the mean value theorem, the first term in (38) is rewritten as

$$\frac{N^{\gamma-1/2}|N^{\gamma}-c_m^{\gamma}|}{c_m^{\gamma}N^{\gamma}}\left|\sum_{t=m+k^*}^{m+c_m}(x_t-d)'\Delta\right|=\frac{N^{\gamma-1/2}O(c_m^{2\gamma-1}c_m^{1/2-\gamma}/\delta)}{c_m^{\gamma}N^{\gamma}}o(1)=o(1),$$

(see p.186 of Aue *et al.* (2009)). For the second term in (38), using $c_m^2/m = o(1)$ and (30),

$$N^{\gamma-1/2}|R(N,c_m) - R(N,N)| = \frac{1}{\sqrt{N}}\left|\sum_{t=m+c_m+1}^{m+N}(x_t-d)'\Delta\right| = \frac{N^2-c_m^2+N-c_m}{2\sqrt{N}m} = o(1).\blacksquare$$

Lemma 2.3. Under Assumptions 1 and 2,

$$\left(\frac{N}{m}\right)^{\gamma-1/2}\left(\max_{1\le k<k^*}\frac{|\tilde{\Gamma}(m,k)|}{h(m,k)} - \left|\frac{1}{\sqrt{3m}(N/m)^\gamma}S_m(k^*,N)\right|\right) \xrightarrow{p} -\infty, \tag{39}$$

where $h(m,k) \coloneqq g(m,k)/c$.

Proof. $\tilde{\Gamma}(m,k)$ can be written as

$$\tilde{\Gamma}(m,k) = \sum_{t=m+1}^{m+k}\epsilon_t + \sum_{t=m+1}^{m+k}x_t'(\beta_0-\hat{\beta}_m) + S_m(k^*,k). \tag{40}$$

For the first term of (40), we have

$$\begin{aligned}\left(\frac{N}{m}\right)^{\gamma-1/2}\max_{1\le k<k^*}\frac{|\sum_{t=m+1}^{m+k}\epsilon_t - \sigma W_m(k)|}{h(m,k)} &= O_p(1)\max_{1\le k<k^*}\frac{k^{(1/\nu-\gamma)}}{N^{1/2-\gamma}} \\ &= O_p\left(\left(\frac{k^*}{N}\right)^{1/2-\gamma}\right) = o_p(1)\end{aligned}$$

and we next find that as $m\to\infty$,

$$\left(\frac{N}{m}\right)^{\gamma-1/2}\left|\max_{1\le k<k^*}\frac{W_m(k)}{h(m,k)}\right| \le \left|\sup_{0<t<k^*}\frac{W_m(t)}{\sqrt{N}(t/N)^\gamma}\right| \overset{\mathrm{D}}{=} \left|\sup_{0<t<k^*/N}\frac{W(t)}{t^\gamma}\right| = o_p(1), \tag{41}$$

which implies that $(N/m)^{\gamma-1/2}\max_{1\le k<k^*}|\sum_{t=m+1}^{m+k}\epsilon_t|/h(m,k) = o_p(1)$ holds.

The second term of (40) can be rewritten as

$$\begin{aligned}&\left(\frac{N}{m}\right)^{\gamma-1/2}\max_{1\le k<k^*}\frac{\left|\sum_{t=m+1}^{m+k}x_t'(\beta_0-\hat{\beta}_m)\right|}{h(m,k)} \\ &= \left(\frac{N}{m}\right)^{\gamma-1/2}\max_{1\le k<k^*}\frac{\left\|\sum_{t=m+1}^{m+k}x_t\right\|}{h(m,k)}O_p\left(\frac{1}{\sqrt{m}}\right) \\ &= O_p(1)\left(\frac{N}{m}\right)^{\gamma-1/2}\left(\max_{1\le k<k^*}\frac{\left\|\sum_{t=m+1}^{m+k}d\right\|}{\sqrt{3}m(k/m)^\gamma(1+k/m)^{2-\gamma}} + \max_{1\le k<k^*}\frac{\left\|\sum_{t=m+1}^{m+k}(x_t-d)\right\|}{\sqrt{3}m(k/m)^\gamma(1+k/m)^{2-\gamma}}\right) \\ &= O_p(1)\left(\frac{N}{m}\right)^{\gamma-1/2}O\left(\left(\frac{k^*}{m}\right)^{1-\gamma}\right) = o_p(1).\end{aligned}$$

The third term of (40) is zero when $k < k^*$. Thus, we prove that the first component of (39) tends to 0 and we next show that the second component diverges as $m \to \infty$. By applying (31), we have

$$\begin{aligned}\frac{S_m(k^*, N)}{\sqrt{3m}(N/m)^\gamma} &= \frac{\sum_{t=m+k^*}^{m+N} d'\Delta}{\sqrt{3m}(N/m)^\gamma} + \frac{\sum_{t=m+k^*}^{m+N} (x_t - d)'\Delta}{\sqrt{3m}(N/m)^\gamma} \\ &= \frac{(N - k^* + 1)\delta}{\sqrt{3m}(N/m)^\gamma} + O\left(\frac{N^{2-\gamma}}{m^{3/2-\gamma}}\right) \\ &= \frac{(N - k^* + 1)\delta}{\sqrt{3m}(N/m)^\gamma} + o(1).\end{aligned}$$

Since, for $\delta > 0$,

$$\lim_{m\to\infty} \frac{(N - k^* + 1)\delta}{\sqrt{3m}(N/m)^\gamma} = \lim_{m\to\infty} \frac{c_m^{1-\gamma}\delta}{\sqrt{3}m^{1/2-\gamma}} = c\sigma > 0,$$

which is obtained from the definition of c_m and (30), the second term in (39) diverges to infinity and thus we obtain Lemma 2.3.■

Lemma 2.4. Under Assumptions 1 and 2,

$$\left(\frac{N}{m}\right)^{\gamma-1/2} \max_{k^*\le k\le N} \left|\tilde{\Gamma}(m, k) - (\sigma W_m(k) + S_m(k^*, k))\right| \Big/ h(m, k) = o_p(1). \tag{42}$$

Proof. From (40), the left-hand side of (42) is decomposed into two terms, one of which is

$$\left(\frac{N}{m}\right)^{\gamma-1/2} \max_{k^*\le k\le N} \frac{\left|\sum_{t=m+1}^{m+k} \epsilon_t - \sigma W_m(k)\right|}{\sqrt{3}m(k/m)^\gamma(1 + k/m)^{2-\gamma}} = O_p(1) \max_{k^*\le k\le N} \frac{k^{1/\nu-\gamma}}{N^{1/2-\gamma}} = o_p(1),$$

since

$$\max_{k^*\le k\le N} \frac{k^{1/\nu-\gamma}}{N^{1/2-\gamma}} \le \left\{ \begin{array}{ll} N^{1/\nu-1/2} & \text{if} \quad 1/\nu \ge \gamma \\ N^{\gamma-1/2}k^{*1/\nu-\gamma} & \text{if} \quad 1/\nu < \gamma \end{array} \right\} = o_p(1),$$

while the other term becomes

$$\begin{aligned}&\left(\frac{N}{m}\right)^{\gamma-1/2} \max_{k^*\le k\le N} \frac{\left|\sum_{t=m+1}^{m+k} x_t'(\beta_0 - \hat{\beta}_m)\right|}{h(m, k)} \\ &= O_p(1)\left(\frac{N}{m}\right)^{\gamma-1/2} \left(\max_{k^*\le k\le N} \frac{\left\|\sum_{t=m+1}^{m+k} d\right\|}{\sqrt{3}m(k/m)^\gamma(1 + k/m)^{2-\gamma}} + \max_{k^*\le k\le N} \frac{\left\|\sum_{t=m+1}^{m+k}(x_t - d)\right\|}{\sqrt{3}m(k/m)^\gamma(1 + k/m)^{2-\gamma}}\right) \\ &= O_p(1)\left\{\left(\frac{N}{m}\right)^{\gamma-1/2} O\left(\frac{N^{1-\gamma}}{m^{1-\gamma}}\right) + \left(\frac{N}{m}\right)^{\gamma-1/2} O\left(\frac{N^{2-\gamma}}{m^{2-\gamma}}\right)\right\} = o_p(1).■\end{aligned}$$

Lemma 2.5. Under Assumptions 1 and 2,

$$\left(\frac{N}{m}\right)^{\gamma-1/2} \max_{k^*\le k\le N}\left|\frac{\sigma W_m(k)+S_m(k^*,k)}{h(m,k)}-\frac{\sigma W_m(k)+S_m(k^*,k)}{\sqrt{3m}(k/m)^\gamma}\right| = o_p(1). \tag{43}$$

Proof. The left-hand side of (43) is bounded by

$$\begin{aligned}&\left(\frac{N}{m}\right)^{\gamma-1/2} \max_{k^*\le k\le N}\frac{|\sigma W_m(k)|}{\sqrt{3m}(k/m)^\gamma}\left|\frac{\sqrt{3m}(k/m)^\gamma}{h(m,k)}-1\right| \\ &+\left(\frac{N}{m}\right)^{\gamma-1/2} \max_{k^*\le k\le N}\frac{|S_m(k^*,k)|}{\sqrt{3m}(k/m)^\gamma}\left|\frac{\sqrt{3m}(k/m)^\gamma}{h(m,k)}-1\right|.\end{aligned} \tag{44}$$

The mean value theorem yields that

$$\max_{k^*\le k\le N}\left|\frac{\sqrt{3m}(k/m)^\gamma}{\sqrt{3m}(k/m)^\gamma(1+k/m)^{2-\gamma}}-1\right| = \max_{k^*\le k\le N}\left|\left(1+\frac{k}{m}\right)^{\gamma-2}-1\right| = O\left(\frac{N}{m}\right)=o(1),$$

and then the first term of (44) is shown to be $o_p(1)$ as proven by Lemma 3.3 in Aue and Horváth (2004). For the second component of (44), we have

$$\begin{aligned}&N^{\gamma-1/2}\max_{k^*\le k\le N}\frac{1}{\sqrt{3}k^\gamma}\left|\sum_{t=m+k^*}^{m+k}(x_t-d)'\Delta+\sum_{t=m+k^*}^{m+k}d'\Delta)\right|\left|\left(1+\frac{k}{m}\right)^{\gamma-2}-1\right| \\ =\ & N^{\gamma-1/2}\left(O\left(\frac{N^{2-\gamma}}{m}\right)+O(N^{1-\gamma})\right)O\left(\frac{k^*}{m}\right)=o(1).\blacksquare\end{aligned}$$

Lemma 2.6. Under Assumptions 1 and 2,

$$\lim_{m\to\infty} P\left(\left(\frac{N}{m}\right)^{\gamma-1/2}\max_{k^*\le k\le N}\left(\frac{|\sigma W_m(k)+S_m(k^*,k)|}{\sqrt{3m}(k/m)^\gamma}-\frac{|S_m(k^*,N)|}{\sqrt{3m}(N/m)^\gamma}\right)\le\beta_m(\gamma)\right)=\Phi(x), \tag{45}$$

$$\text{where}\qquad \beta_m(\gamma)=\left(\frac{N}{m}\right)^{\gamma-1/2}\left(c\sigma-\frac{|S_m(k^*,N)|}{\sqrt{3m}(N/m)^\gamma}\right).$$

Proof. We can see that

$$\begin{aligned}&\max_{k^*\le k\le N}\frac{\sigma W_m(k)+S_m(k^*,k)}{\sqrt{3m}(k/m)^\gamma} \\ =\ & \max_{k^*\le k\le N}\frac{1}{\sqrt{3m}(k/m)^\gamma}\left(\sigma W_m(k)+\sum_{t=m+k^*}^{m+k}(x_t-d)'\Delta+(k-k^*+1)\delta\right).\end{aligned} \tag{46}$$

We find that the order of the second term of (46) becomes

$$\max_{k^*\le k\le N}\frac{\left|\sum_{t=m+k^*}^{m+k}(x_t-d)'\Delta\right|}{\sqrt{3}m^{1/2-\gamma}k^\gamma}=O_p\left(m^{\gamma-1/2}\frac{N^{2-\gamma}}{m}\right)=o_p(m^{\gamma-1/2}N^{1-\gamma}),$$

while the order of the first term is given by

$$\max_{k^*\le k\le N}\frac{\sigma|W_m(k)|}{\sqrt{3m}(k/m)^\gamma}=O_p\left(\left(\frac{N}{m}\right)^{1/2-\gamma}\right)=o_p(m^{\gamma-1/2}N^{1-\gamma}).$$

On the contrary, the last term is bounded by

$$O(m^{\gamma-1/2}N^{1-\gamma})=\frac{(N-k^*+1)\delta}{\sqrt{3m}(N/m)^\gamma}\le\max_{k^*\le k\le N}\frac{(k-k^*+1)\delta}{\sqrt{3m}(k/m)^\gamma}\le O(m^{\gamma-1/2}N^{1-\gamma}),$$

which implies that the last term dominates the others and thus the maximum of (46) is achieved at k close to N. Hence, for all $\varepsilon\in(0,1)$,

$$\lim_{m\to\infty}P\left(\max_{k^*\le k\le N}\frac{|\sigma W_m(k)+S_m(k^*,k)|}{\sqrt{3m}(k/m)^\gamma}=\max_{(1-\varepsilon)N\le k\le N}\frac{|\sigma W_m(k)+S_m(k^*,k)|}{\sqrt{3m}(k/m)^\gamma}\right)=1.$$

Exactly in the same way as Lemma 7.6 of Aue *et al*. (2009), we can show that the maximum of (46) is attained at $k=N$. Therefore, because $S_m(k^*,N)$ dominates $\sigma W_m(N)$ and $S_m(k^*,N)$ is positive for a large m when $\delta>0$, we have, because $\beta_m(\gamma)\to\sigma x/\sqrt{3}$ by Lemma 2.2,

$$\begin{aligned}&\lim_{m\to\infty}P\left(\left(\frac{N}{m}\right)^{\gamma-1/2}\max_{k^*\le k\le N}\left(\frac{|\sigma W_m(k)+S_m(k^*,k)|}{\sqrt{3m}(k/m)^\gamma}-\frac{|S_m(k^*,N)|}{\sqrt{3m}(N/m)^\gamma}\right)\le\beta_m(\gamma)\right)\\&=\lim_{m\to\infty}P\left(\frac{\sigma}{\sqrt{3}}\frac{W_m(N)}{\sqrt{N}}\le\beta_m(\gamma)\right)=\Phi(x).\blacksquare\end{aligned}$$

Proof of Theorem 2: By combining Lemmas 2.3–2.6, we can see that

$$\lim_{m\to\infty}P(\tau_m\ge N)=\lim_{m\to\infty}P\left(\max_{1\le k\le N}\frac{|\Gamma(m,k)|}{g(m,k)}\le 1\right)=\Phi(x).$$

Because $\Phi(x)$ is symmetric around 0, we have

$$\begin{aligned}\Phi(x)&=1-\Phi(-x)\\&=1-\lim_{m\to\infty}P(\tau_m\ge N(m,-x))\\&=1-\lim_{m\to\infty}P\left(\tau_m^{1-\gamma}\ge a_m^{1-\gamma}+x\frac{(1-\gamma)b_m}{c_m^\gamma}\right)\\&=\lim_{m\to\infty}P\left(\frac{c_m^\gamma}{1-\gamma}\frac{\tau_m^{1-\gamma}-a_m^{1-\gamma}}{b_m}\le x\right).\end{aligned}$$

This implies that $\tau_m/a_m\xrightarrow{p}1$ because $a_m^{1-\gamma}c_m^\gamma/b_m\to\infty$. Applying the result in the proof of Theorem 3.1 of Aue *et al*. (2009), we obtain

$$\lim_{m\to\infty}P\left(\frac{\tau_m-a_m}{b_m}\le x\right)=\lim_{m\to\infty}P\left(\frac{c_m^\gamma}{1-\gamma}\frac{\tau_m^{1-\gamma}-a_m^{1-\gamma}}{b_m}\le x\right)=\Phi(x),$$

and hence complete the proof. ■

We next derive the asymptotic distribution of the stopping time based on the maximal-type fluctuation test through Lemmas 3.1–3.6. We define the sequence $N^{FL}(m,x)$ as

$$\begin{aligned}(N^{FL})^2 &:= \frac{\sigma c^{FL} m^{3/2}}{|(1/\eta-1)(\delta-\Delta_2/2)|} - \sigma x \frac{\sqrt{\eta-1}m\sqrt{a_m^{FL}}}{|(1/\eta-1)(\delta-\Delta_2/2)|} \\ &= (a_m^{FL})^2 - 2x a_m^{FL} b_m^{FL}, \end{aligned} \tag{47}$$

where a_m^{FL} and b_m^{FL} are defined in Theorem 3. The following derivation is considered under the condition that $\delta - \Delta_2/2$ is negative and the proof for positive $\delta - \Delta_2/2$ follows similarly and is omitted here. Allowing for an abuse of notation, let $N := N^{FL}$, $a_m := a_m^{FL}$, $b_m := b_m^{FL}$, and $c := c^{FL}$. The detector is rewritten as

$$\begin{aligned}\tilde{\Gamma}_m^{FL}(k,\ell) = & \sum_{t=m+1}^{m+k} \epsilon_t - \frac{k}{m}\sum_{t=1}^{m}\epsilon_t - \frac{k(m+k)}{\ell(m+\ell)}\left(\sum_{t=m+1}^{m+\ell}\epsilon_t - \frac{\ell}{m}\sum_{t=1}^{m}\epsilon_t\right) \\ & + \sum_{t=m+k^*}^{m+k} x_t'\Delta 1_{\{k\geq k^*\}} - \frac{k(m+k)}{\ell(m+\ell)}\sum_{t=m+k^*}^{m+\ell} x_t'\Delta 1_{\{\ell\geq k^*\}}.\end{aligned}$$

Lemma 3.1. Under Assumptions 2(a) and 3, as $m \to \infty$,

(i) $\dfrac{k^*}{\sqrt{a_m}} = o(1)$.

(ii) $\dfrac{N}{a_m} \to 1$.

(iii) $\dfrac{N}{m} \to 0$.

(iv) $\dfrac{N^{3/2}}{m} \to \infty$.

Proof. (i) Since $k^* = O(m^\theta)$ with $0 \leq \theta < 1/2$, $k^* \big/ \sqrt{a_m} = O(m^{\theta-3/8}) = o(1)$.
(ii) From the definition of N, we have

$$\left(\frac{N}{a_m}\right)^2 = 1 + \frac{\eta m k^* \delta}{(\delta-\Delta_2/2)a_m^2} - \sigma x \frac{\sqrt{\eta-1}}{|(1/\eta-1)(\delta-\Delta_2/2)|}\frac{m}{a_m^{3/2}}.$$

The second and third terms tend to 0 since

$$\frac{m}{a_m^{3/2}} = m\left(\frac{|(1/\eta-1)(\delta-\Delta_2/2)|}{c\sigma m^{3/2}}\right)^{3/4} = o(1).$$

(iii) Using (ii) and

$$\frac{a_m}{m} = \frac{c\sqrt{\sigma} m^{3/4-1}}{\sqrt{|(1/\eta - 1)(\delta - \Delta_2/2)|}} = o(1),$$

we find that $N/m = o(1)$.

(iv) From the definition of N, we have $N^{3/2}/m = O(m^{1/8})$.■

Lemma 3.2. Under Assumptions 2(a) and 3, for all real x,

$$\lim_{m\to\infty} \left(\frac{N}{m}\right)^{-1/2} (c\sigma - |J_m(k^*, N)|) = \sqrt{\eta - 1}\sigma x, \tag{48}$$

$$\text{where} \qquad J_m(k^*, a) := \frac{1}{\sqrt{m}} \left(\sum_{t=m+k^*}^{m+a} x_t'\Delta - \frac{a(m+a)}{[\frac{a+1}{\eta}]\left(m + \left[\frac{a+1}{\eta}\right]\right)} \sum_{t=m+k^*}^{m+\left[\frac{a+1}{\eta}\right]} x_t'\Delta \right). \tag{49}$$

Proof. Let

$$C(\eta) := \frac{N(m+N)}{\left[\frac{N+1}{\eta}\right]\left(m + \left[\frac{N+1}{\eta}\right]\right)}. \tag{50}$$

Since

$$\sum_{t=m+a}^{m+b} x_t'\Delta = \sum_{t=m+a}^{m+b} (x_t - d)'\Delta + \sum_{t=m+a}^{m+b} d'\Delta = \frac{(b+a)(b-a+1)}{2m}\Delta_2 + (b-a+1)\delta,$$

$(N/m)^{-1/2} J_m(k^*, k)$ can be expressed as

$$\begin{aligned} \left(\frac{N}{m}\right)^{-1/2} J_m(k^*, k) &= \left(\frac{N}{m}\right)^{-1/2} \frac{1}{\sqrt{m}} \left[\frac{(k+k^*)(k-k^*+1)}{2m}\Delta_2 + (k-k^*+1)\delta \right. \\ &\quad \left. - \frac{k(m+k)}{\ell(m+\ell)} \left\{ \frac{(\ell+k^*)(\ell-k^*+1)}{2m}\Delta_2 + (\ell-k^*+1)\delta \right\} \right] \\ &= J_1 + J_2 + J_3, \end{aligned}$$

where

$$\begin{aligned} J_1 &:= \left(\frac{N}{m}\right)^{-1/2} \frac{1}{\sqrt{m}} \left\{ \frac{k^2}{2m}\Delta_2 + k\delta - \frac{k(m+k)}{\ell(m+\ell)} \left(\frac{\ell^2}{2m}\Delta_2 + \ell\delta \right) \right\}, \\ J_2 &:= \left(\frac{N}{m}\right)^{-1/2} \frac{1}{\sqrt{m}} \left\{ \frac{k}{2m}\Delta_2 - \frac{k(m+k)}{\ell(m+\ell)} \frac{\ell}{2m}\Delta_2 \right\}, \\ J_3 &:= \left(\frac{N}{m}\right)^{-1/2} \frac{1}{\sqrt{m}} \left[\frac{-k^{*2}+k^*}{2m}\Delta_2 + (-k^*+1)\delta - \frac{k(m+k)}{\ell(m+\ell)} \left\{ \frac{-k^{*2}+k^*}{2m}\Delta_2 + (-k^*+1)\delta \right\} \right]. \end{aligned}$$

We investigate the order of each term. First, we have

$$\max_{k^*\le k\le N} J_1 = \frac{1}{\sqrt{N}}\max_{k^*\le k\le N}\frac{k(k-\ell)}{m+\ell}\left(-\delta+\frac{\Delta_2}{2}\right)$$
$$= \frac{1}{\sqrt{N}}\max_{k^*\le k\le N}\frac{k(k-\ell)}{m}\left(-\delta+\frac{\Delta_2}{2}\right)+o(1), \quad (51)$$

where the second equality holds because $1/(m+\ell) = 1/m+O(\ell/m^2)$ and $O(N^{5/2}/m^2) = o(1)$. Similarly, we have

$$\max_{k^*\le k\le N} J_2 = \frac{1}{\sqrt{N}}\max_{k^*\le k\le N}\frac{k(\ell-k)}{m+\ell}\frac{\Delta_2}{2m} = O\left(\frac{N^{3/2}}{m^2}\right) = o(1), \quad (52)$$

$$\max_{k^*\le k\le N} J_3 = \left(\frac{N}{m}\right)^{-1/2}\max_{k^*\le k\le N}\left\{1-\frac{k(m+k)}{\ell(m+\ell)}\right\}\frac{-k^*\delta}{\sqrt{m}}+o(1) = O\left(\frac{k^*}{\sqrt{N}}\right) = o(1). \quad (53)$$

From (51)–(53), we have

$$\left(\frac{N}{m}\right)^{-1/2}\max_{k^*\le k\le N} J_m(k^*,k) = \frac{1}{\sqrt{N}}\max_{k^*\le k\le N}\frac{k(k-\ell)}{m}\left(-\delta+\frac{\Delta_2}{2}\right)+o(1), \quad (54)$$

which implies that, because $\ell = [(k+1)/\eta]$,

$$\left(\frac{N}{m}\right)^{-1/2} J_m(k^*,N) = \left(\frac{N}{m}\right)^{-1/2}\left\{\frac{N\left(N-\left[\frac{N+1}{\eta}\right]\right)}{m^{3/2}}\left(-\delta+\frac{\Delta_2}{2}\right)\right\}+o(1). \quad (55)$$

On the other hand, from the definition of N, we find that

$$c\sigma = \left(1-\frac{1}{\eta}\right)\left(\frac{\Delta_2}{2}-\delta\right)\frac{N^2}{m^{3/2}}+\sqrt{\eta-1}\sigma x\sqrt{\frac{a_m}{m}}. \quad (56)$$

Note that as is seen in (55) and (60), $|J_m(k^*,N)| = J_m(k^*,N)$ for a large m when $\delta-\Delta_2/2<0$. Then, by using (55) and (56), (48) can be written as

$$\left(\frac{N}{m}\right)^{-1/2}\left\{\left(1-\frac{1}{\eta}\right)\left(\frac{\Delta_2}{2}-\delta\right)\frac{N^2}{m^{3/2}}-\left(N-\left[\frac{N+1}{\eta}\right]\right)\left(\frac{\Delta_2}{2}-\delta\right)\frac{N}{m^{3/2}}\right.$$
$$\left.+\sqrt{\eta-1}\sigma x\sqrt{\frac{a_m}{m}}\right\}+o(1). \quad (57)$$

From Lemma 3.1, we can see that

$$\left(\frac{N}{m}\right)^{-1/2}\left\{\left(1-\frac{1}{\eta}\right)N-\left(N-\left[\frac{N+1}{\eta}\right]\right)\right\}\left(\frac{\Delta_2}{2}-\delta\right)\frac{N}{m^{3/2}}$$
$$= \left(\frac{N}{m}\right)^{-1/2}\left(\left[\frac{N+1}{\eta}\right]-\frac{N}{\eta}\right)\left(\frac{\Delta_2}{2}-\delta\right)\frac{N}{m^{3/2}}$$
$$= O\left(\left(\frac{N}{m}\right)^{-1/2}\frac{N}{m^{3/2}}\right) = O\left(\frac{\sqrt{N}}{m}\right) = o(1), \quad (58)$$

and

$$\lim_{m\to\infty}\left(\frac{N}{m}\right)^{-1/2}\sqrt{\eta-1}\sigma x\sqrt{\frac{a_m}{m}}=\sqrt{\eta-1}\sigma x.$$

Thus, we complete the proof.■

Lemma 3.3. Under Assumptions 1, 2, and 3, then

$$\left(\frac{N}{m}\right)^{-1/2}\left(\max_{1\le k<k^*}\frac{|\tilde{\Gamma}_m^{FL}(k,\ell)|}{h^{FL}(m,k)}-|J_m(k^*,N)|\right)\xrightarrow{p}-\infty, \tag{59}$$

where $h^{FL}(m,k):=g^{FL}(m,k)/c^{FL}$.

Proof. We simplify the notations as $h(m,k)=h^{FL}(m,k)$ and in the case of $k\le k^*$,

$$\tilde{\Gamma}_m^{FL}(k,\ell)=\sum_{t=m+1}^{m+k}\epsilon_t-\frac{k}{m}\sum_{t=1}^{m}\epsilon_t-\frac{k(m+k)}{\ell(m+\ell)}\left(\sum_{t=m+1}^{m+\ell}\epsilon_t-\frac{\ell}{m}\sum_{t=1}^{m}\epsilon_t\right),$$

and consequently we obtain

$$\begin{aligned}\max_{1\le k<k^*}\frac{|\tilde{\Gamma}_m^{FL}(k,\ell)|}{h(m,k)} &\le \left(\frac{N}{m}\right)^{-1/2}\max_{1\le k<k^*}\frac{\left|\sum_{t=m+1}^{m+k}\epsilon_t\right|}{h(m,k)}+\max_{1\le \ell<k^*}\frac{|k\bar{\epsilon}_m|}{h(m,k)}\\ &+ O(1)\left(\max_{1\le k<k^*}\frac{\left|\sum_{t=m+1}^{m+\ell}\epsilon_t\right|}{h(m,k)}+\max_{1\le \ell<k^*}\frac{|\ell\bar{\epsilon}_m|}{h(m,k)}\right)\\ &=: B_1+B_2+B_3+B_4.\end{aligned}$$

Then, the first term is bounded by

$$\begin{aligned}\left(\frac{N}{m}\right)^{-1/2}B_1 &\le \left(\frac{N}{m}\right)^{-1/2}\max_{1\le k<k^*}\frac{\left|\sum_{t=m+1}^{m+k}\epsilon_t-\sigma W_m(k)\right|}{\sqrt{m}(1+k/m)^2}+\left(\frac{N}{m}\right)^{-1/2}\max_{1\le k<k^*}\frac{|\sigma W_m(k)|}{\sqrt{m}(1+k/m)^2}\\ &= O_p(1)O\left(\max_{1\le k<k^*}\frac{k^{1/\nu}}{\sqrt{N}}\right)+O(1)\max_{1\le k<k^*}\frac{|\sigma W_m(k)|}{\sqrt{N}}\\ &= O_p(1),\end{aligned}$$

where the last equality is derived by Lemma 2.1 (i) and $\sup_{0<t<k^*/N}|W_m(t)|=O_p(1)$. Then, for the second term, we have

$$\left(\frac{N}{m}\right)^{-1/2}B_2=\left(\frac{N}{m}\right)^{-1/2}\max_{1\le k<k^*}\frac{k\left|\frac{1}{m}\sum_{t=1}^{m}\epsilon_t\right|}{\sqrt{m}(1+k/m)^2}=O\left(\frac{k^*}{\sqrt{N}m}\right)O_p(\sqrt{m})=o_p(1).$$

Similarly, we can derive that $(N/m)^{-1/2}(B_3+B_4)=O_p(1)+o_p(1)$. We have proven that the first term related to the detector in (59) is bounded. We next show that $(N/m)^{-1/2}J_m(k^*,N)$

diverges as $m \to \infty$. Applying (55), and (58), we have

$$\begin{aligned}\left(\frac{N}{m}\right)^{-1/2} J_m(k^*, N) &= \left(\frac{N}{m}\right)^{-1/2} \left(1 - \frac{1}{\eta}\right)\left(\frac{\Delta_2}{2} - \delta\right)\frac{N^2}{m^{3/2}} + o(1) \\ &= \left(1 - \frac{1}{\eta}\right)\left(\frac{\Delta_2}{2} - \delta\right)\frac{N^{3/2}}{m} + o(1). \qquad (60)\end{aligned}$$

If the term $\delta - \Delta_2/2$ is nonzero, we can show that $(N/m)^{-1/2}|J_m(k^*, N)|$ tends to positive infinity and hence we finish the proof of Lemma 3.3.■

Lemma 3.4. Under Assumptions 1, 2, and 3, we have

$$\left(\frac{N}{m}\right)^{-1/2} \max_{k^* \le k \le N} \left|\tilde{\Gamma}_m^{FL}(k, \ell) - \tilde{W}_m(k, \ell)\right| \Big/ h(m, k) = o_p(1), \qquad (61)$$

where

$$\tilde{W}_m(k, \ell) := \tilde{W}_Q(k) - \frac{k(m+k)}{\ell(m+\ell)}\tilde{W}_Q(\ell),$$

$$\tilde{W}_Q(j) := \sigma W_m(j) + \sum_{t=m+k^*}^{m+j} x_t'\Delta 1_{\{j \ge k^*\}}.$$

Proof. Let

$$Q(m, j) := \sum_{t=m+1}^{m+j} \epsilon_t - \frac{j}{m}\sum_{t=1}^{m} \epsilon_t + \sum_{t=m+k^*}^{m+j} x_t'\Delta 1_{\{j \ge k^*\}}.$$

Then, $\tilde{\Gamma}_m^{FL}(k, \ell) = Q(m, k) - \{k(m+k)\}/\{\ell(m+\ell)\}Q(m, \ell)$. We have

$$\begin{aligned}&\left(\frac{N}{m}\right)^{-1/2} \max_{k^* \le k \le N} \frac{|Q(m, k) - \tilde{W}_Q(k)|}{h(m, k)} \\ &\le \left(\frac{N}{m}\right)^{-1/2} \max_{k^* \le k \le N} \frac{|\sum_{t=m+1}^{m+k} \epsilon_t - \sigma W_m(k)|}{\sqrt{m}(1 + k/m)^2} + \left(\frac{N}{m}\right)^{-1/2} \max_{k^* \le k \le N} \frac{|\frac{k}{m}\sum_{t=1}^{m} \epsilon_t|}{\sqrt{m}(1 + k/m)^2} \\ &= O_p\left(\max_{k^* \le k \le N} \frac{k^{1/\nu}}{\sqrt{N}}\right) + O_p\left(\max_{k^* \le k \le N} \frac{k}{\sqrt{Nm}}\right) = o_p(1).\end{aligned}$$

Similarly, we can also show that

$$\left(\frac{N}{m}\right)^{-1/2} \frac{k(m+k)}{\ell(m+\ell)} \max_{k^* \le k \le N} \frac{|Q(m, \ell) - \tilde{W}_Q(\ell)|}{h(m, k)} = o_p(1),$$

since $\ell = [(k+1)/\eta]$ implies that $k(m+k)/\{\ell(m+\ell)\} = O(1)$. Hence, the proof is complete.■

Lemma 3.5. Under Assumptions 1, 2, and 3, we have

$$\left(\frac{N}{m}\right)^{-1/2} \max_{k^* \le k \le N} \left|\frac{\tilde{W}_m(k, \ell)}{h(m, k)} - \frac{\tilde{W}_m(k, \ell)}{\sqrt{m}}\right| = o_p(1). \qquad (62)$$

Proof. The left-hand side of (62) is bounded by

$$\begin{aligned}&\left(\frac{N}{m}\right)^{-1/2}\max_{k^*\le k\le N}\frac{|\sigma W_m(k)|}{\sqrt{m}}\left|\frac{\sqrt{m}}{h(m,k)}-1\right|+\left(\frac{N}{m}\right)^{-1/2}\max_{k^*\le k\le N}\frac{k(m+k)}{\ell(m+\ell)}\frac{|\sigma W_m(\ell)|}{\sqrt{m}}\left|\frac{\sqrt{m}}{h(m,k)}-1\right|\\&+\left(\frac{N}{m}\right)^{-1/2}\frac{1}{\sqrt{m}}\max_{k^*\le k\le N}\left|\sum_{t=m+k^*}^{m+k}x_t'\Delta-\frac{k(m+k)}{\ell(m+\ell)}\sum_{t=m+k^*}^{m+\ell}x_t'\Delta 1_{\{\ell\ge k^*\}}\right|\left|\frac{\sqrt{m}}{h(m,k)}-1\right|\\&=: C_1+C_2+C_3.\end{aligned}$$

It is easily seen that

$$\max_{k^*\le k\le N}\left|\frac{\sqrt{m}}{h(m,k)}-1\right|=\max_{k^*\le k\le N}\left|\frac{1}{(1+k/m)^2}-1\right|=O\left(\frac{N}{m}\right),$$

and

$$\left(\frac{N}{m}\right)^{-1/2}\max_{k^*\le k\le N}\frac{|W_m(k)|}{\sqrt{m}}\le\max_{1\le k\le N}\frac{|W_m(k)|}{\sqrt{N}}\overset{\mathrm{D}}{=}\max_{0<t\le 1}|W(t)|=O_p(1). \tag{63}$$

Thus, C_1 tends to 0 and similarly, we can show that the term C_2 is $o_p(1)$.

For C_3, it tends to zero in the case of $\ell<k^*$ from (66). When $\ell\ge k^*$, we have, from (54),

$$\begin{aligned}C_3 &\le \left(\frac{N}{m}\right)^{-1/2}\max_{k^*\le k\le N}|J_m(k^*,k)|\max_{k^*\le k\le N}\left|\frac{\sqrt{m}}{h(m,k)}-1\right|\\&=\left\{\frac{1}{\sqrt{N}}\max_{k^*\le k\le N}\left|\frac{k(k-\ell)}{m}\left(-\delta+\frac{\Delta_2}{2}\right)\right|+o(1)\right\}O\left(\frac{N}{m}\right)=O\left(\frac{N^{5/2}}{m^2}\right)=o(1).\end{aligned}$$

Thus, the proof is complete.■

Lemma 3.6. Under Assumptions 1, 2, and 3, we have

$$\lim_{m\to\infty}P\left(\left(\frac{N}{m}\right)^{-1/2}\max_{k^*\le k\le N}\left(\frac{\left|\tilde{W}_m(k,\ell)\right|}{\sqrt{m}}-|J_m(k^*,N)|\right)\le\beta_m^{FL}(\gamma)\right)=\Phi(x), \tag{64}$$

$$\text{where}\qquad \beta_m^{FL}(\gamma)=\left(\frac{N}{m}\right)^{-1/2}\left(c^{FL}\sigma-|J_m(k^*,N)|\right).$$

Proof. We first show that the drift term in $\tilde{W}_Q(k)/\sqrt{m}$ is dominant. From (63), we have

$$\max_{k^*\le k\le N}\frac{W_m(k)}{\sqrt{m}}=O_p\left(\left(\frac{N}{m}\right)^{1/2}\right)=o_p\left(\frac{N^2}{m^{3/2}}\right),$$

and thus, because $k(m+k)/\{\ell(m+\ell)\}=O(1)$, the stochastic terms are $o_p(N^2/m^{3/2})$.

We next investigate the order of the magnitude of the following term:

$$\left(\frac{N}{m}\right)^{-1/2}\frac{1}{\sqrt{m}}\left|\sum_{t=m+k^*}^{m+k} x_t'\Delta - \frac{k(m+k)}{\ell(m+\ell)}\sum_{t=m+k^*}^{m+\ell} x_t'\Delta 1_{\{\ell\geq k^*\}}\right|. \tag{65}$$

It is easily seen that (65) tends to zero in the case of $\ell < k^*$ because

$$\begin{aligned}\max_{k^*\leq k\leq N}\frac{1}{\sqrt{m}}\sum_{t=m+k^*}^{m+k} x_t'\Delta &= \max_{k^*\leq k\leq N}\frac{1}{\sqrt{m}}\left\{\sum_{t=m+k^*}^{m+k}(x_t-d)'\Delta + \sum_{t=m+k^*}^{m+k} d'\Delta\right\}\\ &= O\left(\frac{N^2}{m^{3/2}}\right) + O\left(\frac{N}{\sqrt{m}}\right).\end{aligned} \tag{66}$$

For $\ell \geq k^*$, (65) is equal to $(N/m)^{-1/2}|J_m(k^*,k)|$. From (54), it is easily seen that the denominating term in $(N/m)^{-1/2}\max_{k^*\leq k\leq N}|J_m(k^*,k)|$ is

$$\max_{k^*\leq k\leq N}\frac{k(k-\ell)}{\sqrt{N}m}\left|-\Delta_1 - \frac{\Delta_2}{2}\right|. \tag{67}$$

Then, (65) is bounded below by

$$\left(\frac{N}{m}\right)^{-1/2}\max_{k^*\leq k\leq N}|J_m(k^*,k)| \geq \frac{N\left(N-\left[\frac{N+1}{\eta}\right]\right)}{\sqrt{N}m}\left|-\Delta_1-\frac{\Delta_2}{2}\right| + o(1) = O\left(\frac{N^{3/2}}{m}\right).$$

On the contrary, because (67) is a strictly increasing function of k and thus it is bounded above by

$$\begin{aligned}\left(\frac{N}{m}\right)^{-1/2}\max_{k^*\leq k\leq N}|J_m(k^*,k)| &\leq \max_{k^*\leq k\leq N}\frac{k}{\sqrt{N}m}\frac{(\eta-1)(k+1)}{\eta}\left|-\Delta_1-\frac{\Delta_2}{2}\right| + o(1)\\ &= O\left(\frac{N^{3/2}}{m}\right),\end{aligned}$$

because

$$\frac{k+1}{\eta} - 1 < \ell \leq \frac{k+1}{\eta} \quad \text{and thus} \quad \frac{(\eta-1)k-1}{\eta} - 1 \leq k-\ell < \frac{(\eta-1)(k+1)}{\eta}.$$

This implies that the maximum of $|\tilde{W}_m(k,\ell)|/\sqrt{N}$ will be determined by the term (65) and then achieved close to N. Further, we have proven that $(N/m)^{-1/2}|J_m(k^*,N)|$ tends to positive infinity in Lemma 3.3. We thus find that for all $\varepsilon \in (0,1)$,

$$\lim_{m\to\infty} P\left(\left(\frac{N}{m}\right)^{-1/2}\max_{k^*\leq k\leq N}\frac{|\tilde{W}_m(k,\ell)|}{\sqrt{m}} = \left(\frac{N}{m}\right)^{-1/2}\max_{(1-\varepsilon)N\leq k\leq N}\frac{|\tilde{W}_m(k,\ell)|}{\sqrt{m}}\right).$$

Next, we show that

$$
\begin{aligned}
&\left(\frac{N}{m}\right)^{-1/2} \max_{(1-\varepsilon)N\le k\le N}\left|W_m(k)-\frac{k(m+k)W_m(\ell)}{\ell(m+\ell)}-W_m(N)+\frac{N(m+N)W_m\left(\left[\frac{N+1}{\eta}\right]\right)}{\left[\frac{N+1}{\eta}\right]\left(m+\left[\frac{N+1}{\eta}\right]\right)}\right|\Big/\sqrt{m}\\
\le\ &\sup_{(1-\varepsilon)N\le t\le N}\frac{1}{\sqrt{N}}\left|W_m(t)-\frac{t(m+t)W_m(\tilde{\ell})}{\tilde{\ell}(m+\tilde{\ell})}-W_m(N)+\frac{N(m+N)W_m\left(\left[\frac{N+1}{\eta}\right]\right)}{\left[\frac{N+1}{\eta}\right]\left(m+\left[\frac{N+1}{\eta}\right]\right)}\right|\\
\stackrel{\mathrm{D}}{=}\ &\sup_{1-\varepsilon\le s\le 1}\left|W_m(s)-\frac{s\left(\frac{m}{N}+s\right)W_m\left(\frac{1}{\eta}s+c_1\right)}{\left(\frac{1}{\eta}s+c_1\right)\left(\frac{m}{N}+\frac{1}{\eta}s+c_1\right)}-W_m(1)+\frac{\left(\frac{m}{N}+1\right)W_m\left(\frac{1}{\eta}+c_2\right)}{\left(\frac{1}{\eta}+c_2\right)\left(\frac{m}{N}+\frac{1}{\eta}+c_2\right)}\right|\\
\le\ &\sup_{1-\varepsilon\le s\le 1}|W_m(s)-W_m(1)|\\
&+\sup_{1-\varepsilon\le s\le 1}\left|\frac{s\left(\frac{m}{N}+s\right)}{\left(\frac{1}{\eta}s+c_1\right)\left(\frac{m}{N}+\frac{1}{\eta}s+c_1\right)}-\frac{\left(\frac{m}{N}+1\right)}{\left(\frac{1}{\eta}+c_2\right)\left(\frac{m}{N}+\frac{1}{\eta}+c_2\right)}\right|\left|W_m\left(\frac{1}{\eta}s+c_1\right)\right|\\
&+\sup_{1-\varepsilon\le s\le 1}\left|\frac{\left(\frac{m}{N}+1\right)}{\left(\frac{1}{\eta}+c_2\right)\left(\frac{m}{N}+\frac{1}{\eta}+c_2\right)}\right|\left|W_m\left(\frac{1}{\eta}s+c_1\right)-W_m\left(\frac{1}{\eta}+c_2\right)\right|\\
\xrightarrow{p}\ &0,
\end{aligned}
$$

as $\varepsilon\to 0$, where

$$
\tilde{\ell}:=\left[\frac{t+1}{\eta}\right],\quad s:=\frac{t}{N},\quad c_1:=\left(\left[\frac{t+1}{\eta}\right]-\frac{t}{\eta}\right)\Big/N,\quad c_2:=\left(\left[\frac{N+1}{\eta}\right]-\frac{N}{\eta}\right)\Big/N,
$$

and c_1, c_2 tend to zero as $N\to\infty$, where we used the scale transformation for equality in distribution and the last convergence holds according to almost sure continuity of Brownian motions.

Furthermore, we can see that

$$
\begin{aligned}
&\frac{1}{\sqrt{N}}\left|\left(W(N)-\frac{N(m+N)}{\left[\frac{N+1}{\eta}\right]\left(m+\left[\frac{N+1}{\eta}\right]\right)}W\left(\left[\frac{N+1}{\eta}\right]\right)\right)-\left(W(N)-\eta W\left(\frac{N}{\eta}\right)\right)\right|\\
\le\ &\left|\eta-\frac{N(m+N)}{\left[\frac{N+1}{\eta}\right]\left(m+\left[\frac{N+1}{\eta}\right]\right)}\right|\left|\frac{1}{\sqrt{N}}W\left(\left[\frac{N+1}{\eta}\right]\right)\right|+\left|\frac{\eta}{\sqrt{N}}\left(W\left(\left[\frac{N+1}{\eta}\right]\right)-W\left(\frac{N}{\eta}\right)\right)\right|\\
=\ &o_p(1).
\end{aligned}
$$

Since $\left(W(N) - \eta W\left(\frac{N}{\eta}\right)\right)/\sqrt{N} \overset{\mathrm{D}}{=} W(\eta-1)$, we have

$$\begin{aligned}
&\lim_{m\to\infty} P\left(\left(\frac{N}{m}\right)^{1/2} \max_{k^*\le k\le N}\left(\frac{|W_m(k,\ell)|}{\sqrt{m}} - |J_m(k^*,N)|\right) \le \beta_m(\gamma)\right) \\
=\ & \lim_{m\to\infty} P\left(\left(\frac{N}{m}\right)^{1/2} \left(\frac{|W_m(N,[(N+1)/\eta])|}{\sqrt{m}} - |J_m(k^*,N)|\right) \le \frac{\beta_m(\gamma)}{\sigma}\right) \\
=\ & \lim_{m\to\infty} P\left(W(\eta-1) \le \sqrt{\eta-1}x\right) \\
=\ & \Phi(x).
\end{aligned}$$

The proof is complete.■

Proof of Theorem 3. By combining Lemmas 3.3–3.6, we have

$$\begin{aligned}
\lim_{m\to\infty} P(\tau_m \ge N) &= \lim_{m\to\infty} P\left(\max_{1\le k\le N} \frac{|\Gamma_m^{FL}(k,\ell)|}{g^{FL}(m,k)} \le 1\right) \\
&= \lim_{m\to\infty} P\left(\left(\frac{N}{m}\right)^{1/2} \max_{k^*\le k\le N}\left(\frac{|W_m(k,\ell)|}{\sqrt{m}} - |J_m(k^*,N)|\right) \le \beta_m(\gamma)\right) \\
&= \Phi(x).
\end{aligned}$$

Or we can rewrite it as

$$\begin{aligned}
\lim_{m\to\infty} P\left(\frac{\tau_m^2 - a_m^2}{2a_m b_m} \le x\right) &= 1 - \lim_{m\to\infty} P(\tau_m^2 \ge a_m^2 + 2a_m b_m x) \\
&= 1 - \lim_{m\to\infty} P(\tau_m^2 \ge N^2(m,-x)) \\
&= \Phi(x). \qquad (68)
\end{aligned}$$

Since $a_m^2/(2a_m b_m) \to \infty$, (68) implies that $\tau_m/a_m \xrightarrow{p} 1$, and the mean value theorem yields that

$$\frac{\tau_m - a_m}{b_m} = \frac{(\tau_m^2)^{1/2} - (a_m^2)^{1/2}}{b_m} = \frac{\tau_m^2 - a_m^2}{2a_m b_m}(1 + o_p(1)).$$

From Slutsky's theorem, we thus obtain

$$\lim_{m\to\infty} P\left(\frac{\tau_m - a_m}{b_m} \le x\right) = \lim_{m\to\infty} P\left(\frac{\tau_m^2 - a_m^2}{2c_m b_m} \le x\right) = \Phi(x).$$

The proof is complete. ■

Table 1: Critical values

α	1%	2.5%	5%	10%	1%	2.5%	5%	10%
	(a) $\kappa = 1$				(b) $\kappa = 2$			
$\gamma = 0.05$	1.4641	1.2786	1.1296	0.9626	1.8146	1.5781	1.3866	1.1710
$\gamma = 0.15$	1.5738	1.3774	1.2180	1.0429	1.8918	1.6456	1.4483	1.2267
$\gamma = 0.25$	1.6989	1.4888	1.3213	1.1371	1.9725	1.7185	1.5161	1.2912
$\gamma = 0.35$	1.8400	1.6193	1.4476	1.2603	2.0632	1.8059	1.5979	1.3776
$\gamma = 0.45$	2.0250	1.8149	1.6523	1.4783	2.1776	1.9295	1.7361	1.5453
	(c) $\kappa = 3$				(d) $\kappa = 4$			
$\gamma = 0.05$	2.0005	1.7353	1.5230	1.2803	2.1099	1.8332	1.6073	1.3494
$\gamma = 0.15$	2.0597	1.7875	1.5700	1.3222	2.1582	1.8751	1.6447	1.3837
$\gamma = 0.25$	2.1215	1.8433	1.6220	1.3727	2.2083	1.9194	1.6857	1.4226
$\gamma = 0.35$	2.1878	1.9061	1.6824	1.4389	2.2600	1.9694	1.7351	1.4780
$\gamma = 0.45$	2.2702	1.9969	1.7894	1.5789	2.3288	2.0456	1.8232	1.6024
	(e) $\kappa = 5$				(f) $\kappa = 6$			
$\gamma = 0.05$	2.1886	1.9000	1.6647	1.3935	2.2443	1.9490	1.7051	1.4280
$\gamma = 0.15$	2.2295	1.9353	1.6964	1.4218	2.2792	1.9799	1.7326	1.4531
$\gamma = 0.25$	2.2717	1.9718	1.7301	1.4553	2.3158	2.0121	1.7625	1.4810
$\gamma = 0.35$	2.3146	2.0145	1.7712	1.5055	2.3524	2.0463	1.7981	1.5247
$\gamma = 0.45$	2.3686	2.0755	1.8480	1.6187	2.3965	2.1005	1.8666	1.6302
	(g) $\kappa = 7$				(h) $\kappa = 8$			
$\gamma = 0.05$	2.2866	1.9856	1.7365	1.4540	2.3171	2.0131	1.7603	1.4742
$\gamma = 0.15$	2.3174	2.0126	1.7609	1.4756	2.3446	2.0369	1.7823	1.4931
$\gamma = 0.25$	2.3490	2.0404	1.7864	1.5002	2.3727	2.0617	1.8051	1.5156
$\gamma = 0.35$	2.3811	2.0703	1.8173	1.5397	2.4026	2.0885	1.8339	1.5516
$\gamma = 0.45$	2.4182	2.1179	1.8802	1.6394	2.4370	2.1327	1.8918	1.6470

Table 2: Size of the tests ($\kappa = 1$)

		CUSUM		FL
m	ρ	$\gamma = 0.15$	$\gamma = 0.45$	
50	0.4	0.082	0.076	0.092
	0.8	0.158	0.147	0.128
100	0.4	0.066	0.055	0.061
	0.8	0.105	0.096	0.086
250	0.4	0.053	0.049	0.052
	0.8	0.068	0.064	0.061

Table 3: Power of the tests ($\kappa = 1$, $k^* = 1$)

	$d = [-1, -1]$			$d = [1, 1]$		
	CUSUM		FL	CUSUM		FL
b	$\gamma = 0.15$	$\gamma = 0.45$		$\gamma = 0.15$	$\gamma = 0.45$	
(a) $m = 50$, $\rho = 0.4$						
0.0	0.082	0.076	0.092	0.082	0.076	0.092
0.5	0.700	0.674	0.161	0.702	0.680	0.160
1.0	0.996	0.995	0.384	0.996	0.994	0.389
1.5	1.000	1.000	0.709	1.000	1.000	0.706
2.0	1.000	1.000	0.913	1.000	1.000	0.911
(b) $m = 50$, $\rho = 0.8$						
0.0	0.158	0.147	0.128	0.158	0.147	0.128
0.5	0.823	0.803	0.256	0.829	0.805	0.257
1.0	0.993	0.990	0.571	0.995	0.994	0.570
1.5	1.000	1.000	0.827	0.999	0.999	0.821
2.0	1.000	1.000	0.939	1.000	1.000	0.936
(c) $m = 100$, $\rho = 0.4$						
0.0	0.066	0.055	0.061	0.066	0.055	0.061
0.5	0.920	0.912	0.212	0.926	0.921	0.219
1.0	1.000	1.000	0.698	1.000	1.000	0.695
1.5	1.000	1.000	0.964	1.000	1.000	0.970
2.0	1.000	1.000	1.000	1.000	1.000	0.999
(d) $m = 100$, $\rho = 0.8$						
0.0	0.105	0.096	0.086	0.105	0.096	0.086
0.5	0.937	0.926	0.278	0.939	0.929	0.295
1.0	1.000	1.000	0.740	1.000	1.000	0.741
1.5	1.000	1.000	0.957	1.000	1.000	0.959
2.0	1.000	1.000	0.996	1.000	1.000	0.997
(e) $m = 250$, $\rho = 0.4$						
0.0	0.053	0.049	0.052	0.053	0.049	0.052
0.5	1.000	1.000	0.536	1.000	1.000	0.524
1.0	1.000	1.000	0.994	1.000	1.000	0.993
1.5	1.000	1.000	1.000	1.000	1.000	1.000
2.0	1.000	1.000	1.000	1.000	1.000	1.000
(f) $m = 250$, $\rho = 0.8$						
0.0	0.068	0.064	0.061	0.068	0.064	0.061
0.5	1.000	1.000	0.544	1.000	0.999	0.541
1.0	1.000	1.000	0.988	1.000	1.000	0.988
1.5	1.000	1.000	1.000	1.000	1.000	1.000
2.0	1.000	1.000	1.000	1.000	1.000	1.000

Table 4: Power of the tests ($\kappa = 1$, $k^* = 0.8m$)

	$d = [-1, -1]$			$d = [1, 1]$		
	CUSUM		FL	CUSUM		FL
b	$\gamma = 0.15$	$\gamma = 0.45$		$\gamma = 0.15$	$\gamma = 0.45$	
(a) $m = 50$, $\rho = 0.4$						
0.0	0.082	0.076	0.092	0.082	0.076	0.092
0.5	0.136	0.118	0.170	0.131	0.113	0.170
1.0	0.262	0.217	0.520	0.260	0.211	0.513
1.5	0.455	0.386	0.879	0.452	0.391	0.878
2.0	0.651	0.583	0.989	0.661	0.587	0.988
(b) $m = 50$, $\rho = 0.8$						
0.0	0.158	0.147	0.128	0.158	0.147	0.128
0.5	0.253	0.225	0.315	0.264	0.227	0.313
1.0	0.440	0.390	0.728	0.453	0.404	0.725
1.5	0.648	0.592	0.929	0.657	0.600	0.933
2.0	0.806	0.758	0.984	0.808	0.764	0.982
(c) $m = 100$, $\rho = 0.4$						
0.0	0.066	0.055	0.061	0.066	0.055	0.061
0.5	0.149	0.114	0.225	0.154	0.120	0.210
1.0	0.368	0.303	0.827	0.383	0.316	0.828
1.5	0.659	0.580	0.997	0.669	0.602	0.995
2.0	0.880	0.833	1.000	0.889	0.840	1.000
(d) $m = 100$, $\rho = 0.8$						
0.0	0.105	0.096	0.086	0.105	0.096	0.086
0.5	0.224	0.188	0.349	0.232	0.195	0.332
1.0	0.481	0.417	0.870	0.492	0.429	0.867
1.5	0.755	0.697	0.992	0.756	0.703	0.991
2.0	0.913	0.877	1.000	0.917	0.883	1.000
(e) $m = 250$, $\rho = 0.4$						
0.0	0.053	0.049	0.052	0.053	0.049	0.052
0.5	0.237	0.188	0.557	0.242	0.191	0.570
1.0	0.691	0.625	1.000	0.698	0.624	0.999
1.5	0.969	0.944	1.000	0.970	0.943	1.000
2.0	1.000	0.999	1.000	1.000	0.999	1.000
(f) $m = 250$, $\rho = 0.8$						
0.0	0.068	0.064	0.061	0.068	0.064	0.061
0.5	0.284	0.231	0.604	0.287	0.231	0.616
1.0	0.738	0.670	0.998	0.742	0.668	0.998
1.5	0.971	0.950	1.000	0.974	0.953	1.000
2.0	1.000	0.998	1.000	0.999	0.998	1.000

Table 5: Power of the tests ($\kappa = 1$, $b = 1$)

	$d = [-1, -1]$			$d = [1, 1]$		
	CUSUM		FL	CUSUM		FL
k^*	$\gamma = 0.15$	$\gamma = 0.45$		$\gamma = 0.15$	$\gamma = 0.45$	
(a) $m = 50$, $\rho = 0.4$						
0.1m	0.990	0.983	0.741	0.989	0.982	0.735
0.2m	0.975	0.961	0.954	0.972	0.959	0.949
0.3m	0.946	0.916	0.987	0.940	0.917	0.989
0.4m	0.878	0.837	0.997	0.881	0.841	0.996
0.5m	0.767	0.712	0.999	0.770	0.716	0.999
0.6m	0.620	0.554	0.982	0.623	0.553	0.982
0.7m	0.442	0.376	0.870	0.438	0.379	0.870
0.8m	0.262	0.217	0.520	0.260	0.211	0.513
0.9m	0.141	0.120	0.171	0.137	0.115	0.170
(b) $m = 50$, $\rho = 0.8$						
0.1m	0.988	0.983	0.871	0.992	0.986	0.869
0.2m	0.981	0.974	0.970	0.983	0.974	0.965
0.3m	0.970	0.955	0.988	0.966	0.954	0.984
0.4m	0.939	0.913	0.992	0.937	0.917	0.989
0.5m	0.877	0.845	0.995	0.885	0.848	0.992
0.6m	0.778	0.734	0.981	0.781	0.736	0.979
0.7m	0.630	0.579	0.923	0.639	0.588	0.927
0.8m	0.440	0.390	0.728	0.453	0.404	0.725
0.9m	0.261	0.229	0.330	0.272	0.233	0.321
(c) $m = 100$, $\rho = 0.4$						
0.1m	1.000	1.000	0.946	1.000	1.000	0.946
0.2m	1.000	0.999	0.999	1.000	1.000	0.999
0.3m	0.999	0.997	1.000	0.998	0.997	1.000
0.4m	0.991	0.984	1.000	0.990	0.984	1.000
0.5m	0.959	0.934	1.000	0.958	0.938	1.000
0.6m	0.849	0.804	1.000	0.858	0.809	1.000
0.7m	0.637	0.558	0.996	0.651	0.587	0.994
0.8m	0.368	0.303	0.827	0.383	0.316	0.828
0.9m	0.154	0.119	0.229	0.162	0.125	0.216

Table 5: (continued)

	$d = [-1, -1]$			$d = [1, 1]$		
	CUSUM		FL	CUSUM		FL
k^*	$\gamma = 0.15$	$\gamma = 0.45$		$\gamma = 0.15$	$\gamma = 0.45$	
(d) $m = 100$, $\rho = 0.8$						
0.1m	1.000	0.999	0.944	1.000	1.000	0.950
0.2m	0.999	0.998	0.996	1.000	0.999	0.998
0.3m	0.996	0.995	1.000	0.997	0.995	0.999
0.4m	0.991	0.983	1.000	0.989	0.984	1.000
0.5m	0.962	0.945	1.000	0.966	0.948	1.000
0.6m	0.893	0.857	0.999	0.900	0.862	0.999
0.7m	0.738	0.680	0.990	0.742	0.687	0.990
0.8m	0.481	0.417	0.870	0.492	0.429	0.867
0.9m	0.232	0.194	0.358	0.241	0.202	0.343
(e) $m = 250$, $\rho = 0.4$						
0.1m	1.000	1.000	1.000	1.000	1.000	1.000
0.2m	1.000	1.000	1.000	1.000	1.000	1.000
0.3m	1.000	1.000	1.000	1.000	1.000	1.000
0.4m	1.000	1.000	1.000	1.000	1.000	1.000
0.5m	1.000	1.000	1.000	1.000	1.000	1.000
0.6m	0.999	0.999	1.000	0.999	0.997	1.000
0.7m	0.957	0.934	1.000	0.960	0.934	1.000
0.8m	0.691	0.625	1.000	0.698	0.624	0.999
0.9m	0.249	0.194	0.572	0.250	0.198	0.584
(f) $m = 250$, $\rho = 0.8$						
0.1m	1.000	1.000	1.000	1.000	1.000	1.000
0.2m	1.000	1.000	1.000	1.000	1.000	1.000
0.3m	1.000	1.000	1.000	1.000	1.000	1.000
0.4m	1.000	1.000	1.000	1.000	1.000	1.000
0.5m	1.000	1.000	1.000	1.000	1.000	1.000
0.6m	0.999	0.997	1.000	0.998	0.997	1.000
0.7m	0.964	0.940	1.000	0.968	0.945	1.000
0.8m	0.738	0.670	0.998	0.742	0.668	0.998
0.9m	0.299	0.240	0.628	0.296	0.238	0.632

Table 6: Delay time ($\kappa = 1$, $k^* = 1$)

		$d = [-1, -1]$					$d = [1, 1]$				
b		min	1Q	2Q	3Q	max	min	1Q	2Q	3Q	max
(a) $m = 50$, $\rho = 0.4$											
0.5	CUSUM($\gamma = 0.15$)	3	11	16	23	50	3	11	16	24	50
	CUSUM($\gamma = 0.45$)	1	6	11	17	50	1	7	11	18	50
	FL	3	19	29	39	50	3	19	30	40	50
1.0	CUSUM($\gamma = 0.15$)	3	6	8	10	46	2	6	8	10	50
	CUSUM($\gamma = 0.45$)	1	4	5	7	46	1	4	5	7	48
	FL	3	21	33	42	50	3	23	34	42	50
1.5	CUSUM($\gamma = 0.15$)	2	4	5	6	24	2	4	5	6	23
	CUSUM($\gamma = 0.45$)	1	3	3	4	21	1	3	3	4	21
	FL	2	3	30	40	50	2	3	31	40	50
2.0	CUSUM($\gamma = 0.15$)	2	3	4	5	14	2	3	4	5	11
	CUSUM($\gamma = 0.45$)	1	2	3	3	10	1	2	3	3	9
	FL	2	3	3	34	50	2	3	3	34	50
(b) $m = 50$, $\rho = 0.8$											
0.5	CUSUM($\gamma = 0.15$)	3	7	11	17	50	2	7	11	17	50
	CUSUM($\gamma = 0.45$)	2	5	7	12	50	2	4	7	12	50
	FL	3	23	32	41	50	3	23	33	41	50
1.0	CUSUM($\gamma = 0.15$)	2	4	5	8	46	2	4	5	8	47
	CUSUM($\gamma = 0.45$)	2	3	3	5	48	2	3	3	5	50
	FL	2	3	29	39	50	2	3	29	39	50
1.5	CUSUM($\gamma = 0.15$)	2	3	4	5	37	2	3	4	5	41
	CUSUM($\gamma = 0.45$)	2	2	3	3	47	2	2	3	3	24
	FL	2	3	3	32	50	2	3	3	32	50
2.0	CUSUM($\gamma = 0.15$)	2	3	3	4	31	2	3	3	4	25
	CUSUM($\gamma = 0.45$)	2	2	2	3	32	2	2	2	3	21
	FL	2	3	3	3	50	2	3	3	3	50
(c) $m = 100$, $\rho = 0.4$											
0.5	CUSUM($\gamma = 0.15$)	5	17	23	33	100	5	16	22	32	99
	CUSUM($\gamma = 0.45$)	2	9	14	22	98	2	9	13	22	99
	FL	9	49	67	83	100	7	48	69	84	100
1.0	CUSUM($\gamma = 0.15$)	4	8	10	12	38	4	8	10	12	39
	CUSUM($\gamma = 0.45$)	2	4	5	7	31	2	4	5	7	34
	FL	3	54	69	83	100	5	53	68	82	100
1.5	CUSUM($\gamma = 0.15$)	3	5	6	7	15	3	5	6	7	15
	CUSUM($\gamma = 0.45$)	2	3	3	4	11	2	3	3	4	10
	FL	3	46	58	69	100	3	47	58	70	100
2.0	CUSUM($\gamma = 0.15$)	3	4	5	6	10	3	4	5	6	10
	CUSUM($\gamma = 0.45$)	2	2	3	3	7	2	2	3	3	7
	FL	3	39	48	57	100	3	39	48	57	99

Table 6: (continued)

b		$d = [-1, -1]$					$d = [1, 1]$				
		min	1Q	2Q	3Q	max	min	1Q	2Q	3Q	max
(d) $m = 100,\ \rho = 0.8$											
0.5	CUSUM($\gamma = 0.15$)	4	13	18	27	100	5	13	18	27	100
	CUSUM($\gamma = 0.45$)	2	7	11	18	99	2	7	11	18	95
	FL	11	49	67	83	100	9	49	68	84	100
1.0	CUSUM($\gamma = 0.15$)	3	6	8	10	85	3	6	8	10	45
	CUSUM($\gamma = 0.45$)	2	3	4	6	42	2	3	4	6	39
	FL	3	49	64	79	100	3	49	63	79	100
1.5	CUSUM($\gamma = 0.15$)	3	4	5	7	21	3	4	5	7	26
	CUSUM($\gamma = 0.45$)	2	3	3	4	12	2	3	3	4	18
	FL	3	42	53	66	100	3	41	53	66	100
2.0	CUSUM($\gamma = 0.15$)	2	4	4	5	13	2	4	4	5	16
	CUSUM($\gamma = 0.45$)	2	2	3	3	7	2	2	3	3	10
	FL	2	3	43	54	99	2	3	43	54	99
(e) $m = 250,\ \rho = 0.4$											
0.5	CUSUM($\gamma = 0.15$)	10	24	30	38	160	9	23	29	37	128
	CUSUM($\gamma = 0.45$)	2	10	14	21	183	3	10	14	20	106
	FL	27	130	173	211	250	27	125	170	209	250
1.0	CUSUM($\gamma = 0.15$)	6	11	13	15	29	6	11	13	15	26
	CUSUM($\gamma = 0.45$)	2	4	5	7	19	2	4	5	7	17
	FL	27	103	130	157	250	27	101	128	157	250
1.5	CUSUM($\gamma = 0.15$)	5	7	8	9	15	5	7	8	9	14
	CUSUM($\gamma = 0.45$)	2	3	4	4	8	2	3	4	4	8
	FL	27	82	98	116	221	17	81	97	114	207
2.0	CUSUM($\gamma = 0.15$)	4	6	6	7	11	4	6	6	7	10
	CUSUM($\gamma = 0.45$)	2	3	3	3	6	2	3	3	3	5
	FL	11	71	83	95	151	17	70	82	94	146
(f) $m = 250,\ \rho = 0.8$											
0.5	CUSUM($\gamma = 0.15$)	9	21	27	36	177	8	21	27	36	206
	CUSUM($\gamma = 0.45$)	3	9	13	19	201	3	9	13	19	147
	FL	30	126	169	208	250	30	122	167	207	250
1.0	CUSUM($\gamma = 0.15$)	5	10	12	14	31	5	10	12	14	32
	CUSUM($\gamma = 0.45$)	2	4	5	6	19	2	4	5	6	18
	FL	30	101	127	157	250	30	99	125	156	250
1.5	CUSUM($\gamma = 0.15$)	4	7	8	9	18	4	7	8	9	17
	CUSUM($\gamma = 0.45$)	2	3	3	4	8	2	3	3	4	8
	FL	29	81	97	115	232	30	79	96	113	214
2.0	CUSUM($\gamma = 0.15$)	3	5	6	7	12	3	5	6	7	12
	CUSUM($\gamma = 0.45$)	2	3	3	3	6	2	3	3	3	5
	FL	29	69	81	94	158	19	69	81	93	154

Table 7: Delay time ($\kappa = 1$, $k^* = 0.8m$)

		$d = [-1, -1]$					$d = [1, 1]$				
b		min	1Q	2Q	3Q	max	min	1Q	2Q	3Q	max
(a) $m = 50$, $\rho = 0.4$											
0.5	CUSUM($\gamma = 0.15$)	4	25	42	46	50	4	24	41	46	50
	CUSUM($\gamma = 0.45$)	1	12	31	46	50	1	11	29	45	50
	FL	5	22	41	48	50	5	22	42	48	50
1.0	CUSUM($\gamma = 0.15$)	4	41	45	48	50	4	41	45	48	50
	CUSUM($\gamma = 0.45$)	1	27	45	48	50	1	26	45	48	50
	FL	5	44	47	49	50	5	44	47	49	50
1.5	CUSUM($\gamma = 0.15$)	4	43	46	48	50	4	43	46	48	50
	CUSUM($\gamma = 0.45$)	1	43	46	48	50	1	43	46	49	50
	FL	5	44	46	48	50	5	44	46	48	50
2.0	CUSUM($\gamma = 0.15$)	4	43	46	48	50	4	44	46	48	50
	CUSUM($\gamma = 0.45$)	1	44	46	48	50	1	44	46	48	50
	FL	5	43	45	46	50	5	43	45	46	50
(b) $m = 50$, $\rho = 0.8$											
0.5	CUSUM($\gamma = 0.15$)	4	21	38	46	50	4	22	39	46	50
	CUSUM($\gamma = 0.45$)	2	14	30	45	50	2	14	31	45	50
	FL	7	31	45	48	50	7	31	45	48	50
1.0	CUSUM($\gamma = 0.15$)	4	33	44	47	50	4	34	44	47	50
	CUSUM($\gamma = 0.45$)	2	25	44	47	50	2	26	44	47	50
	FL	7	43	46	48	50	7	43	46	48	50
1.5	CUSUM($\gamma = 0.15$)	4	41	45	47	50	4	42	45	47	50
	CUSUM($\gamma = 0.45$)	2	41	45	48	50	2	41	45	47	50
	FL	7	43	45	46	50	7	43	45	46	50
2.0	CUSUM($\gamma = 0.15$)	4	42	45	47	50	4	42	44	47	50
	CUSUM($\gamma = 0.45$)	2	42	45	47	50	2	42	45	47	50
	FL	7	42	44	45	50	7	42	44	45	50
(c) $m = 100$, $\rho = 0.4$											
0.5	CUSUM($\gamma = 0.15$)	10	66	88	95	100	10	68	88	95	100
	CUSUM($\gamma = 0.45$)	2	40	86	93	100	2	43	87	94	100
	FL	9	82	93	97	100	9	72	93	97	100
1.0	CUSUM($\gamma = 0.15$)	10	86	92	96	100	10	86	92	96	100
	CUSUM($\gamma = 0.45$)	2	86	92	96	100	2	86	92	97	100
	FL	9	89	93	96	100	9	89	93	96	100
1.5	CUSUM($\gamma = 0.15$)	10	87	92	96	100	10	87	92	96	100
	CUSUM($\gamma = 0.45$)	2	88	93	97	100	2	88	92	96	100
	FL	9	87	89	92	100	9	87	89	92	100
2.0	CUSUM($\gamma = 0.15$)	10	87	91	95	100	10	87	91	95	100
	CUSUM($\gamma = 0.45$)	2	87	92	95	100	2	87	91	95	100
	FL	9	85	87	89	98	9	85	87	89	98

Table 7: (continued)

		$d=[-1,-1]$					$d=[1,1]$				
b		min	1Q	2Q	3Q	max	min	1Q	2Q	3Q	max
(d) $m=100$, $\rho=0.8$											
0.5	CUSUM($\gamma=0.15$)	10	59	86	94	100	10	61	87	94	100
	CUSUM($\gamma=0.45$)	5	39	83	93	100	5	41	85	94	100
	FL	9	85	92	97	100	9	84	92	97	100
1.0	CUSUM($\gamma=0.15$)	10	84	91	96	100	10	84	90	95	100
	CUSUM($\gamma=0.45$)	5	83	91	96	100	5	84	90	95	100
	FL	9	87	91	95	100	9	87	91	95	100
1.5	CUSUM($\gamma=0.15$)	10	86	91	95	100	10	85	90	95	100
	CUSUM($\gamma=0.45$)	5	86	91	95	100	5	86	91	95	100
	FL	9	85	88	91	100	9	85	88	91	100
2.0	CUSUM($\gamma=0.15$)	10	85	89	93	100	10	85	89	93	100
	CUSUM($\gamma=0.45$)	5	86	90	94	100	5	85	90	94	100
	FL	9	84	86	88	100	9	84	86	88	100
(e) $m=250$, $\rho=0.4$											
0.5	CUSUM($\gamma=0.15$)	28	212	228	240	250	28	213	228	240	250
	CUSUM($\gamma=0.45$)	5	205	228	240	250	5	207	228	240	250
	FL	27	226	236	244	250	27	225	236	244	250
1.0	CUSUM($\gamma=0.15$)	28	219	230	240	250	28	219	230	240	250
	CUSUM($\gamma=0.45$)	5	220	231	241	250	5	220	232	241	250
	FL	27	217	222	228	250	27	216	222	228	250
1.5	CUSUM($\gamma=0.15$)	28	216	224	233	250	28	215	224	233	250
	CUSUM($\gamma=0.45$)	5	217	226	235	250	5	217	226	234	250
	FL	27	211	215	218	234	27	211	214	218	238
2.0	CUSUM($\gamma=0.15$)	28	212	218	225	247	28	212	218	224	248
	CUSUM($\gamma=0.45$)	5	213	219	226	250	5	213	220	226	250
	FL	27	208	211	214	225	27	208	211	213	226
(f) $m=250$, $\rho=0.8$											
0.5	CUSUM($\gamma=0.15$)	18	210	227	240	250	18	209	227	240	250
	CUSUM($\gamma=0.45$)	8	200	225	239	250	8	198	226	239	250
	FL	30	224	234	243	250	30	223	233	242	250
1.0	CUSUM($\gamma=0.15$)	18	217	228	239	250	18	217	229	239	250
	CUSUM($\gamma=0.45$)	8	218	229	239	250	8	218	230	240	250
	FL	30	215	221	227	250	30	215	220	227	250
1.5	CUSUM($\gamma=0.15$)	18	214	222	231	250	18	214	222	231	250
	CUSUM($\gamma=0.45$)	8	215	224	233	250	8	215	224	233	250
	FL	30	210	214	218	236	30	210	213	217	240
2.0	CUSUM($\gamma=0.15$)	18	211	217	223	250	18	211	217	223	250
	CUSUM($\gamma=0.45$)	8	212	218	225	250	8	212	218	225	250
	FL	30	207	210	213	226	30	207	210	213	229

Table 8: Tests for a unit root

	Training period	T	t-statistic(ADF)	Exp-Wald
Denmark	1995Q2-2001Q3	26	-3.290479^c	1.975
Japan	1990Q2-2007Q4	71	-3.311520^c	1.070
New Zealand	1993Q1-2007Q4	60	-3.516139^b	1.755

[1] a, b, c denote statistical significance at 1%, 5%, and 10% level.

Table 9: Results of the monitoring tests

Training period	Estimated break date	CUSUM $\gamma = 0.15$	CUSUM $\gamma = 0.45$	FL
(a) Denmark				
1995Q2-2001Q3	2001Q4	2002Q3	2002Q2	2007Q4
(b) Japan				
1990Q2-1998Q4	2008Q3	2010Q1	2011Q2	2016Q2
1990Q2-2006Q2		2011Q2	2010Q4	2009Q4
1990Q2-2007Q4		2009Q4	2009Q1	**
(c) New Zealand				
1993Q1-2001Q3	2008Q1	**	**	2010Q3
1993Q1-2006Q3		2010Q3	2010Q1	2010Q2
1993Q1-2007Q4		2009Q4	2009Q1	**

[1] ** means that it cannot reject the null hypothesis of no change.

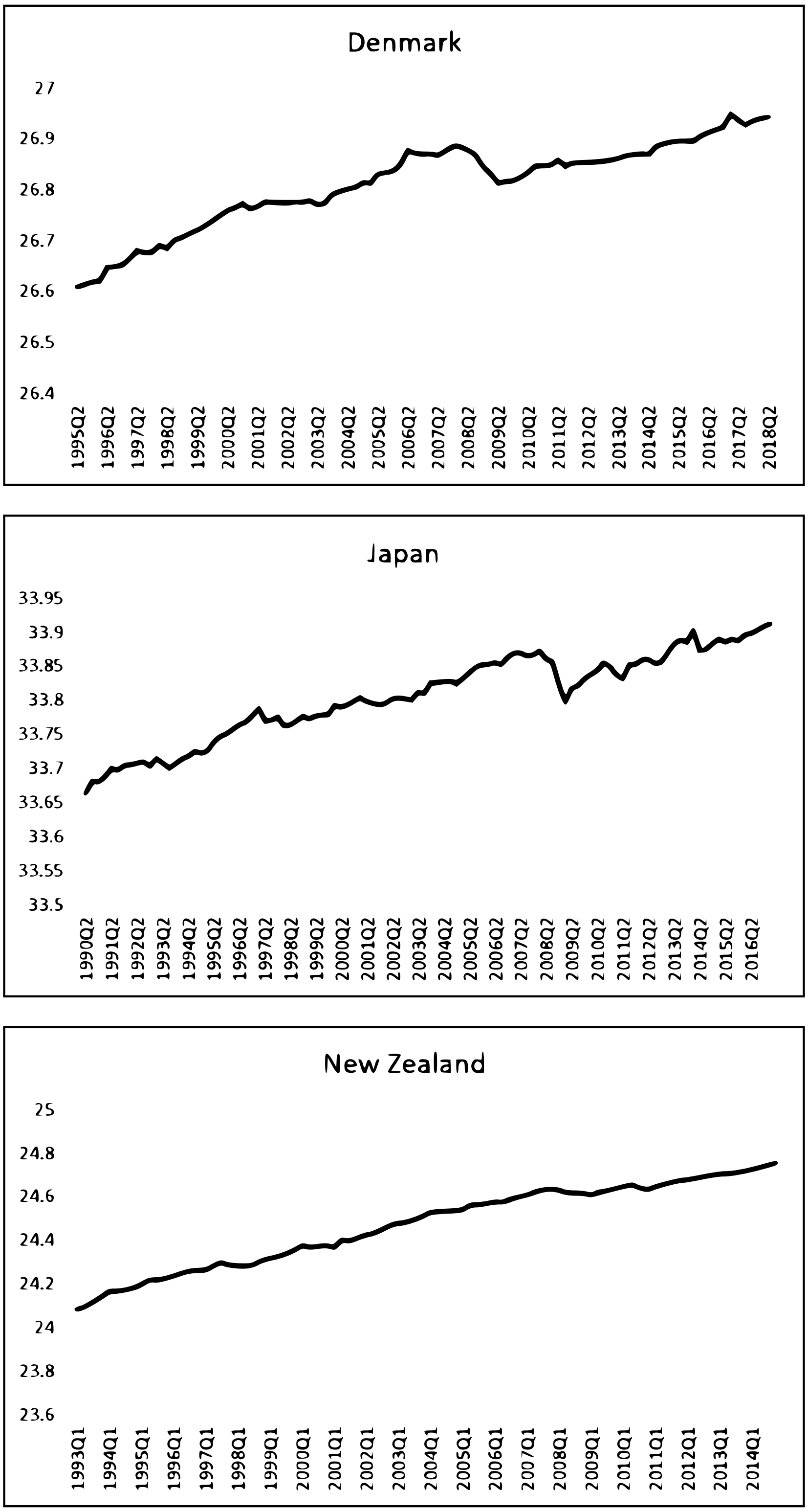

Figure 1: Logarithm of quarterly real GDP