Treatment Effect Models with Strategic Interaction in Treatment Decisions

Tadao Hoshino* and Takahide Yanagi†

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Abstract

This study develops identification and estimation methods for treatment effect models with strategic interaction in treatment decisions. We consider models where one's treatment choice and outcome can be endogenously affected by others' treatment choices. We formulate the interaction of treatment decisions as a two-player complete information game with potential multiple equilibria. For this model, under the assumption of a stochastic equilibrium selection rule, we prove that the marginal treatment effect (MTE) from one's own treatment and that from his/her partner's can be separately point-identified using a latent index framework. Based on our constructive identification results, we propose a two-step semiparametric procedure for estimating the MTE parameters using series approximation. We show that the proposed estimator is uniformly consistent with the optimal convergence rate and has asymptotic normality.

Keywords: binary games; instrumental variables; latent index models; marginal treatment effects; strategic interaction.

JEL Classification: C14, C31, C57.

*School of Political Science and Economics, Waseda University, 1-6-1 Nishi-waseda, Shinjuku-ku, Tokyo 169-8050, Japan. Email: thoshino@waseda.jp.

†Graduate School of Economics, Kyoto University, Yoshida Honmachi, Sakyo, Kyoto, 606-8501, Japan. Email: yanagi@econ.kyoto-u.ac.jp.

1 Introduction

Studies on treatment evaluation commonly rely on the assumption that one's outcome is not affected by others' treatment status, which is the so-called stable unit treatment value assumption (SUTVA). However, SUTVA is often criticized as empirically implausible. For example, in the evaluation of the effect of smoking behavior on health outcomes for couples, SUTVA implies that one's health outcome is not affected by his/her partner's smoking behavior. Other examples include empirical studies on social interactions in education. It is commonly observed that peer behavior, such as delinquent activities, has significant impact on students' academic performance (i.e., a contextual effect; Manski, 1993). For another example, we may consider the effect of increasing/decreasing the minimum wage on unemployment rate in a jurisdiction. As increasing/decreasing the minimum wage in the jurisdiction should affect the unemployment rate in neighboring jurisdictions, SUTVA is violated in this case as well. Given these facts, it has recently been attempted to relax SUTVA (e.g., Hong and Raudenbush, 2006; Hudgens and Halloran, 2008; Ferracci *et al.*, 2014; Aronow and Samii, 2017). However, these studies typically assume (conditional) independence among individual treatment decisions, and such an assumption is inappropriate for the above-mentioned examples.

The objective of this study is to develop identification and estimation methods for treatment effect models that allow the violation of SUTVA (i.e., treatment spillover) and the endogenous interaction of treatment decisions among individuals. In particular, this study focuses on the setup in which the interaction occurs between each pair of individuals (for instance, couples, best friends, twins, or duopoly firms), and postulates that they make decisions on their treatment status simultaneously in a binary game of complete information. Within this framework, we aim to formulate a set of reasonable sufficient conditions under which the treatment effect parameters, such as marginal treatment effects (MTE) and local average treatment effects (LATE), can be point-identified.

To this end, we should consider the possibility of multiple equilibria in the treatment decisions, which is a highly non-trivial task. The presence of multiple equilibria leads to an *incomplete* econometric model (see Tamer, 2003; Lewbel, 2007; Ciliberto and Tamer, 2009; Chesher and Rosen, 2012, among others) in the sense that model-consistent treatment assignment is not unique. The issue of incompleteness has been a common problem in the literature on game model estimation; however, it has not yet been considered in treatment effect models. Moreover, it should be noted that we cannot apply the standard instrumental variable (IV) estimation technique based on the *monotonicity* condition (e.g., Imbens and Angrist, 1994; Heckman and Vytlacil, 1999, 2005). In our model, one's treatment and instrument are not monotonically related in the conventional sense; as one's treatment choice depends on another's treatment through strategic interaction, a change in one's instrumental variable can alter not only his/her treatment but also his/her partner's treatment, which further affects his/her own treatment.

Alternatively, if we view the pair of treatment decisions as a typical element of the treatment set, our situation may be seen as a model with unordered multivalued treatments. Then, one may consider applying the recent IV approaches for the evaluation of multivalued treatments that were proposed in Heckman and Pinto (2018) and Lee and Salanié (2018) and do not require the conventional monotonicity condition. In Heckman and Pinto (2018), an *unordered monotonicity* condition was introduced for treatment effect models with multivalued treatments, and it was demonstrated that several treatment effect parameters can be identified under this condition. The unordered monotonicity requires that a shift in instruments moves all individuals uniformly toward or against each possible treatment. This assumption is excessively restrictive for our case. If a change in instruments increases the probability of treatment for one player but decreases it for his/her partner, the shifts of treatment status are not in one direction only (see also the discussion in Lee and Salanié (2018)). Similarly, in Lee and Salanié (2018), identification of multivalued treatment models was considered. Their identification assumption is that treatment assignment can be uniquely determined by certain threshold-crossing rules, which is more related to our strategy. In the present study, we build on this latter approach and develop an identification strategy different from those in the previously mentioned studies.

More specifically, our identification strategy is based on the combination of the methods of local instrumental variables (LIV) developed in Heckman and Vytlacil (1999, 2005) and a stochastic equilibrium selection rule in the treatment decision game. The key idea is to use local variations of the pair of instrumental variables that alter the players' treatment status but do not directly affect their outcomes. Even though this is a natural extension of the LIV method to a multi-dimensional space, it is still insufficient for point-identification of the treatment effect parameters of interest, owing to the presence of multiple equilibria. We overcome this by explicitly introducing an empirically reasonable equilibrium selection rule; that is, we assume that the treatment decision under multiple equilibria follows a tractable probabilistic model.[1]

The contributions of our study are as follows: First, we can point-identify several MTE parameters that are not identifiable in the conventional framework; for example, *direct MTE*, in which only one's own treatment status switches from untreated to treated, whereas his/her partner's is left unchanged, *indirect MTE*, in which only one's partner's treatment status switches from untreated to treated, whereas one's own status is left unchanged, and *total MTE*, in which the treatment status of both players switches from untreated to treated. Importantly, the identification of these parameters helps us to understand the sources of unobserved heterogeneity in treatment effects in more detail than the conventional MTE framework. The conventional MTE can reveal heterogeneity in the treatment effect only with respect to the individuals' own unobservable characteristics. By contrast, our framework allows us to also examine how one's treatment effect is related to the other's unobservables. Secondly, we show that the presence of multiple equilibria provides a chance of over-identification for the MTEs through our strategy. This is based on the fact that multiple equilibria themselves may have identification power (e.g., de Paula and Tang, 2012; de Paula, 2013). Thirdly, even though treatment evaluation is our main concern, we also provide a new identification result for the game model in our context.

Our identification of the MTEs is constructive, so that we can estimate them directly following the identification strategy. We propose that the MTEs can be estimated by a two-step semiparametric procedure. In the first step, we estimate the parameters in the treatment decision game using a maximum likelihood (ML) approach. Using the estimates from the first step, in the second step, we estimate the MTEs assuming additive separability of the outcome equation, in which we employ semiparametric series (sieve) estimation techniques in the estimation of the outcome equation. We show that the proposed estimator of the MTE is uniformly consistent with the optimal convergence rate and asymptotically normally distributed. In addition, the estimator possesses an oracle property in the sense that its limiting distribution is the same as that for the infeasible estimator, where the parameters in the treatment decision game are assumed to be known. The asymptotic

[1] It should be noted that this assumption does not imply that the players take mixed strategies; rather, it assumes that even though the players are deterministic, there are some unobserved (by econometricians) random variables that determine their decisions on the equilibrium choices. For more details, see Assumption 3.2 below.

results for our two-step semiparametric estimation procedure are interesting in their own right, and indeed, they are novel in the series estimation literature.

To the best of our knowledge, there are few studies addressing the estimation of treatment effects in the presence of strategic interaction where the treatment decisions are explicitly modeled as games. One exception is the recent study Balat and Han (2018), where partial identification of average treatment effects (ATE) was developed. Thus, there is a difference in the main parameters of interest between theirs (ATE) and ours (MTE). In addition, both studies rely on different identification assumptions. The key assumption in Balat and Han (2018) is that the effects of the treatments on the outcomes are symmetric among individuals. Namely, in the smoking example, the symmetry condition requires that the effect of first-hand smoking is identical to that of second-hand smoking, which could be implausible in reality. In general, such an assumption is restrictive for cases where the outcome is specific to each player.[2]

Organization of the paper: The remainder of the paper is organized as follows. In Section 2, we introduce our model and briefly review the approaches for inference on incomplete models of discrete games. In Section 3, the identification of several treatment effect parameters, including MTE and LATE, is established. In Section 4, for the estimation of the MTE parameters, we propose a two-step semiparametric estimator using series approximation. The asymptotic properties of the proposed estimator are also studied in this section. Finally, Section 5 concludes the paper. The proofs of all theorems and technical lemmas are given in Appendices A and B. Other supplementary technical results are provided in Appendices C and D.

Notation: For a natural number n, we use $\mathbf{I}_n$ to denote the $n \times n$ identity matrix. For positive integers a_1 and a_2, $\mathbf{0}_{a_1 \times a_2}$ denotes the $a_1 \times a_2$ zero matrix . $\mathbf{1}(\cdot)$ denotes the indicator function that equals one if its argument is true and zero otherwise. For a random variable X, supp$[X]$ denotes the support of X. For a matrix A, $\|A\|$ denotes its Frobenius norm, i.e., $\|A\| = \sqrt{\mathrm{tr}\{A^\top A\}}$, where $\mathrm{tr}\{\cdot\}$ is the trace of a matrix. When A is a square matrix, we use $\chi_{\max}(A)$ and $\chi_{\min}(A)$ to denote its largest and smallest eigenvalues, respectively. We denote any symmetric generalized inverse of a matrix A by A^-.

2 Model

In this section, we present our treatment effect model with strategic interaction. We denote a player by j and his/her partner (or opponent) by $-j$ throughout the paper, where $j \in \{1, 2\}$. We aim to evaluate the effects of player j's treatment $D_j \in \{0, 1\}$ and/or his/her partner's treatment $D_{-j} \in \{0, 1\}$ on his/her outcome Y_j and/or his/her partner's outcome Y_{-j}. The outcomes may or may not be common to both players, that is, we allow both $Y_j = Y_{-j}$ and $Y_j \neq Y_{-j}$. The treatment D_j may be endogenous in the sense that D_j may be correlated with unobservable determinants of Y_j and Y_{-j}.

Let $Y_j^{(d_j, d_{-j})}$ be the potential outcome for player j when his/her own treatment status is $D_j = d_j$ and his/her partner's is $D_{-j} = d_{-j}$ for $(d_j, d_{-j}) \in \{0, 1\}^2$. This potential outcome notation explicitly allows player j's

[2] Balat and Han (2018) briefly discusses this point in Section 6.1.

outcome to depend on his/her partner's treatment D_{-j}. The observed outcome can be written as

$$Y_j = \sum_{d_j=0}^{1} \sum_{d_{-j}=0}^{1} I^{(d_j, d_{-j})} Y_j^{(d_j, d_{-j})},$$

where $I^{(d_j,d_{-j})} := \mathbf{1}(D_j = d_j, D_{-j} = d_{-j})$. Suppose that player j's potential outcome can be written as

$$Y_j^{(d_j,d_{-j})} = \mu_j^{(d_j,d_{-j})}\left(X_j, U_j^{(d_j,d_{-j})}\right), \tag{2.1}$$

where $X_j \in \mathbb{R}^{\dim(X)}$ is a vector of observable covariates, $U_j^{(d_j,d_{-j})} \in \mathbb{R}$ is an unobservable random variable and $\mu_j^{(d_j,d_{-j})}$ is an unknown structural function. The covariates X_1 and X_2 may contain common elements as well as some player-specific elements. For simplicity, we assume that the dimensions of X_1 and X_2 are both equal to $\dim(X)$, and the same assumption will be made for other variables. The model specification in (2.1) is fairly general in that both $U_j^{(d_j,d_{-j})}$ and $\mu_j^{(d_j,d_{-j})}$ can be dependent on the player type j and the treatment status (d_j, d_{-j}).

2.1 Strategic interaction in treatment decisions

To account for the strategic interaction between players, we presume a natural extension of the latent index model in Heckman and Vytlacil (1999, 2005). Suppose that player j's treatment is determined by

$$D_j = \mathbf{1}\left(\pi_j(D_{-j}, W_j) \geq \varepsilon_j\right), \tag{2.2}$$

where $W_j := (X_j^\top, Z_j^\top)^\top$ is a vector including X_j and the instrument $Z_j \in \mathbb{R}^{\dim(Z)}$, $\varepsilon_j \in \mathbb{R}$ is a continuously distributed unobservable random variable, and π_j is an unknown function. We assume that the instruments include at least one player-specific variable. We do not restrict the dependence structure between ε_j and $U_j^{(d_j,d_{-j})}$, which is the source of the endogeneity.

The existence of the instrument Z_j is crucial to control the endogeneity. The instruments are required to satisfy an exclusion restriction and a relevance condition as in the standard IV framework. Specifically, we assume that Z_j is independent of the unobservables ε_j and $U_j^{(d_j,d_{-j})}$, and that Z_j can cause variations in the value of $\pi_j(D_{-j}, W_j)$. In addition, when using an LIV method to identify the treatment effects, we require that not all elements of Z_j are discrete.[3] More formal statements for the conditions imposed on the instruments will be provided below.

For subsequent analysis, it is convenient to transform the model (2.2) as follows. First, as D_{-j} is a binary variable, without loss of generality, we can write

$$\pi_j(D_{-j}, W_j) = \pi_j^0(W_j) + D_{-j} \cdot \left(\pi_j^1(W_j) - \pi_j^0(W_j)\right), \tag{2.3}$$

[3] Even when only discrete instrumental variables are available, if one makes a parametric functional form assumption on the conditional expectation of $U_j^{(d_j,d_{-j})}$ given (V_1, V_2), where (V_1, V_2) are defined in (2.4) below, there is a chance of identifying the MTE as in Brinch *et al.* (2017).

for some functions π_j^0 and π_j^1. Let $X \coloneqq (X_1, X_2)$, and let

$$V_j \coloneqq F_{\varepsilon_j}(\varepsilon_j|X), \qquad P_j^0 \coloneqq F_{\varepsilon_j}\left(\pi_j^0(W_j)\middle|X\right), \qquad P_j^1 \coloneqq F_{\varepsilon_j}\left(\pi_j^1(W_j)\middle|X\right), \tag{2.4}$$

where $F_{\varepsilon_j}(\cdot|X)$ is the conditional cumulative distribution function (CDF) of ε_j given X. By construction, V_j is distributed as Uniform$[0,1]$ conditional on X. Combining these definitions and (2.3), we can rewrite the model (2.2) as

$$D_j = \mathbf{1}\left(P_j^0 + D_{-j} \cdot (P_j^1 - P_j^0) \geq V_j\right). \tag{2.5}$$

It should be noted that the treatment decision rule (2.5) is almost surely (a.s.) equivalent to

$$D_j = \underset{d \in \{0,1\}}{\operatorname{argmax}}\ d \cdot \{P_j^0 + D_{-j} \cdot (P_j^1 - P_j^0) - V_j\}.$$

That is, we can view D_j as the maximizer of the "payoff function" $d \cdot \{P_j^0 + D_{-j} \cdot (P_j^1 - P_j^0) - V_j\}$. Player j takes the treatment if and only if his/her payoff gain $P_j^0 + D_{-j} \cdot (P_j^1 - P_j^0) - V_j$ is positive. Otherwise, player j does not take the treatment and his/her payoff is zero (for normalization). Here, we assume that (W_j, ε_j) and $(W_{-j}, \varepsilon_{-j})$ are common knowledge to both players, which motivates us to interpret the set of treatments (D_j, D_{-j}) as the result from a simultaneous treatment decision game with complete information.

To proceed with our analysis, we assume that whether the treatment decisions of players j and $-j$ are strategic complements or strategic substitutes is known to us a priori. The treatment decisions are complementary (resp. substitute) if and only if the strategic interaction effect $P_j^1 - P_j^0$ is positive (resp. negative). For expositional simplicity, we hereafter restrict our main focus to the case of complementarity; however, it is not difficult to modify our approach in the case of strategic substitutes.

Assumption 2.1. Strategic complementarity: $P_j^1 - P_j^0 > 0$ (a.s.) for both $j = 1$ and 2.

Remark 2.1 (Players' roles)**.** The roles (types) of the players in a pair are allowed to be either asymmetric or symmetric. If the players' roles are asymmetric, the subscripts j and $-j$ have a specific meaning. For example, when the pair indicates a couple, players 1 and 2 correspond to wife and husband, respectively. For another example, when the pair is composed of best friends, the player indices 1 and 2 might represent the order of their relative ages. By contrast, if the roles are symmetric, the subscripts j and $-j$ have no implication. That is, the players do not have distinguished roles in this case. The identification and estimation methods proposed in this study are applicable to both cases.

2.2 Incompleteness

A major difficulty in our model is the *incompleteness* of the treatment decision model, owing to the presence of multiple equilibria. The realized treatment status based on (2.5) can be viewed as a Nash equilibrium in the complete information game with the payoff matrix given in Table 1.

There are potentially multiple equilibria in this game. Under strategic complementarity, it is easy to see that

Table 1: Payoff matrix

	$D_2 = 0$	$D_2 = 1$
$D_1 = 0$	$(0, 0)$	$(0, P_2^0 - V_2)$
$D_1 = 1$	$(P_1^0 - V_1, 0)$	$(P_1^1 - V_1, P_2^1 - V_2)$

we have the following relationship between the realized treatment (D_1, D_2) and the unobservables (V_1, V_2):

$$\begin{aligned} (D_1, D_2) = (1, 1) &\implies V_1 \leq P_1^1,\ V_2 \leq P_2^1, \\ (D_1, D_2) = (1, 0) &\iff V_1 \leq P_1^0,\ V_2 > P_2^1, \\ (D_1, D_2) = (0, 1) &\iff V_1 > P_1^1,\ V_2 \leq P_2^0, \\ (D_1, D_2) = (0, 0) &\implies V_1 > P_1^0,\ V_2 > P_2^0. \end{aligned} \tag{2.6}$$

The relationship (2.6) is visually summarized in Figure 1.

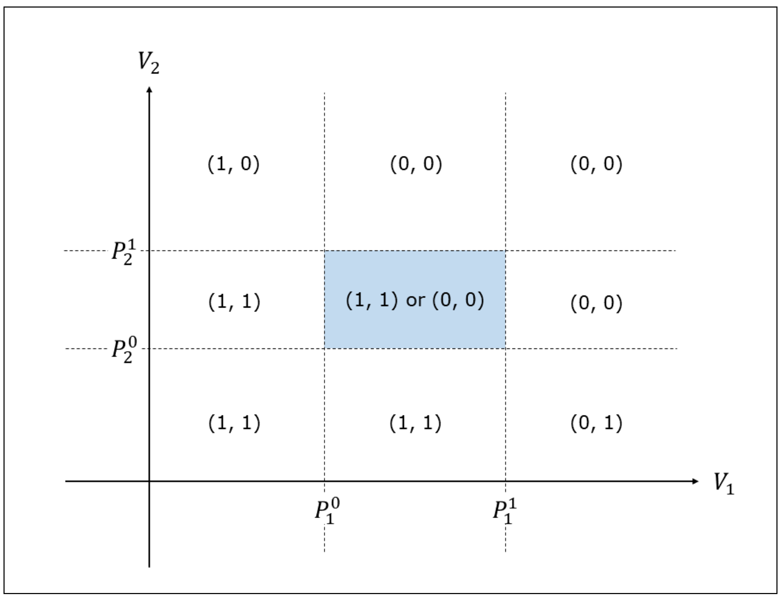

Figure 1: Nash equilibrium under strategic complementarity

As shown in the figure, the space of $V := (V_1, V_2)$ cannot be partitioned into non-overlapping regions associated with the four alternative realizations of $D := (D_1, D_2)$. Both $D = (1, 1)$ and $D = (0, 0)$ can occur when $P_1^0 < V_1 \leq P_1^1$ and $P_2^0 < V_2 \leq P_2^1$, and in this case, the value of D is not uniquely determined. This non-uniqueness of model-consistent decisions is called incompleteness and has been extensively studied in the literature on simultaneous equations models for discrete outcomes.

To handle incompleteness, there are essentially three approaches in the game econometrics literature (see de Paula (2013) for an excellent survey on this topic). The first approach is to focus only on the outcomes that can occur as unique equilibria (e.g., Bresnahan and Reiss, 1990; Berry, 1992). In our case, this happens for $D = (1, 0)$ and $D = (0, 1)$. Then, based on the probability of $D = (1, 0)$, $D = (0, 1)$, and $D \in \{(0, 0), (1, 1)\}$, one can form the likelihood function to estimate the parameters. The second approach is partial identification strategy (e.g., Ciliberto and Tamer, 2009; Chesher and Rosen, 2012). Clearly, the upper (resp. lower) bound of the probability of $D = (1, 1)$ can be attained if $D = (1, 1)$ is always (resp. never) selected when V falls in the multiple equilibria region. A similar argument applies to the case $D = (0, 0)$. By using these inequalities, we can obtain the bounds of the parameters. The third approach is to explicitly introduce a probabilistic (or possibly deterministic) equilibrium selection mechanism (e.g., Bjorn and Vuong, 1984; Kooreman, 1994; Soetevent and Kooreman, 2007; Bajari *et al.*, 2010; Card and Giuliano, 2013). Although the first and second approaches are theoretically less restrictive than the third, they do not provide any information concerning the choice between $D = (0, 0)$ and $(1, 1)$ on the region of multiplicity. As will be discussed later, without additional information on the equilibrium selection, it is quite difficult to point-identify the treatment parameters regarding the cases $D = (0, 0)$ and $(1, 1)$. Thus, this necessitates the adoption of the third approach.

Remark 2.2 (Identification-at-infinity)**.** To identify discrete games, some studies suggest using an identification-at-infinity strategy, which requires the presence of "large support" regressors (e.g., Tamer, 2003; Bajari *et al.*, 2010; Kline, 2015). The identification-at-infinity approach has advantages in that the parameters can be point-identified without explicitly introducing equilibrium selection rules under weaker equilibrium concepts than the Nash equilibrium (see Kline, 2015). However, it is well known that estimators based on identification-at-infinity have slower convergence rate and do not perform well in finite samples (cf., Khan and Tamer, 2010). It should be noted that, whether or not we use the identification-at-infinity approach, we should introduce certain assumptions on the equilibrium selection to "fully" identify the structure of the treatment decision game.

3 Identification

In this section, the identification of the treatment effect parameters in our model is presented. We here refer to the treatment decision game (2.5) as the "first stage", and to the process of outcome realization (2.1) as the "second stage". In this section, for expositional simplicity, we treat the parameters in the first stage as known to us, and focus solely on the identification of the parameters of interest for the second stage. More specifically, we implicitly assume sufficient conditions under which the conditional CDF and the density of V given $X = x$, which we denote by $H(v_1, v_2|x)$ and $h(v_1, v_2|x)$, respectively, and the variables P_1^0, P_1^1, P_2^0, and P_2^1 defined in (2.4) are identified. The identification and consistent estimation of these parameters will be discussed in detail later in Appendix D and Section 4, respectively.

To state the following assumptions, we introduce additional notations as follows: $Y := (Y_1, Y_2)$, $Z := (Z_1, Z_2)$, $W := (W_1, W_2)$, $\varepsilon := (\varepsilon_1, \varepsilon_2)$, and $U^{(d_1,d_2)} := (U_1^{(d_1,d_2)}, U_2^{(d_2,d_1)})$ for $(d_1, d_2) \in \{0, 1\}^2$. Here, we formally introduce the exclusion restriction and the relevance condition for the instruments Z.

Assumption 3.1.

(i) For all $(d_1, d_2) \in \{0, 1\}^2$, the instruments Z are excluded from the structural functions in (2.1), and are independent of the unobservables ε and $U^{(d_1,d_2)}$ conditional on X.

(ii) For both $j = 1$ and 2, the instrument Z_j contains continuous variables such that $\pi_j^0(W_j)$ and $\pi_j^1(W_j)$ are non-degenerate and continuously distributed conditional on X.

These are standard requirements for the LIV method. Assumption 3.1(i) is the so-called exclusion restriction. The instruments Z do not directly affect the outcomes Y and are conditionally independent of the error terms. It should be noted that the transformed error V is also conditionally independent of Z given X by construction.

Thus, the conditional distributions of V and $U^{(d_1,d_2)}$ given W are identical to those given X. Assumption 3.1(ii) is the so-called relevance condition; the instrument Z_j must contain a continuous non-trivial determinant of D_j.

We provide below a series of identification results only for player 1 (the results for player 2 are symmetric and thus are omitted). We first discuss the identification of the conditional means of the potential outcomes:

$$m_1^{(d_1,d_2)}(x,p_1,p_2) := E[Y_1^{(d_1,d_2)}|X=x,V_1=p_1,V_2=p_2] \quad \text{for } (d_1,d_2)\in\{0,1\}^2. \tag{3.1}$$

Once the conditional means are identified, we can identify the MTE parameters straightforwardly. It should be noted that as the total number of treatment patterns is four (i.e., $D\in\{(0,0),(1,0),(0,1),(1,1)\}$), there are six combinations of pairs, that is, six distinct MTE parameters for each player. From the point of view of player 1, the MTEs of interest are, for example, the *direct MTE*, the *indirect MTE*, and the *total MTE*, which are defined as

$$\begin{aligned}
\tau_{direct}^{(d_2)}(x,p_1,p_2) &:= m_1^{(1,d_2)}(x,p_1,p_2) - m_1^{(0,d_2)}(x,p_1,p_2) \quad \text{for } d_2\in\{0,1\},\\
\tau_{indirect}^{(d_1)}(x,p_1,p_2) &:= m_1^{(d_1,1)}(x,p_1,p_2) - m_1^{(d_1,0)}(x,p_1,p_2) \quad \text{for } d_1\in\{0,1\},\\
\tau_{total}(x,p_1,p_2) &:= m_1^{(1,1)}(x,p_1,p_2) - m_1^{(0,0)}(x,p_1,p_2),
\end{aligned}$$

respectively. In the conventional framework, we can identify the source of unobserved heterogeneity in treatment effects only up to the player's own unobservable factor V_1. By contrast, our MTE parameters are also informative about the variation of the treatment effects with his/her partner's unobservable V_2.

3.1 Identification of the conditional means of potential outcomes

We first consider the case where D is equal to either $(1,0)$ or $(0,1)$. In this case, V resides in the regions with unique equilibrium. In the following, we will show that the following quantities can be identified:

$$\begin{aligned}
&m_1^{(1,0)}(x,p_1^0,p_2^1) \quad \text{for } (p_1^0,p_2^1)\in \text{supp}[P_1^0,P_2^1|X=x,D=(1,0)],\\
&m_1^{(0,1)}(x,p_1^1,p_2^0) \quad \text{for } (p_1^1,p_2^0)\in \text{supp}[P_1^1,P_2^0|X=x,D=(0,1)].
\end{aligned}$$

The former is the conditional mean of the potential outcome for player 1 when $D=(1,0)$ is realized and V is located at point B in Figure 2. At this point, players 1 and 2 are indifferent between the actions $D=(1,0)$, $(1,1)$, and $(0,0)$ given that $(P_1^0,P_2^1)=(p_1^0,p_2^1)$ holds. Thus, only a small deviation from (p_1^0,p_2^1) may result in different treatment decisions from $D=(1,0)$. Similarly, the latter is the case when $D=(0,1)$ and V is at point C in the figure, where the players are indifferent between $D=(0,1)$, $(1,1)$, and $(0,0)$.

We define the following function:

$$\psi_1^{(d_1,d_2)}(x,p_1^{d_2},p_2^{d_1}) := E[I^{(d_1,d_2)}Y_1|X=x,P_1^{d_2}=p_1^{d_2},P_2^{d_1}=p_2^{d_1}] \quad \text{for } (d_1,d_2)\in\{0,1\}^2,$$

which can be directly identified from data for $(p_1^{d_2},p_2^{d_1})\in \text{supp}[P_1^{d_2},P_2^{d_1}|X=x,D=(d_1,d_2)]$. Hereafter, when there is no confusion, we suppress the subscript 1 from $m_1^{(d_1,d_2)}$ and $\psi_1^{(d_1,d_2)}$ for notational simplicity. The next theorem formally states that $m^{(1,0)}(x,p_1^0,p_2^1)$ and $m^{(1,0)}(x,p_1^0,p_2^1)$ can be identified.

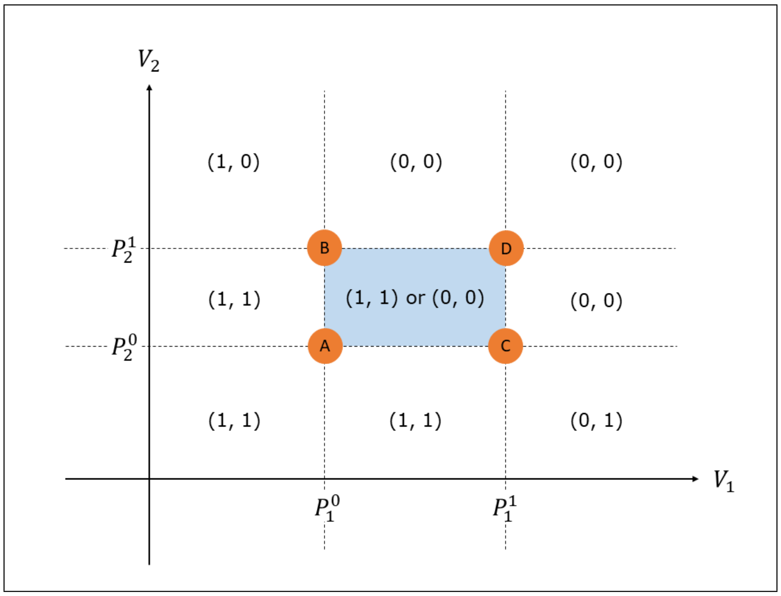

Figure 2: Points at which the conditional means are identified.

Theorem 3.1. Suppose that Assumptions 2.1 and 3.1 hold. Then, if $m^{(1,0)}(x, v_1, v_2)$, $m^{(0,1)}(x, v_1, v_2)$, and $h(v_1, v_2|x)$ are continuous in (v_1, v_2), we have

$$m^{(1,0)}(x, p_1^0, p_2^1) = -\frac{1}{h(p_1^0, p_2^1|x)} \frac{\partial^2 \psi^{(1,0)}(x, p_1^0, p_2^1)}{\partial p_1^0 \partial p_2^1},$$
$$m^{(0,1)}(x, p_1^1, p_2^0) = -\frac{1}{h(p_1^1, p_2^0|x)} \frac{\partial^2 \psi^{(0,1)}(x, p_1^1, p_2^0)}{\partial p_1^1 \partial p_2^0}.$$

This shows that as the quantities on the right-hand sides are identified from data, the parameters on the left-hand sides can be identified. This result is unique and more complicated than the results in previous studies (e.g., Heckman and Vytlacil, 2005; Carneiro and Lee, 2009) in some respects. First, the presence of the joint density $h(v_1, v_2|x)$ in the denominator reflects the possible correlation between the unobservables V_1 and V_2. It should be noted that in the special case where V_1 is independent of V_2 given X, it holds that $h(v_1, v_2|x) = 1$. Secondly, the identification is achieved through the cross partial derivatives with respect to p_1 and p_2 as a consequence of treatment spillover.

A sketch of the proof for $m^{(1,0)}(x, p_1^0, p_2^1)$ would be helpful to illustrate the identification strategy used here. It can be shown that

$$\begin{aligned}\psi^{(1,0)}(x, p_1^0, p_2^1) &= E[Y_1^{(1,0)}|X = x, V_1 \leq p_1^0, V_2 > p_2^1] \cdot \Pr[V_1 \leq p_1^0, V_2 > p_2^1|X = x] \\ &= \int_{p_2^1}^1 \int_0^{p_1^0} m^{(1,0)}(x, v_1, v_2) h(v_1, v_2|x) \mathrm{d}v_1 \mathrm{d}v_2.\end{aligned}$$

Partially differentiating both sides with respect to p_1^0 and p_2^1 yields the desired result. In this derivation, we require strategic complementarity in Assumption 2.1 so that the treatment status $D = (1, 0)$ may be uniquely linked with the region of V (i.e., the upper left region in Figures 1 and 2). Assumption 3.1(i) is required so that the first equality may be derived, and the conditions by P_1^0 and P_2^1 be ignored. Assumption 3.1(ii) ensures that the partial differentiation with respect to p_1^0 and p_2^1 is well-defined.

We move on to the identification in the case where V is possibly in the multiple equilibria region, i.e., the region where $D \in \{(0,0),(1,1)\}$ (i.e., the central shaded area in Figures 1 and 2). As the treatment status D is not uniquely determined in the multiple equilibria region, we introduce an additional assumption on the equilibrium selection that leads to point identification of the parameters of interest. We write the region of V where multiple equilibria occur as

$$\mathcal{V}_{mul}(\mathbf{P}) \coloneqq \left\{(v_1,v_2)\in[0,1]^2 : P_1^0 < v_1 \le P_1^1, P_2^0 < v_2 \le P_2^1\right\},$$

where $\mathbf{P} = (P_1^0, P_1^1, P_2^0, P_2^1)$. Furthermore, we denote the region of V where $D = (0,0)$ is uniquely selected by

$$\mathcal{V}_{uni}^{(0,0)}(\mathbf{P}) \coloneqq \left\{(v_1,v_2)\in[0,1]^2 : P_1^0 < v_1, P_2^0 < v_2\right\} \setminus \mathcal{V}_{mul}(\mathbf{P}).$$

For notational convenience, we often write $\mathcal{V}_{mul} = \mathcal{V}_{mul}(\mathbf{p})$ and $\mathcal{V}_{uni}^{(0,0)} = \mathcal{V}_{uni}^{(0,0)}(\mathbf{p})$ by suppressing their dependence on $\mathbf{P} = \mathbf{p}$, where $\mathbf{p} = (p_1^0, p_1^1, p_2^0, p_2^1)$ are fixed values.

We introduce the following assumption on the selection of treatment status.

Assumption 3.2. There exist an unknown constant $\lambda \in [0,1]$ and an unobservable random variable ϵ distributed as Uniform$[0,1]$ independent of $(W, \varepsilon, U^{(d_1,d_2)})$ such that when $V \in \mathcal{V}_{mul}(\mathbf{P})$, $D = (0,0)$ occurs if and only if $\epsilon \le \lambda$, that is,

$$D = (0,0) \Longleftrightarrow V \in \mathcal{V}_{uni}^{(0,0)}(\mathbf{P}) \vee (V \in \mathcal{V}_{mul}(\mathbf{P}) \wedge \epsilon \le \lambda).$$

That is, the players select action $D = (0,0)$ with probability $\lambda \in [0,1]$ in the multiple equilibria situation. It should be noted that, as mentioned above, we can treat λ as an identified parameter that can be consistently estimated. Even though we introduce Assumption 3.2 for tractability, we can instead consider imposing alternative more general structure on the equilibrium selection (cf., Jun and Pinkse, 2017). For example, we can relax the assumption by allowing λ to depend on X. Even though the assumption does not allow λ or ϵ to depend on the unobservables ε and $U^{(d_1,d_2)}$, in such cases point-identification of the treatment effects would be difficult.

To state the next theorem, we define the following function:

$$\psi^{(d_1,d_2)}(x,\mathbf{p}) \coloneqq E[I^{(d_1,d_2)}Y_1 | X = x, \mathbf{P} = \mathbf{p}],$$

which can be identified from data for $\mathbf{p} \in \text{supp}[\mathbf{P} | X = x, D = (d_1,d_2)]$.

Theorem 3.2. Suppose that Assumptions 2.1, 3.1, and 3.2 hold. Then, if $m^{(0,0)}(x,v_1,v_2)$, $m^{(1,1)}(x,v_1,v_2)$, and $h(v_1, v_2|x)$ are continuous in (v_1, v_2), we have

$$\begin{aligned}
m^{(0,0)}(x, p_1^0, p_2^0) &= \frac{1}{\lambda h(p_1^0, p_2^0|x)} \frac{\partial^2 \psi^{(0,0)}(x, \mathbf{p})}{\partial p_1^0 \partial p_2^0} && \text{for } \lambda > 0, \\
m^{(0,0)}(x, p_1^{d_1}, p_2^{d_2}) &= \frac{1}{(1-\lambda) h(p_1^{d_1}, p_2^{d_2}|x)} \frac{\partial^2 \psi^{(0,0)}(x, \mathbf{p})}{\partial p_1^{d_1} \partial p_2^{d_2}} && \text{for } \lambda < 1 \text{ and } d_1 \neq d_2, \\
m^{(0,0)}(x, p_1^1, p_2^1) &= -\frac{1}{(1-\lambda) h(p_1^1, p_2^1|x)} \frac{\partial^2 \psi^{(0,0)}(x, \mathbf{p})}{\partial p_1^1 \partial p_2^1} && \text{for } \lambda < 1,
\end{aligned}$$

and

$$\begin{aligned}
m^{(1,1)}(x, p_1^0, p_2^0) &= -\frac{1}{\lambda h(p_1^0, p_2^0|x)} \frac{\partial^2 \psi^{(1,1)}(x, \mathbf{p})}{\partial p_1^0 \partial p_2^0} && \text{for } \lambda > 0, \\
m^{(1,1)}(x, p_1^{d_1}, p_2^{d_2}) &= \frac{1}{\lambda h(p_1^{d_1}, p_2^{d_2}|x)} \frac{\partial^2 \psi^{(1,1)}(x, \mathbf{p})}{\partial p_1^{d_1} \partial p_2^{d_2}} && \text{for } \lambda > 0 \text{ and } d_1 \neq d_2, \\
m^{(1,1)}(x, p_1^1, p_2^1) &= \frac{1}{(1-\lambda) h(p_1^1, p_2^1|x)} \frac{\partial^2 \psi^{(1,1)}(x, \mathbf{p})}{\partial p_1^1 \partial p_2^1} && \text{for } \lambda < 1.
\end{aligned}$$

The intuition behind the identification can be visually understood by Figure 2. For example, $m^{(0,0)}(x, p_1^0, p_2^0)$ is the mean of the potential outcome $Y_1^{(0,0)}$ at point A in the figure, where the players are indifferent between the actions $D = (0,0)$ and $(1,1)$ given that $(P_1^0, P_2^0) = (p_1^0, p_2^0)$ holds. Thus, a small fluctuation of (V_1, V_2) around (p_1^0, p_2^0) can change the treatment status of a certain proportion of the players from $D = (0,0)$ to $(1,1)$ only if $\lambda > 0$. It is clear that if $\lambda = 0$, the identification fails because $D = (0,0)$ is never chosen at this point.

For better understanding, we provide an outline of the proof of the result for $m^{(0,0)}(x, p_1^0, p_2^0)$. We can observe that

$$\begin{aligned}
\psi^{(0,0)}(x, \mathbf{p}) &= E[Y_1^{(0,0)}|X = x, V \in \mathcal{V}_{uni}^{(0,0)}] \cdot \Pr[V \in \mathcal{V}_{uni}^{(0,0)}|X = x] + \lambda \cdot E[Y_1^{(0,0)}|X = x, V \in \mathcal{V}_{mul}] \cdot \Pr[V \in \mathcal{V}_{mul}|X = x] \\
&= \int_{p_1^0}^1 \int_{p_2^0}^1 m^{(0,0)}(x, v_1, v_2) h(v_1, v_2|x) \mathrm{d}v_1 \mathrm{d}v_2 - (1-\lambda) \int_{p_1^0}^{p_1^1} \int_{p_2^0}^{p_2^1} m^{(0,0)}(x, v_1, v_2) h(v_1, v_2|x) \mathrm{d}v_1 \mathrm{d}v_2,
\end{aligned}$$

where the second equality follows by direct calculation. Partially differentiating both sides with respect to p_1^0 and p_2^0 leads to

$$m^{(0,0)}(x, p_1^0, p_2^0) = \frac{1}{\lambda h(p_1^0, p_2^0|x)} \frac{\partial^2 \psi^{(0,0)}(x, \mathbf{p})}{\partial p_1^0 \partial p_2^0}.$$

It should be noted that the above identification results are obtained by conditioning the values of all four variables $(P_1^0, P_1^1, P_2^0, P_2^1)$. We recall that Theorem 3.1 has required only two values out of the four as the conditioning variables, as the point at which identification is achieved is well-characterized as the upper left or the lower right corner in the space of V. By contrast, we should fix the values of all four $(P_1^0, P_1^1, P_2^0, P_2^1)$ under multiple equilibria because the four points are required for characterizing the multiple-equilibrium region. Nonetheless, the parameter to be identified here, i.e., $m^{(0,0)}(x, p_1^0, p_2^0)$, relates to only two values p_1^0 and p_2^0, so that it is irrelevant to the other remaining p_1^1 and p_2^1. Indeed, as shown in the next section, the computational effort required for estimating $m^{(0,0)}(x, p_1, p_2)$ and $m^{(1,1)}(x, p_1, p_2)$ is not essentially greater than that for

$m^{(1,0)}(x, p_1, p_2)$ or $m^{(0,1)}(x, p_1, p_2)$, i.e., the estimator is free from the curse-of-dimensionality.

Remark 3.1 (Over-identification)**.** The above theorem states that we can achieve over-identification of parameters. For example, we assume that the conditional support of (P_1^0, P_2^0) given $D = (0, 0)$ overlaps with that of (P_1^1, P_2^1) given $D = (0, 0)$; that is, some $(p_1, p_2) \in \text{supp}[P_1^0, P_2^0 | X = x, D = (0, 0)] \cap \text{supp}[P_1^1, P_2^1 | X = x, D = (0, 0)]$ exists. Then, if λ is bounded away from zero and one (i.e., multiple equilibria do exist), we can identify $m^{(0,0)}(x, p_1, p_2)$ in at least two ways:

$$\begin{aligned} m^{(0,0)}(x, p_1, p_2) &= \frac{1}{\lambda h(p_1, p_2 | x)} \frac{\partial^2 \psi^{(0,0)}(x, p_1, p_1^1, p_2, p_2^1)}{\partial p_1 \partial p_2} \\ &= -\frac{1}{(1-\lambda) h(p_1, p_2 | x)} \frac{\partial^2 \psi^{(0,0)}(x, p_1^0, p_1, p_2^0, p_2)}{\partial p_1 \partial p_2}, \end{aligned}$$

where (p_1^1, p_2^1) and (p_1^0, p_2^0) can be any values as long as they are consistent with the value of (p_1, p_2). This over-identification would be viewed as reminiscent of findings in the game econometrics literature (see, e.g., de Paula and Tang, 2012; de Paula, 2013). Intuitively, when multiple equilibria between $D = (0, 0)$ and $(1, 1)$ exist owing to complementarity, the data would exhibit a positive correlation between D_1 and D_2, and this provides more chances of identifying $m^{(0,0)}$ and $m^{(1,1)}$ than the other two cases. Moreover, the over-identification results provide information on the validity of the assumption of strategic complementarity. This is because if the treatment decisions are not complements but substitutes, the over-identification results for $m^{(0,0)}(x, p_1, p_2)$ and $m^{(1,1)}(x, p_1, p_2)$ in Theorem 3.2 do not hold; rather, similar results hold for $m^{(1,0)}(x, p_1, p_2)$ and $m^{(0,1)}(x, p_1, p_2)$.

3.2 Identification of the treatment effects

Given the results of Theorems 3.1 and 3.2, the identification of the MTE parameters is rather straightforward. First, for the indirect MTE, we note the following equalities:

$$\begin{aligned} \tau_{indirect}^{(0)}(x, p_1, p_2) &= \tau_{total}(x, p_1, p_2) - \tau_{direct}^{(1)}(x, p_1, p_2), \\ \tau_{indirect}^{(1)}(x, p_1, p_2) &= \tau_{total}(x, p_1, p_2) - \tau_{direct}^{(0)}(x, p_1, p_2). \end{aligned}$$

Thus, for the identification of the indirect MTE, it is sufficient to discuss only the identification of the direct MTE and the total MTE. Moreover, for the direct MTE, we focus below only on the case $d_2 = 0$ for expositional simplicity.

By Theorems 3.1 and 3.2, assuming that $\lambda < 1$, we can obtain the direct MTE with $d_2 = 0$ by

$$\tau_{direct}^{(0)}(x, p_1^0, p_2^1) = -\frac{1}{h(p_1^0, p_2^1 | x)} \frac{\partial^2}{\partial p_1^0 \partial p_2^1} \left(\psi^{(1,0)}(x, p_1^0, p_2^1) + \frac{\psi^{(0,0)}(x, \mathbf{p})}{1 - \lambda} \right).$$

We recall that the value of $\psi^{(1,0)}(x, p_1^0, p_2^1)$ cannot be identified from data if $(p_1^0, p_2^1) \notin \text{supp}[P_1^0, P_2^1 | X = x, D = (1, 0)]$. Similarly, if $\mathbf{p} \notin \text{supp}[\mathbf{P} | X = x, D = (0, 0)]$, $\psi^{(0,0)}(x, \mathbf{p})$ is not obtained. However, as p_1^1 and p_2^0 are irrelevant to the value of $m^{(0,0)}(x, p_1^0, p_2^1)$, they can take any values as long as $(p_1^0, p_2^1) \in \text{supp}[P_1^0, P_2^1 | X = x, D = (0, 0)]$. Consequently, the identification of this MTE parameter requires the following common support condition: the support of (P_1^0, P_2^1) for players satisfying $D = (1, 0)$ overlaps with that of (P_1^0, P_2^1) for players satisfying $D = (0, 0)$, that is,

$$(p_1^0, p_2^1) \in \text{supp}[P_1^0, P_2^1 | X = x, D = (1, 0)] \cap \text{supp}[P_1^0, P_2^1 | X = x, D = (0, 0)] \neq \varnothing.$$

This implies that in practice, the observations for which (P_1^0, P_2^1) is not contained in the common support should be dropped when the MTE is computed.

For the identification of the total MTE, assuming that $\lambda > 0$, we have

$$\tau_{total}(x, p_1^0, p_2^0) = -\frac{1}{\lambda h(p_1^0, p_2^0 | x)} \frac{\partial^2}{\partial p_1^0 \partial p_2^0} \left(\psi^{(1,1)}(x, \mathbf{p}) + \psi^{(0,0)}(x, \mathbf{p}) \right).$$

In this case, the common support condition is as follows:

$$(p_1^0, p_2^0) \in \text{supp}[P_1^0, P_2^0 | X = x, D = (1, 1)] \cap \text{supp}[P_1^0, P_2^0 | X = x, D = (0, 0)] \neq \varnothing.$$

Remark 3.2 (Individual specific treatment effects)**.** One may be interested in estimating the treatment effects when only player 1's treatment status switches, whereas that of player 2 is unspecified and subject to change endogenously. The parameter of interest in this situation would be

$$\tau_{indiv}(x, p_1, p_2) \coloneqq E[Y_1^{(1,D_2)} - Y_1^{(0,D_2)} | X = x, V_1 = p_1, V_2 = p_2],$$

where $Y_1^{(d_1,D_2)} \coloneqq (1 - D_2) Y_1^{(d_1,0)} + D_2 Y_1^{(d_1,1)}$. We call this MTE parameter the *individual MTE*.[4] Let

$$m^{(d_1,D_2)}(x, v_1, v_2) \coloneqq E[Y_1^{(d_1,D_2)} | X = x, V_1 = v_1, V_2 = v_2].$$

After some calculations, we can show that

$$\begin{aligned} m^{(0,D_2)}(x, p_1^{d_1}, p_2^{d_2}) &= m^{(0,0)}(x, p_1^{d_1}, p_2^{d_2}) && \text{for } (d_1, d_2) \neq (1, 0), \\ m^{(0,D_2)}(x, p_1^1, p_2^0) &= \frac{1}{\lambda} \left(m^{(0,1)}(x, p_1^1, p_2^0) - (1 - \lambda) \cdot m^{(0,0)}(x, p_1^1, p_2^0) \right) && \text{for } \lambda > 0, \end{aligned}$$

and

$$\begin{aligned} m^{(1,D_2)}(x, p_1^{d_1}, p_2^{d_2}) &= m^{(1,1)}(x, p_1^{d_1}, p_2^{d_2}) && \text{for } (d_1, d_2) \neq (0, 1), \\ m^{(1,D_2)}(x, p_1^0, p_2^1) &= \frac{1}{(1 - \lambda)} \left(m^{(1,0)}(x, p_1^0, p_2^1) - \lambda \cdot m^{(1,1)}(x, p_1^0, p_2^1) \right) && \text{for } \lambda < 1, \end{aligned}$$

under Assumptions 2.1, 3.1, and 3.2. This result implies that the individual MTE can be identified as follows:

$$\begin{aligned} \tau_{indiv}(x, p_1^0, p_2^0) &= \tau_{total}(x, p_1^0, p_2^0), \\ \tau_{indiv}(x, p_1^0, p_2^1) &= \frac{1}{1 - \lambda} \tau_{direct}^{(0)}(x, p_1^0, p_2^1) - \frac{\lambda}{1 - \lambda} \tau_{total}(x, p_1^0, p_2^1), \end{aligned}$$

[4] This is somewhat similar to the framework in Frölich and Huber (2017), where the identification of causal models that allow the presence of an endogenous "mediator" variable was investigated. In our model, D_2 may be regarded as the mediator of D_1.

for example. Thus, for the estimation of the individual MTE, it is sufficient to calculate the direct MTE and the total MTE, so that no additional estimation is required.

Remark 3.3 (Identification of LATE)**.** It is also possible to identify LATE: the average causal effect for individuals whose treatment status is strictly altered by the instrumental variables (e.g., Imbens and Angrist, 1994). To define the LATE parameters in our situation, we consider two values z and z' of the instrument. Suppose that the values of $\mathbf{P}$ are $\mathbf{p} = (p_1^0, p_1^1, p_2^0, p_2^1)^\top$ when $Z = z$, and $\mathbf{p}' = (p_1^{0'}, p_1^{1'}, p_2^{0'}, p_2^{1'})^\top$ when $Z = z'$. For illustrative purpose, we assume that $\mathbf{p} > \mathbf{p}'$ (where the inequality is element-wise), as depicted in Figure 3. Even though we can consider several LATE parameters, as examples, we here focus on the *direct LATE*

$$E[Y_1^{(1,0)} - Y_1^{(0,0)} | X = x, p_1^{0'} < V_1 \leq p_1^0, p_2^1 < V_2 \leq 1],$$

and the *total LATE*

$$E[Y_1^{(1,1)} - Y_1^{(0,0)} | X = x, p_1^{1'} < V_1 \leq p_1^1, p_2^{1'} < V_2 \leq p_2^1].$$

The former and latter indicate the average causal effects for the players in regions [A] and [B], respectively, in Figure 3. The pairs of players in region [A] change their treatment status from $D = (1, 0)$ to $(0, 0)$ as the value of Z shifts from z to z'. Similarly, the pairs of players in region [B] select $D = (0, 0)$ when $Z = z'$, but $D = (1, 1)$ or $(0, 0)$ when $Z = z$. As in Heckman and Vytlacil (2005), we can write the LATE parameters as weighted averages of the MTE parameters:

$$\text{Direct LATE} = \int_{p_2^1}^1 \int_{p_1^{0'}}^{p_1^0} \tau_{direct}^{(0)}(x, v_1, v_2) \frac{h(v_1, v_2 | x)}{\Pr[p_1^{0'} < V_1 \leq p_1^0, p_2^1 < V_2 \leq 1 | X = x]} \mathrm{d}v_1 \mathrm{d}v_2,$$
$$\text{Total LATE} = \int_{p_2^{1'}}^{p_2^1} \int_{p_1^{1'}}^{p_1^1} \tau_{total}(x, v_1, v_2) \frac{h(v_1, v_2 | x)}{\Pr[p_1^{1'} < V_1 \leq p_1^1, p_2^{1'} < V_2 \leq p_2^1 | X = x]} \mathrm{d}v_1 \mathrm{d}v_2.$$

Because the MTE parameters and the weight functions in the integrals are identified, the LATE parameters are also identified.

4 Estimation and Asymptotics

In this section, a two-step semiparametric procedure is proposed for estimating the MTE parameters, given the data $\{\{(Y_{ji}, D_{ji}, W_{ji})\}_{j=1}^2\}_{i=1}^n$ are observed. We first describe the estimation procedure and present the asymptotic properties of the proposed estimator later.

4.1 Two-step estimation

First step: Estimation of the treatment decision game. In the first step, we estimate the parameters of the treatment decision model using a parametric ML approach. To this end, we assume the following.

Assumption 4.1.

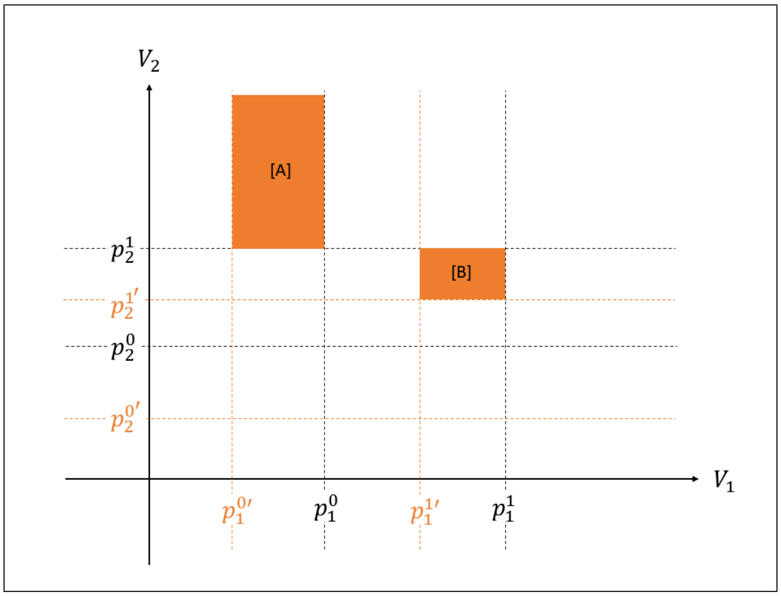

Figure 3: LATE parameters.

(i) Each player j decides his/her treatment by

$$D_{ji} = \mathbf{1}\left(W_{ji}^\top \gamma_0 + D_{-j,i} \cdot \eta(W_{ji}^\top \gamma_1) \geq \varepsilon_{ji}\right), \quad i = 1, \ldots, n.$$

Here, $\gamma \coloneqq (\gamma_0^\top, \gamma_1^\top)^\top \in \mathbb{R}^{2\dim(W)}$ is a vector of parameters, and $\eta(\cdot)$ is a known positive function.

(ii) The random variables $\varepsilon = (\varepsilon_1, \varepsilon_2)$ are independent of W and continuously distributed with known marginal CDFs $F_{\varepsilon_1}(\cdot)$ and $F_{\varepsilon_2}(\cdot)$, respectively, and the joint distribution of ε can be represented by a known copula function $H(\cdot, \cdot; \rho)$ such that $\Pr(\varepsilon_1 \leq a_1, \varepsilon_2 \leq a_2) = H(F_{\varepsilon_1}(a_1), F_{\varepsilon_2}(a_2); \rho)$, where $\rho \in \mathbb{R}$ is a parameter controlling the correlation between ε_1 and ε_2. The copula $H(\cdot, \cdot; \rho)$ has a density function $h(\cdot, \cdot; \rho)$.

In Assumption 4.1(i), we assume a parametric model with strictly positive interaction effect $\eta(\cdot) > 0$ to ensure strategic complementarity. For example, one may use $\eta(\cdot) = \exp(\cdot)$ in practice. In the literature, it has been typically assumed that the interaction effect is a constant value, except for some recent studies (e.g., Kline, 2015). However, such a model specification cannot be adopted here because the constancy of the interaction effect implies that the conditional support of P_j^1 given P_j^0 degenerates to a singleton. We also assume that the coefficients γ are common to the players for simplicity. In Assumption 4.1(ii), we assume that the joint distribution of $(\varepsilon_1, \varepsilon_2)$ is known up to the scalar parameter ρ. A typical example would be a standard bivariate normal distribution. In this case, the copula H corresponds to the Gaussian copula $H(v_1, v_2; \rho) = \Phi_2(\Phi^{-1}(v_1), \Phi^{-1}(v_2); \rho)$, where $\Phi_2(\cdot, \cdot; \rho)$ is the standard bivariate normal CDF with correlation ρ, and $\Phi(\cdot)$ denotes the CDF of the standard normal distribution. As another example, in Aradillas-Lopez (2010), the use of Farlie–Gumbel–Morgenstern (FGM) copula $H(v_1, v_2; \rho) = v_1 v_2[1 + \rho(1 - v_1)(1 - v_2)]$ is considered in a context similar to our game model.

In accordance with the definitions in (2.4), we let $V_{ji} = F_{\varepsilon_j}(\varepsilon_{ji})$, $P_{ji}^0(\gamma) = F_{\varepsilon_j}(W_{ji}^\top \gamma_0)$, $P_{ji}^1(\gamma) = F_{\varepsilon_j}(W_{ji}^\top \gamma_0 + \eta(W_{ji}^\top \gamma_1))$, and $\mathbf{P}_i(\gamma) = (P_{1i}^0(\gamma), P_{1i}^1(\gamma), P_{2i}^0(\gamma), P_{2i}^1(\gamma))$. Let $\theta \coloneqq (\gamma^\top, \rho, \lambda)^\top$ be the vector of the parameters to be estimated. Then, for a given θ, the conditional probability that the i-th pair of players is in the multiple equilibria region is given by

$$\mathcal{L}_{mul}(\mathbf{P}_i(\gamma);\theta) \coloneqq H(P^1_{1i}(\gamma), P^1_{2i}(\gamma);\rho) - H(P^1_{1i}(\gamma), P^0_{2i}(\gamma);\rho) - H(P^0_{1i}(\gamma), P^1_{2i}(\gamma);\rho) + H(P^0_{1i}(\gamma), P^0_{2i}(\gamma);\rho).$$

Further, letting the probability that they choose action $D_i = (d_1, d_2)$ be $\mathcal{L}^{(d_1,d_2)}(\mathbf{P}_i(\gamma);\theta)$ for $(d_1, d_2) \in \{0,1\}^2$, we have

$$\begin{aligned}
\mathcal{L}^{(1,0)}(\mathbf{P}_i(\gamma);\theta) &= P^0_{1i}(\gamma) - H(P^0_{1i}(\gamma), P^1_{2i}(\gamma);\rho),\\
\mathcal{L}^{(0,1)}(\mathbf{P}_i(\gamma);\theta) &= P^0_{2i}(\gamma) - H(P^1_{1i}(\gamma), P^0_{2i}(\gamma);\rho),\\
\mathcal{L}^{(1,1)}(\mathbf{P}_i(\gamma);\theta) &= H(P^1_{1i}(\gamma), P^1_{2i}(\gamma);\rho) - \lambda \cdot \mathcal{L}_{mul}(\mathbf{P}_i(\gamma);\theta),\\
\mathcal{L}^{(0,0)}(\mathbf{P}_i(\gamma);\theta) &= 1 - \mathcal{L}^{(1,0)}(\mathbf{P}_i(\gamma);\theta) - \mathcal{L}^{(0,1)}(\mathbf{P}_i(\gamma);\theta)) - \mathcal{L}^{(1,1)}(\mathbf{P}_i(\gamma);\theta).
\end{aligned} \tag{4.1}$$

Then, the ML estimator $\widehat{\theta}_n$ can be obtained as the maximizer of the log-likelihood function $\sum_{i=1}^n \ell_i(\theta)$ with respect to θ, where

$$\ell_i(\theta) \coloneqq \sum_{d_1=0}^{1}\sum_{d_2=0}^{1} I_i^{(d_1,d_2)} \ln \mathcal{L}^{(d_1,d_2)}(\mathbf{P}_i(\gamma);\theta).$$

Once the ML estimator $\widehat{\theta}_n$ is obtained, letting $\theta^* = (\gamma^{*\top}, \rho^*, \lambda^*)^\top$ be the true value, we can estimate $P^0_{ji} = P^0_{ji}(\gamma^*)$ and $P^1_{ji} = P^1_{ji}(\gamma^*)$ by $\widehat{P}^0_{ji} = P^0_{ji}(\widehat{\gamma}_n)$ and $\widehat{P}^1_{ji} = P^1_{ji}(\widehat{\gamma}_n)$, respectively. Similarly, the true joint CDF and the true joint density of (V_1, V_2), $H(\cdot,\cdot) = H(\cdot,\cdot;\rho^*)$ and $h(\cdot,\cdot) = h(\cdot,\cdot;\rho^*)$, can be estimated as $\widehat{H}(\cdot,\cdot) = H(\cdot,\cdot;\widehat{\rho}_n)$ and $\widehat{h}(\cdot,\cdot) = h(\cdot,\cdot;\widehat{\rho}_n)$. Moreover, we can estimate the probabilities in (4.1) evaluated at the true θ^*, namely, $\mathcal{L}_i^{(d_1,d_2)} = \mathcal{L}^{(d_1,d_2)}(\mathbf{P}_i;\theta^*)$ and $\mathcal{L}_{mul,i} = \mathcal{L}_{mul}(\mathbf{P}_i;\theta^*)$, by $\widehat{\mathcal{L}}_i^{(d_1,d_2)} = \mathcal{L}^{(d_1,d_2)}(\widehat{\mathbf{P}}_i;\widehat{\theta}_n)$ and $\widehat{\mathcal{L}}_{mul,i} = \mathcal{L}_{mul}(\widehat{\mathbf{P}}_i;\widehat{\theta}_n)$, respectively.

The ML estimator $\widehat{\theta}_n$ is standard, so that its asymptotic properties can be derived in a straightforward manner if θ^* can be globally identified. Since the identification of discrete game models is itself an important research area, we formally state the identification conditions for our model in Appendix D.

Second step: Estimation of the MTEs. As discussed in Section 3, there are a variety of MTE (and also LATE) parameters that can be identified. Below, in parallel with subsection 3.2, we specifically discuss the estimation of the direct MTE $\tau^{(0)}_{direct}(x, p^0_1, p^1_2)$ and the total MTE $\tau_{total}(x, p^0_1, p^0_2)$.

To facilitate the analysis, we assume a linear model for the potential outcomes.

Assumption 4.2.

(i) For each player j, the potential outcome $Y_{ji}^{(d_j,d_{-j})}$ is additively separable in the unobservable, such that

$$Y_{ji}^{(d_j,d_{-j})} = X_{ji}^\top \beta_j^{(d_j,d_{-j})} + U_{ji}^{(d_j,d_{-j})} \quad \text{for } i = 1,\ldots,n.$$

Here, $\beta_j^{(d_j,d_{-j})} \in \mathbb{R}^{\dim(X)}$ is a vector of parameters.

(ii) The random variables $U^{(d_1,d_2)} = (U_1^{(d_1,d_2)}, U_2^{(d_2,d_1)})$ are independent of W for all $(d_1, d_2) \in \{0,1\}^2$.

We recall that the direct MTE $\tau^{(0)}_{direct}(x, p^0_1, p^1_2)$ is the difference between the conditional means $m^{(1,0)}(x, p^0_1, p^1_2)$ and $m^{(0,0)}(x, p^0_1, p^1_2)$. Thus, under Assumption 4.2, the observed heterogeneity in the direct MTE with respect to x is derived from the difference between $\beta^{(1,0)}_1$ and $\beta^{(0,0)}_1$. By Assumption 4.2,

$$m^{(1,0)}(x, p^0_1, p^1_2) = x^\top_1 \beta^{(1,0)}_1 + E[U^{(1,0)}_1 | V_1 = p^0_1, V_2 = p^1_2].$$

Note that $E[U^{(1,0)}_1 | V_1 = p^0_1, V_2 = p^1_2] \neq 0$ in general because $U^{(1,0)}_1$ may be correlated with (V_1, V_2). Below, we discuss the estimation of $\beta^{(1,0)}_1$ and $E[U^{(1,0)}_1 | V_1 = p^0_1, V_2 = p^1_2]$.

First, by the law of iterated expectations, we have

$$\begin{aligned} E[I^{(1,0)}_i Y_{1i} | X_i, P^0_{1i}, P^1_{2i}] &= E[I^{(1,0)}_i | X_i, P^0_{1i}, P^1_{2i}] X^\top_{1i} \beta^{(1,0)}_1 + E[I^{(1,0)}_i U^{(1,0)}_{1i} | X_i, P^0_{1i}, P^1_{2i}] \\ &= \mathcal{L}^{(1,0)}_i X^\top_{1i} \beta^{(1,0)}_1 + \mathcal{L}^{(1,0)}_i E[U^{(1,0)}_{1i} | V_{1i} \leq P^0_{1i}, V_{2i} > P^1_{2i}]. \end{aligned}$$

Noting that $\mathcal{L}^{(1,0)}_i$ is a function of (P^0_{1i}, P^1_{2i}), the above equation leads to the following semiparametric partially linear regression model:

$$I^{(1,0)}_i Y_{1i} = \mathcal{L}^{(1,0)}_i X^\top_{1i} \beta^{(1,0)}_1 + g^{(1,0)}_1 (P^0_{1i}, P^1_{2i}) + e^{(1,0)}_{1i}, \tag{4.2}$$

where $g^{(1,0)}_1(P^0_{1i}, P^1_{2i}) \coloneqq \mathcal{L}^{(1,0)}_i E[U^{(1,0)}_{1i} | V_{1i} \leq P^0_{1i}, V_{2i} > P^1_{2i}]$ and the error $e^{(1,0)}_{1i}$ satisfies $E[e^{(1,0)}_{1i} | X_i, P^0_{1i}, P^1_{2i}] = 0$ by construction. Based on (4.2), we estimate the coefficient vector $\beta^{(1,0)}_1$ using the series (sieve) estimation method. Compared with the widely used kernel smoothing method, the series method has the advantage of computational simplicity. This becomes rather prominent when the parameters are estimated for the cases $D = (0, 0)$ and $(1, 1)$, as discussed below.

Let $b_K(p_1, p_2) \coloneqq (b_{1K}(p_1, p_2), \ldots, b_{KK}(p_1, p_2))^\top$ be a $K \times 1$ vector of bivariate basis functions. We assume that $g^{(1,0)}_1(\cdot, \cdot)$ can be well approximated by a linear combination of the basis functions, i.e., $g^{(1,0)}_1(\cdot, \cdot) \approx b_K(\cdot, \cdot)^\top \alpha^{(1,0)}_1$ for some coefficient vector $\alpha^{(1,0)}_1$ with sufficiently large K (a more formal statement will be given later in Assumption 4.7).

We define the following vectors and matrices: $\mathbf{Y}^{(1,0)}_1 \coloneqq \left(I^{(1,0)}_1 Y_{11}, \ldots, I^{(1,0)}_n Y_{1n} \right)^\top$,

$$\begin{aligned} \mathbf{X}^{(1,0)}_1 &\coloneqq \left(\mathcal{L}^{(1,0)}_1 X_{11}, \ldots, \mathcal{L}^{(1,0)}_n X_{1n} \right)^\top, & \widehat{\mathbf{X}}^{(1,0)}_1 &\coloneqq \left(\widehat{\mathcal{L}}^{(1,0)}_1 X_{11}, \ldots, \widehat{\mathcal{L}}^{(1,0)}_n X_{1n} \right)^\top, \\ \mathbf{b}^{(1,0)}_K &\coloneqq \left(b_K \left(P^0_{11}, P^1_{21} \right), \ldots, b_K \left(P^0_{1n}, P^1_{2n} \right) \right)^\top, & \widehat{\mathbf{b}}^{(1,0)}_K &\coloneqq \left(b_K \left(\widehat{P}^0_{11}, \widehat{P}^1_{21} \right), \ldots, b_K \left(\widehat{P}^0_{1n}, \widehat{P}^1_{2n} \right) \right)^\top, \\ \mathbf{R}^{(1,0)}_K &\coloneqq \left(\mathbf{X}^{(1,0)}_1, \mathbf{b}^{(1,0)}_K \right), & \widehat{\mathbf{R}}^{(1,0)}_K &\coloneqq \left(\widehat{\mathbf{X}}^{(1,0)}_1, \widehat{\mathbf{b}}^{(1,0)}_K \right). \end{aligned}$$

Then, we can obtain an estimator of $\delta^{(1,0)}_1 = \left(\beta^{(1,0)\top}_1, \alpha^{(1,0)\top}_1 \right)^\top$ by the least squares regression of $\mathbf{Y}^{(1,0)}_1$ on $\mathbf{R}^{(1,0)}_K$ or $\widehat{\mathbf{R}}^{(1,0)}_K$:

$$\widetilde{\delta}^{(1,0)}_{1n} = \left(\widetilde{\beta}^{(1,0)\top}_{1n}, \widetilde{\alpha}^{(1,0)\top}_{1n} \right)^\top \coloneqq \left[\mathbf{R}^{(1,0)\top}_K \mathbf{R}^{(1,0)}_K \right]^- \mathbf{R}^{(1,0)\top}_K \mathbf{Y}^{(1,0)}_1, \tag{4.3}$$

$$\widehat{\delta}^{(1,0)}_{1n} = \left(\widehat{\beta}^{(1,0)\top}_{1n}, \widehat{\alpha}^{(1,0)\top}_{1n} \right)^\top \coloneqq \left[\widehat{\mathbf{R}}^{(1,0)\top}_K \widehat{\mathbf{R}}^{(1,0)}_K \right]^- \widehat{\mathbf{R}}^{(1,0)\top}_K \mathbf{Y}^{(1,0)}_1. \tag{4.4}$$

It should be noted that $\widetilde{\delta}_{1n}^{(1,0)}$ is an infeasible estimator, where the true parameters of the first stage are treated as known, whereas $\widehat{\delta}_{1n}^{(1,0)}$ is a feasible estimator.

The feasible estimator of $g_1^{(1,0)}(p_1^0, p_2^1)$ can be obtained by $\widehat{g}_1^{(1,0)}(p_1^0, p_2^1) := b_K(p_1^0, p_2^1)^\top \widehat{\alpha}_{1n}^{(1,0)}$. For the estimation of $E[U_1^{(1,0)}|V_1 = p_1^0, V_2 = p_2^1]$, we note that the same argument as in Theorem 3.1 implies that

$$E[U_1^{(1,0)}|V_1 = p_1^0, V_2 = p_2^1] = -\frac{1}{h(p_1^0, p_2^1)} \frac{\partial^2 g_1^{(1,0)}(p_1^0, p_2^1)}{\partial p_1^0 \partial p_2^1}.$$

Thus, by replacing the unknown parameters by their estimates, we can estimate $E[U_1^{(1,0)}|V_1 = p_1^0, V_2 = p_2^1]$ by

$$\begin{aligned} \widehat{E}_n[U_1^{(1,0)}|V_1 = p_1^0, V_2 = p_2^1] &:= -\frac{1}{\widehat{h}(p_1^0, p_2^1)} \frac{\partial^2 \widehat{g}_1^{(1,0)}(p_1^0, p_2^1)}{\partial p_1^0 \partial p_2^1} \\ &= -\frac{1}{\widehat{h}(p_1^0, p_2^1)} \ddot{b}_K(p_1^0, p_2^1)^\top \widehat{\alpha}_{1n}^{(1,0)}, \end{aligned} \tag{4.5}$$

where $\ddot{b}_K(p_1, p_2) := \partial^2 b_K(p_1, p_2)/(\partial p_1 \partial p_2)$. Finally, by combining the estimators (4.4) and (4.5), the estimator of $m^{(1,0)}(x, p_1^0, p_2^1)$ is given by

$$\widehat{m}^{(1,0)}(x, p_1^0, p_2^1) := x_1^\top \widehat{\beta}_{1n}^{(1,0)} + \widehat{E}_n[U_1^{(1,0)}|V_1 = p_1^0, V_2 = p_2^1]. \tag{4.6}$$

The infeasible version of the estimators of $g_1^{(1,0)}(p_1^0, p_2^1)$, $E[U_1^{(1,0)}|V_1 = p_1^0, V_2 = p_2^1]$, and $m^{(1,0)}(x, p_1^0, p_2^1)$ can be defined as above, which we denote by $\widetilde{g}_1^{(1,0)}(p_1^0, p_2^1)$, $\widetilde{E}_n[U_1^{(1,0)}|V_1 = p_1^0, V_2 = p_2^1]$, and $\widetilde{m}^{(1,0)}(x, p_1^0, p_2^1)$, respectively.

We now describe the estimation of $m^{(0,0)}(x, p_1^0, p_2^1)$. Here, we consider only the feasible estimator for brevity. As above, we can observe that

$$m^{(0,0)}(x, p_1^0, p_2^1) = x_1^\top \beta_1^{(0,0)} + E[U_1^{(0,0)}|V_1 = p_1^0, V_2 = p_2^1],$$

under Assumption 4.2; moreover,

$$\begin{aligned} E[I_i^{(0,0)} Y_{1i}|X_i, \mathbf{P}_i] &= E[I_i^{(0,0)}|X_i, \mathbf{P}_i] X_{1i}^\top \beta_1^{(0,0)} + E[I_i^{(0,0)} U_{1i}^{(0,0)}|X_i, \mathbf{P}_i] \\ &= \mathcal{L}_i^{(0,0)} X_{1i}^\top \beta_1^{(1,0)} + \mathcal{L}_i^{(0,0)} E[U_{1i}^{(0,0)}|\mathbf{P}_i, D_i = (0,0)]. \end{aligned}$$

Further, if $E[U_1^{(0,0)}|V_1 = v_1, V_2 = v_2] h(v_1, v_2)$ is continuous in (v_1, v_2), there exist bivariate real-valued functions $g_{11}^{(0,0)}$, $g_{12}^{(0,0)}$, $g_{13}^{(0,0)}$, and $g_{14}^{(0,0)}$ satisfying

$$\begin{aligned} &\mathcal{L}_i^{(0,0)} E[U_{1i}^{(0,0)}|\mathbf{P}_i, D_i = (0,0)] \\ &= E[U_{1i}^{(0,0)}|V_i \in \mathcal{V}_{uni}^{(0,0)}(\mathbf{P}_i)] \cdot \Pr[V_i \in \mathcal{V}_{uni}^{(0,0)}(\mathbf{P}_i)] + \lambda \cdot E[U_{1i}^{(0,0)}|V_i \in \mathcal{V}_{mul}(\mathbf{P}_i)] \cdot \Pr[V_i \in \mathcal{V}_{mul}(\mathbf{P}_i)] \\ &= \int_{P_{1i}^0}^1 \int_{P_{2i}^0}^1 E[U_{1i}^{(0,0)}|V_{1i} = v_1, V_{2i} = v_2] h(v_1, v_2) \mathrm{d}v_1 \mathrm{d}v_2 - (1-\lambda) \int_{P_{1i}^0}^{P_{1i}^1} \int_{P_{2i}^0}^{P_{2i}^1} E[U_{1i}^{(0,0)}|V_{1i} = v_1, V_{2i} = v_2] h(v_1, v_2) \mathrm{d}v_1 \mathrm{d}v_2 \\ &= g_{11}^{(0,0)}(P_{1i}^0, P_{2i}^0) + g_{12}^{(0,0)}(P_{1i}^1, P_{2i}^0) + g_{13}^{(0,0)}(P_{1i}^0, P_{2i}^1) + g_{14}^{(0,0)}(P_{1i}^1, P_{2i}^1). \end{aligned} \tag{4.7}$$

Hence, as with the case of (4.2), we obtain the following semiparametric partially linear additive regression model:

$$I_i^{(0,0)}Y_{1i} = \mathcal{L}_i^{(0,0)}X_{1i}^\top\beta_1^{(0,0)} + g_{11}^{(0,0)}(P_{1i}^0, P_{2i}^0) + g_{12}^{(0,0)}(P_{1i}^1, P_{2i}^0) + g_{13}^{(0,0)}(P_{1i}^0, P_{2i}^1) + g_{14}^{(0,0)}(P_{1i}^1, P_{2i}^1) + e_{1i}^{(0,0)},$$

where $E[e_{1i}^{(0,0)}|X_i, \mathbf{P}_i] = 0$ holds by construction.

We define $\mathbf{Y}_1^{(0,0)}$ and $\widehat{\mathbf{X}}_1^{(0,0)}$ analogously. Furthermore, let

$$\begin{aligned}
\widehat{\mathbf{b}}_{1K}^{(0,0)} &:= \left(b_K\left(\widehat{P}_{11}^0, \widehat{P}_{21}^0\right), \dots, b_K\left(\widehat{P}_{1n}^0, \widehat{P}_{2n}^0\right)\right)^\top, &\quad \widehat{\mathbf{b}}_{2K}^{(0,0)} &:= \left(b_K\left(\widehat{P}_{11}^1, \widehat{P}_{21}^0\right), \dots, b_K\left(\widehat{P}_{1n}^1, \widehat{P}_{2n}^0\right)\right)^\top, \\
\widehat{\mathbf{b}}_{3K}^{(0,0)} &:= \left(b_K\left(\widehat{P}_{11}^0, \widehat{P}_{21}^1\right), \dots, b_K\left(\widehat{P}_{1n}^0, \widehat{P}_{2n}^1\right)\right)^\top, &\quad \widehat{\mathbf{b}}_{4K}^{(0,0)} &:= \left(b_K\left(\widehat{P}_{11}^1, \widehat{P}_{21}^1\right), \dots, b_K\left(\widehat{P}_{1n}^1, \widehat{P}_{2n}^1\right)\right)^\top, \\
\widehat{\mathbf{b}}_{K}^{(0,0)} &:= \left(\widehat{\mathbf{b}}_{1K}^{(0,0)}, \widehat{\mathbf{b}}_{2K}^{(0,0)}, \widehat{\mathbf{b}}_{3K}^{(0,0)}, \widehat{\mathbf{b}}_{4K}^{(0,0)}\right), &\quad \widehat{\mathbf{R}}_{K}^{(0,0)} &:= \left(\widehat{\mathbf{X}}_1^{(0,0)}, \widehat{\mathbf{b}}_{K}^{(0,0)}\right).
\end{aligned}$$

It should be noted that the "locations" of the functions $g_{1l}^{(0,0)}$, $l = 1, \dots 4$, are not identified without further restrictions. To handle this in practice, we may include an intercept term in only one of the $\widehat{\mathbf{b}}_{lK}^{(0,0)}$'s, $l = 1, \dots 4$. To simplify our presentation, we postulate that such a normalization is made implicitly. Assuming again that each $g_{1l}^{(0,0)}(\cdot,\cdot)$ can be approximated by $g_{1l}^{(0,0)}(\cdot,\cdot) \approx b_K(\cdot,\cdot)^\top\alpha_{1l}^{(0,0)}$ for a coefficient vector $\alpha_{1l}^{(0,0)}$,[5] the estimator of $\delta_1^{(0,0)} = \left(\beta_1^{(0,0)\top}, \alpha_{11}^{(0,0)\top}, \dots, \alpha_{14}^{(0,0)\top}\right)^\top$ is obtained by

$$\widehat{\delta}_{1n}^{(0,0)} = \left(\widehat{\beta}_{1n}^{(0,0)\top}, \widehat{\alpha}_{11n}^{(0,0)\top}, \dots, \widehat{\alpha}_{14n}^{(0,0)\top}\right)^\top := \left[\widehat{\mathbf{R}}_K^{(0,0)\top}\widehat{\mathbf{R}}_K^{(0,0)}\right]^- \widehat{\mathbf{R}}_K^{(0,0)\top}\mathbf{Y}_1^{(0,0)}, \tag{4.8}$$

and that of $g_{1l}^{(0,0)}(p_1, p_2)$ by $\widehat{g}_{1l}^{(0,0)}(p_1, p_2) := b_K(p_1, p_2)^\top\widehat{\alpha}_{1ln}^{(0,0)}$ for each l. By Theorem 3.2 and (4.7), we have

$$\begin{aligned}
E[U_1^{(0,0)}|V_1 = p_1^0, V_2 = p_2^1] &= \frac{1}{(1-\lambda)h(p_1^0, p_2^1)}\frac{\partial^2 E[I_i^{(0,0)}U_{1i}^{(0,0)}|X_i, \mathbf{P}_i = \mathbf{p}]}{\partial p_1^0 \partial p_2^1} \\
&= \frac{1}{(1-\lambda)h(p_1^0, p_2^1)}\frac{\partial^2 g_{13}^{(0,0)}(p_1^0, p_2^1)}{\partial p_1^0 \partial p_2^1},
\end{aligned}$$

and thus we can estimate $E[U_1^{(0,0)}|V_1 = p_1^0, V_2 = p_2^1]$ by

$$\begin{aligned}
\widehat{E}_n[U_1^{(0,0)}|V_1 = p_1^0, V_2 = p_2^1] &:= \frac{1}{(1-\widehat{\lambda}_n)\widehat{h}(p_1^0, p_2^1)}\frac{\partial^2 \widehat{g}_{13}^{(0,0)}(p_1^0, p_2^1)}{\partial p_1^0 \partial p_2^1} \\
&= \frac{1}{(1-\widehat{\lambda}_n)\widehat{h}(p_1^0, p_2^1)}\ddot{b}_K(p_1^0, p_2^1)^\top\widehat{\alpha}_{13n}^{(0,0)}.
\end{aligned} \tag{4.9}$$

Combining (4.8) and (4.9) yields the estimator of $m^{(0,0)}(x, p_1^0, p_2^1)$ as

$$\widehat{m}^{(0,0)}(x, p_1^0, p_2^1) := x_1^\top\widehat{\beta}_{1n}^{(0,0)} + \widehat{E}_n[U_1^{(0,0)}|V_1 = p_1^0, V_2 = p_2^1]. \tag{4.10}$$

[5] Note that it is possible to use different orders of basis terms to approximate each component of the functions $g_{1l}^{(0,0)}$'s, $l = 1, \dots 4$. However, we use the same order K for all, for simplicity.

Finally, by (4.6) and (4.10), the direct MTE $\tau^{(0)}_{direct}(x, p_1^0, p_2^1)$ can be estimated by

$$\widehat{\tau}^{(0)}_{direct}(x, p_1^0, p_2^1) \coloneqq \widehat{m}^{(1,0)}(x, p_1^0, p_2^1) - \widehat{m}^{(0,0)}(x, p_1^0, p_2^1). \tag{4.11}$$

We can estimate the total MTE $\tau_{total}(x, p_1^0, p_2^0)$ in the same manner as above. The estimator of $m^{(0,0)}(x, p_1^0, p_2^0)$ can be obtained by

$$\widehat{m}^{(0,0)}(x, p_1^0, p_2^0) \coloneqq x_1^\top \widehat{\beta}^{(0,0)}_{1n} + \widehat{E}_n[U_1^{(0,0)} | V_1 = p_1^0, V_2 = p_2^0],$$

where

$$\begin{aligned}\widehat{E}_n[U_1^{(0,0)} | V_1 = p_1^0, V_2 = p_2^0] &\coloneqq \frac{1}{\widehat{\lambda}_n \widehat{h}(p_1^0, p_2^0)} \frac{\partial^2 \widehat{g}^{(0,0)}_{11}(p_1^0, p_2^0)}{\partial p_1^0 \partial p_2^0} \\ &= \frac{1}{\widehat{\lambda}_n \widehat{h}(p_1^0, p_2^0)} \ddot{b}_K(p_1^0, p_2^0)^\top \widehat{\alpha}^{(0,0)}_{11n}.\end{aligned}$$

For the estimation of $m^{(1,1)}(x, p_1^0, p_2^0)$, we first estimate the following partially linear additive regression model:

$$I_i^{(1,1)} Y_{1i} = \mathcal{L}_i^{(1,1)} X_{1i}^\top \beta_1^{(1,1)} + g^{(1,1)}_{11}(P^0_{1i}, P^0_{2i}) + g^{(1,1)}_{12}(P^1_{1i}, P^0_{2i}) + g^{(1,1)}_{13}(P^0_{1i}, P^1_{2i}) + g^{(1,1)}_{14}(P^1_{1i}, P^1_{2i}) + e^{(1,1)}_{1i}.$$

Let $\widehat{\beta}^{(1,1)}_{1n}$ and $\widehat{g}^{(1,1)}_{11}(p_1^0, p_2^0) \coloneqq b_K(p_1^0, p_2^0)^\top \widehat{\alpha}^{(1,1)}_{11n}$ be the resulting estimators of $\beta_1^{(1,1)}$ and $g^{(1,1)}_{11}(p_1^0, p_2^0)$, respectively, where $\widehat{\beta}^{(1,1)}_{1n}$ and $\widehat{\alpha}^{(1,1)}_{11n}$ can be obtained analogously to (4.8). Then, we can estimate $m^{(1,1)}(x, p_1^0, p_2^0)$ by

$$\widehat{m}^{(1,1)}(x, p_1^0, p_2^0) \coloneqq x_1^\top \widehat{\beta}^{(1,1)}_{1n} + \widehat{E}_n[U_1^{(1,1)} | V_1 = p_1^0, V_2 = p_2^0],$$

where

$$\begin{aligned}\widehat{E}_n[U_1^{(1,1)} | V_1 = p_1^0, V_2 = p_2^0] &\coloneqq -\frac{1}{\widehat{\lambda}_n \widehat{h}(p_1^0, p_2^0)} \frac{\partial^2 \widehat{g}^{(1,1)}_{11}(p_1^0, p_2^0)}{\partial p_1^0 \partial p_2^0} \\ &= -\frac{1}{\widehat{\lambda}_n \widehat{h}(p_1^0, p_2^0)} \ddot{b}_K(p_1^0, p_2^0)^\top \widehat{\alpha}^{(1,1)}_{11n}.\end{aligned}$$

Finally, the total MTE $\tau_{total}(x, p_1^0, p_2^0)$ can be estimated by

$$\widehat{\tau}_{total}(x, p_1^0, p_2^0) \coloneqq \widehat{m}^{(1,1)}(x, p_1^0, p_2^0) - \widehat{m}^{(0,0)}(x, p_1^0, p_2^0). \tag{4.12}$$

4.2 Asymptotics

We derive the asymptotic properties of the proposed estimation method. We mainly investigate the estimator of the conditional mean for $D = (1, 0)$. Analogous arguments apply to the other cases, and the asymptotic properties of the MTE estimators follow immediately.

In addition to the assumptions in the previous sections, we impose conditions on the first stage treatment decision game.

Assumption 4.3.

(i) $\|\widehat{\theta}_n - \theta^*\| = O_P(n^{-1/2})$.

(ii) The CDFs $F_{\varepsilon_j}(\cdot)$ for $j = 1, 2$ and the interaction function $\eta(\cdot)$ are continuously differentiable with bounded derivatives.

(iii) $H(p_1, p_2; \rho)$ is Lipschitz continuous with respect to $(p_1, p_2) \in [0, 1]^2$ and ρ in the neighborhood of ρ^*.

(iv) Uniformly in $(p_1, p_2) \in [0, 1]^2$, $h(p_1, p_2; \rho)$ is bounded away from zero and infinity, and is Lipschitz continuous with respect to ρ in the neighborhood of ρ^*.

Assumption 4.3(i) is the $\sqrt{n}$-consistency of the ML estimator $\widehat{\theta}_n$. It is a high-level condition; however, it is a standard result for parametric ML estimation. Assumption 4.3(iv) implies that $\widehat{h}(\cdot, \cdot)$ is uniformly bounded away from zero and infinity with probability approaching one (w.p.a.1) in conjunction with (i).

We define $R_K(P_1^0, P_2^1) \coloneqq (\mathcal{L}^{(1,0)} X_1^\top, b_K(P_1^0, P_2^1)^\top)^\top$ and

$$\Psi_K^{(1,0)} \coloneqq E\left[R_K(P_1^0, P_2^1) R_K(P_1^0, P_2^1)^\top\right], \qquad \Psi_{nK}^{(1,0)} \coloneqq \mathbf{R}_K^{(1,0)\top} \mathbf{R}_K^{(1,0)}/n, \qquad \widehat{\Psi}_{nK}^{(1,0)} \coloneqq \widehat{\mathbf{R}}_K^{(1,0)\top} \widehat{\mathbf{R}}_K^{(1,0)}/n,$$
$$\Sigma_K^{(1,0)} \coloneqq E\left[\left(e_1^{(1,0)}\right)^2 R_K(P_1^0, P_2^1) R_K(P_1^0, P_2^1)^\top\right].$$

Assumption 4.4.

(i) The data $\{\{(Y_{ji}, D_{ji}, X_{ji}, Z_{ji})\}_{j=1}^2\}_{i=1}^n$ are independent and identically distributed.

(ii) The support of W is a compact subset of $\mathbb{R}^{2\dim(W)}$.

Assumption 4.5.

(i) There exist positive constants $\underline{c}_\Psi$ and $\bar{c}_\Psi$ such that

$$0 < \underline{c}_\Psi \le \chi_{\min}\left(\Psi_K^{(1,0)}\right) \le \chi_{\max}\left(\Psi_K^{(1,0)}\right) \le \bar{c}_\Psi < \infty,$$

uniformly in K.

(ii) There exist positive constants $\underline{c}_\Sigma$ and $\bar{c}_\Sigma$ such that

$$0 < \underline{c}_\Sigma \le \chi_{\min}\left(\Sigma_K^{(1,0)}\right) \le \chi_{\max}\left(\Sigma_K^{(1,0)}\right) \le \bar{c}_\Sigma < \infty,$$

uniformly in K.

Assumption 4.6. $E\left[\left(e_1^{(1,0)}\right)^4 \middle| W, D\right]$ is bounded.

Assumption 4.4 is a standard and relatively weak condition for microeconomic applications. We do not restrict the dependence between the variables for j and those for $-j$. Assumption 4.5(i) is a standard non-singularity condition that ensures the existence of the inverse matrices $\left[\Psi_{nK}^{(1,0)}\right]^{-1}$ and $\left[\widehat{\Psi}_{nK}^{(1,0)}\right]^{-1}$ w.p.a.1.[6] Assumption 4.6 is introduced to conveniently derive the limiting distribution of our estimator.

To state the next assumption, we introduce the following notation. For a sufficiently smooth function $g(p_1, p_2)$ and a vector of non-negative integers $\mathbf{a} = (a_1, a_2)$, let

$$\partial^{\mathbf{a}} g(p_1, p_2) := \frac{\partial^{|\mathbf{a}|} g(p_1, p_2)}{\partial^{a_1} p_1 \partial^{a_2} p_2},$$

where $|\mathbf{a}| = a_1 + a_2$. If $|\mathbf{a}| = 0$, then $\partial^{\mathbf{a}} g(p_1, p_2) = g(p_1, p_2)$.

Assumption 4.7. For some integer $s \geq 2$, the functions $g_1^{(1,0)}(\cdot, \cdot)$ and $b_K(\cdot, \cdot)$ are at least s-times continuously differentiable, and there exists some fixed $\alpha_1^{(1,0)} \in \mathbb{R}^K$ such that

$$\sup_{(p_1,p_2)\in[0,1]^2} \left| \partial^{\mathbf{a}} g_1^{(1,0)}(p_1, p_2) - \partial^{\mathbf{a}} b_K(p_1, p_2)^\top \alpha_1^{(1,0)} \right| = O\left(K^{(|\mathbf{a}|-s)/2}\right).$$

Assumption 4.7 clearly restricts the choice of the basis functions. For example, Lemma 2 in Holland (2017) shows that when $g_1^{(1,0)}(\cdot, \cdot)$ is s-times continuously differentiable on $[0, 1]^2$, Assumption 4.7 is satisfied by tensor product B-splines of order r (degree $r - 1$) for $r - 2 \geq s$. For slightly more refined results, when $g_1^{(1,0)}(\cdot, \cdot)$ is in a Hölder space of smoothness s, B-splines, wavelets, and Cattaneo and Farrell's local polynomial partitioning series can satisfy Assumption 4.7 (for details, see Chen and Christensen, 2018, Corollary 3.1, and Cattaneo and Farrell, 2013, Lemma A.2).

Assumption 4.8. As $n \to \infty$, (i) $\zeta_0(K)\sqrt{(\log K)/n} \to 0$, and (ii) $\zeta_1(K)/\sqrt{n} \to 0$, where

$$\zeta_d(K) := \max_{|\mathbf{a}|\leq d} \sup_{(p_1,p_2)\in[0,1]^2} \|\partial^{\mathbf{a}} b_K(p_1, p_2)\|.$$

Assumption 4.8(i) is used to prove the convergence of the matrix $\Psi_{nK}^{(1,0)}$ to $\Psi_K^{(1,0)}$, and 4.8(ii) is additionally introduced to ensure the convergence of $\widehat{\Psi}_{nK}^{(1,0)}$ to $\Psi_K^{(1,0)}$ as well. Assumptions similar to the latter have often been employed in the literature of semiparametric two-step series estimation (e.g., Lee, 2007; Newey, 2009; Hoshino, 2017). The bound of $\zeta_0(K)$ is well known for several basis functions, which is typically (except for power series) $\zeta_0(K) = O(\sqrt{K})$ (see, e.g., Chen, 2007; Belloni *et al.*, 2015). By contrast, there are fewer readily available results for the bound of $\zeta_1(K)$. For example, Cattaneo and Farrell's local polynomial partitioning series satisfies $\zeta_1(K) = O(K)$ (see Cattaneo and Farrell, 2013, Lemma A.1). In Appendix C, we show that the tensor product B-spline also satisfies $\zeta_1(K) = O(K)$.

The following theorem proves the convergence rate of the estimators of $\delta_1^{(1,0)}$, feasible as well as infeasible.

[6] Then, w.p.a.1. the infeasible and feasible estimator of $\delta_1^{(1,0)}$ can be written as

$$\widetilde{\delta}_{1n}^{(1,0)} = \left[\Psi_{nK}^{(1,0)}\right]^{-1} \mathbf{R}_K^{(1,0)\top} \mathbf{Y}_1^{(1,0)}/n \quad \text{and} \quad \widehat{\delta}_{1n}^{(1,0)} = \left[\widehat{\Psi}_{nK}^{(1,0)}\right]^{-1} \widehat{\mathbf{R}}_K^{(1,0)\top} \mathbf{Y}_1^{(1,0)}/n,$$

respectively.

Theorem 4.1. Suppose that Assumptions 4.1, 4.2, 4.3(i)–(iii), 4.4, 4.5(i), and 4.6–4.8 hold. If in addition $\sqrt{n}K^{-s/2} = O(1)$ holds, then we have

$$\text{(i)} \quad \left\| \widetilde{\beta}_{1n}^{(1,0)} - \beta_1^{(1,0)} \right\| = O_P(n^{-1/2}), \quad \text{(ii)} \quad \left\| \widetilde{\alpha}_{1n}^{(1,0)} - \alpha_1^{(1,0)} \right\| = O_P(\sqrt{K/n}),$$
$$\text{(iii)} \quad \left\| \widehat{\beta}_{1n}^{(1,0)} - \beta_1^{(1,0)} \right\| = O_P(n^{-1/2}), \quad \text{(iv)} \quad \left\| \widehat{\alpha}_{1n}^{(1,0)} - \alpha_1^{(1,0)} \right\| = O_P(\sqrt{K/n}).$$

For the estimators of $\beta_1^{(1,0)}$, in addition to the $\sqrt{n}$-consistency, it is possible to derive their asymptotic normality under an additional "undersmoothing" condition ensuring that the series approximation bias is of order $o(n^{-1/2})$. It should be noted that the asymptotic distribution of $\widehat{\beta}_{1n}^{(1,0)}$ has more complicated and larger asymptotic variance than that of the infeasible estimator $\widetilde{\beta}_{1n}^{(1,0)}$ (cf., Newey and McFadden, 1994, Section 6). As our primary goal is to estimate the MTEs, we do not discuss any further the asymptotic distribution of the estimators of $\beta_1^{(1,0)}$.

We move on to the nonparametric part. Let $\mathcal{P}_{nK}$ and $\widehat{\mathcal{P}}_{nK}$ be the empirical projection operators onto the sieve space, namely,

$$\mathcal{P}_{nK}[g(p_1, p_2)] = b_K(p_1, p_2)^\top \mathbb{S}_K \left[\Psi_{nK}^{(1,0)} \right]^{-1} \frac{1}{n} \sum_{i=1}^n R_K(P_{1i}^0, P_{2i}^1) g(P_{1i}^0, P_{2i}^1),$$
$$\widehat{\mathcal{P}}_{nK}[g(p_1, p_2)] = b_K(p_1, p_2)^\top \mathbb{S}_K \left[\widehat{\Psi}_{nK}^{(1,0)} \right]^{-1} \frac{1}{n} \sum_{i=1}^n R_K(\widehat{P}_{1i}^0, \widehat{P}_{2i}^1) g(\widehat{P}_{1i}^0, \widehat{P}_{2i}^1),$$

where $\mathbb{S}_K \coloneqq (\mathbf{0}_{K \times \dim(X)}, \mathbf{I}_K)$. We define the sup-operator norm of $\mathcal{P}_{nK}$ as

$$\|\mathcal{P}_{nK}\|_\infty \coloneqq \sup \left\{ \frac{\sup_{(p_1,p_2) \in [0,1]^2} |\mathcal{P}_{nK}[g(p_1,p_2)]|}{\sup_{(p_1,p_2) \in [0,1]^2} |g(p_1,p_2)|} : g \in \mathscr{L}^\infty([0,1]^2), \sup_{(p_1,p_2) \in [0,1]^2} |g(p_1,p_2)| \neq 0 \right\},$$

where $\mathscr{L}^\infty([0,1]^2)$ is the space of uniformly bounded functions on $[0,1]^2$. The sup norm of the operator $\widehat{\mathcal{P}}_{nK}$ is similarly defined.

Assumption 4.9. (i) $\|\mathcal{P}_{nK}\|_\infty = O_P(1)$. (ii) $\|\widehat{\mathcal{P}}_{nK}\|_\infty = O_P(1)$.

Assumption 4.10. For any $\alpha \in \mathbb{R}^K$, $\sup_{(p_1,p_2) \in [0,1]^2} \left| \partial^{\mathbf{a}} b_K(p_1,p_2)^\top \alpha \right| = O(K^{|\mathbf{a}|/2}) \sup_{(p_1,p_2) \in [0,1]^2} \left| b_K(p_1,p_2)^\top \alpha \right|$.

Assumption 4.9 is another condition that restricts the choice of the basis functions. In Huang (2003), it was shown that Assumption 4.9(i) holds true for spline bases under some mild regularity conditions (such as quasi-uniform knot partition). In addition, wavelets can also satisfy this assumption, as shown in Theorem 5.2 in Chen and Christensen (2015). For the verification of Assumption 4.9(ii), see Appendix C. In the proof of Corollary 3.1 in Chen and Christensen (2018), it is shown that Assumption 4.10 holds for splines and wavelets.

Assumption 4.11.

(i) There exist finite constants $c_b > 0$ and $\omega \geq 0$ such that $\|b_K(p_1, p_2) - b_K(p_1', p_2')\| \leq c_b K^\omega \|(p_1, p_2) - (p_1', p_2')\|$ for all $(p_1, p_2), (p_1', p_2') \in [0,1]^2$.

(ii) $\zeta_0^2(K)\sqrt{(\log n)/n} \to 0$.

The next theorem establishes the uniform convergence rate of the estimators for the conditional mean function of the potential outcome.

Theorem 4.2. Suppose that Assumptions 4.1–4.4, 4.5(i), and 4.6–4.11 hold. Then, we have

$$\text{(i)} \quad \sup_{(p_1,p_2)\in[0,1]^2} \left| \widetilde{m}^{(1,0)}(x,p_1,p_2) - m^{(1,0)}(x,p_1,p_2) \right| = O_P(\zeta_0(K)K\sqrt{\log n/n}) + O_P(K^{(2-s)/2}),$$

$$\text{(ii)} \quad \sup_{(p_1,p_2)\in[0,1]^2} \left| \widehat{m}^{(1,0)}(x,p_1,p_2) - m^{(1,0)}(x,p_1,p_2) \right| = O_P(\zeta_0(K)K\sqrt{\log n/n}) + O_P(K^{(2-s)/2}).$$

The proof of the theorem is straightforward by Theorem 4.1(i), (iii), and Lemma B.4, and thus it is omitted. When $\zeta_0(K) \asymp \sqrt{K}$, by choosing $K \asymp (\log n/n)^{-1/(1+s)}$, we can obtain

$$\sup_{(p_1,p_2)\in[0,1]^2} \left| \widehat{m}^{(1,0)}(x,p_1,p_2) - m^{(1,0)}(x,p_1,p_2) \right| = O_P\left((\log n/n)^{\frac{s-2}{2s+2}} \right).$$

This implies that our estimator of the conditional mean function (and also the estimator of the MTE) can converge at the optimal uniform rate given in Stone (1982).

The next theorem provides the asymptotic distributions of the feasible estimator and the infeasible oracle estimator for the conditional mean $m^{(1,0)}(x,p_1,p_2)$.

Theorem 4.3. Suppose that Assumptions 4.1–4.11 hold. In addition, if $K \cdot \|\ddot{b}_K(p_1^0,p_2^1)\|^{-1} \to 0$, $\zeta_0(K)\sqrt{K/n} \to 0$, $\sqrt{n}K^{(2-s)/2} = O(1)$, and $\zeta_0(K)\zeta_1(K)/\sqrt{n} = O(1)$ hold, then we have

$$\text{(i)} \quad \frac{\sqrt{n}\left(\widetilde{m}^{(1,0)}(x,p_1^0,p_2^1) - m^{(1,0)}(x,p_1^0,p_2^1)\right)}{\sigma_K^{(1,0)}(p_1^0,p_2^1)} \xrightarrow{d} N(0,1),$$

$$\text{(ii)} \quad \frac{\sqrt{n}\left(\widehat{m}^{(1,0)}(x,p_1^0,p_2^1) - m^{(1,0)}(x,p_1^0,p_2^1)\right)}{\sigma_K^{(1,0)}(p_1^0,p_2^1)} \xrightarrow{d} N(0,1),$$

where

$$\sigma_K^{(1,0)}(p_1^0,p_2^1) \coloneqq \frac{\sqrt{\ddot{b}_K(p_1^0,p_2^1)^\top \mathbb{S}_K \left[\Psi_K^{(1,0)}\right]^{-1} \Sigma_K^{(1,0)} \left[\Psi_K^{(1,0)}\right]^{-1} \mathbb{S}_K^\top \ddot{b}_K(p_1^0,p_2^1)}}{h(p_1^0,p_2^1)}.$$

Theorem 4.3 indicates that the asymptotic distribution of the feasible estimator is equivalent to that of the infeasible oracle estimator.

Remark 4.1 (Choice of K)**.** Assume that $\zeta_0(K) = O(\sqrt{K})$ and $\zeta_1(K) = O(K)$ hold, as in the case of tensor product splines. Then, for deriving the limiting distribution of $\widehat{m}^{(1,0)}(x,p_1^0,p_2^1)$ in Theorem 4.3, we need both $\sqrt{n}K^{(2-s)/2} = O(1)$ and $K^3/n = O(1)$. Thus, when selecting $K \asymp n^\nu$ for some positive number $\nu > 0$, ν should satisfy $1/(s-2) \le \nu \le 1/3$ in estimating the conditional mean $m^{(1,0)}(x,p_1^0,p_2^1)$, implying that s must be larger than or equal to 5.

The standard deviation $\sigma_K^{(1,0)}(p_1^0, p_2^1)$ can be easily estimated by a sample analog, replacing the true values and functions with their estimates. Specifically, it can be estimated by

$$\widehat{\sigma}_K^{(1,0)}(p_1^0, p_2^1) \coloneqq \frac{\sqrt{\ddot{b}_K(p_1^0, p_2^1)^\top \mathbb{S}_K \left[\widehat{\Psi}_{nK}^{(1,0)}\right]^- \widehat{\Sigma}_{nK}^{(1,0)} \left[\widehat{\Psi}_{nK}^{(1,0)}\right]^- \mathbb{S}_K^\top \ddot{b}_K(p_1^0, p_2^1)}}{\widehat{h}(p_1^0, p_2^1)},$$

where $\widehat{\Sigma}_{nK}^{(1,0)} \coloneqq n^{-1} \sum_{i=1}^n \left(\widehat{e}_{1i}^{(1,0)}\right)^2 R_K(\widehat{P}_{1i}^0, \widehat{P}_{2i}^1) R_K(\widehat{P}_{1i}^0, \widehat{P}_{2i}^1)^\top$ with $\widehat{e}_{1i}^{(1,0)} \coloneqq I_i^{(1,0)} Y_{1i} - \widehat{\mathcal{L}}_i^{(1,0)} X_{1i}^\top \widehat{\beta}_{1n}^{(1,0)} - \widehat{g}_1^{(1,0)}(\widehat{P}_{1i}^0, \widehat{P}_{2i}^1)$.

Under conditions similar to those in Theorem 4.3, we can also show that

$$\frac{\sqrt{n}\left(\widehat{m}^{(0,0)}(x, p_1^0, p_2^0) - m^{(0,0)}(x, p_1^0, p_2^0)\right)}{\sigma_K^{(0,0)}(p_1^0, p_2^0)} \xrightarrow{d} N(0,1), \qquad \frac{\sqrt{n}\left(\widehat{m}^{(0,0)}(x, p_1^0, p_2^1) - m^{(0,0)}(x, p_1^0, p_2^1)\right)}{\sigma_K^{(0,0)}(p_1^0, p_2^1)} \xrightarrow{d} N(0,1),$$

$$\frac{\sqrt{n}\left(\widehat{m}^{(1,1)}(x, p_1^0, p_2^0) - m^{(1,1)}(x, p_1^0, p_2^0)\right)}{\sigma_K^{(1,1)}(p_1^0, p_2^0)} \xrightarrow{d} N(0,1),$$

where the asymptotic standard errors are given by

$$\sigma_K^{(0,0)}(p_1^0, p_2^0) \coloneqq \frac{\sqrt{\ddot{b}_K(p_1^0, p_2^0)^\top \mathbb{S}_{1K} \left[\Psi_K^{(0,0)}\right]^{-1} \Sigma_K^{(0,0)} \left[\Psi_K^{(0,0)}\right]^{-1} \mathbb{S}_{1K}^\top \ddot{b}_K(p_1^0, p_2^0)}}{\lambda h(p_1^0, p_2^0)},$$

$$\sigma_K^{(0,0)}(p_1^0, p_2^1) \coloneqq \frac{\sqrt{\ddot{b}_K(p_1^0, p_2^1)^\top \mathbb{S}_{3K} \left[\Psi_K^{(0,0)}\right]^{-1} \Sigma_K^{(0,0)} \left[\Psi_K^{(0,0)}\right]^{-1} \mathbb{S}_{3K}^\top \ddot{b}_K(p_1^0, p_2^1)}}{(1-\lambda) h(p_1^0, p_2^1)},$$

$$\sigma_K^{(1,1)}(p_1^0, p_2^0) \coloneqq \frac{\sqrt{\ddot{b}_K(p_1^0, p_2^0)^\top \mathbb{S}_{1K} \left[\Psi_K^{(1,1)}\right]^{-1} \Sigma_K^{(1,1)} \left[\Psi_K^{(1,1)}\right]^{-1} \mathbb{S}_{1K}^\top \ddot{b}_K(p_1^0, p_2^0)}}{\lambda h(p_1^0, p_2^0)}.$$

Here, $\mathbb{S}_{1K} \coloneqq (\mathbf{0}_{K \times \dim(X)}, \mathbf{I}_K, \mathbf{0}_{K \times 3K})$ and $\mathbb{S}_{3K} \coloneqq (\mathbf{0}_{K \times \dim(X)}, \mathbf{0}_{K \times 2K}, \mathbf{I}_K, \mathbf{0}_{K \times K})$. The definitions of the matrices $\Psi_K^{(0,0)}$, $\Sigma_K^{(0,0)}$, $\Psi_K^{(1,1)}$, and $\Sigma_K^{(1,1)}$ should be clear from the context. Consequently, the limiting distributions of the MTE estimators $\widehat{\tau}_{direct}^{(0)}(x, p_1^0, p_2^1)$ and $\widehat{\tau}_{total}(x, p_1^0, p_2^0)$ can be characterized as follows:

$$\frac{\sqrt{n}\left(\widehat{\tau}_{direct}^{(0)}(x, p_1^0, p_2^1) - \tau_{direct}^{(0)}(x, p_1^0, p_2^1)\right)}{\sqrt{\left[\sigma_K^{(1,0)}(p_1^0, p_2^1)\right]^2 + \left[\sigma_K^{(0,0)}(p_1^0, p_2^1)\right]^2}} \xrightarrow{d} N(0,1), \qquad \frac{\sqrt{n}\left(\widehat{\tau}_{total}(x, p_1^0, p_2^0) - \tau_{total}(x, p_1^0, p_2^0)\right)}{\sqrt{\left[\sigma_K^{(1,1)}(p_1^0, p_2^0)\right]^2 + \left[\sigma_K^{(0,0)}(p_1^0, p_2^0)\right]^2}} \xrightarrow{d} N(0,1).$$

The limiting distributions of the other MTE estimators can be derived similarly.

5 Conclusion

This paper proposed identification and estimation methods for treatment effect models that admit both treatment spillover and the dependence between treatment decisions within a pair of individuals. We first demonstrated that the interaction in treatment decisions can be modeled as a binary game of complete information with potential multiple equilibria. Treatment evaluation in the presence of strategic interaction is a highly non-trivial problem because the presence of multiple equilibria precludes the use of conventional identification strategies. Assuming an empirically reasonable equilibrium selection rule, we showed that several treatment effect parameters, such as MTEs and LATEs, can be point-identified by using an extended version of the LIV method. Based on our constructive identification results, we proposed a two-step semiparametric series estimation procedure for estimating the MTEs. We showed that the proposed estimator of MTE is uniformly consistent and achieves the optimal convergence rate. We also derived the asymptotic normality of the estimator, and showed that its limiting distribution is the same as that of the infeasible oracle estimator.

The results of this study suggest several extensions that would be promising to investigate. First, it would be worthwhile to extend our results to the case of strategic interaction among more than two players. In the literature of game econometrics, there are only a few studies concerned with identification and estimation of game models with more than two players, primarily because the characterization of equilibrium is highly complicated compared to the case of two-player games. Accordingly, the direct applicability of our approach to such cases is unclear. Secondly, it may be beneficial to develop treatment evaluation techniques under treatment decision games of incomplete information. We assumed that the realized values of all random variables are common knowledge to the pair of players; however, in some situations it would be more realistic that the players do not have full information regarding their partner's preferences. Finally, while we attained point-identification of several treatment effect parameters by explicitly assuming an equilibrium selection rule, even when the selection rule is violated, we may be able to establish partial identification of the parameters as in Ciliberto and Tamer (2009) or Chesher and Rosen (2012). These topics are left for future research.

A Appendix: Proofs of Theorems

This appendix collects the proofs of the theorems. The technical lemmas used in the proofs are given in Appendix B.

A.1 Proofs of theorems in Section 3

A.1.1 Proof of Theorem 3.1

We provide the proof for $m^{(1,0)}(x, p_1^0, p_2^1)$ only, as the proof for $m^{(0,1)}(x, p_1^1, p_2^0)$ is analogous. By the law of iterated expectations, the relationship (2.6) implies that

$$\begin{aligned}
&\psi^{(1,0)}(x, p_1^0, p_2^1) \\
&= E[I^{(1,0)} Y_1 | X = x, P_1^0 = p_1^0, P_2^1 = p_2^1] \\
&= E[Y_1^{(1,0)} | X = x, P_1^0 = p_1^0, P_2^1 = p_2^1, D = (1,0)] \cdot \Pr[D = (1,0) | X = x, P_1^0 = p_1^0, P_2^1 = p_2^1] \\
&= E[Y_1^{(1,0)} | X = x, P_1^0 = p_1^0, P_2^1 = p_2^1, V_1 \le p_1^0, V_2 > p_2^1] \cdot \Pr[V_1 \le p_1^0, V_2 > p_2^1 | X = x, P_1^0 = p_1^0, P_2^1 = p_2^1] \\
&= E[Y_1^{(1,0)} | X = x, V_1 \le p_1^0, V_2 > p_2^1] \cdot \Pr[V_1 \le p_1^0, V_2 > p_2^1 | X = x],
\end{aligned}$$

where we used Assumption 3.1(i) in the last equality. Here, it holds that

$$E[Y_1^{(1,0)} | X = x, V_1 \le p_1^0, V_2 > p_2^1] = \frac{1}{\Pr[V_1 \le p_1^0, V_2 > p_2^1 | X = x]} \int_{p_2^1}^1 \int_0^{p_1^0} m^{(1,0)}(x, v_1, v_2) h(v_1, v_2 | x) \mathrm{d}v_1 \mathrm{d}v_2.$$

As a result, the cross partial differentiation of $\psi^{(1,0)}(x, p_1^0, p_2^1)$ with respect to p_1^0 and p_2^1 leads to

$$\begin{aligned}
\psi^{(1,0)}(x, p_1^0, p_2^1) &= \int_{p_2^1}^1 \int_0^{p_1^0} m^{(1,0)}(x, v_1, v_2) h(v_1, v_2 | x) \mathrm{d}v_1 \mathrm{d}v_2 \\
\Longrightarrow \frac{\partial^2 \psi^{(1,0)}(x, p_1^0, p_2^1)}{\partial p_1^0 \partial p_2^1} &= -m^{(1,0)}(x, p_1^0, p_2^1) h(p_1^0, p_2^1 | x),
\end{aligned}$$

by the Leibniz integral rule provided that $m^{(1,0)}(x, v_1, v_2) h(v_1, v_2 | x)$ is continuous in (v_1, v_2). By rearranging the above equation, we obtain the desired result for $m^{(1,0)}(x, p_1^0, p_2^1)$. □

A.1.2 Proof of Theorem 3.2

We prove only the case $m_1^{(0,0)}(x, p_1^0, p_2^0)$, as the proofs for the other cases are similar. By the law of iterated expectations,

$$\begin{aligned}
\psi^{(0,0)}(x, \mathbf{p}) &= E[I^{(0,0)} Y_1 | X = x, \mathbf{P} = \mathbf{p}] \\
&= E[Y_1^{(0,0)} | X = x, \mathbf{P} = \mathbf{p}, D = (0,0)] \cdot \Pr[D = (0,0) | X = x, \mathbf{P} = \mathbf{p}].
\end{aligned}$$

For notational simplicity, we write $\mathcal{V}_{uni}^{(0,0)} = \mathcal{V}_{uni}^{(0,0)}(\mathbf{p})$ and $\mathcal{V}_{mul} = \mathcal{V}_{mul}(\mathbf{p})$ by suppressing the dependence on $\mathbf{p}$. As $\mathcal{V}_{uni}^{(0,0)}$ and $\mathcal{V}_{mul}$ are disjoint, by Assumptions 3.1(i) and 3.2, it holds that

$$\begin{aligned}
&E[Y_1^{(0,0)}|X = x, \mathbf{P} = \mathbf{p}, D = (0,0)] \\
&= E[Y_1^{(0,0)}|X = x, V \in \mathcal{V}_{uni}^{(0,0)} \vee (V \in \mathcal{V}_{mul} \wedge \epsilon \leq \lambda)] \\
&= E[Y_1^{(0,0)}|X = x, V \in \mathcal{V}_{uni}^{(0,0)}] \cdot \frac{\Pr[V \in \mathcal{V}_{uni}^{(0,0)}|X = x]}{\Pr[V \in \mathcal{V}_{uni}^{(0,0)} \vee (V \in \mathcal{V}_{mul} \wedge \epsilon \leq \lambda)|X = x]} \\
&\quad + E[Y_1^{(0,0)}|X = x, (V \in \mathcal{V}_{mul} \wedge \epsilon \leq \lambda)] \cdot \frac{\Pr[(V \in \mathcal{V}_{mul} \wedge \epsilon \leq \lambda)|X = x]}{\Pr[V \in \mathcal{V}_{uni}^{(0,0)} \vee (V \in \mathcal{V}_{mul} \wedge \epsilon \leq \lambda)|X = x]} \\
&= E[Y_1^{(0,0)}|X = x, V \in \mathcal{V}_{uni}^{(0,0)}] \cdot \frac{\Pr[V \in \mathcal{V}_{uni}^{(0,0)}|X = x]}{\Pr[V \in \mathcal{V}_{uni}^{(0,0)}|X = x] + \lambda \cdot \Pr[V \in \mathcal{V}_{mul}|X = x]} \\
&\quad + E[Y_1^{(0,0)}|X = x, V \in \mathcal{V}_{mul}] \cdot \frac{\lambda \cdot \Pr[V \in \mathcal{V}_{mul}|X = x]}{\Pr[V \in \mathcal{V}_{uni}^{(0,0)}|X = x] + \lambda \cdot \Pr[V \in \mathcal{V}_{mul}|X = x]}.
\end{aligned}$$

Similarly, we also have

$$\Pr[D = (0,0)|X = x, \mathbf{P} = \mathbf{p}] = \Pr[V \in \mathcal{V}_{uni}^{(0,0)}|X = x] + \lambda \cdot \Pr[V \in \mathcal{V}_{mul}|X = x].$$

As a result, we obtain

$$\begin{aligned}
\psi^{(0,0)}(x, \mathbf{p}) = &\, E[Y_1^{(0,0)}|X = x, V \in \mathcal{V}_{uni}^{(0,0)}] \cdot \Pr[V \in \mathcal{V}_{uni}^{(0,0)}|X = x] \\
&+ \lambda \cdot E[Y_1^{(0,0)}|X = x, V \in \mathcal{V}_{mul}] \cdot \Pr[V \in \mathcal{V}_{mul}|X = x].
\end{aligned}$$

Further, it holds that

$$\begin{aligned}
&E[Y_1^{(0,0)}|X = x, V \in \mathcal{V}_{uni}^{(0,0)}] \cdot \Pr[V \in \mathcal{V}_{uni}^{(0,0)}|X = x] \\
&= \int_{p_2^0}^1 \int_{p_1^0}^1 m^{(0,0)}(x, v_1, v_2) h(v_1, v_2|x) \mathrm{d}v_1 \mathrm{d}v_2 - \int_{p_2^0}^{p_2^1} \int_{p_1^0}^{p_1^1} m^{(0,0)}(x, v_1, v_2) h(v_1, v_2|x) \mathrm{d}v_1 \mathrm{d}v_2,
\end{aligned}$$

and

$$E[Y_1^{(0,0)}|X = x, V \in \mathcal{V}_{mul}] \cdot \Pr[V \in \mathcal{V}_{mul}|X = x] = \int_{p_2^0}^{p_2^1} \int_{p_1^0}^{p_1^1} m^{(0,0)}(x, v_1, v_2) h(v_1, v_2|x) \mathrm{d}v_1 \mathrm{d}v_2.$$

Hence, we have

$$\psi^{(0,0)}(x, \mathbf{p}) = \int_{p_2^0}^1 \int_{p_1^0}^1 m^{(0,0)}(x, v_1, v_2) h(v_1, v_2|x) \mathrm{d}v_1 \mathrm{d}v_2 - (1 - \lambda) \int_{p_2^0}^{p_2^1} \int_{p_1^0}^{p_1^1} m^{(0,0)}(x, v_1, v_2) h(v_1, v_2|x) \mathrm{d}v_1 \mathrm{d}v_2.$$

By partially differentiating both sides with respect to p_1^0 and p_2^0 and rearranging the equation, the Leibniz integral rule and the continuity of $m^{(0,0)}(x, v_1, v_2)h(v_1, v_2|x)$ in (v_1, v_2) lead to the desired result:

$$m^{(0,0)}(x, p_1^0, p_2^0) = \frac{1}{\lambda h(p_1^0, p_2^0|x)} \frac{\partial^2 \psi^{(0,0)}(x, \mathbf{p})}{\partial p_1^0 \partial p_2^0}.$$

□

A.2 Proofs of theorems in Section 4

For notational simplicity, when there is no confusion, we often suppress the superscript $(1, 0)$ in the following proofs. In the proofs, we often refer to lemmas and equations in Appendix B.

Here, we introduce additional notations as follows. Let $\mathbb{S}_a$ be a selection matrix of dimension $a \times (\dim(X) + K)$ such that $\mathbb{S}_a \delta_1^{(1,0)}$ is the corresponding $a \times 1$ subvector of $\delta_1^{(1,0)}$. Specifically, we write $\mathbb{S}_X \delta_1^{(1,0)} = \beta_1^{(1,0)}$ and $\mathbb{S}_K \delta_1^{(1,0)} = \alpha_1^{(1,0)}$ with $\mathbb{S}_X = \mathbb{S}_{\dim(X)} := (\mathbf{I}_{\dim(X)}, \mathbf{0}_{\dim(X)\times K})$ and $\mathbb{S}_K := (\mathbf{0}_{K\times\dim(X)}, \mathbf{I}_K)$. Let $\mathbf{e}_1^{(1,0)} := (e_{11}^{(1,0)}, \ldots, e_{1n}^{(1,0)})^\top$ be the vector of error terms in the partially linear model (4.2). Further, we define the vectors $\mathbf{u}_1^{(1,0)} := \mathbf{g}_1^{(1,0)} - \mathbf{b}_K^{(1,0)}\alpha_1^{(1,0)}$ and $\widehat{\mathbf{u}}_1^{(1,0)} := \widehat{\mathbf{g}}_1^{(1,0)} - \widehat{\mathbf{b}}_K^{(1,0)}\alpha_1^{(1,0)}$ of series approximation errors with $\mathbf{g}_1^{(1,0)} := (g_1^{(1,0)}(P_{11}^0, P_{21}^1), \ldots, g_1^{(1,0)}(P_{1n}^0, P_{2n}^1))^\top$ and $\widehat{\mathbf{g}}_1^{(1,0)} := (g_1^{(1,0)}(\widehat{P}_{11}^0, \widehat{P}_{21}^1), \ldots, g_1^{(1,0)}(\widehat{P}_{1n}^0, \widehat{P}_{2n}^1))^\top$.

We recall the definitions of the infeasible and feasible estimators $\widetilde{\delta}_{1n}^{(1,0)}$ and $\widehat{\delta}_{1n}^{(1,0)}$, which are given in (4.3) and (4.4), respectively. The infeasible estimator $\mathbb{S}_a\widetilde{\delta}_{1n}^{(1,0)}$ can be decomposed as follows:

$$\mathbb{S}_a \left(\widetilde{\delta}_{1n}^{(1,0)} - \delta_1^{(1,0)}\right) = \mathbb{S}_a \Psi_{nK}^{-1} \mathbf{R}_K^\top \mathbf{u}_1/n + \mathbb{S}_a \Psi_{nK}^{-1} \mathbf{R}_K^\top \mathbf{e}_1/n. \tag{A.1}$$

Similarly, the feasible estimator $\mathbb{S}_a\widehat{\delta}_{1n}^{(1,0)}$ has the following decomposition:

$$\begin{aligned}\mathbb{S}_a \left(\widehat{\delta}_{1n}^{(1,0)} - \delta_1^{(1,0)}\right) = {}& \mathbb{S}_a \widehat{\Psi}_{nK}^{-1} \widehat{\mathbf{R}}_K^\top \left(\mathbf{g}_1^{(1,0)} - \widehat{\mathbf{g}}_1^{(1,0)}\right)/n + \mathbb{S}_a \widehat{\Psi}_{nK}^{-1} \widehat{\mathbf{R}}_K^\top \left(\mathbf{X}_1^{(1,0)} - \widehat{\mathbf{X}}_1^{(1,0)}\right)\beta_1^{(1,0)}/n \\ &+ \mathbb{S}_a \widehat{\Psi}_{nK}^{-1} \widehat{\mathbf{R}}_K^\top \widehat{\mathbf{u}}_1/n + \mathbb{S}_a \widehat{\Psi}_{nK}^{-1} \widehat{\mathbf{R}}_K^\top \mathbf{e}_1/n.\end{aligned} \tag{A.2}$$

A.2.1 Proof of Theorem 4.1

Proof of (i)–(ii). We first show that the first term on the right-hand side of (A.1) is of order $O_P(n^{-1/2})$ for any choice of $\mathbb{S}_a$. We observe that

$$\begin{aligned}\|\mathbb{S}_a \Psi_{nK}^{-1} \mathbf{R}_K^\top \mathbf{u}_1/n\|^2 &= \mathrm{tr}\left\{\mathbf{u}_1^\top \mathbf{R}_K \Psi_{nK}^{-1} \mathbb{S}_a^\top \mathbb{S}_a \Psi_{nK}^{-1} \mathbf{R}_K^\top \mathbf{u}_1\right\}/n^2 \\ &\leq \mathrm{tr}\left\{\mathbf{u}_1^\top \mathbf{R}_K \Psi_{nK}^{-2} \mathbf{R}_K^\top \mathbf{u}_1\right\}/n^2 \\ &\leq [\underline{c}_\Psi + o_P(1)]^{-2} \cdot \mathrm{tr}\left\{\mathbf{u}_1^\top \Psi_{nK} \mathbf{u}_1\right\}/n = O_P(1) \cdot \|\mathbf{u}_1\|^2/n,\end{aligned}$$

where in the second inequality and last equality, we have used the following result:

$$0 < \underline{c}_\Psi + o_P(1) \leq \chi_{\min}(\Psi_{nK}) \leq \chi_{\max}(\Psi_{nK}) \leq \bar{c}_\Psi + o_P(1) < \infty, \quad \text{w.p.a.1.}, \tag{A.3}$$

which is implied by Lemma B.2(i) and Assumption 4.5(i). By Assumption 4.7,

$$\|\mathbf{u}_1\|^2 \leq n \cdot \left[\sup_{(p_1,p_2)\in[0,1]^2} \left| g_1^{(1,0)}(p_1,p_2) - b_K(p_1,p_2)^\top \alpha_1^{(1,0)} \right| \right]^2 = O(nK^{-s}),$$

implying that

$$\|\mathbb{S}_a \Psi_{nK}^{-1} \mathbf{R}_K^\top \mathbf{u}_1/n\| = \underbrace{O_P(K^{-s/2})}_{O_P(n^{-1/2})}. \tag{A.4}$$

For the second term in (A.1), Assumptions 4.4(i) and 4.6 and (A.3) imply that

$$\begin{aligned} E\left[\left\| \mathbb{S}_a \Psi_{nK}^{-1} \mathbf{R}_K^\top \mathbf{e}_1/n \right\|^2 \middle| \{W_i\}_{i=1}^n \right] &= \operatorname{tr}\left\{ \mathbb{S}_a \Psi_{nK}^{-1} \mathbf{R}_K^\top E\left[\mathbf{e}_1 \mathbf{e}_1^\top \middle| \{W_i\}_{i=1}^n \right] \mathbf{R}_K \Psi_{nK}^{-1} \mathbb{S}_a^\top \right\} /n^2 \\ &= O_P(1) \cdot \operatorname{tr}\left\{ \mathbb{S}_a \Psi_{nK}^{-1} \mathbb{S}_a^\top \right\} /n = O_P\left(a/n\right). \end{aligned}$$

Hence, we have

$$\left\| \mathbb{S}_a \Psi_{nK}^{-1} \mathbf{R}_K^\top \mathbf{e}_1/n \right\| = O_P\left(\sqrt{a/n} \right), \tag{A.5}$$

by Markov's inequality. We have (i) and (ii) by noting that they are the cases when $a = \dim(X)$ and $a = K$, respectively.

Proof of (iii)–(iv). By Lemma B.2(ii) and Assumption 4.5(i), we obtain

$$0 < \underline{c}_\Psi + o_P(1) \leq \chi_{\min}\left(\widehat{\Psi}_{nK} \right) \leq \chi_{\max}\left(\widehat{\Psi}_{nK} \right) \leq \bar{c}_\Psi + o_P(1) < \infty, \quad \text{w.p.a.1.} \tag{A.6}$$

Then, by using the same argument as in (A.4), it holds that $\left\| \mathbb{S}_a \widehat{\Psi}_{nK}^{-1} \widehat{\mathbf{R}}_K^\top \widehat{\mathbf{u}}_1/n \right\| = \underbrace{O_P(K^{-s/2})}_{O_P(n^{-1/2})}$ for any $\mathbb{S}_a$. Therefore, by (A.2),

$$\begin{aligned} \mathbb{S}_a \left(\widehat{\delta}_{1n}^{(1,0)} - \delta_1^{(1,0)} \right) &= \mathbb{S}_a \widehat{\Psi}_{nK}^{-1} \widehat{\mathbf{R}}_K^\top \left(\mathbf{g}_1^{(1,0)} - \widehat{\mathbf{g}}_1^{(1,0)} \right)/n + \mathbb{S}_a \widehat{\Psi}_{nK}^{-1} \widehat{\mathbf{R}}_K^\top \left(\mathbf{X}_1^{(1,0)} - \widehat{\mathbf{X}}_1^{(1,0)} \right) \beta_1^{(1,0)}/n \\ &\quad + \mathbb{S}_a \widehat{\Psi}_{nK}^{-1} \widehat{\mathbf{R}}_K^\top \mathbf{e}_1/n + O_P(n^{-1/2}). \end{aligned} \tag{A.7}$$

By the mean value expansion and Lemma B.1, we have

$$g_1^{(1,0)}(\widehat{P}_{1i}^0, \widehat{P}_{2i}^1) - g_1^{(1,0)}(P_{1i}^0, P_{2i}^1) = \left\{ \frac{\partial g_1^{(1,0)}\left(\bar{P}_{1i}^0, \bar{P}_{2i}^1 \right)}{\partial p_1} + \frac{\partial g_1^{(1,0)}\left(\bar{P}_{1i}^0, \bar{P}_{2i}^1 \right)}{\partial p_2} \right\} \cdot O_P(n^{-1/2}),$$

where $\bar{P}^0_{1i} \in [\widehat{P}^0_{1i}, P^0_{1i}]$ and $\bar{P}^1_{2i} \in [\widehat{P}^1_{2i}, P^1_{2i}]$. Thus, by the triangle inequality and (A.6), we have

$$\begin{aligned}\left\|\mathbb{S}_a\widehat{\Psi}^{-1}_{nK}\widehat{\mathbf{R}}^\top_K\left(\mathbf{g}^{(1,0)}_1 - \widehat{\mathbf{g}}^{(1,0)}_1\right)/n\right\|^2 &\le O_P(n^{-1})\cdot\left\{\left\|\mathbb{S}_a\widehat{\Psi}^{-1}_{nK}\widehat{\mathbf{R}}^\top_K\partial_1\bar{\mathbf{g}}^{(1,0)}_1/n\right\|^2 + \left\|\mathbb{S}_a\widehat{\Psi}^{-1}_{nK}\widehat{\mathbf{R}}^\top_K\partial_2\bar{\mathbf{g}}^{(1,0)}_1/n\right\|^2\right\}\\ &\le O_P(n^{-1})\cdot\operatorname{tr}\left\{\mathbb{S}_a\widehat{\Psi}^{-1}_{nK}\mathbb{S}^\top_a\right\} = O_P(a/n),\end{aligned} \tag{A.8}$$

where $\partial_j\bar{\mathbf{g}}^{(1,0)}_1 = (\partial g^{(1,0)}_1(\bar{P}^0_{11}, \bar{P}^1_{21})/\partial p_j, \ldots, \partial g^{(1,0)}_1(\bar{P}^0_{1n}, \bar{P}^1_{2n})/\partial p_j)^\top$ for $j = 1, 2$, implying that the first term in (A.7) is of order $O_P(\sqrt{a/n})$. For the second term, it can be similarly verified that $||\mathbb{S}_a\widehat{\Psi}^{-1}_{nK}\widehat{\mathbf{R}}^\top_K(\mathbf{X}^{(1,0)}_1 - \widehat{\mathbf{X}}^{(1,0)}_1)\beta^{(1,0)}_1/n|| = O_P(\sqrt{a/n})$ from (B.1). For the third term in (A.7), noting that $\widehat{\Psi}^{-1}_{nK}$ and $\widehat{\mathbf{R}}_K$ are functions of $\{W_i, D_i\}^n_{i=1}$, we have

$$\begin{aligned}E\left[\left\|\mathbb{S}_a\widehat{\Psi}^{-1}_{nK}\widehat{\mathbf{R}}^\top_K\mathbf{e}_1/n\right\|^2\middle|\{W_i, D_i\}^n_{i=1}\right] &= \operatorname{tr}\left\{\mathbb{S}_a\widehat{\Psi}^{-1}_{nK}\widehat{\mathbf{R}}^\top_K E\left[\mathbf{e}_1\mathbf{e}^\top_1\middle|\{W_i, D_i\}^n_{i=1}\right]\widehat{\mathbf{R}}_K\widehat{\Psi}^{-1}_{nK}\mathbb{S}^\top_a\right\}/n^2\\ &= O_P(1)\cdot\operatorname{tr}\left\{\mathbb{S}_a\widehat{\Psi}^{-1}_{nK}\mathbb{S}^\top_a\right\}/n = O_P\left(a/n\right),\end{aligned}$$

by Assumptions 4.4(i) and 4.6 and (A.6), which leads to the desired result by Markov's inequality. This completes the proof. □

A.2.2 Proof of Theorem 4.3

To prove (i) and (ii) of the theorem, we first show that there exists a constant $0 < c_\sigma < \infty$ such that

$$\sigma^{(1,0)}_K(p^0_1, p^1_2) \ge c_\sigma\cdot\|\ddot{b}_K(p^0_1, p^1_2)\|. \tag{A.9}$$

By Assumptions 4.3(iv) and 4.5(ii), and noting that $\mathbb{S}_K\mathbb{S}^\top_K = \mathbf{I}_K$, we observe

$$\begin{aligned}\left(\sigma^{(1,0)}_K(p^0_1, p^1_2)\right)^2 &= \frac{1}{h^2(p^0_1, p^1_2)}\ddot{b}_K(p^0_1, p^1_2)^\top\mathbb{S}_K\Psi^{-1}_K\Sigma_K\Psi^{-1}_K\mathbb{S}^\top_K\ddot{b}_K(p^0_1, p^1_2)\\ &\ge \underbrace{\frac{\underline{c}_\Sigma}{\bar{c}^2_\Psi\cdot h^2(p^0_1, p^1_2)}}_{>0}\cdot\|\ddot{b}_K(p^0_1, p^1_2)\|^2,\end{aligned}$$

which implies (A.9). Note that this implies $K\cdot[\sigma^{(1,0)}_K(p^0_1, p^1_2)]^{-1} \to 0$ by the assumption $K\cdot\|\ddot{b}_K(p^0_1, p^1_2)\|^{-1} \to 0$.

Proof of Result (i). By the definition of the infeasible estimator $\widetilde{m}^{(1,0)}(x, p^0_1, p^1_2)$, we observe that

$$\begin{aligned}&\widetilde{m}^{(1,0)}(x, p^0_1, p^1_2) - m^{(1,0)}(x, p^0_1, p^1_2)\\ &= x^\top_1\left(\widetilde{\beta}^{(1,0)}_{1n} - \beta^{(1,0)}_1\right) - \frac{1}{h(p^0_1, p^1_2)}\ddot{b}_K(p^0_1, p^1_2)^\top\left(\widetilde{\alpha}^{(1,0)}_{1n} - \alpha^{(1,0)}_1\right) + \frac{1}{h(p^0_1, p^1_2)}\left(\frac{\partial^2 g^{(1,0)}_1(p^0_1, p^1_2)}{\partial p_1\partial p_2} - \ddot{b}_K(p^0_1, p^1_2)^\top\alpha^{(1,0)}_1\right)\\ &= -\frac{1}{h(p^0_1, p^1_2)}\ddot{b}_K(p^0_1, p^1_2)^\top\left(\widetilde{\alpha}^{(1,0)}_{1n} - \alpha^{(1,0)}_1\right) + O_P(n^{-1/2}) + O(K^{(2-s)/2})\end{aligned}$$

$$= A_{1n} + A_{2n} + O_P(n^{-1/2}),$$

by Theorem 4.1(i) and Assumption 4.7, where

$$A_{1n} \coloneqq -\frac{1}{h(p_1^0, p_2^1)} \ddot{b}_K(p_1^0, p_2^1)^\top \mathbb{S}_K \Psi_{nK}^{-1} \mathbf{R}_K^\top \mathbf{u}_1 / n,$$
$$A_{2n} \coloneqq -\frac{1}{h(p_1^0, p_2^1)} \ddot{b}_K(p_1^0, p_2^1)^\top \mathbb{S}_K \Psi_{nK}^{-1} \mathbf{R}_K^\top \mathbf{e}_1 / n.$$

By the Cauchy–Schwarz inequality, we have

$$\begin{aligned} |A_{1n}| &\leq O(1) \cdot \|\ddot{b}_K(p_1^0, p_2^1)\| \cdot \left\| \mathbb{S}_K \Psi_{nK}^{-1} \mathbf{R}_K^\top \mathbf{u}_1 / n \right\| \\ &= \|\ddot{b}_K(p_1^0, p_2^1)\| \cdot \underbrace{O_P(K^{-s/2})}_{o_P(n^{-1/2})}, \end{aligned} \tag{A.10}$$

where the equality follows from (A.4). We define

$$A'_{2n} \coloneqq -\frac{1}{h(p_1^0, p_2^1)} \ddot{b}_K(p_1^0, p_2^1)^\top \mathbb{S}_K \Psi_K^{-1} \mathbf{R}_K^\top \mathbf{e}_1 / n.$$

It is easy to see that $\|A_{2n} - A'_{2n}\| = O(1) \cdot \|\ddot{b}_K(p_1^0, p_2^1)\| \cdot \|\mathbb{S}_K(\Psi_{nK}^{-1} - \Psi_K^{-1})\mathbf{R}_K^\top \mathbf{e}_1/n\| = \|\ddot{b}_K(p_1^0, p_2^1)\| \cdot o_P(n^{-1/2})$ by Lemma B.2(iii) and Markov's inequality. Thus, by (A.9), we obtain

$$\begin{aligned} \frac{\sqrt{n}\left(\widetilde{m}^{(1,0)}(x, p_1^0, p_2^1) - m^{(1,0)}(x, p_1^0, p_2^1)\right)}{\sigma_K^{(1,0)}(p_1^0, p_2^1)} &= \frac{\sqrt{n}(A_{1n} + A_{2n})}{\sigma_K^{(1,0)}(p_1^0, p_2^1)} + o_P(1) \\ &= \frac{\sqrt{n} A'_{2n}}{\sigma_K^{(1,0)}(p_1^0, p_2^1)} + o_P(1). \end{aligned}$$

We now show the asymptotic normality of $\sqrt{n} A'_{2n} / \sigma_K^{(1,0)}(p_1^0, p_2^1)$. Let

$$\xi_i \coloneqq -\Pi_K(p_1^0, p_2^1) R_K(P_{1i}^0, P_{2i}^1) e_{1i} / \sqrt{n},$$

where

$$\Pi_K(p_1^0, p_2^1) \coloneqq \left[\sigma_K^{(1,0)}(p_1^0, p_2^1) \cdot h(p_1^0, p_2^1)\right]^{-1} \ddot{b}_K(p_1^0, p_2^1)^\top \mathbb{S}_K \Psi_K^{-1},$$

so that $\sum_{i=1}^n \xi_i = \sqrt{n} A'_{2n} / \sigma_K^{(1,0)}(p_1^0, p_2^1)$. By construction, $E[\xi_i] = 0$ and $E[\xi_i^2] = n^{-1}$ hold. Let $R_{K,i} = R_K(P_{1i}^0, P_{2i}^1)$ for simplicity. Then, Assumption 4.6 and the law of iterated expectations yield

$$\begin{aligned} E\left[\xi_i^4\right] &= n^{-2} E\left[\Pi_K(p_1^0, p_2^1) R_{K,i} R_{K,i}^\top \Pi_K(p_1^0, p_2^1)^\top \Pi_K(p_1^0, p_2^1) R_{K,i} R_{K,i}^\top \Pi_K(p_1^0, p_2^1)^\top E\left[e_{1,i}^4 | W_i\right]\right] \\ &\leq O(n^{-2}) \cdot E\left[\Pi_K(p_1^0, p_2^1) R_{K,i} R_{K,i}^\top \Pi_K(p_1^0, p_2^1)^\top \Pi_K(p_1^0, p_2^1) R_{K,i} R_{K,i}^\top \Pi_K(p_1^0, p_2^1)^\top\right] \\ &= O(n^{-2}) \cdot E\left[\operatorname{tr}\left\{R_{K,i} R_{K,i}^\top \Pi_K(p_1^0, p_2^1)^\top \Pi_K(p_1^0, p_2^1) R_{K,i} R_{K,i}^\top \Pi_K(p_1^0, p_2^1)^\top \Pi_K(p_1^0, p_2^1)\right\}\right] \\ &\leq O(n^{-2}) \cdot \chi_{\max}\left(\Pi_K(p_1^0, p_2^1)^\top \Pi_K(p_1^0, p_2^1)\right) E\left[\operatorname{tr}\left\{R_{K,i} R_{K,i}^\top \Pi_K(p_1^0, p_2^1)^\top \Pi_K(p_1^0, p_2^1) R_{K,i} R_{K,i}^\top\right\}\right] \end{aligned}$$

$$\leq O(n^{-2}) \cdot \left[\chi_{\max}\left(\Pi_K(p_1^0,p_2^1)^\top \Pi_K(p_1^0,p_2^1)\right)\right]^2 E\left[\mathrm{tr}\left\{R_{K,i}R_{K,i}^\top R_{K,i}R_{K,i}^\top\right\}\right]$$
$$= O(\zeta_0^2(K)K/n^2) \cdot \left[\chi_{\max}\left(\Pi_K(p_1^0,p_2^1)^\top \Pi_K(p_1^0,p_2^1)\right)\right]^2,$$

where the last equality follows from $E[\mathrm{tr}\{R_{K,i}R_{K,i}^\top R_{K,i}R_{K,i}^\top\}] \leq \zeta_0^2(K)\mathrm{tr}\{E[R_{K,i}R_{K,i}^\top]\} = O(\zeta_0^2(K)K)$ under Assumption 4.5(i). Since (A.9) and Assumption 4.3(iv) imply that

$$\chi_{\max}\left(\Pi_K(p_1^0,p_2^1)^\top \Pi_K(p_1^0,p_2^1)\right) \leq \mathrm{tr}\left\{\Pi_K(p_1^0,p_2^1)^\top \Pi_K(p_1^0,p_2^1)\right\}$$
$$\leq O(1)\cdot \|\ddot{b}_K(p_1^0,p_2^1)\|^{-2} \cdot \ddot{b}_K(p_1^0,p_2^1)^\top \mathbb{S}_K \Psi_K^{-2} \mathbb{S}_K^\top \ddot{b}_K(p_1^0,p_2^1) = O(1),$$

we have $\sum_{i=1}^n E\left[\xi_i^4\right] = O(\zeta_0^2(K)K/n) = o(1)$. Hence, the result (i) follows from Lyapunov's central limit theorem.

Proof of Result (ii). By the definition of the feasible estimator $\widehat{m}^{(1,0)}(x,p_1^0,p_2^1)$, we observe that

$$\begin{aligned}
&\widehat{m}^{(1,0)}(x,p_1^0,p_2^1) - m^{(1,0)}(x,p_1^0,p_2^1)\\
&= x_1^\top\left(\widehat{\beta}_{1n}^{(1,0)} - \beta_1^{(1,0)}\right) - \frac{1}{\widehat{h}(p_1^0,p_2^1)}\ddot{b}_K(p_1^0,p_2^1)^\top\left(\widehat{\alpha}_{1n}^{(1,0)} - \alpha_1^{(1,0)}\right) + \frac{1}{\widehat{h}(p_1^0,p_2^1)}\left(\frac{\partial^2 g_1^{(1,0)}(p_1^0,p_2^1)}{\partial p_1 \partial p_2} - \ddot{b}_K(p_1^0,p_2^1)^\top \alpha_1^{(1,0)}\right)\\
&\quad + \left(\frac{1}{h(p_1^0,p_2^1)} - \frac{1}{\widehat{h}(p_1^0,p_2^1)}\right)\frac{\partial^2 g_1^{(1,0)}(p_1,p_2)}{\partial p_1 \partial p_2}\\
&= -\frac{1}{\widehat{h}(p_1^0,p_2^1)}\ddot{b}_K(p_1^0,p_2^1)^\top\left(\widehat{\alpha}_{1n}^{(1,0)} - \alpha_1^{(1,0)}\right) + O_P(n^{-1/2}) + O_P(K^{(2-s)/2})\\
&= C_{1n} + C_{2n} + C_{3n} + C_{4n} + O_P(n^{-1/2}),
\end{aligned}$$

by Theorem 4.1(iii), Assumption 4.7, and (B.4), where

$$\begin{aligned}
C_{1n} &:= -\frac{1}{\widehat{h}(p_1^0,p_2^1)}\ddot{b}_K(p_1^0,p_2^1)^\top \mathbb{S}_K \widehat{\Psi}_{nK}^{-1}\widehat{\mathbf{R}}_K^\top\left(\mathbf{g}_1^{(1,0)} - \widehat{\mathbf{g}}_1^{(1,0)}\right)/n,\\
C_{2n} &:= -\frac{1}{\widehat{h}(p_1^0,p_2^1)}\ddot{b}_K(p_1^0,p_2^1)^\top \mathbb{S}_K \widehat{\Psi}_{nK}^{-1}\widehat{\mathbf{R}}_K^\top\left(\mathbf{X}_1^{(1,0)} - \widehat{\mathbf{X}}_1^{(1,0)}\right)\beta_1^{(1,0)}/n,\\
C_{3n} &:= -\frac{1}{\widehat{h}(p_1^0,p_2^1)}\ddot{b}_K(p_1^0,p_2^1)^\top \mathbb{S}_K \widehat{\Psi}_{nK}^{-1}\widehat{\mathbf{R}}_K^\top \widehat{\mathbf{u}}_1/n,\\
C_{4n} &:= -\frac{1}{\widehat{h}(p_1^0,p_2^1)}\ddot{b}_K(p_1^0,p_2^1)^\top \mathbb{S}_K \widehat{\Psi}_{nK}^{-1}\widehat{\mathbf{R}}_K^\top \mathbf{e}_1/n.
\end{aligned}$$

The fact that $\widehat{h}(p_1^0,p_2^1) > 0$ w.p.a.1 and Assumption 4.10 imply that

$$\begin{aligned}
|C_{1n}| &\leq O_P(1)\cdot O(K)\cdot \sup_{(p_1,p_2)\in[0,1]^2}\left|b_K(p_1,p_2)^\top \mathbb{S}_K \widehat{\Psi}_{nK}^{-1}\widehat{\mathbf{R}}_K^\top\left(\mathbf{g}_1^{(1,0)} - \widehat{\mathbf{g}}_1^{(1,0)}\right)/n\right|\\
&= \underbrace{O_P(K)\cdot O_P\left(\zeta_0(K)\zeta_1(K)/n + n^{-1/2}\right)}_{O_P(K/\sqrt{n})},
\end{aligned}$$

where the last equality follows from (B.3). Analogously, we can observe that $|C_{2n}| = O_P(K/\sqrt{n})$. In addition, the same argument as in (A.10) implies that $|C_{3n}| = \|\ddot{b}_K(p_1^0, p_2^1)\| \cdot o_P(n^{-1/2})$. Further,

$$\begin{aligned}
C_{4n} &= -\frac{1}{h(p_1^0, p_2^1)} \ddot{b}_K(p_1^0, p_2^1)^\top \mathbb{S}_K \Psi_{nK}^{-1} \mathbf{R}_K^\top \mathbf{e}_1 / n \\
&\quad - \frac{1}{\widehat{h}(p_1^0, p_2^1)} \left(\ddot{b}_K(p_1^0, p_2^1)^\top \mathbb{S}_K \widehat{\Psi}_{nK}^{-1} \widehat{\mathbf{R}}_K^\top \mathbf{e}_1 / n - \ddot{b}_K(p_1^0, p_2^1)^\top \mathbb{S}_K \Psi_{nK}^{-1} \mathbf{R}_K^\top \mathbf{e}_1 / n \right) \\
&\quad - \left(\frac{1}{\widehat{h}(p_1^0, p_2^1)} - \frac{1}{h(p_1^0, p_2^1)} \right) \ddot{b}_K(p_1^0, p_2^1)^\top \mathbb{S}_K \Psi_{nK}^{-1} \mathbf{R}_K^\top \mathbf{e}_1 / n \\
&= \underbrace{-\frac{1}{h(p_1^0, p_2^1)} \ddot{b}_K(p_1^0, p_2^1)^\top \mathbb{S}_K \Psi_{nK}^{-1} \mathbf{R}_K^\top \mathbf{e}_1 / n}_{= A_{2n}} + \underbrace{O_P(K \zeta_0(K) \zeta_1(K) / n)}_{O_P(K/\sqrt{n})} + \|\ddot{b}_K(p_1^0, p_2^1)\| \cdot O_P(n^{-1}),
\end{aligned}$$

by Lemma B.2(ii), (B.2), (B.4), and Markov's inequality.

Summarizing these results, we obtain

$$\begin{aligned}
\frac{\sqrt{n}\left(\widehat{m}^{(1,0)}(x, p_1^0, p_2^1) - m^{(1,0)}(x, p_1^0, p_2^1)\right)}{\sigma_K^{(1,0)}(p_1^0, p_2^1)} &= \frac{\sqrt{n}(C_{1n} + C_{2n} + C_{3n} + C_{4n})}{\sigma_K^{(1,0)}(p_1^0, p_2^1)} + o_P(1) \\
&= \frac{\sqrt{n} A_{2n}}{\sigma_K^{(1,0)}(p_1^0, p_2^1)} + o_P(1),
\end{aligned}$$

by (A.9). Then, the remaining part of the proof follows by the same argument as in (i). □

B Appendix: Technical Lemmas

Lemma B.1. Suppose that Assumptions 4.1, 4.2, 4.3(i), (ii), and 4.4(ii) hold. Then

$$\max_{j \in \{1,2\}} \max_{1 \le i \le n} |\widehat{P}_{ji}^0 - P_{ji}^0| = O_P(n^{-1/2}), \qquad \max_{j \in \{1,2\}} \max_{1 \le i \le n} |\widehat{P}_{ji}^1 - P_{ji}^1| = O_P(n^{-1/2}).$$

Proof. We prove the result for P_{ji}^0 only, as the proof for P_{ji}^1 is similar. For any i, the mean value expansion leads to

$$\widehat{P}_{ji}^0 = P_{ji}^0 + f_{\varepsilon_j}(W_{ji}^\top \bar{\gamma}_n) W_{ji}^\top (\widehat{\gamma}_{0n} - \gamma_0^*),$$

where f_{ε_j} is the marginal density function of ε_j and $\bar{\gamma}_n \in [\widehat{\gamma}_{0n}, \gamma_0^*]$. Thus, by Assumptions 4.3(i), (ii), and 4.4(ii), we have

$$\begin{aligned}
\max_{1 \le i \le n} |\widehat{P}_{ji}^0 - P_{ji}^0| &\le \left| f_{\varepsilon_j}(W_{ji}^\top \bar{\gamma}_n) \right| \cdot \|W_{ji}\| \cdot \|\widehat{\gamma}_{0n} - \gamma_0^*\| \\
&= O_P(n^{-1/2}),
\end{aligned}$$

for both $j = 1$ and 2. □

Lemma B.2. Suppose that Assumptions 4.1, 4.2, 4.3(i)–(iii), 4.4, 4.5(i), and 4.8 hold. Then, we have

$$\text{(i)} \quad \left\|\Psi_{nK}^{(1,0)} - \Psi_K^{(1,0)}\right\| = O_P(\zeta_0(K)\sqrt{(\log K)/n}) = o_P(1),$$

$$\text{(ii)} \quad \left\|\widehat{\Psi}_{nK}^{(1,0)} - \Psi_K^{(1,0)}\right\| = O_P(\zeta_0(K)\sqrt{(\log K)/n}) + O_P(\zeta_1(K)/\sqrt{n}) = o_P(1),$$

$$\text{(iii)} \quad \left\|\left[\Psi_{nK}^{(1,0)}\right]^{-1} - \left[\Psi_K^{(1,0)}\right]^{-1}\right\| = O_P(\zeta_0(K)\sqrt{(\log K)/n}) = o_P(1),$$

$$\text{(iv)} \quad \left\|\left[\widehat{\Psi}_{nK}^{(1,0)}\right]^{-1} - \left[\Psi_K^{(1,0)}\right]^{-1}\right\| = O_P(\zeta_0(K)\sqrt{(\log K)/n}) + O_P(\zeta_1(K)/\sqrt{n}) = o_P(1),$$

Proof. (i) See Lemma 2.1 in Chen and Christensen (2015) and Theorem 4.6 in Belloni *et al.* (2015).

(ii) The triangle inequality leads to

$$\|\widehat{\Psi}_{nK} - \Psi_K\| \leq \|\widehat{\Psi}_{nK} - \Psi_{nK}\| + \|\Psi_{nK} - \Psi_K\|.$$

The second term is $O_P(\zeta_0(K)\sqrt{(\log K)/n})$ by (i). For the first term, the triangle inequality implies

$$\left\|\widehat{\Psi}_{nK} - \Psi_{nK}\right\| \leq \left\|\left(\widehat{\mathbf{R}}_K^\top - \mathbf{R}_K^\top\right)\left(\widehat{\mathbf{R}}_K - \mathbf{R}_K\right)/n\right\| + 2\left\|\mathbf{R}_K^\top\left(\widehat{\mathbf{R}}_K - \mathbf{R}_K\right)/n\right\|.$$

Assumptions 4.3(i), (iii), and Lemma B.1 yield

$$\begin{aligned}\widehat{\mathcal{L}}_i^{(1,0)} X_{1i} - \mathcal{L}_i^{(1,0)} X_{1i} &= \left(\widehat{P}_{1i}^0 - P_{1i}^0 - H(\widehat{P}_{1i}^0, \widehat{P}_{2i}^1; \widehat{\rho}_n) + H(P_{1i}^0, P_{2i}^1; \rho^*)\right) X_{1i} \\ &= X_{1i} \cdot O_P(n^{-1/2}).\end{aligned} \tag{B.1}$$

Further, the mean value expansion and Lemma B.1 lead to

$$b_K(\widehat{P}_{1i}^0, \widehat{P}_{2i}^1) - b_K(P_{1i}^0, P_{2i}^1) = \left\{\frac{\partial b_K\left(\bar{P}_{1i}^0, \bar{P}_{2i}^1\right)}{\partial p_1} + \frac{\partial b_K\left(\bar{P}_{1i}^0, \bar{P}_{2i}^1\right)}{\partial p_2}\right\} \cdot O_P(n^{-1/2}),$$

where $\bar{P}_{1i}^0 \in [\widehat{P}_{1i}^0, P_{1i}^0]$ and $\bar{P}_{2i}^1 \in [\widehat{P}_{2i}^1, P_{2i}^1]$. Thus, by Assumption 4.4(ii) and the triangle inequality,

$$\begin{aligned}\left\|\left(\widehat{\mathbf{R}}_K - \mathbf{R}_K\right)/\sqrt{n}\right\| &\leq \left\|\left(\widehat{\mathbf{X}}_1^{(1,0)} - \mathbf{X}_1^{(1,0)}\right)/\sqrt{n}\right\| + \left\|\left(\widehat{\mathbf{b}}_K - \mathbf{b}_K\right)/\sqrt{n}\right\| \\ &\leq O_P(n^{-1/2}) \cdot \{\|\mathbf{X}_1/\sqrt{n}\| + \|\partial_1 \bar{\mathbf{b}}_K/\sqrt{n}\| + \|\partial_2 \bar{\mathbf{b}}_K/\sqrt{n}\|\} \\ &= O_P(\zeta_1(K)/\sqrt{n}),\end{aligned} \tag{B.2}$$

where $\mathbf{X}_1 := (X_{11}, \ldots X_{1n})^\top$ and $\partial_j \bar{\mathbf{b}}_K := (\partial b_K(\bar{P}_{11}^0, \bar{P}_{21}^1)/\partial p_j, \ldots, \partial b_K(\bar{P}_{1n}^0, \bar{P}_{2n}^1)/\partial p_j)^\top$ for $j = 1, 2$. Hence, the first term satisfies $\|(\widehat{\mathbf{R}}_K^\top - \mathbf{R}_K^\top)(\widehat{\mathbf{R}}_K - \mathbf{R}_K)/n\| \leq \|(\widehat{\mathbf{R}}_K - \mathbf{R}_K)/\sqrt{n}\|^2 = O_P(\zeta_1^2(K)/n)$.

For the second term, by (A.3), we obtain

$$\left\|\mathbf{R}_K^\top\left(\widehat{\mathbf{R}}_K - \mathbf{R}_K\right)/n\right\|^2 = \operatorname{tr}\left\{\left(\widehat{\mathbf{R}}_K^\top - \mathbf{R}_K^\top\right)\mathbf{R}_K\mathbf{R}_K^\top\left(\widehat{\mathbf{R}}_K - \mathbf{R}_K\right)/n^2\right\}$$

$$\leq [\bar{c}_\Psi + o_P(1)] \cdot \mathrm{tr}\left\{ \left(\widehat{\mathbf{R}}_K^\top - \mathbf{R}_K^\top \right) \left(\widehat{\mathbf{R}}_K - \mathbf{R}_K \right) / n \right\}$$
$$= [\bar{c}_\Psi + o_P(1)] \cdot \| (\widehat{\mathbf{R}}_K - \mathbf{R}_K)/\sqrt{n} \|^2 = O_P(\zeta_1^2(K)/n).$$

(iii) We first note that $\Psi_{nK}^{-1} - \Psi_K^{-1} = \Psi_K^{-1} \left(\Psi_K - \Psi_{nK} \right) \Psi_{nK}^{-1}$. Then, Assumption 4.5(i) and (A.3) imply the desired result:

$$\begin{aligned}
\left\| \Psi_{nK}^{-1} - \Psi_K^{-1} \right\|^2 &= \mathrm{tr}\left\{ \Psi_K^{-1} \left(\Psi_K - \Psi_{nK} \right) \Psi_{nK}^{-2} \left(\Psi_K - \Psi_{nK} \right) \Psi_K^{-1} \right\} \\
&\leq [\underline{c}_\Psi + o_P(1)]^{-2} \cdot \mathrm{tr}\left\{ \left(\Psi_K - \Psi_{nK} \right) \Psi_K^{-2} \left(\Psi_K - \Psi_{nK} \right) \right\} \\
&= O_P(1) \cdot \underbrace{\| \Psi_{nK} - \Psi_K \|^2}_{O_P(\zeta_0^2(K)(\log K)/n)} .
\end{aligned}$$

(iv) The proof is the same as in (iii) by noting (A.6). □

Lemma B.3. Suppose that Assumptions 4.1, 4.2, 4.3(i)–(iii), 4.4, 4.5(i), and 4.6–4.11 hold. Then, we have

(i) $$\sup_{(p_1,p_2)\in[0,1]^2} \left| \widetilde{g}_1^{(1,0)}(p_1,p_2) - g_1^{(1,0)}(p_1,p_2) \right| = O_P(\zeta_0(K)\sqrt{(\log n)/n}) + O_P(K^{-s/2}),$$

(ii) $$\sup_{(p_1,p_2)\in[0,1]^2} \left| \frac{\partial^2}{\partial p_1 \partial p_2} \left(\widetilde{g}_1^{(1,0)}(p_1,p_2) - g_1^{(1,0)}(p_1,p_2) \right) \right| = O_P(\zeta_0(K) K \sqrt{(\log n)/n}) + O_P(K^{(2-s)/2}),$$

(iii) $$\sup_{(p_1,p_2)\in[0,1]^2} \left| \widehat{g}_1^{(1,0)}(p_1,p_2) - g_1^{(1,0)}(p_1,p_2) \right| = O_P(\zeta_0(K)\sqrt{(\log n)/n}) + O_P(K^{-s/2}),$$

(iv) $$\sup_{(p_1,p_2)\in[0,1]^2} \left| \frac{\partial^2}{\partial p_1 \partial p_2} \left(\widehat{g}_1^{(1,0)}(p_1,p_2) - g_1^{(1,0)}(p_1,p_2) \right) \right| = O_P(\zeta_0(K) K \sqrt{(\log n)/n}) + O_P(K^{(2-s)/2}).$$

Proof. (i) By the triangle inequality and Assumptions 4.7 and 4.9(i), we observe

$$\begin{aligned}
&\sup_{(p_1,p_2)\in[0,1]^2} \left| \widetilde{g}_1^{(1,0)}(p_1,p_2) - g_1^{(1,0)}(p_1,p_2) \right| \\
&\leq \sup_{(p_1,p_2)\in[0,1]^2} \left| b_K(p_1,p_2)^\top (\widetilde{\alpha}_{1n}^{(1,0)} - \alpha_1^{(1,0)}) \right| + \sup_{(p_1,p_2)\in[0,1]^2} \left| g_1^{(1,0)}(p_1,p_2) - b_K(p_1,p_2)^\top \alpha_1^{(1,0)} \right| \\
&= \sup_{(p_1,p_2)\in[0,1]^2} \left| b_K(p_1,p_2)^\top (\widetilde{\alpha}_{1n}^{(1,0)} - \alpha_1^{(1,0)}) \right| + O(K^{-s/2}) \\
&\leq \sup_{(p_1,p_2)\in[0,1]^2} \left| b_K(p_1,p_2)^\top \mathbb{S}_K \Psi_{nK}^{-1} \mathbf{R}_K^\top \mathbf{e}_1 / n \right| + \sup_{(p_1,p_2)\in[0,1]^2} \left| \mathcal{P}_{nK}[u_1(p_1,p_2)] \right| + O(K^{-s/2}) \\
&\leq \sup_{(p_1,p_2)\in[0,1]^2} \left| b_K(p_1,p_2)^\top \mathbb{S}_K \Psi_{nK}^{-1} \mathbf{R}_K^\top \mathbf{e}_1 / n \right| + \| \mathcal{P}_{nK} \|_\infty \cdot \sup_{(p_1,p_2)\in[0,1]^2} |u_1(p_1,p_2)| + O(K^{-s/2}) \\
&= \sup_{(p_1,p_2)\in[0,1]^2} \left| b_K(p_1,p_2)^\top \mathbb{S}_K \Psi_{nK}^{-1} \mathbf{R}_K^\top \mathbf{e}_1 / n \right| + O_P(K^{-s/2}),
\end{aligned}$$

where $u_1(p_1,p_2) := g_1^{(1,0)}(p_1,p_2) - b_K(p_1,p_2)^\top \alpha_1^{(1,0)}$.

Following the proof of Lemma 3.1(ii) of Chen and Christensen (2018), we will show that the first term on the right-hand side is of order $O_P(\zeta_0(K)\sqrt{(\log n)/n})$. First, we partition the interval $[0,1]$ into countably many sub-intervals of equal length, and let the set of the partitioning points (including 0 and 1) be $\mathcal{T}_n$. It should be noted that we can construct the partition such that for any $(p_1,p_2) \in [0,1]^2$ there exists a point $(t_{p_1}, t_{p_2}) \in \mathcal{T}_n^2$ satisfying

$$\|(p_1,p_2) - (t_{p_1}, t_{p_2})\| \le c_p \zeta_0(K) K^{-(\omega+1/2)},$$

for some positive constant $c_p > 0$, where ω is as in Assumption 4.11(i). Then, by Assumption 4.11(i), we have

$$\begin{aligned}
&\sup_{(p_1,p_2)\in[0,1]^2} \left| b_K(p_1,p_2)^\top \mathbb{S}_K \Psi_{nK}^{-1} \mathbf{R}_K^\top \mathbf{e}_1/n \right| \\
&\le \max_{(t_1,t_2)\in\mathcal{T}_n^2} \left| b_K(t_1,t_2)^\top \mathbb{S}_K \Psi_{nK}^{-1} \mathbf{R}_K^\top \mathbf{e}_1/n \right| + \sup_{(p_1,p_2)\in[0,1]^2} \left| \{b_K(p_1,p_2) - b_K(t_{p_1},t_{p_2})\}^\top \mathbb{S}_K \Psi_{nK}^{-1} \mathbf{R}_K^\top \mathbf{e}_1/n \right| \\
&\le \max_{(t_1,t_2)\in\mathcal{T}_n^2} \left| b_K(t_1,t_2)^\top \mathbb{S}_K \Psi_{nK}^{-1} \mathbf{R}_K^\top \mathbf{e}_1/n \right| + O(\zeta_0(K) K^{-1/2}) \cdot \left\| \mathbb{S}_K \Psi_{nK}^{-1} \mathbf{R}_K^\top \mathbf{e}_1/n \right\| \\
&= \max_{(t_1,t_2)\in\mathcal{T}_n^2} \left| b_K(t_1,t_2)^\top \mathbb{S}_K \Psi_{nK}^{-1} \mathbf{R}_K^\top \mathbf{e}_1/n \right| + \underbrace{O_P(\zeta_0(K)/\sqrt{n})}_{o_P(\zeta_0(K)\sqrt{\log n/n})},
\end{aligned}$$

where the last equality follows from (A.5).

To examine the first term in the above equation, we decompose $e_{1i} = e_{11i} + e_{12i}$, where

$$\begin{aligned}
e_{11i} &:= e_{1i}\mathbf{1}\{|e_{1i}| \le M_n\} - E[e_{1i}\mathbf{1}\{|e_{1i}| \le M_n\}|W_i], \\
e_{12i} &:= e_{1i}\mathbf{1}\{|e_{1i}| > M_n\} - E[e_{1i}\mathbf{1}\{|e_{1i}| > M_n\}|W_i],
\end{aligned}$$

and M_n is a sequence of positive numbers diverging to ∞. Let $\mathbf{e}_{11} = (e_{111}, \ldots, e_{11n})^\top$ and $\mathbf{e}_{12} = (e_{121}, \ldots, e_{12n})^\top$. Then, we observe

$$\begin{aligned}
&b_K(t_1,t_2)^\top \mathbb{S}_K \Psi_{nK}^{-1} \mathbf{R}_K^\top \mathbf{e}_1/n \\
&= b_K(t_1,t_2)^\top \mathbb{S}_K \Psi_K^{-1} \mathbf{R}_K^\top \mathbf{e}_1/n + b_K(t_1,t_2)^\top \mathbb{S}_K \left(\Psi_{nK}^{-1} - \Psi_K^{-1}\right) \mathbf{R}_K^\top \mathbf{e}_1/n \\
&= b_K(t_1,t_2)^\top \mathbb{S}_K \Psi_K^{-1} \mathbf{R}_K^\top \mathbf{e}_{11}/n + b_K(t_1,t_2)^\top \mathbb{S}_K \Psi_K^{-1} \mathbf{R}_K^\top \mathbf{e}_{12}/n + b_K(t_1,t_2)^\top \mathbb{S}_K \left(\Psi_{nK}^{-1} - \Psi_K^{-1}\right) \mathbf{R}_K^\top \mathbf{e}_1/n \\
&= b_K(t_1,t_2)^\top \mathbb{S}_K \Psi_K^{-1} \mathbf{R}_K^\top \mathbf{e}_{11}/n + b_K(t_1,t_2)^\top \mathbb{S}_K \Psi_K^{-1} \mathbf{R}_K^\top \mathbf{e}_{12}/n + \underbrace{O_P(\zeta_0^2(K)\sqrt{(\log K)}/n)}_{o_P(\zeta_0(K)\sqrt{(\log n)/n})},
\end{aligned}$$

where the last equality follows from Lemma B.2(iii) and Markov's inequality.

Let $q_i(t_1,t_2) := b_K(t_1,t_2)^\top \mathbb{S}_K \Psi_K^{-1} R_K(P_{1i}^0, P_{2i}^1)$, so that

$$\frac{1}{n}\sum_{i=1}^n q_i(t_1,t_2) e_{11i} = b_K(t_1,t_2)^\top \mathbb{S}_K \Psi_K^{-1} \mathbf{R}_K^\top \mathbf{e}_{11}/n.$$

Note that $E[q_i(t_1,t_2)e_{11i}] = 0$. Furthermore, it is straightforward by the Cauchy–Schwarz inequality that there exist positive constants $c_1, c_2 > 0$ such that

$$|q_i(t_1,t_2)| \le \|b_K(t_1,t_2)\| \cdot \left\|\Psi_K^{-1} R_K(P_{1i}^0, P_{2i}^1)\right\| \le c_1 \zeta_0^2(K),$$

and that $E[q_i^2(t_1,t_2)] \le c_2 \zeta_0^2(K)$. Therefore, for all $(t_1,t_2) \in \mathcal{T}_n^2$, we have $|q_i(t_1,t_2)e_{11i}| \le c_1' \zeta_0^2(K) M_n$ and $E[q_i^2(t_1,t_2)e_{11i}^2] = E\left[q_i^2(t_1,t_2)E[e_{11i}^2|W_i]\right] \le c_2'\zeta_0^2(K)$ for some $c_1', c_2' > 0$ by Assumption 4.6. By Bernstein's inequality with any non-negative $\varrho_n \ge 0$, we now have

$$\begin{aligned}\Pr\left(\max_{(t_1,t_2)\in\mathcal{T}_n^2}\left|b_K(t_1,t_2)^\top \mathbb{S}_K \Psi_K^{-1}\mathbf{R}_K^\top \mathbf{e}_{11}/n\right| > \varrho_n\right) &\le |\mathcal{T}_n^2| \max_{(t_1,t_2)\in\mathcal{T}_n^2} \Pr\left(\left|\frac{1}{n}\sum_{i=1}^n q_i(t_1,t_2)e_{11i}\right| > \varrho_n\right)\\ &\le 2\exp\left\{\log|\mathcal{T}_n^2| - \frac{1}{4}\frac{\varrho_n^2}{c_2'\zeta_0^2(K)/n + c_1'\zeta_0^2(K)M_n\varrho_n/n}\right\}\\ &\le 2\exp\left\{\log|\mathcal{T}_n^2| - \frac{\varrho_n^2}{c_3(\zeta_0^2(K)/n)[1+M_n\varrho_n]}\right\},\end{aligned}$$

for some positive constant $c_3 > 0$, where $|\mathcal{T}_n^2|$ denotes the cardinality of the set $\mathcal{T}_n^2$. Then, setting $\varrho_n = C\zeta_0(K)\sqrt{(\log n)/n}$ for a large constant $C > 0$, provided that $|\mathcal{T}_n^2|$ and M_n grow sufficiently slowly so that $M_n\varrho_n = o(1)$, we have

$$\log|\mathcal{T}_n^2| - \frac{\varrho_n^2}{c_3(\zeta_0^2(K)/n)[1+M_n\varrho_n]} = \log|\mathcal{T}_n^2| - \frac{C^2\zeta_0^2(K)(\log n)/n}{c_3(\zeta_0^2(K)/n)[1+o(1)]} \asymp \log\left(\frac{|\mathcal{T}_n^2|}{n^{C^2}}\right) \to -\infty,$$

as $C \to \infty$, implying that $\max_{(t_1,t_2)\in\mathcal{T}_n^2}\left|b_K(t_1,t_2)^\top \mathbb{S}_K \Psi_K^{-1}\mathbf{R}_K^\top \mathbf{e}_{11}/n\right| = O_P(\zeta_0(K)\sqrt{(\log n)/n})$.

Next, by Markov's inequality and Assumption 4.6, it holds that

$$\begin{aligned}\Pr\left(\max_{(t_1,t_2)\in\mathcal{T}_n^2}\left|b_K(t_1,t_2)^\top \mathbb{S}_K \Psi_K^{-1}\mathbf{R}_K^\top \mathbf{e}_{12}/n\right| > \varrho_n\right) &\le \Pr\left(\zeta_0(K)\left\|\Psi_K^{-1}\mathbf{R}_K^\top \mathbf{e}_{12}/n\right\| > \varrho_n\right)\\ &\le \Pr\left(c_4\zeta_0^2(K)\left|\frac{1}{n}\sum_{i=1}^n e_{12i}\right| > \varrho_n\right)\\ &\le \frac{2c_4\zeta_0^2(K)}{\varrho_n}E[|e_{1i}|\mathbf{1}\{|e_{1i}| > M_n\}]\\ &\le \frac{2c_4\zeta_0^2(K)}{\varrho_n M_n^3}E[e_{1i}^4\mathbf{1}\{|e_{1i}| > M_n\}] = O\left(\frac{\zeta_0^2(K)}{\varrho_n M_n^3}\right).\end{aligned}$$

Again, setting $\varrho_n = C\zeta_0(K)\sqrt{(\log n)/n}$ for a large constant $C > 0$, if $\zeta_0(K)/\sqrt{(\log n)/n} = O(M_n^3)$, we have

$$\frac{\zeta_0^2(K)}{\varrho_n M_n^3} = \frac{1}{C}\frac{\zeta_0(K)}{\sqrt{(\log n)/n}M_n^3} \to 0,$$

as $C \to \infty$, which implies $\max_{(t_1,t_2)\in\mathcal{T}_n^2}\left|b_K(t_1,t_2)^\top \mathbb{S}_K \Psi_K^{-1}\mathbf{R}_K^\top \mathbf{e}_{12}/n\right| = O_P(\zeta_0(K)\sqrt{(\log n)/n})$. It should be noted that $\zeta_0(K)/\sqrt{(\log n)/n} = O(M_n^3)$ is not inconsistent with the requirement $M_n\zeta_0(K)\sqrt{(\log n)/n} = o(1)$ under Assumption 4.11(ii). By combining these results, the proof is completed.

(ii) By the triangle inequality and Assumption 4.7, we observe

$$
\begin{aligned}
&\sup_{(p_1,p_2)\in[0,1]^2}\left|\frac{\partial^2}{\partial p_1\partial p_2}\left(\widetilde{g}_1^{(1,0)}(p_1,p_2)-g_1^{(1,0)}(p_1,p_2)\right)\right| \\
&\le \sup_{(p_1,p_2)\in[0,1]^2}\left|\frac{\partial^2}{\partial p_1\partial p_2}\left(b_K(p_1,p_2)^\top(\widetilde{\alpha}_{1n}^{(1,0)}-\alpha_1^{(1,0)})\right)\right| + \sup_{(p_1,p_2)\in[0,1]^2}\left|\frac{\partial^2}{\partial p_1\partial p_2}\left(g_1^{(1,0)}(p_1,p_2)-b_K(p_1,p_2)^\top\alpha_1^{(1,0)}\right)\right| \\
&\le \sup_{(p_1,p_2)\in[0,1]^2}\left|\frac{\partial^2}{\partial p_1\partial p_2}\left(b_K(p_1,p_2)^\top(\widetilde{\alpha}_{1n}^{(1,0)}-\alpha_1^{(1,0)})\right)\right| + O(K^{(2-s)/2}).
\end{aligned}
$$

Further, by Assumption 4.10, we have

$$
\begin{aligned}
\sup_{(p_1,p_2)\in[0,1]^2}\left|\frac{\partial^2}{\partial p_1\partial p_2}\left(b_K(p_1,p_2)^\top(\widetilde{\alpha}_{1n}^{(1,0)}-\alpha_1^{(1,0)})\right)\right| &= O(K)\sup_{(p_1,p_2)\in[0,1]^2}\left|b_K(p_1,p_2)^\top(\widetilde{\alpha}_{1n}^{(1,0)}-\alpha_1^{(1,0)})\right| \\
&= O_P(\zeta_0(K)K\sqrt{(\log n)/n}) + O_P(K^{(2-s)/2}),
\end{aligned}
$$

where the second equality follows from (i). This completes the proof.

(iii) By the decomposition for $\widehat{\delta}_{1n}^{(1,0)}$ in (A.2), the triangle inequality and Assumptions 4.7 and 4.9(ii) imply

$$
\begin{aligned}
&\sup_{(p_1,p_2)\in[0,1]^2}\left|\widehat{g}_1^{(1,0)}(p_1,p_2)-g_1^{(1,0)}(p_1,p_2)\right| \\
&\le \sup_{(p_1,p_2)\in[0,1]^2}\left|b_K(p_1,p_2)^\top(\widehat{\alpha}_{1n}^{(1,0)}-\alpha_1^{(1,0)})\right| + \sup_{(p_1,p_2)\in[0,1]^2}\left|g_1^{(1,0)}(p_1,p_2)-b_K(p_1,p_2)^\top\alpha_1^{(1,0)}\right| \\
&\le \sup_{(p_1,p_2)\in[0,1]^2}\left|b_K(p_1,p_2)^\top\mathbb{S}_K\widehat{\Psi}_{nK}^{-1}\widehat{\mathbf{R}}_K^\top\left(\mathbf{g}_1^{(1,0)}-\widehat{\mathbf{g}}_1^{(1,0)}\right)/n\right| \\
&\quad + \sup_{(p_1,p_2)\in[0,1]^2}\left|b_K(p_1,p_2)^\top\mathbb{S}_K\widehat{\Psi}_{nK}^{-1}\widehat{\mathbf{R}}_K^\top\left(\mathbf{X}_1^{(1,0)}-\widehat{\mathbf{X}}_1^{(1,0)}\right)\beta_1^{(1,0)}/n\right| \\
&\quad + \sup_{(p_1,p_2)\in[0,1]^2}\left|b_K(p_1,p_2)^\top\mathbb{S}_K\widehat{\Psi}_{nK}^{-1}\widehat{\mathbf{R}}_K^\top\mathbf{e}_1/n\right| + \sup_{(p_1,p_2)\in[0,1]^2}\left|\widehat{\mathcal{P}}_{nK}[u_1(p_1,p_2)]\right| + O(K^{-s/2}) \\
&\le \sup_{(p_1,p_2)\in[0,1]^2}\left|b_K(p_1,p_2)^\top\mathbb{S}_K\widehat{\Psi}_{nK}^{-1}\widehat{\mathbf{R}}_K^\top\left(\mathbf{g}_1^{(1,0)}-\widehat{\mathbf{g}}_1^{(1,0)}\right)/n\right| \\
&\quad + \sup_{(p_1,p_2)\in[0,1]^2}\left|b_K(p_1,p_2)^\top\mathbb{S}_K\widehat{\Psi}_{nK}^{-1}\widehat{\mathbf{R}}_K^\top\left(\mathbf{X}_1^{(1,0)}-\widehat{\mathbf{X}}_1^{(1,0)}\right)\beta_1^{(1,0)}/n\right| \\
&\quad + \sup_{(p_1,p_2)\in[0,1]^2}\left|b_K(p_1,p_2)^\top\mathbb{S}_K\widehat{\Psi}_{nK}^{-1}\widehat{\mathbf{R}}_K^\top\mathbf{e}_1/n\right| + O_P(K^{-s/2}).
\end{aligned}
$$

By the triangle inequality,

$$
\begin{aligned}
\left|b_K(p_1,p_2)^\top\mathbb{S}_K\widehat{\Psi}_{nK}^{-1}\widehat{\mathbf{R}}_K^\top\left(\mathbf{g}_1^{(1,0)}-\widehat{\mathbf{g}}_1^{(1,0)}\right)/n\right| &\le \left|b_K(p_1,p_2)^\top\mathbb{S}_K\left\{\widehat{\Psi}_{nK}^{-1}-\Psi_{nK}^{-1}\right\}\widehat{\mathbf{R}}_K^\top\left(\mathbf{g}_1^{(1,0)}-\widehat{\mathbf{g}}_1^{(1,0)}\right)/n\right| \\
&\quad + \left|b_K(p_1,p_2)^\top\mathbb{S}_K\Psi_{nK}^{-1}\left(\widehat{\mathbf{R}}_K-\mathbf{R}_K\right)^\top\left(\mathbf{g}_1^{(1,0)}-\widehat{\mathbf{g}}_1^{(1,0)}\right)/n\right| \\
&\quad + \left|b_K(p_1,p_2)^\top\mathbb{S}_K\Psi_{nK}^{-1}\mathbf{R}_K^\top\left(\mathbf{g}_1^{(1,0)}-\widehat{\mathbf{g}}_1^{(1,0)}\right)/n\right|.
\end{aligned}
$$

Similarly to (A.8), we can easily see that

$$\begin{aligned}\left|b_K(p_1,p_2)^\top \mathbb{S}_K \left\{\widehat{\Psi}_{nK}^{-1} - \Psi_{nK}^{-1}\right\} \widehat{\mathbf{R}}_K^\top \left(\mathbf{g}_1^{(1,0)} - \widehat{\mathbf{g}}_1^{(1,0)}\right)/n\right| &\overset{p}{\precsim} O_P(n^{-1/2}) \cdot ||b_K(p_1,p_2)|| \cdot \left\|\widehat{\Psi}_{nK}^{-1} - \Psi_{nK}^{-1}\right\| \\ &= \underbrace{O_P(\zeta_0(K)\zeta_1(K)/n)}_{o_P(\zeta_0(K)\sqrt{(\log n)/n})}, \\ \left|b_K(p_1,p_2)^\top \mathbb{S}_K \Psi_{nK}^{-1}\left(\widehat{\mathbf{R}}_K - \mathbf{R}_K\right)^\top \left(\mathbf{g}_1^{(1,0)} - \widehat{\mathbf{g}}_1^{(1,0)}\right)/n\right| &\overset{p}{\precsim} O_P(n^{-1/2}) \cdot ||b_K(p_1,p_2)|| \cdot \left\|\left(\widehat{\mathbf{R}}_K - \mathbf{R}_K\right)/\sqrt{n}\right\| \\ &= \underbrace{O_P(\zeta_0(K)\zeta_1(K)/n)}_{o_P(\zeta_0(K)\sqrt{(\log n)/n})},\end{aligned}$$

uniformly in $(p_1,p_2) \in [0,1]^2$, where the equalities follow from (B.2).

Here, for arbitrary constants C_1 and C_2, let

$$\breve{g}_1^{(1,0)}(P_1^0, P_2^1) \coloneqq g_1^{(1,0)}(P_1^0, P_2^1) - g_1^{(1,0)}(P_1^0 + C_1 n^{-1/2}, P_2^1 + C_2 n^{-1/2}),$$

and $\breve{\mathbf{g}}_1^{(1,0)} \coloneqq (\breve{g}_1^{(1,0)}(P_{11}^0, P_{21}^1), \ldots, \breve{g}_1^{(1,0)}(P_{1n}^0, P_{2n}^1))^\top$. Then, we obtain

$$\begin{aligned}\sup_{(p_1,p_2)\in[0,1]^2}\left|b_K(p_1,p_2)^\top \mathbb{S}_K \Psi_{nK}^{-1}\mathbf{R}_K^\top \left(\mathbf{g}_1^{(1,0)} - \widehat{\mathbf{g}}_1^{(1,0)}\right)/n\right| &\overset{p}{\precsim} \sup_{(p_1,p_2)\in[0,1]^2}\left|b_K(p_1,p_2)^\top \mathbb{S}_K \Psi_{nK}^{-1}\mathbf{R}_K^\top \breve{\mathbf{g}}_1^{(1,0)}/n\right| \\ &= \sup_{(p_1,p_2)\in[0,1]^2} |\mathcal{P}_{nk}[\breve{g}_1^{(1,0)}(p_1,p_2)]| \\ &\le \|\mathcal{P}_{nk}\|_\infty \cdot O(n^{-1/2}) = O_P(n^{-1/2}),\end{aligned}$$

under Assumption 4.9(i). Summarizing these results, we obtain

$$\sup_{(p_1,p_2)\in[0,1]^2}\left|b_K(p_1,p_2)^\top \mathbb{S}_K \widehat{\Psi}_{nK}^{-1}\widehat{\mathbf{R}}_K^\top \left(\mathbf{g}_1^{(1,0)} - \widehat{\mathbf{g}}_1^{(1,0)}\right)/n\right| = \underbrace{O_P(\zeta_0(K)\zeta_1(K)/n + n^{-1/2})}_{o_P(\zeta_0(K)\sqrt{(\log n)/n})}. \tag{B.3}$$

By applying the same argument as above, it can be verified that

$$\left|b_K(p_1,p_2)^\top \mathbb{S}_K \widehat{\Psi}_{nK}^{-1}\widehat{\mathbf{R}}_K^\top \left(\mathbf{X}_1^{(1,0)} - \hat{\mathbf{X}}_1^{(1,0)}\right)\beta_1^{(1,0)}/n\right| = o_P(\zeta_0(K)\sqrt{(\log n)/n}),$$

and that

$$\left|b_K(p_1,p_2)^\top \mathbb{S}_K \widehat{\Psi}_{nK}^{-1}\widehat{\mathbf{R}}_K^\top \mathbf{e}_1/n\right| \le \left|b_K(p_1,p_2)^\top \mathbb{S}_K \Psi_{nK}^{-1}\mathbf{R}_K^\top \mathbf{e}_1/n\right| + o_P(\zeta_0(K)\sqrt{(\log n)/n}),$$

uniformly in $(p_1,p_2) \in [0,1]^2$. Thus, we obtain

$$\begin{aligned}\sup_{(p_1,p_2)\in[0,1]^2}\left|\widehat{g}_1^{(1,0)}(p_1,p_2) - g_1^{(1,0)}(p_1,p_2)\right| &\le \sup_{(p_1,p_2)\in[0,1]^2}\left|b_K(p_1,p_2)^\top \mathbb{S}_K \Psi_{nK}^{-1}\mathbf{R}_K^\top \mathbf{e}_1/n\right| \\ &\quad + o_P(\zeta_0(K)\sqrt{(\log n)/n}) + O_P(K^{-s/2}).\end{aligned}$$

Finally, the result follows from the fact that the first term on the right-hand side is of order $O_P(\zeta_0(K)\sqrt{\log n/n})$, as shown in the proof of (i).

(iv) The proof of (iv) is analogous to that of (ii). □

Lemma B.4. Suppose that Assumptions 4.1–4.4, 4.5(i) and 4.6–4.11 hold. Then, we have

$$\text{(i)}\quad \sup_{(p_1,p_2)\in[0,1]^2}\left|\widetilde{E}_n[U_1^{(1,0)}|V_1=p_1,V_2=p_2]-E[U_1^{(1,0)}|V_1=p_1,V_2=p_2]\right| = O_P(\zeta_0(K)K\sqrt{\log n/n})+O_P(K^{(2-s)/2}),$$

$$\text{(ii)}\quad \sup_{(p_1,p_2)\in[0,1]^2}\left|\widehat{E}_n[U_1^{(1,0)}|V_1=p_1,V_2=p_2]-E[U_1^{(1,0)}|V_1=p_1,V_2=p_2]\right| = O_P(\zeta_0(K)K\sqrt{\log n/n})+O_P(K^{(2-s)/2}).$$

Proof. (i) The proof of (i) is immediate from Lemma B.3(ii).

(ii) Assumptions 4.3(i) and (iv) imply that $\widehat{h}(p_1,p_2)$ is uniformly consistent for $h(p_1,p_2)$ and that $\widehat{h}(p_1,p_2)$ is uniformly bounded away from zero w.p.a.1. Thus, uniformly in $(p_1,p_2)\in[0,1]^2$, we have

$$\begin{aligned}\left|\frac{1}{h(p_1,p_2)}-\frac{1}{\widehat{h}(p_1,p_2)}\right| &= \left|\frac{\widehat{h}(p_1,p_2)-h(p_1,p_2)}{h(p_1,p_2)\cdot\widehat{h}(p_1,p_2)}\right| \\ &\le O_P(1)\cdot\left|\widehat{h}(p_1,p_2)-h(p_1,p_2)\right| = O_P(1)\cdot|\widehat{\rho}_n-\rho^*| = O_P(n^{-1/2}).\end{aligned} \tag{B.4}$$

By the triangle and Cauchy–Schwarz inequalities, (B.4), and Lemma B.3(iv), it holds that

$$\begin{aligned}&\left|\widehat{E}_n[U_1^{(1,0)}|V_1=p_1,V_2=p_2]-E[U_1^{(1,0)}|V_1=p_1,V_2=p_2]\right| \\ &\le \left|\frac{1}{\widehat{h}(p_1,p_2)}\frac{\partial^2}{\partial p_1\partial p_2}\left(g_1^{(1,0)}(p_1,p_2)-\widehat{g}_1^{(1,0)}(p_1,p_2)\right)\right| + \left|\left(\frac{1}{h(p_1,p_2)}-\frac{1}{\widehat{h}(p_1,p_2)}\right)\frac{\partial^2 g_1^{(1,0)}(p_1,p_2)}{\partial p_1\partial p_2}\right| \\ &= O_P(\zeta_0(K)K\sqrt{\log n/n})+O_P(K^{(2-s)/2})+O_P(n^{-1/2}),\end{aligned}$$

uniformly in $(p_1,p_2)\in[0,1]^2$. □

C Appendix: Verification of Assumptions for Asymptotic Results

Verification of $\zeta_1(K)=O(K)$ for tensor-product B-splines. We consider univariate B-splines of order r with quasi-uniform k internal knots, i.e., $b_r(p):=(b_{r,1}(p),\dots b_{r,k+r}(p))^\top$ for $p\in[0,1]$, where the length of each knot interval is proportional to $1/k$. Note that, in the notation of the main text, $b_K(p_1,p_2)=b_r(p_1)\otimes b_r(p_2)$ such that $k^2\asymp K$. As is well known, the derivatives of B-spline functions can be simply expressed in terms of lower order B-spline functions. Specifically, the first derivative of $b_r(p)$ can be written as (see, e.g., Zhou and Wolfe, 2000)

$$\frac{\partial b_r(p)}{\partial p} = (r-1)\Delta_r b_{r-1}(p),$$

where

$$\Delta_r := \underbrace{\begin{pmatrix} \frac{-1}{t_1 - t_{2-r}} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{t_1 - t_{2-r}} & \frac{-1}{t_2 - t_{3-r}} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{t_2 - t_{3-r}} & \frac{-1}{t_3 - t_{4-r}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{t_{k+r-1} - t_k} \end{pmatrix}}_{(k+r)\times(k+r-1)}.$$

Then, since $\chi_{\max}(\Delta_r^\top \Delta_r) = O(k^2)$, we have

$$\begin{aligned} \left\| \frac{\partial b_r(p)}{\partial p} \right\|^2 &= (r-1)^2 \cdot b_{r-1}^\top(p) \Delta_r^\top \Delta_r b_{r-1}(p) \\ &\le O(k^2) \underbrace{\|b_{r-1}(p)\|^2}_{O(k)} = O(k^3). \end{aligned}$$

uniformly in p. Hence,

$$\begin{aligned} \left\| \frac{\partial b_K(p_1, p_2)}{\partial p_1} \right\|^2 = \left\| \frac{\partial b_r(p_1)}{\partial p_1} \otimes b_r(p_2) \right\|^2 &= \left\| \frac{\partial b_r(p_1)}{\partial p_1} \right\|^2 \cdot \|b_r(p_2)\|^2 \\ &\le O(k^4) = O(K^2), \end{aligned}$$

uniformly in $(p_1, p_2) \in [0,1]^2$. This implies the desired result.

Verification of Assumption 4.9(ii). We assume that (A.6) holds. Letting $\widehat{\mathbf{g}} := (g(\widehat{P}_{11}^0, \widehat{P}_{21}^1), \ldots, g(\widehat{P}_{1n}^0, \widehat{P}_{2n}^1))^\top$ uniformly in $(p_1, p_2) \in [0,1]^2$, we can observe that

$$\begin{aligned} \left| \widehat{\mathcal{P}}_{nK}[g(p_1, p_2)] \right|^2 &= n^{-2} \cdot \mathrm{tr}\left\{ \widehat{\Psi}_{nK}^{-1} \widehat{\mathbf{R}}_K^\top \widehat{\mathbf{g}} \widehat{\mathbf{g}}^\top \widehat{\mathbf{R}}_K \widehat{\Psi}_{nK}^{-1} \mathbb{S}_K^\top b_K(p_1, p_2) b_K(p_1, p_2)^\top \mathbb{S}_K \right\} \\ &\le O_P(n^{-2}) \cdot \mathrm{tr}\left\{ \widehat{\mathbf{R}}_K^\top \widehat{\mathbf{g}} \widehat{\mathbf{g}}^\top \widehat{\mathbf{R}}_K \widehat{\Psi}_{nK}^{-1} \mathbb{S}_K^\top b_K(p_1, p_2) b_K(p_1, p_2)^\top \mathbb{S}_K \right\} \\ &\le O_P(n^{-2}) \cdot \mathrm{tr}\left\{ \mathbb{S}_K^\top b_K(p_1, p_2) b_K(p_1, p_2)^\top \mathbb{S}_K \widehat{\mathbf{R}}_K^\top \widehat{\mathbf{g}} \widehat{\mathbf{g}}^\top \widehat{\mathbf{R}}_K \right\} \\ &= O_P(1) \cdot \left| \frac{1}{n} \sum_{i=1}^n b_K(p_1, p_2)^\top \mathbb{S}_K R_K(\widehat{P}_{1i}^0, \widehat{P}_{2i}^1) g(\widehat{P}_{1i}^0, \widehat{P}_{2i}^1) \right|^2. \end{aligned}$$

Here, we consider basis functions with equally-spaced local polynomial structure, such as B-splines and the partitioning polynomial series in Cattaneo and Farrell (2013). Further, we assume that (P_1^0, P_2^1) is quasi-uniformly distributed on $[0,1]^2$ so that $\Pr(b_K(p_1, p_2)^\top b_K(P_1^0, P_2^1) \neq 0) \asymp O(1/K)$ holds uniformly in

$(p_1, p_2) \in [0,1]^2$, and let $\widehat{\mathbf{1}}_i(p_1,p_2) = \mathbf{1}(b_K(p_1,p_2)^\top b_K(\widehat{P}^0_{1i}, \widehat{P}^1_{2i}) \neq 0)$. In view of Lemma B.1, we can expect that $\frac{1}{n}\sum_{i=1}^n \widehat{\mathbf{1}}_i(p_1,p_2) = O_P(1/K)$. Then, the above expression yields

$$
\begin{aligned}
\left|\widehat{\mathcal{P}}_{nK}[g(p_1,p_2)]\right| &= O_P(1)\cdot\left|\frac{1}{n}\sum_{i=1}^n b_K(p_1,p_2)^\top \mathbb{S}_K R_K(\widehat{P}^0_{1i}, \widehat{P}^1_{2i}) g(\widehat{P}^0_{1i}, \widehat{P}^1_{2i})\right| \\
&\leq O_P(1)\cdot\frac{1}{n}\sum_{i=1}^n \widehat{\mathbf{1}}_i(p_1,p_2)\underbrace{\left|b_K(p_1,p_2)^\top \mathbb{S}_K R_K(\widehat{P}^0_{1i}, \widehat{P}^1_{2i})\right|}_{\zeta_0^2(K)=O(K)}\sup_{(p_1,p_2)\in[0,1]^2}|g(p_1,p_2)| \\
&\overset{p}{\precsim} \sup_{(p_1,p_2)\in[0,1]^2}|g(p_1,p_2)|.
\end{aligned}
$$

Then, $\|\widehat{\mathcal{P}}_{nK}\|_\infty = O_P(1)$ follows from the definition of the sup-operator norm used here.

D Appendix: Identification of the Treatment Decision Game

We examine the global identification of the parameter in the parametric treatment decision game developed in Section 4. In Bjorn and Vuong (1984), identification was established in a similar game model by examining the Fisher information matrix. Our analysis below complements their result by using a different proof strategy.

As shown in (4.1), we have the following system of equations for the probabilities $\mathcal{L}^{(d_1,d_2)}(w) := \Pr(D = (d_1,d_2)|W=w)$ for given $w = (w_1, w_2)$, which are identifiable from observed data:

$$
\begin{cases}
\mathcal{L}^{(1,0)}(w) = p_1^0 - H(p_1^0, p_2^1;\rho) \\
\mathcal{L}^{(0,1)}(w) = p_2^0 - H(p_1^1, p_2^0;\rho) \\
\mathcal{L}^{(1,1)}(w) = H(p_1^1, p_2^1;\rho) - \lambda\cdot\mathcal{L}_{mul}(\mathbf{p};\rho)
\end{cases},
$$

where $p_j^0 = F_{\varepsilon_j}(w_j^\top\gamma_0)$, $p_j^1 = F_{\varepsilon_j}(w_j^\top\gamma_0 + \eta(w_j^\top\gamma_1))$, $\mathbf{p} = (p_1^0, p_1^1, p_2^0, p_2^1)$, and $\mathcal{L}_{mul}(\mathbf{p};\rho) = H(p_1^1,p_2^1;\rho) - H(p_1^1,p_2^0;\rho) - H(p_1^0,p_2^1;\rho) + H(p_1^0,p_2^0;\rho)$ for a given parameter value $\theta = (\gamma^\top, \rho, \lambda)^\top$. Note that $\mathcal{L}^{(0,0)}(w)$ is redundant for identification, as it is a linear combination of the above probabilities: $\mathcal{L}^{(0,0)}(w) = 1 - \mathcal{L}^{(1,0)}(w) - \mathcal{L}^{(0,1)}(w) - \mathcal{L}^{(1,1)}(w)$.

In addition to Assumptions 2.1, 3.2, and 4.1, we introduce the following assumptions. Below, we consider a general parameter value $\theta = (\gamma^\top, \rho, \lambda)^\top$ (treating the true value $\theta^* = (\gamma^{*\top}, \rho^*, \lambda^*)^\top$ as a special case) that belongs to a parameter space $\Theta \subseteq \mathbb{R}^{2\dim(W)} \times (\underline{c}_\rho, \bar{c}_\rho) \times (0,1)$, where $\underline{c}_\rho$ and $\bar{c}_\rho$ are real numbers whose values depend on the choice of the copula function.

Assumption D.1.

(i) The interaction function $\eta(\cdot)$ is strictly increasing, positive, and continuous.

(ii) The marginal CDFs of $(\varepsilon_1, \varepsilon_2)$, $F_{\varepsilon_1}(\cdot)$ and $F_{\varepsilon_2}(\cdot)$, are strictly increasing and continuous.

(iii) The copula $H(\cdot,\cdot;\rho)$ is twice differentiable in its arguments and ρ.

Assumption D.2.

(i) There exist pairs of values $w = (w_1, w_2)$ and $\widetilde{w} = (\widetilde{w}_1, \widetilde{w}_2)$ such that $\widetilde{w}_1^\top \gamma_0 = 0$, $H(p_1^0, p_2^1; \rho) = H(\widetilde{p}_1^0, \widetilde{p}_2^1; \rho)$, and $p_1^0 \neq \widetilde{p}_1^0$, where $\widetilde{p}_j^0 = F_{\varepsilon_j}(\widetilde{w}_j^\top \gamma_0)$ and $\widetilde{p}_j^1 = F_{\varepsilon_j}(\widetilde{w}_j^\top \gamma_0 + \eta(\widetilde{w}_j^\top \gamma_1))$. The set of the values of W_1 satisfying these conditions, that is,

$$\bigcup_{w_2 \in \text{supp}[W_2]} \left\{ w_1 \in \text{supp}[W_1 | W_2 = w_2] : \exists \widetilde{w} \in \text{supp}[W], \widetilde{w}_1^\top \gamma_0 = 0, H(p_1^0, p_2^1; \rho) = H(\widetilde{p}_1^0, \widetilde{p}_2^1; \rho), p_1^0 \neq \widetilde{p}_1^0 \right\},$$

is non-empty and does not lie in a proper linear subspace of $\mathbb{R}^{\dim(W)}$ a.s.

(ii) The set of the values of W_1

$$\bigcup_{w_2, \ddot{w}_2 \in \text{supp}[W_2]} \left\{ w_1 \in \text{supp}[W_1 | W_2 = w_2] \cap \text{supp}[W_1 | W_2 = \ddot{w}_2] : w_2^\top \gamma_0 \neq \ddot{w}_2^\top \gamma_0 \right\},$$

is non-empty and does not lie in a proper linear subspace of $\mathbb{R}^{\dim(W)}$ a.s.

Assumption D.1 is a set of regularity conditions that partly overlap with Assumption 4.3.

Assumption D.2(i) requires that the support of W is sufficiently rich to vary the values of $w_1^\top \gamma_0$ and $H(p_1^0, p_2^1; \rho)$. It implicitly requires that γ_0 and γ_1 are non-zero vectors and that W_1 and W_2 contain some player-specific elements. It is important to note that, in practice, we need not detect exact values of w and $\widetilde{w}$ satisfying the condition. Note also that this assumption may be regarded as a replacement for the condition for using the identification-at-infinity strategy. The identification-at-infinity requires that (at least) one variable in W_2 can tend to $-\infty$ or ∞, which can reduce the model into a single-agent decision problem and allows us to identify γ easily. Instead of using the identification-at-infinity approach, as in condition (i), we focus on a situation where $H(p_1^0, p_2^1; \rho)$ in $\mathcal{L}^{(1,0)}(w)$ and $H(\widetilde{p}_1^0, \widetilde{p}_2^1; \rho)$ in $\mathcal{L}^{(1,0)}(\widetilde{w})$ balance out.

Assumption D.3. The copula function $H(p_1, p_2; \rho)$ is strictly more stochastically increasing in joint distribution with respect to ρ (see Definition 3.3 of Han and Vytlacil (2017)).

The assumption restricts a dependence ordering of the copula function in terms of stochastic monotonicity. In Han and Vytlacil (2017), this property was introduced to identify generalized bivariate probit models, and it was shown that several commonly used copula functions satisfy it (e.g., Gaussian copula and FGM copula). The reader is referred to that study for further discussions on the dependence ordering properties of copula functions.

Theorem D.1. Suppose that Assumptions 2.1, 3.2, 4.1, D.1, D.2, and D.3 hold for a given parameter value $\theta = (\gamma^\top, \rho, \lambda)^\top \in \Theta$. Further, we assume that $\{(w_1^\top \gamma_0 + \eta(w_1^\top \gamma_1), \rho) : \theta \in \Theta\}$ is open and simply connected for any given w_1. Then, θ is globally identified if γ_0 and γ_1 are non-zero vectors and $\mathcal{L}_{mul}(\mathbf{p}; \rho) > 0$ for some $\mathbf{p} \in \text{supp}[\mathbf{P}]$.

Proof. To show the identifiability of γ_0, we examine the following system for w and $\widetilde{w}$ given in Assumption D.2(i):

$$\begin{cases} \mathcal{L}^{(1,0)}(w) = p_1^0 - H(p_1^0, p_2^1; \rho) \\ \mathcal{L}^{(1,0)}(\widetilde{w}) = \widetilde{p}_1^0 - H(\widetilde{p}_1^0, \widetilde{p}_2^1; \rho) \end{cases},$$

so that $\mathcal{L}^{(1,0)}(w) - \mathcal{L}^{(1,0)}(\widetilde{w}) = p_1^0 - \widetilde{p}_1^0$ under the condition $H(p_1^0, p_2^1; \rho) = H(\widetilde{p}_1^0, \widetilde{p}_2^1; \rho)$. Since $\widetilde{w}_1^\top \gamma_0 = 0$ by assumption, $\widetilde{p}_1^0 = F_{\varepsilon_1}(0)$ does not depend on γ_0. Hence, the strict monotonicity of F_{ε_1} implies that $w_1^\top \gamma_0 = F_{\varepsilon_1}^{-1}(\mathcal{L}^{(1,0)}(w) - \mathcal{L}^{(1,0)}(\widetilde{w}) + F_{\varepsilon_1}(0))$. This equation identifies $w_1^\top \gamma_0$, implying the identifiability of γ_0 under Assumption D.2(i).

We now show the identification of (γ_1, ρ). We consider any pair of values (w_1, w_2) and $(w_1, \ddot{w}_2)$ such that $p_2^0 \neq \ddot{p}_2^0$, where $\ddot{p}_2^0 = F_{\varepsilon_2}(\ddot{w}_2^\top \gamma_0)$. Note that such w and $\ddot{w}$ exist by Assumption D.2(ii) and the strict monotonicity of F_{ε_2}. We now have the following system:

$$\begin{cases} \mathcal{L}^{(0,1)}(w_1, w_2) = p_2^0 - H(p_1^1, p_2^0; \rho) \\ \mathcal{L}^{(0,1)}(w_1, \ddot{w}_2) = \ddot{p}_2^0 - H(p_1^1, \ddot{p}_2^0; \rho) \end{cases}.$$

Here, the parameter to be identified is (p_1^1, ρ), as p_2^0 and $\ddot{p}_2^0$ are already identified by the identification of γ_0. If this system has a unique solution, we achieve identification of (p_1^1, ρ). To proceed, we define the following function:

$$G(\vartheta_{w_1}) := \begin{pmatrix} p_2^0 - H(p_1^1, p_2^0; \rho) \\ \ddot{p}_2^0 - H(p_1^1, \ddot{p}_2^0; \rho) \end{pmatrix},$$

where $\vartheta_{w_1} := (p_1^1, \rho)$. The Jacobian of $G(\vartheta_{w_1})$ is given by

$$J_G(\vartheta_{w_1}) := \frac{\partial G(\vartheta_{w_1})}{\partial \vartheta_{w_1}^\top} = \begin{pmatrix} -H_1(p_1^1, p_2^0; \rho) & -H_\rho(p_1^1, p_2^0; \rho) \\ -H_1(p_1^1, \ddot{p}_2^0; \rho) & -H_\rho(p_1^1, \ddot{p}_2^0; \rho) \end{pmatrix},$$

where H_1 and H_ρ are the partial derivatives of the copula H with respect to the first argument and ρ, respectively. Its determinant is given by

$$|J_G(\vartheta_{w_1})| = H_1(p_1^1, p_2^0; \rho) H_1(p_1^1, \ddot{p}_2^0; \rho) \left(\frac{H_\rho(p_1^1, \ddot{p}_2^0; \rho)}{H_1(p_1^1, \ddot{p}_2^0; \rho)} - \frac{H_\rho(p_1^1, p_2^0; \rho)}{H_1(p_1^1, p_2^0; \rho)} \right),$$

which is positive for any $p_2^0 > \ddot{p}_2^0$ and is negative for any $p_2^0 < \ddot{p}_2^0$ under Assumptions D.1(iii) and D.3 (see Lemma 4.1 of Han and Vytlacil, 2017). This implies that $J_G(\vartheta_{w_1})$ is of full rank when $p_2^0 \neq \ddot{p}_2^0$. Hence, the same arguments as in the proof of Theorem 5.1 in Han and Vytlacil (2017) lead to the identification of ϑ_{w_1} under the assumption that $\{(w_1^\top \gamma_0 + \eta(w_1^\top \gamma_1), \rho) : \theta \in \Theta\}$ is open and simply connected. Moreover, the strict monotonicity of F_{ε_j} and η implies that $w_1^\top \gamma_1 = \eta^{-1}(F_{\varepsilon_1}^{-1}(p_1^1) - w_1^\top \gamma_0)$ and thus that γ_1 is identified from p_1^1 and γ_0 under Assumption D.2(ii).

Finally, λ can be identified by $\lambda = (H(p_1^1, p_2^1; \rho) - \mathcal{L}^{(1,1)}(w)) / \mathcal{L}_{mul}(\mathbf{p}; \rho)$ under the assumption that $\mathcal{L}_{mul}(\mathbf{p}; \rho) > 0$ for $\mathbf{p} \in \text{supp}[\mathbf{P}]$. □

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