A Pigouvian Approach to Congestion in Matching Markets*

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Abstract

Matching markets often require recruiting agents – "programs" – to conduct a costly screening of "applicants", who are agents on the other side. A market becomes congested if programs must screen too many applicants. The cost associated with application submission is a Pigouvian tax that mitigates the negative externality that applicants impose on programs. A higher cost reduces congestion by discouraging applicants from applying to certain programs; however, match quality may be in jeopardy. We measure the effects of such Pigouvian taxes by studying variants of the Gale-Shapley Deferred-Acceptance mechanism with differential application costs. Using data collected in a multiple-elicitation experiment conducted in a real-life matching market, we show that a (low) application cost effectively reduces congestion without sacrificing matching quality.

JEL Codes: C78, D47, D50, D61, I21

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1 Introduction

In many commonly observed matching markets, agents on one side, called "programs" (e.g., employers or colleges), actively recruit agents on the other side, called "applicants" (e.g., employees or students). The matching process usually includes an application stage, in which applicants submit their candidacy to programs of their choice, and a screening stage, in which each program screen and rank its candidates.

As a result of the massive advancements in both information technology and market design in recent years, there has been a largely welcomed trend toward applicants applying to an increasing number of programs. For example, in the job market for new graduates of PhD programs in economics, the centralized platforms – EconJobMarket and AEA Job Opening for Economists – have dramatically reduced the application cost associated with submitting a job application. As a result, hiring departments now screen many more candidates (Nguyen, Peters and Poitevin, forthcoming).

While the traditional matching theory considers application cost as a market friction (see, for example Rogerson, Shimer and Wright, 2005) and welcomes these new developments, a practical issue has emerged: Programs must screen too many applicants and pay a high cost, which amounts to a negative externality on programs as a result of applicant behavior. Costly screening to form preferences is not uncommon in real-life scenarios. Recruiting programs have enough information to rank applicants only after reviewing application files or conducting interviews. When programs receive too many applications, congestion occurs, and matching outcomes can be inefficient (Arnosti, Johari and Kanoria, 2016).

This externality motivates us to consider application cost as a Pigouvian tax. Intuitively, applicants apply to fewer programs when the application cost is higher, which mitigates market congestion. However, the total welfare effects, including those on match quality, are ambiguous because a high application cost may preclude some efficient matches.

In a setting of many-to-one matching without transfers, we provide the first empirical evidence on the performance of a set of popular market designs with differential application costs. Similar to implementations of Pigouvian taxes in other settings (Pigou, 1920), we hypothesize that when designed appropriately, an application cost can enhance welfare by reducing negative externalities.

Inspired by practical market designs, we consider two forms of application cost: A positive marginal cost and an application limit, both of which are used in real-life centralized and decentralized matching markets (see examples in Section 6, especially Table XVI). The latter imposes an upper bound on the number of applications that each applicant can submit and therefore creates a discontinuity in the marginal cost that is zero below the limit and infinite above the limit.

We implement a multiple-elicitation experiment for the real-life matching of 129 applicants to the 7 master's programs at the Toulouse School of Economics (TSE). Because the number of programs in our setting is low, we assume that applicants have full knowledge of their true preferences, or equivalently, that the cost to applicants to learn their preferences is negligible. While they can rank applicants however they prefer, programs must pay some costs to form their preferences.

The experiment considers market designs that are characterized by three variants of the Gale-Shapley Deferred-Acceptance mechanism (DA) encountered in practice: The traditional DA, under which applicants can apply to all programs without any cost; DA with truncation (DA-T), under which applicants can apply to no more than 4 programs; and DA with cost (DA-C), under which applicants must write a motivation letter for each additional application beyond the first three applications. Under each mechanism, every applicant is required to rank her applications and submit a rank-ordered list of programs (ROL), although programs do not observe the submitted ROLs. Applicants are informed that one of the mechanisms will be implemented, and they are required to provide one ROL under each mechanism. Therefore, they have incentives to behave optimally under each mechanism.

The experiment was conducted in May 2013 for admission in the 2013-14 academic year, and the experimental design allows us to evaluate the effects of application cost in two ways. The first is a direct evaluation of the three market designs, as every applicant "experiences" all three market designs. Measuring screening costs by the number of applications each program receives, we find that both DA-T and DA-C reduce the screening cost of programs without significantly changing the matching outcome. On average, relative to DA without cost which results in 75 applications per program, each program receives 13 fewer applications under DA-T and 16 fewer under DA-C.

The second approach uses a structural analysis of counterfactuals. Because of the strategy-proofness of DA (Dubins and Freedman, 1981; Roth, 1982), we take applicants' submitted ROLs under DA as their true ordinal preferences. Furthermore, we announce in the experiment that it is optimal for applicants to report their true preferences under DA. We also supplement the ROL data with a survey in which each applicant reveals whether a program is unacceptable to her. With all this information, we then estimate applicant preferences in an extended version of the exploded logit model.

In order to conduct counterfactual equilibrium analysis, we also need information on program preferences and applicant beliefs. Program preferences are estimated using the programs' observed ranking of applicants. Applicant beliefs are derived from an applicant's (incomplete) information on other applicants' preferences and on how each program ranks her. Assuming rational expectations and a common prior, we calculate applicant beliefs using the estimates of the parameters governing program and applicant preferences.

We then simulate the equilibrium outcomes of various configurations of DA-T and DA-C. DA-T is associated with degree K, where $K \in \{1, ..., 7\}$ is the number of choices an applicant is allowed to submit. DA-C can have a constant marginal cost, taking various values over a wide range for every application after the first. In equilibrium, applicants choose an optimal strategy under each market design, while programs are assumed to be non-strategic. That is, as a simplifying assumption, programs always announce their true number of positions, screen all applications they receive, and rank them truthfully.

Using a Gibbs sampler to simulate applicant preferences, we answer the following question: *Based on all the information observed by the researcher, what is the best prediction of the counterfactual outcome if a counterfactual market design had been implemented at TSE in 2013?* We focus on the trade-off between screening cost, measured as the number of applicants to screen, and match quality, measured as the numbers of blocking pairs and unmatched applicants. An applicant-program pair (i, j) can form a blocking pair in a given matching if both would be better off by being matched together after leaving their current matches.[1] The literature shows that the absence of a blocking pair in a matching outcome, also known as the *stability* of the matching, is the key to the success

[1]For applicant i, leaving her current match means leaving the currently matched program if there is one; otherwise, i's status changes from being unmatched to being matched with j. For program j, leaving the current match means that j expels the least preferred applicant among those who are currently matched with j if j has no vacancy left; otherwise, j accepts i and reduces its vacancy by one seat.

of matching markets (Roth, 1991).

The results show that relative to DA without cost, DA-C with a low cost reduces screening costs by half without harming match quality; a less-restrictive DA-T, e.g., DA-T-5 and DA-T-6, does not affect match quality, either, but the reduction in congestion is not as substantial. As expected, a high-cost DA-C or a highly restrictive DA-T results in lower match quality because a prohibitively high application cost prevents applicants from applying to a sufficient number of programs, which leads to a high number of blocking pairs and unmatched applicants.

The findings show DA-C's potential to combat congestion, when the application cost involves a dead-weight loss (e.g., writing an otherwise useless motivation letter) and even more so when it is a monetary transfer. If the market designer has rich information about the market participants, the magnitude of the marginal cost can be optimized to balance congestion and match quality. Otherwise, by setting a very low cost, the market designer runs a negligible risk of reducing match quality but still has the opportunity to significantly reduce congestion.

Although our study focuses on a centralized market, we provide examples in which versions of DA-T and DA-C are implemented in both centralized and decentralized markets (see additional details in Section 6). For example, academic journals usually prohibit simultaneous submissions to multiple journals, echoing DA-T. In the same spirit as DA-C, colleges and graduate programs charge application fees and/or require applicants to submit essays with their applications. Our results thus provide a plausible explanation for these observed practices: The screening costs for journals and colleges are notoriously high, and therefore, it is important to reduce congestion by imposing an application cost. Our findings thus shed light on how to improve the market design in these settings.

Related literature. Congestion in matching markets has been studied extensively, but often theoretically. One of the earliest papers on the topic is Roth and Xing (1997), who show that thick markets may suffer from congestion. Search and screening costs hinder the evaluation of all potential matches by both sides.

Most related to our research, Arnosti et al. (2016) study the regulation of market congestion using application costs and provide a comprehensive survey of the literature. Their paper considers a dynamic model of one-to-one matching in which agents (applicants and employers) arrive and depart over time. All applicants and employers are homogenous. As in our paper – but in a decentralized setting – each applicant applies to programs at a cost, and each employer pays a screening cost to verify whether an applicant is compatible. If the applicant is compatible, the employer makes her an offer without knowing if she has already been hired by someone else. If she receives an offer, the applicant accepts or rejects it. These authors also consider imposing application limits that restrict the maximum number of applications allowed per applicant.

Application limits are also considered by Che and Koh (2016), who model a college admissions game in which colleges manage yield by strategically targeting applicants who are less likely to receive other offers. Application costs are also present in Chade, Lewis and Smith (2014). More recently, in a model of matching economics PhDs to university positions, Nguyen et al. (forthcoming) note that a reduced application cost increases the probability of application. Furthermore, they show that this reduced cost causes an increase in the probability that offers are turned down by applicants and an increase in the probability that positions remain unmatched. As a result, some universities' welfare may decrease.

Another way educational institutions mitigate screening costs is to organize entrance exams. For example, in the 2000s, certain universities in Brazil used a low-cost first-round exam to help to screen out applicants who were unlikely to qualify, while a second-round exam, which was costlier and more informative, refined the selection (Carvalho, Magnac and Xiong, 2017). This two-tier screening device, a centralized exam followed by a costlier decentralized exam, has also been used in Japan (Hafalir, Hakimov, Kübler and Kurino, 2016).

Our paper also sheds light on the design of online platforms, an extensively studied topic (Fradkin, 2014; Halaburda, Piskorski and Yildirim, 2015; Horton, 2015). That literature often shows in different contexts that limiting the number of potential matches might have desirable properties in terms of aggregate welfare because of decreased screening costs.

Signaling in matching markets also relates to the issues we study (Coles, Kushnir and Niederle, 2013; Lee and Schwarz, 2017). Programs can target applicants by letting applicants signal their interests in specific programs. To avoid cheap talk, signals are made costly or are limited in terms of the total number of signals permitted. This step may improve welfare in equilibrium because offers are made to applicants who have a higher probability of accepting the offer (Coles et al., 2013). In decentralized settings, pre-match interviews can be organized between applicants and programs (Lee and Schwarz, 2017), which are costly for both sides in terms of application costs and screening costs.

Our study is also related to the literature on labor markets with costly search (for a survey, see Rogerson et al., 2005). However, the externality between applicants and programs goes in the opposite direction in that setting, as decreasing individual search costs leads to better prospects for programs, while in our setting, decreasing these costs can be detrimental to program welfare because of screening costs.

The use of Pigouvian taxes to correct negative externalities is originally attributable to Pigou (1920) and has been extensively studied theoretically (see, for example, Baumol, 1972; Sandmo, 1975). Recently, it has received increased attention, and in particular, the effectiveness of Pigouvian taxes in regulating pollution has been empirically evaluated; see for example Muller and Mendelsohn (2009) and Knittel and Sandler (2013).

In terms of theoretical tools, the two-sided matching framework we study is well analyzed and summarized in Roth and Sotomayor (1990). More specifically, our model builds on the theoretical results derived by Haeringer and Klijn (2009) and Fack, Grenet and He (2015) in which the properties of the DA mechanism with truncation and/or application costs are investigated.

The estimation of applicant preferences using data on applicants' submitted ROLs is an important building block of our empirical analysis. In a strand of the recent empirical school choice literature, applicants are assumed to report their true preferences, as in our experiment under DA. Standard discrete choice methods are extended – as in an exploded logit, for example – to utilize the identifying information contained in ROLs (Hastings, Kane and Staiger, 2008; Abdulkadiroğlu, Agarwal and Pathak, Forthcoming). Another strand of this literature explicitly considers possible strategic applicant behavior, for example, using data from the Boston immediate-acceptance mechanism (Agarwal and Somaini, Forthcoming; Calsamiglia, Fu and Güell, 2014; He, 2015; Hwang, 2017) or from DA with truncation (Ajayi, 2013; Carvalho et al., 2017; Fack et al., 2015).

More generally, this paper builds on a growing body of literature in which structural methods are applied to experimental data and thus enlarge the span of counterfactual policy analyses (for a survey, see Blundell, 2017).

Organization of the paper. Section 2 below formalizes the many-to-one matching and summarizes the theoretical predictions. The experiment conducted at TSE is described in Section 3, and Section 4 summarizes the data and provides a direct evaluation. Our structural evaluation is presented in Section 5. The paper concludes in Section 6 with a discussion of our findings and their implications for practical market design.

2 Many-to-one Matching: Set-up

A many-to-one matching market in which applicants are to be matched with programs is denoted by

$$\left\{ [v_{i,j}]_{i\in\mathcal{I},j\in\mathcal{J}\cup\{0\}}, [s_{i,j}]_{i\in\mathcal{I}\cup\{0\},j\in\mathcal{J}}, [q_j]_{j\in\mathcal{J}}, C_A(\cdot), C_P(\cdot) \right\},$$

where $\mathcal{I} \equiv \{1,...,I\}$ is the set of applicants and $\mathcal{J} \equiv \{1,...,J\}$ is the set of programs, with the addition of 0 as an outside option or the option of being unmatched. Applicant i receives a von Neumann-Morgenstern utility $v_{i,j} \in \mathbb{R}$ if matched with j; and $v_i \equiv (v_{i,0}, v_{i,1}, \ldots, v_{i,J})$. Programs have responsive preferences, and more specifically, program j values its match with i at $s_{i,j} \in \mathbb{R}$ regardless of who else is matched with j. Let $s_i = (s_{i,0}, \ldots, s_{i,J})$. Each applicant can be matched with at most one program, and each program has a capacity of q_j identical positions. There is no indifference in preferences on either side. Program j is acceptable to i if $v_{i,j} \geqslant v_{i,0}$, and otherwise, it is unacceptable. Program j finds applicant to be *qualified* if $s_{i,j} \geqslant s_{0,j}$; otherwise, i is unqualified for j. Moreover, applicant i is qualified for program j if and only if i meets j's prerequisites.

Deviating from the previous literature, we assume that it is costly for an applicant to "apply" to programs, $C_A(\cdot) \geqslant 0$, while every program has to conduct a costly screening of its candidates to form its preferences, $C_P(\cdot) \geqslant 0$. Both cost functions, which are defined below, are homogeneous across i and j, respectively.

We consider a centralized market. Programs first announce their capacities and prerequisites, and every applicant then submits a rank-ordered list (ROL) of $K_i \leqslant J$ programs, denoted by $L_i = \left(l_i^1, ..., l_i^{K_i}\right)$, where $l_i^k \in \mathcal{J}$ is i's k^{th} choice. An ROL defines a relationship $\succ_{L_i}$ such that $j \succ_{L_i} j'$ if and only if j is ranked above j' in L_i. The set of all possible ROLs, $\mathcal{L}$, includes all ROLs that rank at least one program. We define program

j's *candidates* as the set of applicants who include project j in their submitted ROLs.

Application cost and screening cost. When submitting ROL L, an applicant incurs a cost that is assumed to depend on the number of programs being ranked in L, denoted by $|L|$, but not on how they are ranked, $C_A(|L|) : \{0, \ldots, J\} \to \overline{\mathbb{R}}^+ \equiv [0, +\infty]$. Similarly, upon receiving an application, a program pays a screening cost to learn its value of being matched with the applicant. The total screening cost of forming preferences over the subset of applicants $\mathcal{I}_j$ is $C_P(|\mathcal{I}_j|) : \{0, \ldots, I\} \to \overline{\mathbb{R}}^+$. For simplicity, we assume that a program always pays the cost of forming preferences over all its candidates. In other words, programs do not strategically choose to remain uninformed about their preferences over (a subset of) its candidates.

Both the application cost and the screening cost monotonically increase such that for all $n \geqslant 0$, $C_A(n+1) \geqslant C_A(n)$ and $C_P(n+1) \geqslant C_P(n)$. These specifications are rather flexible, and in particular, such application cost captures many common practices of matching markets, as we show shortly.

2.1 Matching and Matching Mechanisms

We define a matching $\mu : \mathcal{I} \to \mathcal{J} \cup \{0\}$ such that (i) $\mu(i) = j$ if applicant i is matched with j; (ii) $\mu(i) = 0$ if applicant i is unmatched; and (iii) $\mu^{-1}(j)$ is the set of applicants matched with j, while $|\mu^{-1}(j)|$ is the number of applicants matched with j and does not exceed j's capacity.

A matching μ is **individually rational** if each applicant prefers her current match to remaining unmatched and if each program prefers each of its currently matched applicants to having one more position unmatched. Given matching μ, (i, j) form a **blocking pair** if i prefers j over her matched program $\mu(i)$, while i is more preferred by j to either having a vacant position or keeping the least-preferred applicant of the currently matched ones, $s_{i,j} > \min_{\{i' \in \mu^{-1}(j)\}}(s_{i',j})$. μ is **stable** if there is no blocking pair and if it is individually rational.

Stability implies Pareto efficiency when both sides have strict preferences,[2] and sta-

[2]Suppose that μ' Pareto dominates a stable matching μ. There must exist a program or an applicant who is strictly better off in μ'. Begin by assuming that applicant i strictly prefers $\mu'(i)$ to $\mu(i)$. Because μ is stable and because program preferences are strict, program $\mu'(i)$ must be strictly worse off in matching μ' than in matching μ; otherwise, i and $\mu'(i)$ could form a blocking pair in μ. This situation contradicts the Pareto domination presumption. We would reach the same contradiction had we begun by assuming

bility is essential to the success of many matching markets (Roth, 1991).

Deferred Acceptance and its variants. The applicant-program match is solved by a matching mechanism that is a function of applicants' submitted ROLs and programs' preferences over applicants. As a computerized algorithm, the applicant-proposing DA works as follows:

Round 1. Every applicant applies to the first choice listed in her ROL. Each program rejects unqualified applicants and the least-preferred applicants in excess of its capacity and tentatively holds the other applicants.

Generally, in

Round k. Every applicant who is rejected in Round $(k-1)$ applies to the next choice on her ROL. Each program, pooling new applicants and those who were held in Round $(k-1)$, rejects unqualified applicants and the least preferred applicants in excess of its capacity. Those who are not rejected are tentatively held by the programs.

The process ends after any Round k when no rejections are issued. Each program is then matched with the applicants it is currently holding.

The three variants of DA that we consider differ only in their application costs. In the traditional DA mechanism, denoted by DA, the application cost is always zero. Therefore, $C(|L|) = 0$ for all $L \in \mathcal{L}$. The DA with truncation, denoted by DA-T, does not allow applicants to apply to more than $K \in \{1, \ldots, J\}$ programs, which is defined as DA-T of degree K and denoted by DA-T-K. Therefore, $C(|L|) = \infty \times \mathbb{1}\{|L| > K\}$, with the convention that $\infty \times 0 = 0$. Finally, we consider DA with costs, denoted by DA-C, under which applicants must pay a cost for each program beyond their top K choices. In this case, $C(|L|) = c \times (|L| - K) \times \mathbb{1}(|L| > K)$ if the marginal cost of application beyond K is c, a positive constant.

2.2 Timeline, Information, Strategy, and Equilibrium Concept

We consider the matching game with incomplete information and a timeline depicted in Figure I. First, at the "announcement" stage, a mechanism is chosen and made public to both programs and applicants; in addition, programs reveal their capacities and pre-

that there exists program j that strictly prefers $\mu'^{-1}(j)$ to $\mu^{-1}(j)$. Therefore, the stable matching μ is Pareto efficient.

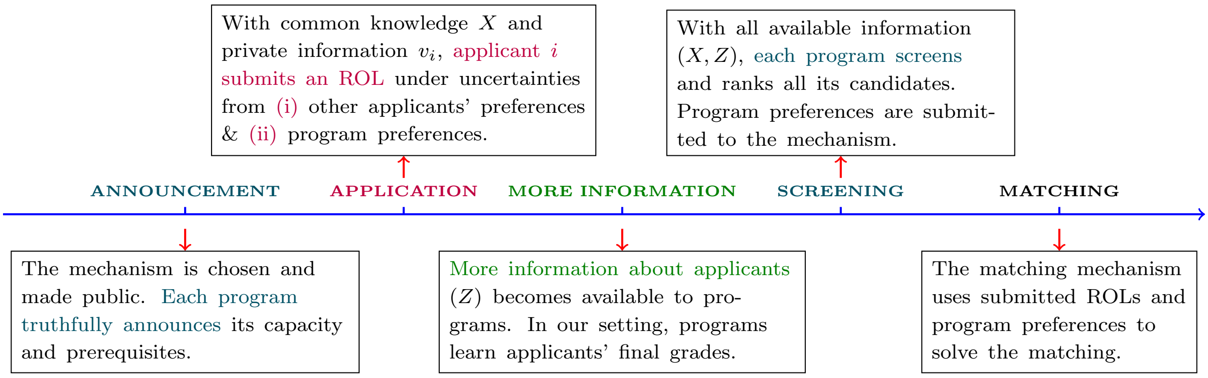

Figure I: Timeline of the Matching Game under a Given Mechanism

requisites. Prerequisites define the necessary and sufficient conditions for qualification such that an applicant is qualified for a program if and only if she meets the program's prerequisites.

Second, at the "application" stage, applicant preferences (v_i) are private information, while the distribution of applicant preferences – conditional on some common-knowledge applicant and program characteristics (denoted by $X \equiv [x_i]_{i \in \mathcal{I}}$) – is common knowledge. For example, X includes information previously announced by programs, particularly their prerequisites.

After applications are submitted in the form of ROLs, more information about applicants, denoted by Z, can become available to programs. For example, in our experiment, applicants' final grades are realized and learned by programs after applicants have submitted their ROLs. With information on (X, Z), programs always behave truthfully at the "screening" stage, in which they screen their candidates who are the applicants including them in their ROLs. Each program then submits its preferences over its candidates.

Finally, at the "matching" stage, the mechanism finds a matching by considering submissions from both sides.

Strategies and equilibrium. First, in summary, the following assumption is imposed on programs' behavior.[3]

Assumption 1 *Programs do not behave strategically in the following sense: (i) Each program truthfully reports its capacity and prerequisites, (ii) screens all its candidates,*

[3]It is known that programs may have incentives to be strategic with their capacity and preference reporting under the applicant-proposing DA (Roth and Sotomayor, 1990; Sönmez, 1997). Kojima and Pathak (2009) present a model in which the proposal-receiving side of the market has diminishing incentives to be strategic when the market size grows.

and (iii) reports to the mechanism its true preference ranking over all its candidates.

To mitigate the issue of multiple equilibria, we also impose several assumptions on applicant strategy.

Assumption 2 *Applicants do not apply to any unacceptable program or to any program for which they are not qualified.*

Because applicants always have access to their outside options, the marginal benefit of applying to unacceptable programs is zero; similarly, applying to a program for which an applicant does not qualify or meet the prerequisites brings zero payoff because the program will never accept an unqualified applicant. Assumption 2 therefore regulates how applicants behave when they have multiple best responses and is binding only when the marginal application cost is zero.

We focus on symmetric pure strategies such that $\sigma(v_i, x_i) : \mathbb{R}^{J+1} \times \mathbb{R}^{|x_i|} \to \mathcal{L}_i$, where x_i is i's characteristics/information that are common knowledge and $\mathcal{L}_i \subseteq \mathcal{L}$ are all the ROLs consistent with Assumption 2.[4] When playing the game, an applicant's beliefs are equated with her probability of acceptance by each program and depend on her information set. The probability that applicant i is accepted by program j when submitting list L_i is denoted by $\pi_j(L_i|X, \sigma)$, which is conditional on common knowledge X and on others' strategy σ. A Bayesian Nash equilibrium is σ^* such that for all (v_i, x_i),

$$\sigma^*(v_i, x_i) \in \underset{\sigma(v_i,x_i)\in\mathcal{L}_i}{\arg\max} \sum_{j\in\mathcal{J}} \pi_j(\sigma(v_i, x_i)|X, \sigma^*) \max(v_{i,j}, v_{i,0}) - C_A(|\sigma(v_i, s_i)|),$$

where $\pi_j(\cdot|X, \sigma^*)$ is the probability of acceptance by program j consistent with σ^*;[5] only $\max(v_{i,j}, v_{i,0})$ matters because i can always take the outside option whenever matched with an unacceptable program. The existence of Bayesian Nash equilibrium in pure strategies can be established by applying Theorem 4 (Purification Theorem) in Milgrom and Weber (1985). Moreover, we can solve for equilibrium, and thus, $\pi_j(\cdot|X, \sigma^*)$ following the steps in Appendix D.1.

[4]For example, x_i may include i's academic performance in the first semester. We assume this information is common knowledge because there are 129 applicants only and because applicants are informed about their academic rankings. Such an x_i determines i's action because it affects i's probability of acceptance by each program.

[5]In other words, given that everyone plays strategy σ^*, when i submits a list L, the probability that i is accepted by j is exactly $\pi_j(L|X, \sigma^*)$. This calculation takes into account the distribution of other applicants' preferences and that of program preferences. A step-by-step formalization of $\pi_j(L|X, \sigma^*)$ can be found in, e.g., Fack et al. (2015).

Before discussing equilibrium properties, we introduce the following assumption.

Assumption 3 *Equilibrium acceptance probabilities are non-degenerate for qualified applicants:* $\pi_j(L|X, \sigma^*) \in (0, 1)$ *for every* $j \in L$ *and for all* i *and* $L \in \mathcal{L}_i$.

In other words, there is sufficient uncertainty in other applicants' preferences and in program preferences such that every applicant has some chance of being accepted by any program to which she applies as long as she meets the prerequisites. This assumption is satisfied in our econometric model because we assume that both applicant and program preferences (v_i, s_i) have full support on the real line (see Section 5).

Given Assumptions 1, 2, and 3, the literature provides the following results:

(i) *Under DA*, i.e., $C_A(|L|) = 0$ for all L, there is a unique Bayesian Nash equilibrium in which every applicant truthfully ranks all of the acceptable programs for which she qualifies (Dubins and Freedman, 1981; Roth, 1982; Fack et al., 2015). The equilibrium outcome is the applicant-optimal stable matching that Pareto dominates all other stable matchings in terms of applicant welfare (Gale and Shapley, 1962).

(ii) *Under DA-T-K for* $K \in \{1, \ldots, J\}$, it is a dominated strategy if one submits ROL L that does not truthfully rank programs included in L (Haeringer and Klijn, 2009). An equilibrium matching outcome is not necessarily stable; for any applicant-program pair (i, j), the probability of (i, j) forming a blocking pair is bounded above by a function that is decreasing in K and equal to zero when $K = J$ (Fack et al., 2015).

(iii) *Under DA-C*, it is a dominated strategy if one submits ROL L that does not truthfully rank programs included in L. An equilibrium matching outcome is not necessarily stable; in the case of a constant marginal cost (c), for any applicant-program pair (i, j), the probability that (i, j) forms a blocking pair is bounded above by a function that is increasing in c and equal to zero when $c = 0$ (Fack et al., 2015).

We use these results to guide our experimental and research design and, in particular, the mechanisms considered in counterfactual analysis.

3 Experiment at the Toulouse School of Economics

3.1 Background and Experimental Design

TSE organizes its master's programs into two years of study, M1 and M2. In the first year, it admits approximately 150 students, who are placed into three M1 programs: Law and economics, statistics and econometrics, and economics. Having successfully finished their M1 study, students are allowed to apply to the seven M2 programs for their second year of study. The names of the programs, which indicate their differentiated foci, are described in Table I. In the rest of the paper, the programs are randomly ordered and labeled as P1 to P7. Our study focuses on the matching between applicants and programs.

Table I: Program Names and Prerequisites

Program Acronym	**Prerequisites**	**Program Name** in English *(French)*
ECL	Yes	Economics and Competition Law *(Economie et Droit de la Concurrence)*
EMO	No	Economics of Markets and Organizations *(Economie des Marchés et des Organisations)*
ERNA	No	Environmental and Natural Resources Economics *(Economie de l'environnement et des ressources naturelles)*
ETE	No	Economic Theory and Econometrics – PhD Track *(Economie Mathématique et Econométrie)*
MIF	No	Financial Markets and Intermediaries *(Marchés et Intermédiaires Financiers)*
PPD	No	Development Economics and Public Policies *(Politique Publique et Développement)*
STA	Yes	Statistics and Econometrics *(Statistique et Econométrie)*

Notes: The programs are listed in alphabetical order, according to their acronyms. This order does not correspond to P1-P7. The prerequisites for Program STA include some courses in statistics and econometrics; those for Program ECL include certain courses in law.

In partnership with the TSE administration, we conducted an experiment in May 2013 aimed at improving the market design for the applicant-program match. Previously, for example, in 2011 and 2012, the match was organized in a semi-centralized but rather ad-hoc fashion. Applicants submitted two ranked choices without being provided explicit explanations on how these two choices would be used. Given the applicants' choices and academic files, the program directors met to decide who was matched with which program. During the meeting (and sometimes multiple meetings), each program director screened its candidates and decided whether or not to accept each candidate. Because every applicant was guaranteed the opportunity to continue his/her study at TSE, the costliest part of this process was finding solutions for applicants who were ranked low

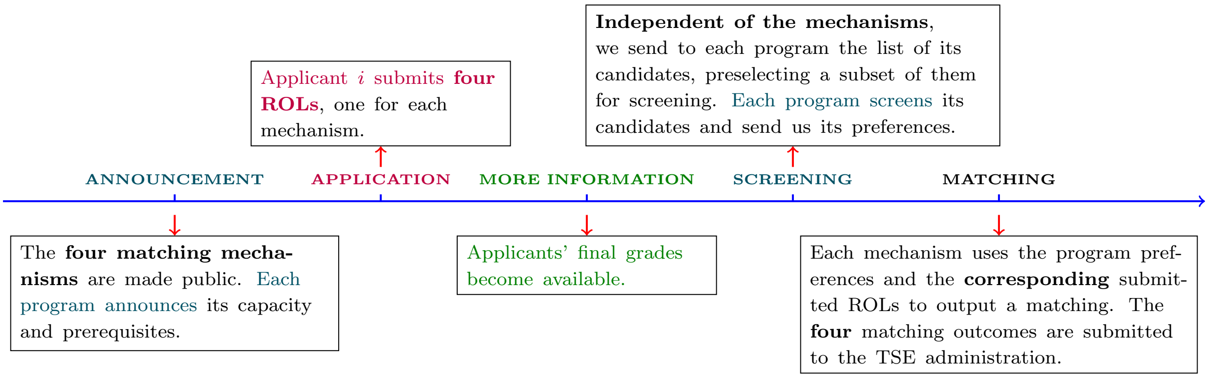

Figure II: Timeline of Matching Applicants with M2 Programs at TSE in the Experiment

and thus rejected by their two choices.

In our experiment, the "subjects" come from two sides of the matching market in 2013: The seven M2 programs, which are represented by their directors, and the 129 applicants who were finishing their M1 study at TSE and were applying for admission to TSE's M2 programs. As depicted in Figure II, the matching game in the experiment is played in five stages:

(i) **Announcement:** The programs announce their capacity and prerequisites (in terms of courses that an applicant must have taken during M1 study). Only two programs have such prerequisites. The programs also note that an applicant's final grade – a weighted average of all courses taken during the academic year – will be used in the screening process. Applicants are informed that one of the three aforementioned mechanisms (DA, DA-T, and DA-C) and the Immediate Acceptance (IA) will be chosen to match applicants with programs.[6] The version of DA-T is DA-T-4; under the designed DA-C, applicants can freely rank their top-3 choices and must write a motivation letter for each additional choice.

(ii) **Application:** *Under each mechanism*, applicants submit an ROL on an official university website that is also used for all coursework.

(iii) **More Information.** After applicants submit their ROLs, they take their final exams and obtain their final grades, which are weighted averages of all courses

[6]The Immediate Acceptance mechanism, also known as the Boston mechanism, is sometimes used for school choice in practice. The definition of the mechanism is given in Appendix A.1. Although also being included in the experiment that we conduct, this mechanism is not the focus of the current study. We therefore do not present the results of this mechanism.

taken during the academic year.

(iv) **Screening.** To each program, we send a list of the applicants who have included the program in at least one of her four submitted ROLs. We also attach the information on their final grades and their grades for each individual course, but we do not inform the programs about how each applicant ranks them. To save the programs screening costs and convince them that engaging in the experiment is not too costly for them, we, as the market designer and the clearinghouse, pre-select a subset of the program's candidates for screening. Meanwhile, the program directors are explicitly encouraged to screen all applicants. The programs then rank the applicants and send us their rankings.

(v) **Matching.** After receiving the programs' rankings over applicants, we calculate the matching under each of the four mechanisms and send the outcomes to the administration. The TSE administration chooses one of the four matchings to be implemented (which happens to be the one from DA).

The above information and the instructions of the experiment are all explained on the application website, and we provide screenshots in Appendix A.

"Sub-game perfect equilibrium." It is announced that one of the mechanisms will be chosen for the final matching, so the applicants should play a "sub-game perfect equilibrium." That is, under each mechanism, it is in applicants' best interest to behave optimally as if that mechanism were actually implemented. On the website, we also make this point explicit (see the screenshots in Figures E.1–E.5 in Appendix A.

One may be concerned that, under DA-C, applicants must pay the cost before knowing if the mechanism will be chosen. In other words, the return to writing a motivation letter is discounted by the probability that DA-C is chosen, which amounts to inflating the application cost. Taking this possibility into account, the following analysis does not directly use the data from DA-C for structural estimation, and instead, we use the data from DA to estimate applicant preferences.

Information on the mechanisms. We provide applicants with definitions of the mechanisms and explicit tips on how to play the game under each mechanism. Under DA, we emphasize that truth-telling, i.e., ranking programs according to their true preferences, is a dominant strategy. Under DA-T-4 and DA-C, it is noted that while truth-telling is no longer a dominant strategy, it is still in the applicants' best interest to order the ranked programs truthfully. The screenshots in Appendix A (Figures E.1–E.5) show what applicants could see on the website.

Pre-selecting candidates to be screened. To save each program screening costs, we use applicants' submitted ROLs to identify a subset of applicants over whom the program's strict preferences are required to calculate the matching. This identification is achieved by running each of the four mechanisms with a series of approximations of programs' preferences—first by sorting applicants by their average M1 grades and then by perturbing the ranking numerous times by adding independent noises. In this process, we also take into account whether an applicant meets the prerequisites of a program, if there are any. The variance of noises is significant, leading to a large enough subset of applicants to be screened by each program.

The number of pre-selected applicants is program-specific and ranges from 27 to 52, with an average of 41; Appendix A.3 provides more details. When the mechanisms were actually run, no program had to accept or reject an applicant who was not pre-selected for that program. That is, there was no need for the programs to screen more applicants than the pre-selected ones. Additionally, no program screened more candidates beyond the pre-selected ones, even though they were encouraged to do so. In our estimation and counterfactual analyses, we sometimes need the complete preference ranking of each program over all applicants. For this, we expand the ranking over the pre-selected applicants to all applicants following the procedure explained in Appendix B.

Randomized orders of programs and mechanisms. To prevent potential framing effects, we randomly assign the 129 applicants into 7 groups, each of which has a unique order of programs presented on the website. Each of the 7 programs is presented as the first to a unique group of applicants and presented as the last to another. Moreover, applicants in a given group play the four mechanisms in one of the following four sequences (DA, DA-T-4, DA-C, IA), (DA, DA-C, DA-T-4, IA), (IA, DA, DA-T-4, DA-C), and (IA, DA, DA-C, DA-T-4). We decide to have DA-T-4 and DA-C immediately follow DA in order to avoid potential confusion about the mechanisms.

3.2 Length of ROL and Unacceptable Programs

In the experiment, applicants are required to rank all 7 programs under DA, 4 programs under DA-T, and at least 3 programs under DA-C, as the web design makes it cumbersome to allow applicants to rank a flexible number of programs. To alleviate this constraint, we also elicit information from every applicant on whether a program is acceptable to him/her. When evaluating the programs' screening costs, we assume that applicants never apply to unacceptable programs (Assumption 2).

Before beginning the application process, every applicant must answer the following question for each program:

> "*If you are accepted (and only accepted) by* **program name**, *will you stay at TSE and register for the program in September 2013?*"

In the survey, "**program name**" is replaced by the full name of one of the seven programs. An applicant must tick one of the three possible answers, "Yes," "No," and "I Don't Know." We relabel these responses as "definitely acceptable," "unacceptable," and "possibly acceptable", respectively, whereby a possibly acceptable program is less preferred than a definitely acceptable one but more preferred than an unacceptable one. The same question is repeated for all seven programs, and we clarify that this information is not used in the actual matching process. In the following sections, we sometimes refer to the "definitely acceptable" and "possibly acceptable" categories collectively as "acceptable".

The responses to these questions are summarized in Table II. Two findings stand out from the table, which shows the heterogeneity of applicant preferences: Program P4 is definitely acceptable to 81% of the applicants and unacceptable to only 7% of them; in contrast, program P6, which is more specialized and has prerequisites, is unacceptable to the majority of applicants (78%).

When assuming that applicants do not apply to unacceptable programs, we notice that one applicant is matched with an unacceptable program under all mechanisms (and is enrolled in this program in September 2013). Since dropping the applicant's unacceptable programs from his/her ROLs would change the matching outcome under every mechanism, we instead re-categorize the program to be "possibly acceptable" to this applicant. Furthermore, there are 9 applicants whose ROLs under DA are not consistent

Table II: How the Applicants Consider the Programs: Percentages

Program	Unacceptable	Possibly Acceptable	Definitely Acceptable
P1	22	10	67
	(3.63)	(2.62)	(4.06)
P2	29	19	53
	(3.98)	(3.43)	(4.39)
P3	38	18	44
	(4.30)	(3.34)	(4.40)
P4	7	12	81
	(2.23)	(2.89)	(3.46)
P5	33	24	43
	(4.15)	(3.74)	(4.38)
P6	78	7	16
	(3.69)	(2.26)	(3.17)
P7	40	18	43
	(4.27)	(3.34)	(4.30)

Notes: This table shows the percentages of 129 applicants who consider each program as "unacceptable," "possibly acceptable," and "definitely acceptable". The results are calculated from the applicants' responses to survey questions that are not used in the actual matching process. Bootstrap standard errors from $10,000$ samples are in parentheses.

with the survey responses, provided that they report true preferences under DA. For example, one ranks an unacceptable program before a possibly acceptable one. For these applicants, we update some of the programs' acceptability to restore consistency. More details on this data cleaning are available in Appendix B.

4 Experimental Data and Direct Evaluation

In addition to observing applicant behaviors in the experiment, we also collect administrative data on applicants' grades from M1 courses (from the first semester and from both semesters), demographic information, and scholarship status. The key variables are summarized in Table III.

Table III: Summary Statistics

Variable	Mean	S.D.	Min	Max
M1 Grades: First semester	11.66	2.21	6.01	17.40
M1 Grades: Final (2 semesters)	12.27	2.04	8.02	17.68
Age	24.47	1.73	20.96	33.57
M1 Program: Economics	0.57	0.50	0	1
M1 Program: Statistics	0.34	0.48	0	1
Female	0.45	0.50	0	1
Scholarship	0.28	0.45	0	1

Notes: In total, there are 129 observations. Scholarship equals one if the applicant holds a scholarship from TSE or from the French government. Out of a total of 20 points, the grades are the credit-weighted average of grades from the courses taken during their first-semester or two-semester M1 study.

4.1 Direct Evaluation

We now turn to applicant strategies and matching outcomes under each mechanism. The two types of measures on which we focus are matching quality and screening costs. The first measure of matching quality is stability, which has been shown to be key to the success of matching markets (Roth, 1991). Additionally, we also consider applicant and program welfare. Screening cost is measured by the number of candidates to be screened.

Under the assumption that applicants report truthfully under DA, we take applicants' reported ROLs under DA as their true ordinal preferences. In this section, we do not make further assumptions on applicants' strategies under DA-T-4 or DA-C and instead evaluate what is observed. Recall that we assume that applicants do not apply to the programs that are deemed unacceptable or for which they do not meet the prerequisites. Therefore, we remove these programs from the ROLs.

4.1.1 Statistical Inferences with "One Observation"

The focus of the study is the matching game consisting of agents on both sides, applicants and programs. Therefore, in an ideal experiment, we would have the same game independently played multiple times by different sets of agents under different mechanisms. In other words, the unit of observation in the experiment is a game play or a matching market. However, our data contain outcomes from only one observation or from only one market under several mechanisms. Thus, careful consideration is necessary when making statistical inferences.

This leads us to resampling: We randomly resample applicants with replacement; the size of the resample is equal to the size of the original dataset; and in each resample, the programs are always kept the same. Implicitly, we assume that the empirical distribution of applications (types and strategies) is a good approximation of the theoretical distribution; by drawing from the empirical distribution independently, we create an independent play of the same game in each resample. In a given resample, multiple observations may originate from the same applicant in the original data; when programs rank the applicants, the ties among them are broken randomly.[7] Standard errors and testing results in this section are based on $10,000$ resamples.

[7]We use a lottery to break ties for each program. Because ties involve the same applicant being sampled multiple times in a resample, how ties are broken does not affect our outcome.

4.1.2 Applicant Strategies

Table IV presents the distribution of the number of programs ranked under each mechanism. By imposing a cost, both DA-T-4 and DA-C discourage the applicants from ranking too many programs. As a result, every applicant ranks 0.7 (0.9) programs fewer on average under DA-T-4 (DA-C) than under DA. The differences between these numbers of applications are all statistically significant (p-value < 0.01), as is the difference between DA-T-4 and DA-C (p-value < 0.05).

Table IV: Number of Programs Ranked by Applicants (in percentages)

Mechanism	# of programs ranked							Average number of programs ranked[a]
	1	2	3	4	5	6	7	
DA	4	10	17	26	28	15	0	4.09
	(1.69)	(2.62)	(3.31)	(3.87)	(3.95)	(3.14)	0.00	(0.12)
DA-T-4	4	11	29	57	-	-	-	3.38
	(1.69)	(2.70)	(3.98)	(4.34)	-	-	-	(0.07)
DA-C	5	14	56	12	9	5	0	3.21
	(1.84)	(3.02)	(4.42)	(2.87)	(2.54)	(1.85)	0.00	(0.10)

Notes: This table shows the distribution of the numbers of choices in the submitted ROLs under each mechanism. Each number – except those in the last column – represents the percentages of the applicants who rank that many programs in their ROLs. We remove from the ROLs the unacceptable programs and those for which the applicant is not qualified. Bootstrap standard errors from $10,000$ samples are shown in parentheses. [a] From t-tests, the differences in the numbers of applications under DA, DA-T-4, and DA-C are all statistically significant (p-value < 0.05).

DA-T-4 allows applicants to freely rank four programs, and therefore, the mode of the number of ranked programs is four, accounting for 57 percent of the applicants. Recall that it is free to rank the first three programs under DA-C, while the applicant has to pay a cost by writing a motivation letter for each of the fourth to seventh choices. Unsurprisingly, the mode is three; however, 26 percent of the applicants choose to pay some costs and rank more than three programs.

For DA-C, there is a negative but insignificant correlation between applicant first-semester grades and the number of applications. The correlation coefficient is -0.04 (p-value $= 0.41$).

4.1.3 Match Quality

Applicants' submitted ROLs and programs' rankings over applicants allow us to calculate the matching outcome under each mechanism. Table V reports the number of matched applicants by program and mechanism. Programs P5-P7 do not meet their capacities mainly because the total capacity (142) exceeds the total number of applicants (129).

Only under DA-C is an applicant unmatched.

Table V: Number of Matched Applicants by Program under Each Mechanism

Program	Capacity	#Matched Applicants under		
		DA	DA-T-4	DA-C
P1	14	14	14	14
		(0.06)	(0.12)	(0.06)
P2	22	22	22	22
		(1.30)	(1.28)	(1.48)
P3	22	22	22	22
		(2.30)	(2.01)	(2.00)
P4	28	28	28	28
		(0.02)	(0.02)	(0.02)
P5	22	12	12	11
		(3.08)	(3.21)	(3.09)
P6	12	10	10	10
		(2.31)	(2.31)	(2.31)
P7	22	21	21	21
		(2.07)	(2.15)	(2.35)
Total Unmatched		0	0	1
		(1.27)	(1.90)	(1.99)

Notes: This table shows the number of applicants matched with each program under each mechanism. The total capacity is 142. Three programs, P5-P7, do not meet their capacities. Bootstrap standard errors from 10,000 samples are in parentheses.

In terms of applicant welfare, 85 percent of the applicants are assigned to their most preferred program, as shown in Table VI. DA and DA-T-4 perform equally well on this dimension, while DA-C performs worse due to the applicant left unmatched with this mechanism. The ordinal welfare distribution barely varies across the 10,000 resamples under either mechanism.

Table VI: Ordinal Welfare of the Applicants under Each Mechanism

	Fraction of Applicants Matched with				**Unmatched**
	Most Preferred	2nd Preferred	3rd Preferred	4th Preferred	
DA	85	10	4	1	0
DA-T-4	85	10	4	1	0
DA-C	85	10	4	0	1

Notes: This table shows the numbers of applicants matched with their most preferred program, second most preferred one, etc., under each mechanism. The applicants' true ordinal preferences are what they reveal under DA. No applicant is matched with his/her 5th or less preferred program. Bootstrap standard errors from 10,000 samples are not reported, but are all less than 0.05.

Table VII further investigates individual matches. The left side of the table reports the frequency of blocking pairs under the two mechanisms. DA-T-4 leads to one blocking pair only, and DA-C results in two blocking pairs formed by two applicants and two programs.

Table VII: Deviation from the Optimal Stable Matching under Each Mechanism

Mechanism	Blocking Pairs			App. w/ Diff. Match			Prog. w/ Diff. Match		
	#pairs	#app.	#prog.	Total	Worse off	Better off	Total	Worse off[a]	Better off[a]
DA-T-4	1	1	1	3	2	1	3	2	1
	(1.87)	(1.73)	(0.85)	(3.36)	(2.24)	(1.45)	(1.74)		
DA-C	2	2	2	4	3	1	4	3	1
	(2.64)	(1.92)	(1.07)	(3.41)	(2.32)	(1.46)	(1.62)		

Notes: This table shows how the matching outcome under each mechanism is different from the applicant-optimal stable matching (i.e., the DA outcome). There are 129 applicants in total. Blocking pairs are defined with respect to applicants' true ordinal preferences that are elicited under DA. Bootstrap standard errors from $10,000$ samples are in parentheses. [a] Standard errors are not calculated for these two statistics because a program's welfare change cannot always be labelled as better off or worse off in every bootstrap sample (although it is feasible in the experimental data). Recall that a program is better off if all the matched applicants are (weakly) better than those matched in the old matching.

Furthermore, we investigate individual welfare across mechanisms. The middle part of Table VII compares the matchings from DA-T-4 and DA-C to that from DA. DA gives us the applicant-optimal stable matching, which also happens to coincide with the program-optimal matching in our data. DA-T-4 and DA-C lead to 3 and 4 applicants having different outcomes, respectively. Moreover, the number of applicants who are worse off is greater than the number of those who are better off.

The right side of Table VII shows the number of programs that have different matching outcomes under DA-T-4/DA-C relative to DA. 3 (4) programs' outcomes are affected by DA-T-4 (DA-C), and more programs are worse off than better off under DA-T-4/DA-C.[8]

In summary, both applicant welfare and program welfare differ across the mechanisms, but the magnitude is small or even negligible. However, as we emphasize, programs' screening cost is another important factor, and we now investigate the screening costs that programs must pay under each mechanism.

4.1.4 Programs' Screening Costs

Screening cost depends on whether the programs can screen applicants round by round, and crucially, this possibility relies on how programs can use information about applicants' submitted ROLs.

For various reasons, some programs may prefer to take into account the applicants' preferences or their ROLs when screening them. This would, however, create incentive problems for the applicants. To preserve the strategy-proofness of DA, it is usually

[8]Given two matchings, μ and μ', program j is better off if the newly matched applicants, $\mu'^{-1}(j)\backslash(\mu^{-1}(j) \cap \mu'^{-1}(j))$, dominate the displaced applicants, $\mu^{-1}(j)\backslash(\mu^{-1}(j) \cap \mu'^{-1}(j))$, i.e., element-wise $\mu'^{-1}(j)\backslash(\mu^{-1}(j) \cap \mu'^{-1}(j)) \succ_j \mu^{-1}(j)\backslash(\mu^{-1}(j) \cap \mu'^{-1}(j))$ when the matched applicants in these two sets are ordered according to j's ordinal preferences, $\succ_j$.

required that applicants' ROLs are not seen by the programs. For example, the National Resident Matching Program (NRMP) – the clearinghouse that matches medical residents with hospitals – states that "*[y]our rank order list is confidential and never will be shared with the programs.*" One way to keep ROLs confidential is to ask the programs to screen the applicants all at once before running the mechanisms, which is the practice in, for example, university admissions in Germany and Victoria, Australia and the NRMP, where certain variants of DA are used.

Furthermore, recall that we make no assumptions about screening cost except that it strictly increases with the number of candidates to be screened (see Section 2 for more details). Given this assumption, Table VIII shows, on average, how many candidates a program screens under each mechanism or, equivalently, how many applicants include a given program in their submitted ROLs under each mechanism. When the application cost increases from DA to DA-T-4 (DA-C), the number decreases from 75 to 62 (59), while the differences are all significant at the 1% level based on t-tests.

Table VIII: Summary Statistics: Number of Candidates to Be Screened

	mean	s.d.	min	max
DA	75.29	37.39	11	120
DA-T-4	62.29	33.97	11	117
DA-C	59.14	31.14	11	107

Notes: Each row of this table shows the summary statistics of the number of candidates to be screened across the seven programs. Treating each program under a given mechanism as an observation, t-tests show that the difference between any two mechanisms is significant at the 1% level.

We further normalize each program's screening cost by its capacity. Figure III shows the number of candidates per opening (i.e., per available seat) to be screened by each program under each mechanism. Again, the screening cost is negatively correlated with the application cost. Under DA, a program has to screen 3.69 candidates per opening on average. DA-T-4 reduces the cost by approximately 0.65 candidate per opening; DA-C is even more effective. Moreover, when moving from DA-T-4 to DA-C, this reduction occurs for every program except P6.

In conclusion, among the three mechanisms, the increases in application cost does not significantly affect match quality but greatly reduces programs' screening costs.

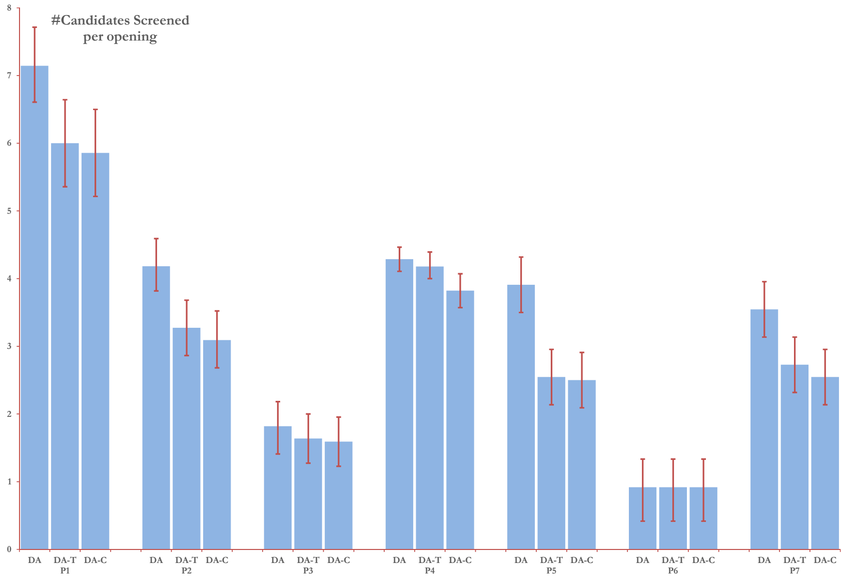

Figure III: Number of Candidates Screened by Each Program per Opening

Notes: This figure shows the total number of candidates screened by each program per opening under each mechanism. Programs must screen all their candidates at once before the mechanism is implemented. The average number of candidates screened across the seven programs is 3.69 under DA 3.04 under DA-T-4, and 2.90 under DA-C. The error bars indicate the 90% confidence intervals from $10,000$ bootstrap samples.

5 Evaluation Based on Structural Estimation

The set of market designs that we can directly evaluate is necessarily restricted because we implement only one instance of DA-T and one instance of DA-C in the experiment. In order to draw a complete picture of the trade-offs between screening costs and matching quality, we would like to evaluate many other market designs with different parameters governing truncation and application costs. Building on the rich information on both programs and applicants we observe, this section implements a structural estimation and conducts evaluations using counterfactual analysis.

To achieve this goal, our empirical strategy contains several stages. When DA is replaced by DA-T or DA-C, an applicant's optimal strategy is no longer truthful reporting, and instead, it depends on other applicants' actions. Recovering the cardinal preferences of all agents becomes necessary. First, we use experimental data to estimate applicants' cardinal preferences. In particular, we combine applicants' submitted ROLs under DA with the survey data on program acceptability; we also make use of the fact that truthfully ranking acceptable, qualified programs is the unique equilibrium (Assumption 3). Second, we estimate program preferences over applicants from programs' submitted rankings of applicants, given the assumption that programs truthfully report their preferences

(Assumption 1).

Next, we use data on final grades to estimate applicants' beliefs about their probabilities of acceptance by each program. Their optimal strategies, however, depend on the solution concept that we retain. We specify the structure of incomplete information of the game that applicants are assumed to play, leading to use a Bayesian Nash equilibrium as defined in Section 2. That is, when submitting ROLs, we assume that each applicant knows her own preferences but not others', even though she is aware of their distribution. Moreover, in spite of knowing what criteria programs use to rank applicants, she is uncertain about how she is ranked by programs.

Finally, we simulate the counterfactual equilibria attained when different versions of DA-T and DA-C are used.

The rest of this section formalizes the game of incomplete information. We first specify the following key elements: (i) The estimation of applicants' cardinal preferences, (ii) the estimation of program preferences, (iii) the information structure and solution concept, and (iv) applicants' beliefs about program preferences. For counterfactual analyses, we also explain the procedure for computing equilibria under each of these mechanisms.

5.1 Applicant Preferences

For each applicant, we observe a submitted ROL under DA, $(l_i^{*,1}, ., l_i^{*,J})$, which corresponds to her true ordinal preferences and from the survey, we observe whether each program is definitely acceptable, possibly acceptable, or unacceptable to the applicant.

Econometric model. Suppressing subscript i, we first postulate that the utility function associated with each program, $j \in \{1, ., J\}$, is:

$$v_j = x\beta_j + \varepsilon_j, \tag{1}$$

in which ε_j is extreme-value distributed and independent across programs. Furthermore, we model the acceptability using an outside option whose value is written as the sum of two terms. The first term is observed at the moment of the application decision and is written as

$$v_0 = x\beta_0 + \varepsilon_0. \tag{2}$$

To model programs as "definitely acceptable", "possibly acceptable", or "unacceptable", we posit that final acceptability is determined by adding to v_0 the value of another random variable, $\eta \in [x\beta_f + \underline{\eta},\ x\beta_f + \overline{\eta}]$, which is revealed after the matching process.[9] Program j is said to be **definitely acceptable** if and only if it has a value higher than the best possible outside option:

$$v_j > v_0 + x\beta_f + \eta \text{ for any } \eta,$$

that is,

$$v_j > v_0 + x\beta_f + \overline{\eta}.$$

Similarly, the program is deemed (definitely) **unacceptable** if and only if it is always dominated by the worst possible outside option:

$$v_j < v_0 + x\beta_f + \underline{\eta}.$$

Otherwise, the applicant considers the program **possibly acceptable** if

$$v_j \in [v_0 + x\beta_f + \underline{\eta}, v_0 + x\beta_f + \overline{\eta}]. \tag{3}$$

Let the ranking information be described as the ROL complemented with the max/min outside options $(\overline{O}, \underline{O})$ so that the observation is now an extended ROL:

$$(l^{*,1}, ..., l^{*,\overline{J}}, \overline{O}, l^{*,\overline{J}+1}, ..., \underline{O}, l^{*,\underline{J}}, ..., l^{*,J}),$$

in which $l^{*,\overline{J}} \in \mathcal{J}$ for $1 \leqslant \overline{J} \leqslant J$ is the lowest-ranked definitely acceptable program. When $\overline{J} = 0$, there is no definitely acceptable program, and when $\overline{J} = J$, all programs are definitely acceptable. Similarly, $l^{*,\underline{J}} \in \mathcal{J}$ for $1 \leqslant \underline{J} \leqslant J$ is the highest-ranked unacceptable program so that $\underline{J} > \overline{J}$ and that $\underline{J} > J$ implies an absence of unacceptable programs.

Identification and likelihood function. Location normalization is needed, and the simplest normalization sets the lower bound of the outside option to zero:

$$(\beta_0 + \beta_f) = 0, \underline{\eta} = 0.$$

[9] This formalization to take into account the three types of acceptability differs from our theoretical framework in Section 2. However, as long as we categorize both "definitely acceptable" and "possibly acceptable" as acceptable, our theoretical results still hold true.

The upper bound, $\overline{\eta}$, is to be estimated. The scale normalization is given by the usual logit assumption. The parameter vector is thus described by $\theta = ((\beta_j)_{j=1,.,J}, \overline{\eta})$. If the survey information on program acceptability were not available, the choice probability of ROL $(l^1, ., l^J)$ would be described by an exploded logit (Beggs, Cardell and Hausman, 1981):

$$\Pr(v_{l^1} = \max_{j \in \{1,\ldots,J\}} v_j, v_{l^2} = \max_{j \in \{2,\ldots,J\}} v_{l^j}, \ldots, v_{l^{J-1}} = \max_{j \in \{J-1,J\}} v_{l^j}) = \prod_{j=1,\ldots,J-1} \frac{\exp(x\beta_{l^j})}{\sum_{k=j}^{J} \exp(x\beta_{l^k})}.$$

When we introduce information on program acceptability, the likelihood function can be written as a sum of exploded logit terms (see Appendix C). We can then estimate θ by maximum likelihood.

Results. We consider various sets of explanatory variables taken from administrative information on grades, M1 program (Economics, Econometrics and Statistics, Economics and Law), gender, age, and having a scholarship or not. There are many course grades that can be used at a fine level of detail. Because of the timing of the application decision, we opt to use the M1 first-semester grade as the main explanatory variable. At the time of application, no applicant knows his/her final grades, the weighted sum of his/her first- and second-semester grades.

The results are presented in Table IX. A clear pattern is that one's M1 program have some power to predict her preferences over the M2 programs, as the coefficients of "M1 Economics" and "M1 Statistics" have different signs across M2 programs.

5.2 Program Preferences

Program j sets a latent score for each applicant, $s_{i,j}$, according to which applicants are ranked. Conditional on meeting prerequisites $p_{i,j}$ (a binary variable), $s_{i,j}$ depends only on the final grade (the credit-weighted average of grades from two semesters), $FinalGrade_i$, and some noise,

$$s_{i,j} = FinalGrade_i + \sigma\xi_{i,j}, \tag{4}$$

in which $\xi_{i,j}$ is extreme value distributed. The term $\xi_{i,j}$ stands for the additional information that programs can use beyond final grades. $\xi_{i,j}$ is assumed to be independent of $FinalGrade_i$ and the covariates, x_i.

Program prerequisites are common knowledge, and applicants condition their decision

Table IX: Estimation of Applicant Preferences

	P1	P2	P3	P4	P5	P6	P7
Grade: First semester	0.28	0.10	0.00	0.10	0.00	-0.04	-0.01
	(0.08)	(0.07)	(0.07)	(0.06)	(0.06)	(0.08)	(0.08)
M1 Program: Economics	0.58	-0.37	0.83	-0.12	-0.027	-6.65	0.016
	(0.50)	(0.49)	(0.98)	(0.78)	(0.55)	(1.16)	(0.51)
M1 Program: Statistics	1.13	-0.71	2.73	-0.31	-0.02	-6.43	0.58
	(0.51)	(0.52)	(1.02)	(0.81)	(0.59)	(1.18)	(0.54)
Female	-0.39	0.95	0.55	-0.54	0.672	0.51	-0.04
	(0.31)	(0.30)	(0.30)	(0.31)	(0.27)	(0.36)	(0.32)
Scholarship	-0.57	-0.99	-0.024	-0.58	0.00	-0.32	-0.70
	(0.36)	(0.37)	(0.27)	(0.35)	(0.27)	(0.40)	(0.33)
Intercept	1.11	0.71	0.377	1.88	0.55	-0.87	0.40
	(0.16)	(0.16)	(0.18)	(0.16)	(0.14)	(0.19)	(0.16)
Upper bound of shocks to the outside option:			0.707				
			(0.07)				

Notes: Estimation is based on applicants' submitted ROLs under DA supplemented with survey information on program acceptability. The likelihood function is specified in Appendix C and can be considered as an extended version of the multinomial logit with rank-ordered data combined with acceptability. The McFadden pseudo-R-squared is equal to 0.078.

on $p_{i,j}$. Program preferences are therefore lexicographical and rank the subsample $p_{i,j} = 1$ first, the subsample $p_{i,j} = 0$ second, and according to $s_{i,j}$ within the first subsample.[10]

Given the structure of the experiment, we can estimate the parameters governing program preferences using a rank-ordered logit on the subsample of applicants satisfying the prerequisites, $p_{i,j} = 1$. Nonetheless, the rank-ordered logit procedure is fragile when many alternatives – or applicants – are considered, as we do here, in a small sample of seven programs. Therefore, we consider a limited information procedure using only length-K rankings among applicants with $K \geqslant 2$.

Specifically, let the full ranking of applicants for program j be $(i_1^{(j)}, ., i_I^{(j)})$, and partition this length-I vector into $I(K)$ vectors starting with $(i_{1,}, ., i_K)$, then $(i_{K}, ., i_{2K-1})$, etc. We write the pseudo-likelihood function corresponding to these $J \times I(K)$ observations treated as ROLs while neglecting possible correlations across observations. This process does not affect the consistency of the estimates (Avery, Hansen and Hotz, 1983), although standard errors should be computed using a sandwich formula.[11]

[10]Because no one with $p_{i,j} = 0$ is accepted by program j, how we rank the applicants with $p_{i,j} = 0$ does not matter, as long as they are ranked after those with $p_{i,j} = 1$.

[11]An alternative is to consider sequences $(i_{1,}, ., i_K)$, $(i_{K+1,}, ., i_{2K})$, etc., in which independence is satisfied but information is more limited.

Results. The estimation of the latent score, equation (4), depends on how we choose K to form length-K ROLs. We present results for three different K's, while the results from $K = 2$ are used in the counterfactual analysis. When $K = 2$, the estimates are as follows:

$$\tilde{s}_{i,j} = FinalGrade_i + \underset{(.022)}{0.058} \times \xi_{i,j}, \text{McFadden pseudo } R^2 = 0.337;$$

when $K = 3$:

$$\tilde{s}_{i,j} = FinalGrade_i + \underset{(.020)}{0.056} \times \xi_{i.j}, \text{McFadden pseudo } R^2 = 0.391;$$

when $K = 5$:

$$\tilde{s}_{i,j} = FinalGrade_i + \underset{(.015)}{0.049} \times \xi_{i,j}, \text{McFadden pseudo } R^2 = 0.496.$$

5.3 Solution Concept and Information Structure

We first clarify the solution concept and evaluation approach that we adopt to conduct counterfactual analysis. Solution concepts depend on the information that each applicant has, as Panel A of Table X shows. A possibility is to assume complete information in our matching game. That is, both applicant and program preferences are common knowledge to every applicant, which leads us to Nash equilibrium as the solution concept. However, the complete-information assumption may be too restrictive in our setting because it is unlikely that every applicant knows everyone else's cardinal preferences or can predict her final grade and program preferences perfectly (cf. Figure I). Indeed, for this reason, our model in Section 2 specifies a game of incomplete information, which leads to Bayesian Nash equilibrium.

In addition, multiple methods are available to evaluate counterfactual outcomes, depending on the use of information available to researchers. First, we always use common-knowledge information X. As Panel B of Table X shows, the outcome under any given market design can be evaluated with or without information on applicants' and programs' observed ordinal preferences. If the information in the data on ordinal preferences is utilized, we have a "sample-specific" evaluation; otherwise, when this information is not used, the method is "population-specific".

In addition to choosing Bayesian Nash equilibrium (A2) as our solution concept, we

Table X: Solution Concepts and Evaluation Methods

	Panel A. Solution concepts
Complete information?	
A1. Yes (Nash equilibrium)	Applicant and program preferences are common knowledge. Every applicant best responds to others' actions in equilibrium.
A2. No (Bayesian Nash)	Applicant and program preferences are private information, but their distributions and X are common knowledge. Every applicant best responds to the distribution of others' actions in equilibrium, conditional on common knowledge.
Panel B. Methods for Outcome Evaluation (conditional on common knowledge X)	
Conditional on observed applicant & program ordinal preferences?	
B1. Yes (sample-specific)	An outcome is evaluated conditional on X and on the observed ordinal preferences of applicants and programs. That is, we consider only random draws of preference shocks consistent with the observed ordinal preferences. Each simulated applicant (with simulated cardinal preferences) plays an equilibrium strategy according to either Nash equilibrium (A1) or Bayesian Nash (A2).
B2. No (population-specific)	An outcome is evaluated conditional on X but unconditional on the observed ordinal preferences of applicants and programs. That is, we consider random draws of shocks from the assumed distribution. Each simulated applicant (with simulated cardinal preferences) plays an equilibrium strategy according to either Nash equilibrium (A1) or Bayesian Nash (A2).

adopt the "sample-specific" evaluation (B1). This evaluation facilitates the comparison between our experimental results with the counterfactual analyses, as the former is obviously conditional on applicants' and programs' realized ordinal preferences.

To summarize, when solving for a Bayesian Nash equilibrium, we need every applicant to best respond to others with incomplete information on other applicants' and programs' preferences. The distributions of applicant and program preferences are modeled and estimated in the previous two subsections. However, programs form preferences with information on applicant final grades that is not available at the time of application (cf. Figure II). It is therefore necessary to consider the uncertainty in final grades when we estimate applicants' beliefs about program preferences, which is what we turn to next.

5.4 Applicant Beliefs on Program Preferences

When forming beliefs about program preferences, applicant i observes her own characteristics, x_i, including her grades during the first semester and the prerequisites of each program ($p_{i,j} = 1$ if i satisfies the prerequisites of program j and 0 otherwise). These prerequisites consist of choices of courses that are made well in advance and are partially determined by one's M1 program. We therefore consider them as fixed in the statistical model.

We assume that applicants have rational expectations and expect that

(i) The information structure detailed above is common knowledge and, in particular, program preferences are formed according to the model above.

(ii) Applicant i predicts her final grade $FinalGrade_i$ as a function of characteristics x_i, using the following equation:

$$FinalGrade_i = x_i\gamma + \nu_i, \tag{5}$$

in which x_i includes the characteristics used in the modified exploded logit model above and also the first-semester grade in particular; ν_i is assumed to have a normal distribution.

The specifications of applicant and program preferences and the grade equation guarantee that any possible applicant preference ranking and program preference ranking among qualified applicants have a positive probability of occurrence. It is straightforward to show that equilibrium acceptance probabilities are non-degenerate for any qualified applicant, and therefore, Assumption 3 is satisfied.

Estimation. The rational-expectation assumption implies that applicant beliefs are consistent with the distribution of final grades and program preferences.

The grade equation (5) is estimated by standard ordinary least squares (OLS), and the results are shown in Table XI. Not surprisingly, the main determinant of the final grade is the first-semester grade. One may notice that the coefficient on "Scholarship" is significantly negative, which may be explained by the fact that scholarship is often need-based. For the residual ν_i in the grade equation, the assumption of normal distribution is reasonable, as evident in a quantile-quantile plot of the residuals, which is available upon request.

Based on these results, we simulate in the counterfactual analysis applicant beliefs about program preferences by drawing ν_i in a normal distribution and $\xi_{i,j}$ in extreme value distributions. Furthermore, we take as constant the prerequisites of P3 and P6

Table XI: Estimation of the Grade Equation

	Estimate	Std. Error	t Statistics	Pr(>|t|)
Grade: First semester	0.86	0.03	27.65	0.00
M1 Program: Economics	-0.42	0.25	-1.70	0.09
M1 Program: Statistics	-0.73	0.26	-2.77	0.02
Female	0.08	0.14	0.57	0.57
Scholarship	-0.37	0.15	-2.46	0.02
Intercept	2.83	0.42	6.77	0.00

Notes: Number of observations: 129; Residual standard error: 0.76 on 123 degrees of freedom; R-squared: 0.87; Adjusted R-squared: 0.86; F-statistic: 158.1 on 5 and 123 degrees of freedom, p-value: $< 2.2e - 16$;

because they are decided well in advance. Recall that we assume away programs' strategic behaviors, and this therefore describes our set-up for simulating applicant beliefs on program preferences.

5.5 Counterfactual Analysis of Market Designs

5.5.1 Simulating Equilibrium Outcomes: Procedures

We solve for a Bayesian Nash equilibrium (case A2 in Table X) under each mechanism and then evaluate the sample-specific counterfactual outcomes (case B1 in Table X).

Counterfactuals under DA-T and DA-C. We perform counterfactual analysis under DA-T with degrees of truncation varying from 1 to 6. Under DA-C, the application cost has the following form in the counterfactual analysis:

$$C(|L|) = c \times (|L| - 1) \times \mathbb{1}\{|L| > 1\},$$

in which c is the constant marginal cost for second and further choices and c can take one of the 11 values in $\{10^{-5}, 3.1 \times 10^{-5}, 10^{-4}, 3.1 \times 10^{-4}, ..., 0.01, 0.31, 1\}$, increasing on a logarithmic scale. We have $C(|L|) = 0$ for $|L| \leqslant 1$ to avoid distorting the applicants' participation decisions.

To make sense of the magnitude of application costs, one may consider the following thought experiment: A "possibly acceptable" program can be improved to "definitely acceptable" by adding at most 0.707, which is the estimate of the maximum shock to outside option ($\overline{\eta}$ in equation 3); the same program can be demoted to "unacceptable" by subtracting 0.707 at most. Moreover, the variance of utility shocks is $\pi^2/6 (\approx 1.645)$. From this perspective, most of the above application costs, especially those below 0.031, are quite small.

Computation of equilibrium. For both DA-T and DA-C, an equilibrium must be solved as a fixed point in terms of cutoff distribution. A program's *cutoff* in a given matching is the lowest rank among the applicants accepted by the program if it has no vacancy; otherwise, the cutoff is zero. A Bayesian Nash equilibrium can be summarized by the joint distribution of the cutoffs of programs (Azevedo and Leshno, 2016; Fack et al., 2015). Given a particular draw of random shocks, an applicant ranked above a program's cutoff can be accepted by this program. Solving for a Bayesian Nash equilibrium can then be achieved by finding a fixed point of cutoff distribution.

We briefly outline the algorithm here, and further details are provided in Appendix D.1. The algorithm works with M (4000 in our implementation) simulated samples of size I (the number of applicants) in which the random shocks that contribute to the construction of applicant preferences ($\varepsilon_i^{(m)}$), to final grades ($\nu_i^{(m)}$), and to program preferences ($\xi_{i,j}^{(m)}$) are drawn from their respective distributions. These random shocks remain fixed in the whole procedure.

Over the M simulations, our procedure aims at finding a distribution of cutoffs and outcomes whose randomness comes from random applicant preferences, random grades, and random program preferences. We adopt the following iterative procedure:

- *Initialization*: The distribution of cutoffs across the M simulated samples, say, $\Phi_0 = \{\Phi_0^{(1)}, ., \Phi_0^{(M)}\}$, is initialized at the value obtained under DA in the observed sample. Therefore, Φ_0 is degenerate because it is constant across the simulated samples. We also calculate applicants' and programs' preferences in each simulation sample m:

  (i) We compute applicants' cardinal and therefore ordinal preferences by inputting the simulated shocks $\varepsilon_i^{(m)}$, estimated coefficients, and observed characteristics into the utility functions specified in equations (1) and (2). Note that shocks to applicant preferences ($\varepsilon_i^{(m)}$) are drawn **unconditionally** on the ordinal preferences observed in the sample.

  (ii) We attribute to each program the same capacity as in the experiment and compute its "cardinal" preferences over applicants by plugging the simulated shocks ($\nu_i^{(m)}$ and $\xi_{i,j}^{(m)}$), estimated coefficients, and observed first-semester grades into the grade function (equation 5) and the latent score function (equation 4). These preferences are then translated into rankings over applicants in increasing order of preference from 1 to I, denoted by $r_{i,j}^{(m)}$.

- *Iteration*: For $t = 1, 2, \ldots$, iteration t begins with a distribution of cutoffs, Φ_{t-1}, and has two steps to obtain Φ_t:

  (i) In each simulation sample m, each applicant submits the ROL that maximizes her expected utility given the "current" distribution of cutoffs, Φ_{t-1}. Necessarily, the optimal ROL takes into account the application cost in either DA-T or DA-C. This computation is further detailed in Appendix D.1. Recall that no applicant applies to unacceptable programs or programs whose prerequisites are not met by the applicant (Assumption 2).

  (ii) In each simulation sample m, we run the DA algorithm using ROLs submitted by applicants and simulated program preferences ($r_{i,j}^{(m)}$). This process results in a matching outcome and a cutoff for each program, which leads us to the updated cutoff distribution, Φ_t.

- *Stopping rule*: Stop when $\|\Phi_t - \Phi_{t-1}\|$ is small enough.

The resulting distribution of cutoffs, $\Phi_e = \{\Phi_e^{(1)}, ., \Phi_e^{(M)}\}$, characterizes a Bayesian Nash equilibrium under the given mechanism.

Sample-specific counterfactual actions and outcomes. We evaluate the counterfactual welfare conditional on applicants' and programs' observed ordinal preferences, which is case B1 in Table X in Section 5.3. That is, the shocks to applicant preferences (ε) are simulated conditional on the observed ROLs submitted by applicants under DA, and program ordinal preferences are also held constant at their observed values. In this way, we answer the following question: *Based on all the information observed by the researcher, what is the best prediction of the counterfactual outcome if a counterfactual market design had been implemented at TSE in 2013?*[12]

[12]There are certainly many alternative counterfactual questions that one can ask. Our choice makes it easier to compare simulated equilibrium outcomes with the observed outcomes because the latter are certainly conditional on the realized ordinal preferences of applicants and programs.

More specifically, we adopt the following procedure to simulate a new set of samples:

(i) In each simulation sample m, we compute applicants' cardinal and therefore ordinal preferences by inputting the simulated shocks $\varepsilon_i^{(m)}$, estimated coefficients, and observed characteristics into the utility functions specified in equations (1) and (2). As spelled out in Section 5.3, we now compute shocks to applicant preferences ($\varepsilon_i^{(m)}$) **conditionally** on the ordinal preferences observed in the sample using a Gibbs sampler described in Appendix D.2.

(ii) Each applicant submits the ROL that maximizes her expected utility, given the equilibrium distribution of the cutoffs, Φ_e. This process is conducted separately for each version of DA-T and DA-C.[13]

(iii) In each simulation sample, we run the DA algorithm with the optimal ROLs submitted by applicants and the observed program preferences.

Similar to the direct evaluation, we measure the performance of each design on several dimensions: (i) Matching quality, which is summarized by stability (or the incidence of blocking pairs) and unmatched applicants,[14] and (ii) screening costs. In addition to examining applicants' and programs' ordinal welfare, we evaluate their cardinal welfare, given that our preference estimates have cardinal implications.

5.5.2 Counterfactual Analysis: Results

Our first set of results is summarized in Figure IV, which shows the effects of market design on screening costs, measured as the number of candidates screened per opening, and on matching quality, measured as the number of blocking pairs and the number of unmatched applicants.[15]

Subfigure (a) depicts the effects of DA-T of different degrees. DA-T-K, for $K \in \{1, 2, ..., 7\}$, means that applicants are allowed to freely rank only up to K choices; therefore, DA-T-7 is DA (without truncation). Note from the figure that when increasing

[13] Note that applicants' equilibrium actions under DA with no cost are always constant in our sample-specific counterfactual analysis.

[14] Note that the stable matching in our data is unique in all simulation samples, given the ordinal preferences of applicants and programs (c.f., Section 4). In other words, the applicant-optimal stable matching and the program-optimal stable matching coincide.

[15] The confidence intervals shown in Figure IV consider the uncertainty in applicant preferences but not the uncertainty in the estimated coefficients. Our simulations of DA-T that take into account the latter uncertainty show similar results to Figure IVa.

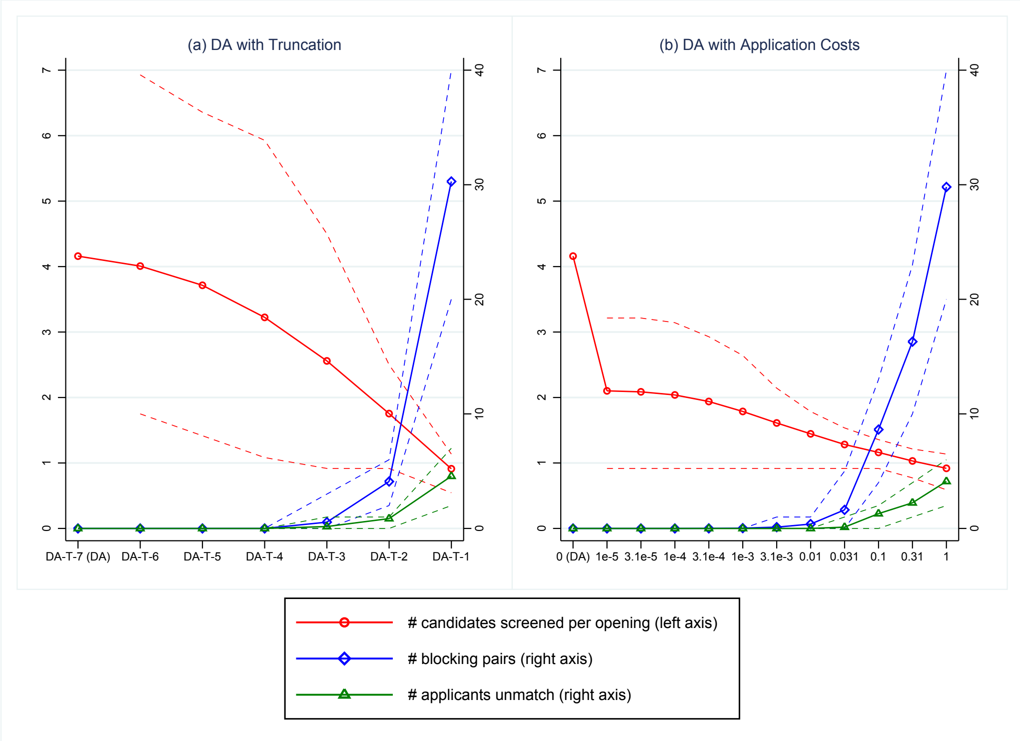

Figure IV: Match Quality under DA with Truncations or Finite Application Costs

Notes: This figure shows the counterfactual analysis of the effects of market design on matching quality (measured by the number of blocking pairs and the number of unmatched applicants) and screening costs (measured by the number of candidates screened per opening). Subfigure (a) depicts the effects of DA-T with different degrees, where DA-T-K, $K \in \{1, 2, ..., 7\}$, meaning that applicants are allowed to freely rank only up to K choices. Therefore, DA-T-7 is DA (without truncation). Subfigure (b) presents the effects of DA-C with a constant marginal cost for second and later choices and a zero cost of ranking the first choice. The constant marginal cost varies from 0 to 1, and it becomes DA when the cost is zero. The dashed lines are the 90% confidence intervals from the 4000 simulation samples. There is no variation in simulating the outcome under DA because all simulated applicant and program preferences are conditional on the observed ordinal preferences.

the application limit from DA-T-7 to DA-T-3, match quality remains (almost) the same as the applicant-optimal stable matching, while screening costs decrease monotonically from screening 4.16 candidates per opening to screening 2.56 per opening. When the application limit tightens further, the screening costs decrease further to 1.75 and 0.91 candidates per opening under DA-T-2 and DA-T-1, respectively. However, this result comes at the expense of matching quality: The number of blocking pairs increases from 0.5 under DA-T-3 to 4.10 under DA-T-2 and then skyrockets to 30.29 under DA-T-1; we also observe some unmatched applicants, 0.86 under DA-T-2 and 4.56 under DA-T-1.

Subfigure (b) examines the effects of DA-C with a constant marginal cost for second and later choices, while there is no cost of submitting the first choice. As explained above, the marginal costs vary from 0 to 1.

Several interesting patterns emerge. First, the screening costs drop dramatically, even when there is a very low application cost: Imposing an application cost of 10^{-5} decreases the number of candidates screened per opening by 50%, while match quality remains the same. Second, the reduction in screening costs by this low application cost is larger than that of DA-T-3 (although smaller than that of DA-T-2). When the cost rises, screening costs further decrease, but the match quality begins to be impacted only when the cost is above 0.0031. Third, when the application cost is 1, the outcome is almost identical to DA-T-1 because the cost is so high that it is not worth ranking a second program.

In summary, Figure IV describes the trade-off between screening costs and match quality; more importantly, imposing a mild application cost in either DA-T or DA-C, especially the latter, can significantly decrease screening costs without harming match quality.

One may be concerned that our analysis of DA-C does not take into account the application costs that are actually paid. We argue that application costs can be designed as a transfer, e.g., as a fee paid to programs, rather than as a deadweight loss, e.g., writing a motivation letter. Therefore, application cost affects only the distribution of total welfare between applicants and programs. Nonetheless, in the following, we study how applicant welfare is affected by such costs.

5.5.3 Applicants Welfare

Cardinal welfare. Table XII provides an evaluation based on the estimated cardinal preferences while explicitly taking into account the application costs. For each given market design, we first calculate cardinal welfare per applicant in each simulation sample and then measure its mean and standard deviation across all simulation samples.

Panel A first presents per-applicant cardinal welfare across the DA-T of different degrees under which an applicant can submit a limited number of choices. The results confirm Figure IV and show that DA-T-6 to DA-T-4 achieve the exact DA outcome, while DA-T-3 incurs a small welfare loss. Surprisingly, DA-T-2 obtains a welfare gain that is comparable to re-assigning 1.5 applicants from an "almost" unacceptable program to an "almost" definitely acceptable one.[16] This outcome occurs even though DA-T-2 leaves 0.86 applicants unmatched on average, as partially reflected in the increase in the

[16] Recall that an applicant considers program j possibly acceptable if and only if $v_j \in [\varepsilon_0, \varepsilon_0 + \overline{\eta}]$. The utility difference between an "almost" unacceptable program and an "almost" definitely acceptable one is $\overline{\eta}$, and the estimate of $\overline{\eta}$ is 0.707.

standard deviation of applicant welfare. DA-T-1 brings the lowest applicant welfare with the highest variance, consistent with the large number of unmatched applicants (4.56). Relative to DA, the welfare loss in DA-T-1 is comparable to moving 2.9 applicants from an almost unacceptable program to an almost definitely acceptable one.

The analysis of DA-C can also be found in Panel A of Table XII. In addition to describing the per-applicant welfare produced by matching outcomes, we also report the application costs paid by applicants and their net welfare. In terms of matching outcomes, per-applicant welfare has a non-linear relationship with application cost: Welfare increases with the application cost when the cost is not too high ($\leqslant 0.31$), but it drops to the lowest when it is prohibitively high ($= 1$). In contrast, the standard deviation of per-applicant welfare monotonically increases with cost, consistent with the monotonically increasing number of unmatched applicants. Relative to DA, the maximum welfare gain is obtained when the cost equals 0.31, equivalent to re-assigning 5.8 applicants from an almost unacceptable program to an almost definitely acceptable one; the largest welfare loss (under DA-T-1) amounts to moving 1.1 applicants from an almost definitely acceptable program to an almost unacceptable one.

The application costs that are actually paid by applicants follow the same pattern as the welfare. Applicants pay higher costs when the cost increases but remains moderate, and they stop applying to additional programs once the cost is prohibitively high. The maximum costs are paid when the marginal cost is 0.31: The magnitude is equivalent to re-assigning 7.8 applicants from an almost definitely acceptable program to an almost unacceptable one.

As discussed earlier, we can consider application cost as a transfer from applicants to programs, e.g., a transfer of monetary fees, and therefore, it does not incur a dead-weight loss, in contrast to writing essays that would otherwise be useless. Nonetheless, one may be interested in applicants' net welfare, given the application cost they pay. Panel A of Table XII shows that the net welfare is monotonically decreasing in application cost. Relative to DA, the maximum welfare loss (at marginal cost = 1) amounts to re-assigning 2.5 applicants from an almost definitely acceptable program to an almost unacceptable one, which is still slightly lower than the loss under DA-T-1.

In summary, Panel A confirms that the welfare loss due to application costs (in the form of DA-T or DA-C) is negligible when the cost is low. Moreover, combined with

Table XII: Cardinal Welfare per Applicant under Different Market Designs

DA-T of diff. degrees			DA-C w/ diff. marginal application cost[c]						
	Welfare per app.		Marginal cost	Welfare per app.		Costs paid		Net welfare	
	mean	s.d.		mean	s.d.	mean	s.d.	mean	s.d.
			A. Counterfactual Analysis[a]						
DA	3.2489	0.1064	0 (DA)	3.2489	0.1064	0	0	3.2489	0.1064
DA-T-6	3.2489	0.1064	0.00001	3.2489	0.1064	0.0000	0.0000	3.2489	0.1064
DA-T-5	3.2489	0.1064	0.000031	3.2489	0.1064	0.0000	0.0000	3.2488	0.1064
DA-T-4	3.2489	0.1064	0.0001	3.2489	0.1064	0.0001	0.0000	3.2488	0.1064
DA-T-3	3.2479	0.1064	0.00031	3.2490	0.1064	0.0004	0.0000	3.2486	0.1064
DA-T-2	3.2558	0.1070	0.001	3.2492	0.1064	0.0010	0.0001	3.2482	0.1064
DA-T-1	3.2328	0.1136	0.0031	3.2499	0.1065	0.0025	0.0003	3.2475	0.1064
			0.01	3.2522	0.1065	0.0061	0.0009	3.2461	0.1065
			0.031	3.2594	0.1068	0.0134	0.0028	3.2460	0.1068
			0.1	3.2721	0.1083	0.0292	0.0092	3.2430	0.1086
			0.31	3.2806	0.1103	0.0425	0.0294	3.2381	0.1135
			1	3.2426	0.1136	0.0075	0.0950	3.2351	0.1477
			B. In the Experiment[b]						
DA-T-4	3.2494	0.1061	DA-C	3.2409	0.1061	-	-	-	-

Notes: The calculations are based on 4000 simulations. In each simulation, the shocks to applicants' cardinal preferences are drawn conditional on the observed true ordinal preferences (i.e., the submitted ROLs under DA, supplemented with the program acceptability information). More details are provided in Appendix D.2. In each simulation sample, we calculate cardinal welfare per application (i.e., the average of applicant welfare); the table reports the means and standard deviations of per-applicant welfare across simulation samples. Therefore, the standard deviation measures the variation across simulation samples but not across applicants. When simulating matching outcomes, program preferences are always fixed at the observed values. [a] In counterfactual analysis, applicants play an equilibrium strategy in each simulation sample, while the equilibrium is solved numerically, as in Appendix D.1. Given the realization of cardinal preferences, applicants may play different actions across samples, as dictated by the equilibrium strategy. [b] For these calculations, applicants always take the actions played in the experiment across all simulation samples. In the experiment, DA-C allows the applicant to rank up to three choices, and a motivation letter is required for each additional choice. However, we do not have a measure of the magnitude of the cost.[c] In the counterfactual analysis, DA-C allows applicants to rank one choice but requires a constant marginal cost for additional choices.

Figure IV, DA-C appears to be a better choice than DA-T: We can set the marginal cost as high as 0.01 so that there is almost no loss in terms of match quality, and the number of candidates screened is 1.44 per opening. To reach a comparable screening cost, DA-T has to be degree 2 (DA-T-2), under which programs screen 1.75 candidates per opening and 0.86 applicants are unmatched.

Panel B of Table XII evaluates the cardinal welfare of the game play observed in the experiment. To simulate these results, we hold constant applicants' actions across the 4000 simulation samples. DA-T-4 reaches almost identical welfare levels both in equilibrium and in the experiment. However, DA-C in the experiment has a different cost configuration from that in the counterfactual. In fact, DA-C in the experiment is the same as DA-T-3 plus a cost for additional choices, and therefore, the former should weakly dominate DA-T-3. While this prediction is true in simulations of equilibrium outcomes, it is not true in the actual game played.[17] The welfare loss is due largely to

[17] When simulating the version of DA-C in the experiment, almost no applicants rank more than three

the unmatched applicant under DA-C in the experiment, which may be explained by the fact that applicant actions, which are held constant across simulation samples, can be suboptimal in some samples. Moreover, this loss may indicate that applicants do not use equilibrium strategies in the experiment.

Welfare effect on individual applicants. In addition to examining cardinal welfare, we further investigate how many applicants are better off or worse off. The advantage of this welfare measure is that it does not compare welfare across applicants. The results are summarized in Table XIII, which has a similar structure as Table XII.

Table XIII: Applicant Ordinal Welfare under Different Market Designs:
Counterfactual Analysis; Relative to the Applicant-Optimal Stable Matching

DA-T of diff. degrees					DA-C w/ diff. marginal application cost				
	Better off		Worse off		Marginal cost	Better off		Worse off	
	Mean	s.d.	Mean	s.d.		Mean	s.d.	Mean	s.d.
DA-T-6	0.00	0.00	0.00	0.00	0.00001	0.00	0.00	0.00	0.00
DA-T-5	0.00	0.00	0.00	0.00	0.000031	0.00	0.04	0.00	0.04
DA-T-4	0.00	0.00	0.00	0.00	0.0001	0.00	0.05	0.00	0.05
DA-T-3	0.04	0.20	0.21	0.41	0.00031	0.01	0.09	0.01	0.11
DA-T-2	1.40	0.88	2.16	0.86	0.001	0.03	0.18	0.03	0.20
DA-T-1	6.38	1.34	10.16	2.12	0.0031	0.11	0.34	0.12	0.38
					0.01	0.36	0.60	0.40	0.67
					0.031	1.13	1.00	1.24	1.11
					0.1	3.56	1.57	3.71	1.47
					0.31	5.55	1.46	6.87	1.89
					1	6.33	1.34	10.25	2.22

Notes: The counterfactual simulation is the same as that in Table XII; the notes therein provide more details. In each of the 4000 simulation samples, we first calculate the number of applicants who are better off (or worse off) relative to the DA outcome, which is the applicant-optimal stable matching. The table then reports the means and standard deviations of these two ordinal welfare measures under a given market design across simulation samples.

In each simulation sample, we calculate how many applicants are better off (or worse off) relative to their individual outcome under DA – the applicant-optimal stable matching. The table then reports the means and standard deviations across simulation samples. We do not consider the application costs paid by applicants.

Consistent with the previous results, applicants (almost) always obtain the applicant-optimal stable outcome when the application cost is low (i.e., under DA-T-3 to DA-T-6 or DA-C with a marginal cost of no more than 0.01). However, whenever the outcomes deviate from the applicant-optimal stable outcome, there are always more "losers" than

choices, even when the cost is negligible. Therefore, this simulation leads to the same outcome as DA-T-3, which obtains (almost) the same outcome as DA.

"winners." This finding thus indicates that the gain in per-applicant welfare under DA-T-2 or DA-C with costs 0.1 or 0.31 comes at the expense of some applicants, as shown in Table XII.

5.5.4 Program Welfare in Matching Outcomes

Given the two-sided nature of the matching game, we are also interested in program welfare from matching outcomes without taking into account screening costs.

We define two measures of program welfare while maintaining the assumption that programs have responsive preferences.[18] The first measure is the sum of the final grades of all of the program's matched applicants. That is, for program j, its welfare given matching μ is $\Sigma_{i\in\mu(j)} FinalGrade_i$. The second measure takes into account programs' ordinal preferences, $\Sigma_{i\in\mu(j)} r_{i,j}$ for each j. Namely, j receives 129 points if matched with its top applicant and 1 point if matched with its lowest-ranked applicant. By construction, both measures take into account the quantity and quality of matched applicants.

Similar to the analysis of applicants' cardinal welfare, we first study per-program welfare (Table XIV) and then investigate the number of "winners" and "losers" according to the two welfare measures (Table XV). It should be emphasized that this welfare analysis does not consider screening costs.

A clear pattern evident in Table XIV is that program welfare decreases when application costs increase under either DA-T or DA-C. This pattern is explained mainly by the number of unmatched applicants. If an average applicant – with a final grade of 12.27 and ranked by the program as 65th – becomes unmatched, the per-program welfare decreases by 1.75 or 9.29, according to the two welfare measures, respectively. For example, under DA-T-1, 4.11 applicants on average are unmatched (Figure IV), which leads to a welfare loss of approximately 7 or 38. The actual loss is smaller because unmatched applicants tend to have lower final grades and/or to be ranked lower by programs.

Nonetheless, Table XIV shows that the loss in program welfare is negligible when the application cost is low. Even in the extreme, under DA-T-1 or DA-C with a marginal cost equal to one, on average, the loss to a program's welfare amounts to losing an applicant

[18] We choose not to use the estimate of the latent score (equation 4) to measure programs' cardinal preferences because (i) only the ordinal information in program preferences matter in the game (not the cardinality) and (ii) the only observable determining the latent score is applicants' final grade. The second justification motivates our first measure of program welfare.

Table XIV: Programs' Cardinal Welfare under Different Market Designs

DA-T of diff. degrees					DA-C w/ diff. marginal application cost				
	Welfare per program					Welfare per program			
	Def. 1: Grades		Def. 2: Rank		Marginal cost	Def. 1: Grades		Def. 2: Rank	
	Mean	s.d.	Mean	s.d.		Mean	s.d.	Mean	s.d.
					A. Counterfactual Analysis				
DA	226.10	0	1456.14	0	0 (DA)	226.10	0	1456.14	0
DA-T-6	226.10	0.00	1456.14	0.00	0.00001	226.10	0.00	1456.14	0.00
DA-T-5	226.10	0.00	1456.14	0.00	0.000031	226.10	0.00	1456.14	0.00
DA-T-4	226.10	0.00	1456.14	0.00	0.0001	226.10	0.00	1456.14	0.10
DA-T-3	225.85	0.55	1455.63	1.03	0.00031	226.10	0.00	1456.14	0.33
DA-T-2	224.83	0.52	1453.44	1.18	0.001	226.10	0.02	1456.12	0.52
DA-T-1	219.49	2.37	1435.68	11.38	0.0031	226.10	0.05	1456.06	0.96
					0.01	226.09	0.10	1455.92	1.50
					0.031	225.90	0.55	1454.52	4.09
					0.1	224.02	1.21	1444.81	7.35
					0.31	222.77	1.61	1442.72	8.55
					1	220.17	2.30	1436.83	10.57
					B. In the Experiment				
DA-T	226.10	0	1448.14	0	DA-C	224.63	0	1445.43	0

Notes: The counterfactual simulation is the same as that in Table XII; the notes therein provide more details. In each of the 4000 simulation samples, we first calculate the per-program welfare according to two measures. The first is the sum of final grades of all applicants matched with the given program, and the second is the sum of matched applicants' rankings by the program (129 points if matched with the top-ranked applicant, 1 point if matched with the lowest-ranked applicant). The table then reports the mean and standard deviation of these two welfare measures under a given market design across simulation samples. The standard deviation is zero for program welfare under DA, DA-T and DT-C in the experiment because there is no variation in matching outcomes across simulation samples.

with a final grade ranging from 5.93 to 6.61 or one ranked 109th. On the other hand, the standard deviation indicates that the effect on program welfare varies more significantly across simulation samples when the application cost increases.

Comparing the welfare of each program under a given market design with that under DA (Table XV), we find there are more "losers" than "winners" when the application cost increases. This pattern holds true with both of the welfare measures. Once again, when the application cost is low, the effects of matching outcomes on program welfare are negligible.

6 Conclusion and Discussion

This paper investigates the market design for a two-sided many-to-one matching market with the application cost as a Pigouvian tax to reduce congestion and lower the screening costs of programs that are recruiting applicants. Both forms of application cost, DA-T (an application limit) and DA-C (a positive marginal cost of application), are effective in

Table XV: Program Welfare under Different Market Designs

Counterfactual Analysis; Relative to the Program-Optimal Stable Matching

DA-T of diff. degrees					DA-C w/ diff. marginal application cost[c]				
	Better off		Worse off		Marginal cost	Better off		Worse off	
	Mean	s.d.	Mean	s.d.		Mean	s.d.	Mean	s.d.
A. Program Welfare Definition 1: Grades[b]									
DA-T-6	0.00	0.00	0.00	0.00	0.00001	0.00	0.00	0.00	0.00
DA-T-5	0.00	0.00	0.00	0.00	0.000031	0.00	0.01	0.00	0.01
DA-T-4	0.00	0.00	0.00	0.00	0.0001	0.00	0.01	0.00	0.01
DA-T-3	0.01	0.03	0.03	0.07	0.00031	0.00	0.01	0.00	0.02
DA-T-2	0.12	0.06	0.29	0.10	0.001	0.00	0.03	0.01	0.04
DA-T-1	0.26	0.13	0.55	0.12	0.0031	0.02	0.05	0.03	0.08
					0.01	0.05	0.08	0.08	0.13
					0.031	0.11	0.10	0.21	0.16
					0.1	0.18	0.11	0.41	0.11
					0.31	0.27	0.12	0.48	0.11
					1	0.28	0.13	0.54	0.11
B. Program Welfare Definition 2: Rank[b]									
DA-T-6	0.00	0.00	0.00	0.00	0.00001	0.00	0.00	0.00	0.00
DA-T-5	0.00	0.00	0.00	0.00	0.000031	0.00	0.01	0.00	0.01
DA-T-4	0.00	0.00	0.00	0.00	0.0001	0.00	0.01	0.00	0.01
DA-T-3	0.01	0.03	0.03	0.07	0.00031	0.00	0.01	0.00	0.02
DA-T-2	0.16	0.09	0.26	0.11	0.001	0.00	0.03	0.01	0.04
DA-T-1	0.36	0.15	0.45	0.12	0.0031	0.02	0.05	0.03	0.08
					0.01	0.05	0.08	0.08	0.13
					0.031	0.12	0.10	0.21	0.16
					0.1	0.20	0.11	0.39	0.11
					0.31	0.33	0.13	0.42	0.12
					1	0.36	0.13	0.45	0.12

Notes: The counterfactual simulation is the same as that in Table XII; the notes therein provide more details. In each of the 4000 simulation samples, we first calculate how many programs are better off (or worse off) relative to the program-optimal matching according to two measures. The first is the sum of final grades of all applicants matched with the given program, and the second is the sum of matched applicants' rankings by the program (129 points if matched with the top-ranked applicant, 1 point if matched with the lowest-ranked applicant). Under a given market design, the table then reports the means and standard deviations of the measures on welfare changes across simulation samples.

reducing congestion. However, some key differences should be noted.

First, under DA-T of degree K, the marginal cost of applying to an additional program is zero when the total number of applications is below K. However, the marginal benefit of applying to an additional acceptable program, which can be small, is always positive because of the full-support assumption on applicant and program preferences. Therefore, applicants always optimally apply to K programs under DA-T-K whenever they have at least K acceptable programs. In comparison, under DA-C, an applicant considers the positive marginal cost for additional applications and therefore may choose not to apply to an acceptable program.

Second, the programs to which an applicant chooses not to apply under DA-C must have marginal benefits lower than the marginal costs, implying that the expected loss in match quality is bounded by application cost. DA-T does not offer such an opportunity for bounding losses, which may explain why, in our data, a negligible applicant cost can potentially reduce congestion significantly with little impact on match quality; a higher degree of DA-T, e.g., DA-T-K for $K \in \{4, 5, 6\}$, does not have a large effect, while a lower degree of DA-T may be detrimental to match quality. This observation implies the possibility that the market designer, with little information on applicants' and programs' preferences, chooses a very low application cost and reduces congestion significantly without sacrificing matching quality.

6.1 Market Design and Congestion in Real-Life Markets

The results of our study have implications for market design in other centralized or decentralized matching markets, even though we focus on a special centralized setting. In the literature, the Gale-Shapley DA mechanism has been used to approximate equilibrium outcomes of decentralized settings for marriage and other markets (Chiappori and Salanié, 2016). It is therefore a natural starting point to focus on variants of DA.

Various forms of application costs are used to combat congestion across matching markets. Table XVI presents examples of application costs used in practice in both centralized and decentralized markets, including DA-T and DA-C.

Table XVI: Examples of Various Market Designs in Practice

Mechanism	**Centralized**	**Decentralized**
DA (cost ≈ 0)	School choice in Boston, MA	Research articles and academic conferences Job market for Econ PhDs Online platforms: Craigslist and Zillow
DA-T	NYC high schools (ranking 12 choices) Lycées in Paris (ranking 8 choices) Univ. admissions in Australia/Chile	Airbnb (1 request to book) "Early Decision" in college admissions (apply to only one college) Research articles and academic journals (except law journals)
DA-C	University admission in Hungary National Resident Matching Program	**Unit cost (marginal cost>0)**: Upwork College admission in the U.S. Graduate admission in the U.S. **Fixed cost**: eHarmony (18-hour questionnaire)

While DA-T in centralized markets usually allows applicants to submit multiple applications (e.g., school choice in New York City and Paris), in decentralized markets, DA-T-1 is the version most often applied in practice when DA-T is adopted: Airbnb allows one "request to book" at any point in time; one can apply to only one college for early decision in the U.S.; academic journals, except those in law, do not permit an article to be considered by multiple journals simultaneously. A possible explanation for these practices is that screening costs are significantly high in these markets.[19]

The usual configuration of DA-C in centralized markets is similar to that developed in the paper: Applicants pay a constant marginal cost if one is willing to apply to more programs beyond a limited number of choices, e.g., NRMP in the U.S. and university admissions in Hungary.

In decentralized markets, however, the application cost in DA-C can be either a marginal cost or a fixed cost. An applicant pays an application fee or makes program-specific investments when applying to an additional college or graduate program, which amounts to paying a marginal cost. Similarly, one may have to pay some fees to apply to an additional job on Upwork, a platform matching freelancers and tasks. An example of a fixed cost is the costly registration to participate in the eHarmony dating platform — it may take up to 18 hours to fill out the questionnaire appropriately.

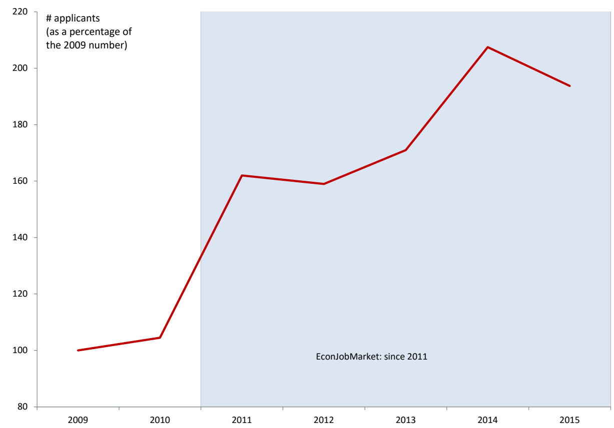

Figure V: Number of Candidates Received by TSE for a Junior Faculty Position

Notes: Every number is expressed as a percentage of the number in 2009. During these years, neither the targeted fields nor the number of positions were announced in advance.

Meanwhile, there are many markets with (almost) zero application cost (Table XVI). This practice can be justified when screening costs are low. For example, academic conferences in economics allow multiple submissions of the same article, while the review process is less demanding than that for academic journals. Other examples are more commonly a result of the recent development of decreasing application costs. For instance,

[19]Interestingly, academic journals in law, which allow an article to be considered simultaneously by several journals, ask law students to review submitted articles, which may reduce screening costs.

the rise of platforms such as EconJobMarket for economics PhDs has led to an increase in the number of applications that economics departments have received. Figure V shows TSE's experience during 2009-15. In 2011, TSE began accepting applications through EconJobMarket instead of its own website. The marginal cost of applying to TSE thus decreased from a non-negligible amount (including the costs of determining the correct procedure and submitting files separately for TSE) to almost zero (i.e., a few more clicks if the applicant already has her files on EconJobMarket). Immediately, the number of applications increased by approximately 60% in 2011 and remained at this high level in subsequent years. This almost-zero "application cost" is also observed on online platforms such as Craigslist and Zillow.

In summary, our results can help us understand the current design in these markets and, more importantly, how to improve them when congestion is a concern due to non-negligible screening costs.

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Appendix A Experiment: Design and Implementation

A.1 The Boston Immediate Acceptance Mechanism

A popular mechanism is the Immediate Acceptance mechanism (IA), also known as the Boston mechanism, which solicits ROLs from applicants, uses programs' strict rankings over applicants as inputs, and includes multiple rounds:

Round 1. Each program considers all the applicants who rank it first and assigns its positions in the order of the program's preferences until either no positions remain or no applicant who has listed it as his/her first choice remains.

Generally, in

Round k ($k > 1$). The k^{th} choice of applicants who have not yet been assigned is considered. Each program that still has available positions assigns the remaining positions to applicants who rank the program as their k^{th} choice in the order of that program's preferences until either no positions remain or no applicant who has listed it as his/her k^{th} choice remains.

The process terminates after any round k in which every applicant is assigned a position at some program or if the only applicants who remain unassigned listed no more than k choices.

A.2 Design: Interface for Applicants

The experiment provides applicants with an interface hosted on the official website that Université Toulouse 1 Capitole uses for coursework. With screenshots from the website (Figures E.1–E.5), this appendix details each step of the experiment involving applicants.

To minimize the potential "framing" effects, we randomize on three dimensions: (1) The order of the programs in the survey questionnaire on acceptability, (2) the order of the programs when applicants are asked to rank them under the four mechanisms, and (3) the order of mechanisms in which applicants play. First, applicants are randomly and (almost) evenly divided into seven groups. Second, for each group, we assign an order of the seven programs to be presented when asking applicants to report acceptability. The seven program orders are such that each program is presented once first, once last, and the rest at random positions. Third, each group faces the "reverse" program order when asked to rank the programs under the four mechanisms. That is, if P1 is the first to be shown in the acceptability survey, it is presented last when the applicant decides her ROL, and vice versa. Moreover, these seven program orders are the same ones used in the survey. Finally, each group of applicants plays the four mechanisms following one of the four orders conditional on DA-C and DA-T always being after DA.

The experiment begins with the welcome page (Figure E.1) to which applicants have access after being notified by email. It describes in both English and French the purpose of the experiment (i.e., trying four mechanisms instead of adopting only one mechanism) and explains important dates and other information. The above content is the same for every applicant; then, every applicant clicks the link that takes her to the next screen, which differs across applicants.

On the second webpage, we implement a survey on program acceptability (Figure E.2). For each given program, applicants reveal their enrollment decision under the assumption that they were only accepted by that program. It is emphasized that the answers to survey questions are not used in the actual matching mechanisms. The order of the programs is randomized to avoid potential framing effects. Similarly, the link to the next step is also randomized such that one may play either IA or DA first.

Once they have reached the webpage to submit an ROL under a given mechanism, applicants are given suggestions on how to play the specific game (Figures E.3-E.5). It is also emphasized that each ROL submitted under a given mechanism is used if and only if the mechanism is selected for the final assignment. Under every mechanism, all the programs are listed for applicants to rank and the order of the programs is randomized.

A.3 Programs' Involvement in the Experiment

When designing the experiment, all the program directors were informed about the experiment and its intended purposes. The seven directors were involved at all stages of the experiment.

Pre-experiment stage. Each director was asked to provide a short description of the program, prerequisites (if any), and a capacity. This information was then uploaded to the website for the experiment.

Screening candidates. We did not simply forward applicants' submitted ROLs under the four mechanisms to the program directors. There were several concerns. First, we intended to prevent the directors from learning applicants' ROLs (i.e., how each applicant ranked the programs). Second, providing all ROLs would require the directors to screen too many candidates (c.f., Table VIII).

Instead, in the files containing all the applicants, their grades and their M1 program enrollment information, we **pre-selected** a subset of applicants for each program. Each program director received a program-specific file and was asked to rank all the applicants within it. We explained that the pre-selected applicants were those who would be considered by the program with a high probability, while the others were unlikely to be considered. At the same time, we strongly encouraged them to screen as many candidates as possible. When ranking candidates, the program directors had flexibility: They did not have to respect the ranking by grades; the ranking could be weak; and if necessary, ties in the mechanisms would be broken by grades.

We pre-selected applicants by running the four mechanisms with "fictitious" program preferences and capacities. We created those program preferences by adding noises to applicants' final grades. For each program, the pre-selected applicants were those who had ever been "considered" (either accepted or rejected) by the program in these simulations.

Among the seven programs, the number of pre-selected candidates ranged from 27 to 52, with a mean of 41. No director screened more candidates beyond the pre-selected ones. This process was proven "successful" in the sense that beyond the pre-selected applicants for a program, no one was ever "considered" by the program under any mechanism.

Motivation letters. The directors did not have access to the motivation letters because receiving a letter from an applicant would imply that the applicant did not rank the program within their top three choices. We planned to transmit the letters to the directors if (a) there were a large number of applicants assigned to the fourth or lower choices under DA-C and (b) if the administration would choose the DA-C outcome upon seeing the tentative DA-C outcome together with the outcomes from the other three mechanisms. Ultimately, all applicants except one – who was unmatched – were matched with one of their top three choices (see Table VI). Therefore, the tentative DA-C outcome was very similar to the "real one" that could have been obtained had the directors had access to the letters.[20] Moreover, upon seeing the results, the TSE administration did not choose DA-C.

[20]Had the directors seen the letters, the unmatched applicant would not have had a different outcome because she/he submitted only three choices.

Appendix B Data

B.1 Programs' Ranking over All Applicants

As discussed in Appendix A.3, the program directors did not rank all applicants, while a complete order that ranks all applicants by each program is needed for counterfactual analysis. We construct the complete order by the following steps:

(i) The partial orders of applicants that are submitted by the program directors are always respected in the constructed complete orders.

(ii) For the two programs with prerequisites, every applicant who does not meet the prerequisites is ranked at the bottom in a descending order of their final grades.

(iii) For each program, the complete order by applicants' final grades is respected whenever it is not in conflict with the program's submitted partial order. When there is a conflict, we do the following adjustment: Suppose applicant i is moved up in the submitted partial order and is just above applicant i^*. This implies that i's final grade is lower than i^*'s. In the constructed complete order, i is also just above i^*. We iterate this process for each conflict between the ranking by final grades and the submitted partial order.

B.2 Data Cleaning

B.2.1 Acceptability and Applicant Strategies

The survey data on program acceptability and applicants' submitted true preferences under DA are not always consistent with each other. Because we emphasize to applicants that the survey data is not used in the matching mechanisms, the DA data should have a greater weight. Below, we describe how we make some modifications, but as few as possible, in the survey data to restore consistency.

According to the original survey data, one applicant is matched with an unacceptable program (the third choice in the submitted DA ROL) under the three versions of the DA mechanism and with another under IA (the fourth choice in the submitted DA ROL). Moreover, we observe that the applicant enrolls in the matched program under DA in September 2013. We therefore change the applicants' third and fourth choices to be "possibly acceptable."

Another type of inconsistency occurs when an applicant ranks an unacceptable program above an acceptable or puts a "possibly acceptable" program above a "definitely acceptable" in the submitted DA ROL. There are in total nine applicants who have this inconsistency. We detail our modifications in Table B.1. The principle is as following: Given an applicant, her submitted ROL is taken as the true preference order;

therefore, the ROL should always rank "definitely acceptable" programs above "possibly acceptable" ones which are above "unacceptables." In other words, program acceptability should never increase when we move down in the ROL. When there is a violation between a pair, say l^k and l^{k+1} such that l^{k+1}'s acceptability is higher than l^k's, we change l^{k+1}'s acceptability to l^k's. It turns out that we only need to make one modification like this in each of the nice ROLs (Table B.1).

Table B.1: Modifications to the Survey Data on Program Acceptability

App.	Submitted ROL under DA	**Original Program Acceptability**							**Modified Program Acceptability**						
		P1	P2	P3	P4	P5	P6	P7	P1	P2	P3	P4	P5	P6	P7
1	(P3, P1, P4, P7, P2, **P6**, P5)	1	-1	1	1	-1	**0**	0	1	-1	1	1	-1	**-1**	0
2	(P3, P4, P7, **P2**, P5, P6, P1)	-1	**1**	1	1	-1	-1	0	-1	**0**	1	1	-1	-1	0
3	(P4, P3, P2, P5, P6, **P1**, P7)	**0**	0	0	1	0	-1	-1	**-1**	0	0	1	0	-1	-1
4	(P3, P1, P7, **P4**, P2, P5, P6)	1	-1	1	**1**	-1	-1	0	1	-1	1	**0**	-1	-1	0
5	(P2, P1, P4, P5, P3, P7, **P6**)	1	1	0	0	0	**0**	-1	1	1	0	0	0	**-1**	-1
6	(P2, P5, P3, P1, **P4**, P6, P5)	0	1	1	**1**	0	0	1	0	1	1	**0**	0	0	1
7	(P4, P5, P1, P2, P7, **P3**, P6)	1	1	**1**	1	1	-1	0	1	1	**0**	1	1	-1	0
8	(P1, P4, P3, P2, **P7**, P5, P6)	1	0	1	1	0	-1	**1**	1	0	1	1	0	-1	**0**
9	(P2, P4, P1, P3, P5, **P6**, P7)	1	1	1	1	-1	**1**	-1	1	1	1	1	-1	**-1**	-1

Notes: Modified program acceptability is in red boldface.

B.2.2 Repeated Choices in DA-C

Under DA-C, the applicant interface on the web site allows one to repeatedly rank the same program multiple times. Indeed, there are two applicants (i and i') following this pattern. We change their strategies in the following way:

$$(l_i^1, l_i^2, l_i^3, l_i^1, l_i^1, l_i^1) \rightarrow (l_i^1, l_i^2, l_i^3),$$
$$(l_{i'}^1, l_{i'}^2, l_{i'}^3, l_{i'}^4, l_{i'}^4, l_{i'}^5) \rightarrow (l_{i'}^1, l_{i'}^2, l_{i'}^3, l_{i'}^4, l_{i'}^5).$$

The two applicants do not provide additional motivation letters for repeated choices, and both of them are assigned to their respective first choice under DA-C.

Appendix C Likelihood Function based on the Extended Rank-Order List

This appendix details the derivation of the likelihood function based on the extended rank-order list which is the ROL submitted under DA supplemented by information on program acceptability.

C.1 Notations and Properties

Let:

$$G(x, \lambda, \mu) = \mu \exp(-\lambda \exp(-x)).$$

Note that a Type I extreme value random variable is such that:

$$\Pr(\varepsilon < y) = G(y, 1, 1) = \int_{-\infty}^{y} \exp(-x)\exp(-\exp(-x))dx,$$

and:

$$G(x + a, \lambda, \mu) = G(x, \lambda \exp(-a), \mu).$$

We also have:

$$G(x, \lambda_1, \mu_1)G(x, \lambda_2, \mu_2) = G(x, \lambda_1 + \lambda_2, \mu_1\mu_2),$$

and

$$G(+\infty, \lambda, \mu) = \mu$$

Furthermore:

$$\begin{aligned}
\int_{-\infty}^{y} \exp(-x)\exp(-\exp(-x))G(x, \lambda, \mu)dx &= \mu \int_{-\infty}^{y} \exp(-x)\exp(-(1+\lambda)\exp(-x))dx, \\
&= \frac{\mu}{1+\lambda}\exp(-(1+\lambda)\exp(-y)) \\
&= G(y, 1+\lambda, \frac{\mu}{1+\lambda}),
\end{aligned}$$

so that:

$$\mathbb{E}_x(G(x, \lambda, \mu)\mathbf{1}\{x \leqslant y\}) = G(y, 1+\lambda, \frac{\mu}{1+\lambda}).$$

These properties are used below.

C.2 The Rank-Ordered Multinomial Logit

To simplify notations, we define $v_j \equiv x\beta_j + \varepsilon_j \equiv b_j + \varepsilon_j$, omitting the index for applicants. Furthermore, consider:

$$\begin{aligned}
\Pr(v_{J-1} &> v_J) = \mathbb{E}_{J-1}\left(\int_{-\infty}^{\varepsilon_{J-1}+b_{J-1}-b_J} \exp(-t)\exp(-\exp(-t))dt\right) \\
&= \mathbb{E}_{J-1}\left(G(\varepsilon_{J-1}+b_{J-1}-b_J,1,1)\right), \\
&= \mathbb{E}_{J-1}\left(G(\varepsilon_{J-1},\exp(b_J-b_{J-1}),1\right), \\
&= G(+\infty, 1+\exp(b_J-b_{J-1}), \frac{1}{1+\exp(b_J-b_{J-1})}) \\
&= \frac{1}{1+\exp(b_J-b_{J-1})} \\
&= \frac{\exp(b_{J-1})}{\exp(b_{J-1})+\exp(b_J)}.
\end{aligned}$$

Iterating:

$$\begin{aligned}
\Pr(v_{J-2} &> v_{J-1}, v_{J-1} > v_J) = \\
&= \mathbb{E}_{J-2}\left(G(\varepsilon_{J-2}+b_{J-2}-b_{J-1}, 1+\exp(b_J-b_{J-1}), \frac{1}{1+\exp(b_J-b_{J-1})}\right)), \\
&= \mathbb{E}_{J-2}\left(G(\varepsilon_{J-2}, \exp(b_{J-1}-b_{J-2})+\exp(b_J-b_{J-2}), \frac{1}{1+\exp(b_J-b_{J-1})}\right) \\
&= G(+\infty, 1+\exp(b_{J-1}-b_{J-2})+\exp(b_J-b_{J-2}), \\
&\quad \frac{1}{1+\exp(b_J-b_{J-1})}.\frac{1}{1+\exp(b_{J-1}-b_{J-2})+\exp(b_J-b_{J-2})}) \\
&= \frac{\exp(b_{J-2})}{\exp(b_{J-2})+\exp(b_{J-1})+\exp(b_J)}\frac{\exp(b_{J-1})}{\exp(b_{J-1})+\exp(b_J)},
\end{aligned}$$

which is the usual rank-ordered logit (or exploded logit) formula. By extension:

$$\begin{aligned}
\Pr(v_{J-K} &> v_{J-K+1}, ., v_{J-1} > v_J) \\
&= \mathbb{E}_{J-K}\left(G(\varepsilon_{J-K}, \lambda_{J-K}, \mu_{J-K})\right) \\
&= G(+\infty, \lambda_{J-K-1}, \mu_{J-K-1}) = \mu_{J-K-1},
\end{aligned}$$

in which $\lambda_j = \exp(-(b_j - b_{j+1}))(1+\lambda_{j+1})$ and $\lambda_J = 0$. This yields:

$$\lambda_j = \sum_{k>j} \exp(b_k - b_j).$$

Furthermore, $\mu_j = \frac{\mu_{j+1}}{1+\lambda_{j+1}}$ and $\mu_{J-1} = 1$ so that:

$$\mu_j = \prod_{m\geqslant j+1} \frac{1}{1+\sum_{k>m}\exp(b_k - b_m)}.$$

Further iteration yields:

$$\Pr(v_1 > v_2, ., v_{J-1} > v_J) = \mu_0 = \prod_{m \geqslant 1} \frac{1}{1 + \sum_{k>m} \exp(b_k - b_m)} = \prod_{m \geqslant 1} \frac{\exp(b_m)}{\sum_{k \geqslant m} \exp(b_k)}.$$

C.3 Derivation of Likelihood Function

To further simplify notations, without loss of generality, we let the extended rank-order list L^f be $(1, ..., \overline{J}, \overline{O}, \overline{J}+1, ..., \underline{J}-1, \underline{O}, \underline{J}, ..., J)$. We derive the following likelihood:

$$L = \Pr(v_1 \geqslant ... \geqslant v_{\bar{J}} \geqslant \overline{v}_0 \geqslant v_{\bar{J}+1} \geqslant ... \geqslant v_{\underline{J}-1} \geqslant \underline{v}_0 \geqslant v_{\underline{J}} \geqslant ... \geqslant v_J),$$

where $\overline{v}_0$ and $\underline{v}_0$ are the max and min value of outside option. Moreover, by normalization, $\underline{v}_0 = \varepsilon_0$. From the above results, the term at the right of $\underline{v}_0$ equals:

$$\Pr(\varepsilon_0 > v_{\underline{J}}, ., v_{J-1} > v_J) = \mathbb{E}_{\varepsilon_0} G(\varepsilon_0, \lambda^{(0)}, \mu^{(0)}),$$

in which:

$$\lambda^{(0)} = \sum_{k \geqslant \underline{J}} \exp(b_k), \mu^{(0)} = \prod_{m \geqslant \underline{J}} \frac{\exp(b_m)}{\sum_{k \geqslant m} \exp(b_k)}.$$

Nonetheless, the term $\Pr(v_{\bar{J}} \geqslant \bar{v} \geqslant v_{\bar{J}+1} \geqslant . \geqslant v_{\underline{J}-1} \geqslant \underline{v} \geqslant v_{\underline{J}} \geqslant . \geqslant v_J)$ requires a different evaluation from above because ε_0 affects both $\overline{v}_0$ and $\underline{v}_0$.

We start from the case in which there are more than two "possibly acceptable" alternatives and show that it applies as well to the case in which there are fewer "possibly acceptable" alternatives.

C.3.1 Two or More "Possibly Acceptable" Alternatives

Consider that there are h "possibly acceptable" alternatives i.e. $\bar{J}+1 = \underline{J}-h, ., \underline{J}-1$ are associated to don't know.

A backward induction mechanism. Start the induction from :

$$\begin{aligned} M(\varepsilon_{\underline{J}-2}) &= \left(\int_{\varepsilon_0 - b_{\underline{J}-1}}^{\varepsilon_{\underline{J}-2} + b_{\underline{J}-2} - b_{\underline{J}-1}} f(\varepsilon_{\underline{J}-1}) d\varepsilon_{\underline{J}-1} . G(\varepsilon_0, \lambda^{(0)}, \mu^{(0)}) \right) \\ &= \left(G(\varepsilon_{\underline{J}-2}, \exp(b_{\underline{J}-1} - b_{\underline{J}-2}), 1) - G(\varepsilon_0, \exp(b_{\underline{J}-1}), 1) \right) G(\varepsilon_0, \lambda^{(0)}, \mu^{(0)}), \\ &= G(\varepsilon_{\underline{J}-2}, \exp(b_{\underline{J}-1} - b_{\underline{J}-2}), 1) G_0(\varepsilon_0, \lambda^{(0)}, \mu^{(0)}) - G(\varepsilon_0, \exp(b_{\underline{J}-1}) + \lambda^{(0)}, \mu^{(0)}), \end{aligned}$$

to posit that, by analogy:

$$\begin{aligned}
& M(\varepsilon_{\underline{J}-1-j}) \\
= & \int_{\varepsilon_0-b_{\underline{J}-j}}^{\varepsilon_{\underline{J}-1-j}+b_{\underline{J}-1-j}-b_{\underline{J}-j}} f(\varepsilon_{\underline{J}-j})d\varepsilon_{\underline{J}-j}\ldots\left(\int_{\varepsilon_0-b_{\underline{J}-1}}^{\varepsilon_{\underline{J}-2}+b_{\underline{J}-2}-b_{\underline{J}-1}} f(\varepsilon_{\underline{J}-1})d\varepsilon_{\underline{J}-1}.G(\varepsilon_0,\lambda^{(0)},\mu^{(0)})\right), \\
= & \sum_{p=1}^{j}\sigma_{\{p=1\}}G(\varepsilon_{\underline{J}-1-j},\lambda_{p,j}^{(\varepsilon)},\mu_{p,j}^{(\varepsilon)})K_{p-1}(\varepsilon_0)-K_j(\varepsilon_0),
\end{aligned}$$

in which:

$$\sigma_{\{p=1\}}=\begin{cases} 1 & \text{if } p=1, \\ -1 & \text{otherwise.}\end{cases}$$

The initialization is such that:

$$\begin{aligned}
\lambda_{1,1}^{(\varepsilon)} &= \exp(b_{\underline{J}-1}-b_{\underline{J}-2}),\mu_{1,1}^{(\varepsilon)}=1, \\
K_0(\varepsilon_0) &= G_0(\varepsilon_0,\lambda^{(0)},\mu^{(0)}), \qquad (6)\\
K_1(\varepsilon_0) &= G(\varepsilon_0,\exp(b_{\underline{J}-1})+\lambda^{(0)},\mu^{(0)}).
\end{aligned}$$

By induction, for $j\geqslant 1$:

$$\begin{aligned}
M(\varepsilon_{\underline{J}-2-j}) &= \int_{\varepsilon_0-b_{\underline{J}-1-j}}^{\varepsilon_{\underline{J}-2-j}+b_{\underline{J}-2-j}-b_{\underline{J}-1-j}} f(\varepsilon_{\underline{J}-1-j})M(\varepsilon_{\underline{J}-1-j})d\varepsilon_{\underline{J}-1-j} \\
&= \sum_{p=1}^{j}\sigma_{\{p=1\}}\begin{pmatrix} G(\varepsilon_{\underline{J}-2-j}+b_{\underline{J}-2-j}-b_{\underline{J}-1-j},1+\lambda_{p,j}^{(\varepsilon)},\frac{\mu_{p,j}^{(\varepsilon)}}{1+\lambda_{p,j}^{(\varepsilon)}}) \\ -G(\varepsilon_0-b_{\underline{J}-1-j},1+\lambda_{p,j}^{(\varepsilon)},\frac{\mu_{p,j}^{(\varepsilon)}}{1+\lambda_{p,j}^{(\varepsilon)}})\end{pmatrix}K_{p-1}(\varepsilon_0) \\
&\quad -\begin{pmatrix} G(\varepsilon_{\underline{J}-2-j}+b_{\underline{J}-2-j}-b_{\underline{J}-1-j},1,1) \\ -G(\varepsilon_0-b_{\underline{J}-1-j},1,1)\end{pmatrix}K_j(\varepsilon_0)
\end{aligned}$$

$$\begin{aligned}
= & \sum_{p=1}^{j}\sigma_{\{p=1\}}\begin{pmatrix} G(\varepsilon_{\underline{J}-2-j},\exp(b_{\underline{J}-1-j}-b_{\underline{J}-2-j})(1+\lambda_{p,j}^{(\varepsilon)}),\frac{\mu_{p,j}^{(\varepsilon)}}{1+\lambda_{p,j}^{(\varepsilon)}}) \\ -G(\varepsilon_0,\exp(b_{\underline{J}-1-j})(1+\lambda_{p,j}^{(\varepsilon)}),\frac{\mu_{p,j}^{(\varepsilon)}}{1+\lambda_{p,j}^{(\varepsilon)}})\end{pmatrix}K_{p-1}(\varepsilon_0) \\
& -\begin{pmatrix} G(\varepsilon_{\underline{J}-2-j},\exp(b_{\underline{J}-1-j}-b_{\underline{J}-2-j}),1) \\ -G(\varepsilon_0,\exp(b_{\underline{J}-1-j}),1)\end{pmatrix}K_j(\varepsilon_0).
\end{aligned}$$

which we equalize to:

$$M(\varepsilon_{\underline{J}-2-j})=\sum_{p=1}^{j+1}\sigma_{\{p=1\}}G(\varepsilon_{\underline{J}-2-j},\lambda_{p,j+1}^{(\varepsilon)},\mu_{p,j+1}^{(\varepsilon)})K_{p-1}(\varepsilon_0)-K_{j+1}(\varepsilon_0)$$

Identifying first the terms a function of $\varepsilon_{\underline{J}-2-j}$, we have, for $0 < p \leqslant j$:

$$\begin{aligned}\lambda^{(\varepsilon)}_{p,j+1} &= \exp(b_{\underline{J}-1-j} - b_{\underline{J}-2-j})(1 + \lambda^{(\varepsilon)}_{p,j}),\\ \mu^{(\varepsilon)}_{p,j+1} &= \frac{\mu^{(\varepsilon)}_{p,j}}{1 + \lambda^{(\varepsilon)}_{p,j}},\end{aligned}$$

and

$$\begin{aligned}\lambda^{(\varepsilon)}_{j+1,j+1} &= \exp(b_{\underline{J}-1-j} - b_{\underline{J}-2-j}),\\ \mu^{(\varepsilon)}_{j+1,j+1} &= 1.\end{aligned}$$

This yields for $0 < p \leqslant j$:

$$1 + \lambda^{(\varepsilon)}_{p,j} = \frac{1}{C(b_{\underline{J}-1-j}; (b_{\underline{J}-k})_{p \leqslant k \leqslant j+1})},$$

and, using the convention that the product $\prod\limits_{p<m\leqslant j+1}$ running over an empty set is equal to 1,

$$\mu^{(\varepsilon)}_{p,j+1} = \prod_{p<m\leqslant j+1} C(b_{\underline{J}-m}; (b_{\underline{J}-k})_{p\leqslant k \leqslant m}).$$

Finally, identifying the terms in ε_0:

$$\begin{aligned}&K_{j+1}(\varepsilon_0)\\ =& \sum_{p=1}^{j} \sigma_{\{p=1\}} G(\varepsilon_0, \exp(b_{\underline{J}-1-j})(1 + \lambda^{(\varepsilon)}_{p,j}), \frac{\mu^{(\varepsilon)}_{p,j}}{1 + \lambda^{(\varepsilon)}_{p,j}}) K_{p-1}(\varepsilon_0) - G(\varepsilon_0, \exp(b_{\underline{J}-1-j}), 1) K_j(\varepsilon_0)\\ =& \sum_{p=1}^{j} \sigma_{\{p=1\}} G(\varepsilon_0, \exp(b_{\underline{J}-1-j})(1 + \lambda^{(\varepsilon)}_{p,j}), \mu^{(\varepsilon)}_{p,j+1}) K_{p-1}(\varepsilon_0) - G(\varepsilon_0, \exp(b_{\underline{J}-1-j}), 1) K_j(\varepsilon_0).\end{aligned}$$

Let us postulate that:

$$K_j(\varepsilon_0) = G(\varepsilon_0, \sum_{1\leqslant k\leqslant j} \exp(b_{\underline{J}-k}) + \lambda^{(0)}, \mu^{(0)}_j).$$

Because of equation (6), this property is true for $j = 0$ and $j = 1$ with:

$$\mu^{(0)}_0 = \mu^{(0)}_1 = \mu^{(0)}.$$

Suppose that it is true for j and write:

$$\begin{aligned}
& K_{j+1}(\varepsilon_0) \\
= & \sum_{p=1}^{j} \sigma_{\{p=1\}} G(\varepsilon_0, \exp(b_{\underline{J}-1-j})(1+\lambda_{p,j}^{(\varepsilon)}), \mu_{p,j+1}^{(\varepsilon)}) G(\varepsilon_0, \sum_{1\leqslant k\leqslant p-1} \exp(b_{\underline{J}-k}) + \lambda^{(0)}, \mu_{p-1}^{(0)}) \\
& - G(\varepsilon_0, \exp(b_{\underline{J}-1-j}), 1) G(\varepsilon_0, \sum_{1\leqslant k\leqslant j} \exp(b_{\underline{J}-k}) + \lambda^{(0)}, \mu_j^{(0)}).
\end{aligned}$$

By the above:

$$\exp(b_{\underline{J}-1-j})(1+\lambda_{p,j}^{(\varepsilon)}) = \frac{\exp(b_{\underline{J}-1-j})}{C(b_{\underline{J}-1-j}; (b_{\underline{J}-k})_{p\leqslant k\leqslant j+1})} = \sum_{p\leqslant k\leqslant j+1} \exp(b_{\underline{J}-k}),$$

so that:

$$\begin{aligned}
& K_{j+1}(\varepsilon_0) \\
= & \sum_{p=1}^{j} \sigma_{\{p=1\}} G(\varepsilon_0, \sum_{p\leqslant k\leqslant j+1} \exp(b_{\underline{J}-k}) + \sum_{1\leqslant k\leqslant p-1} \exp(b_{\underline{J}-k}) + \lambda^{(0)}, \mu_{p,j+1}^{(\varepsilon)} \mu_{p-1}^{(0)}) \\
& - G(\varepsilon_0, \sum_{1\leqslant k\leqslant j+1} \exp(b_{\underline{J}-k}) + \lambda^{(0)}, \mu_j^{(0)}), \\
= & G(\varepsilon_0, \sum_{1\leqslant k\leqslant j+1} \exp(b_{\underline{J}-k}) + \lambda^{(0)}, \mu_{j+1}^{(0)})
\end{aligned}$$

since second arguments of all G elements are equal. Additionally:

$$\begin{aligned}
\mu_{j+1}^{(0)} &= \sum_{p=1}^{j} \sigma_{\{p=1\}} \mu_{p,j+1}^{(\varepsilon)} \mu_{p-1}^{(0)} - \mu_j^{(0)}, \qquad (7) \\
&= \sum_{p=1}^{j} \sigma_{\{p=1\}} \prod_{p<m\leqslant j+1} C(b_{\underline{J}-m}; (b_{\underline{J}-k})_{p\leqslant k\leqslant m}) \mu_{p-1}^{(0)} - \mu_j^{(0)}.
\end{aligned}$$

As a summary, we have:

$$\begin{aligned}
M(\varepsilon_{\underline{J}-1-j}) &= \sum_{p=1}^{j} \sigma_{\{p=1\}} G(\varepsilon_{\underline{J}-1-j}, \lambda_{p,j}^{(\varepsilon)}, \mu_{p,j}^{(\varepsilon)}) G(\varepsilon_0, \sum_{1\leqslant k\leqslant p-1} \exp(b_{\underline{J}-k}) + \lambda^{(0)}, \mu_{p-1}^{(0)}) \\
& \quad - G(\varepsilon_0, \sum_{1\leqslant k\leqslant j} \exp(b_{\underline{J}-k}) + \lambda^{(0)}, \mu_j^{(0)}).
\end{aligned}$$

Integration As in the previous section, we have:

$$
\begin{aligned}
&G^*(\varepsilon_0)\\
=&\int_{\varepsilon_0-b_{\bar{J}+1}}^{\varepsilon_0+\bar{\eta}-b_{\bar{J}+1}} M(\varepsilon_{\bar{J}+1})f(\varepsilon_{\bar{J}+1})d\varepsilon_{\bar{J}+1}\\
=&\int_{\varepsilon_0-b_{\bar{J}+1}}^{\varepsilon_0+\bar{\eta}-b_{\bar{J}+1}} \left[\begin{array}{l} \sum_{p=1}^{h-1}\sigma_{\{p=1\}}G(\varepsilon_{\bar{J}+1},\lambda^{(\varepsilon)}_{p,h-1},\mu^{(\varepsilon)}_{p,h-1})G(\varepsilon_0,\sum_{1\leqslant k\leqslant p-1}\exp(b_{\underline{J}-k})+\lambda^{(0)},\mu^{(0)}_{p-1})\\ -G(\varepsilon_0,\sum_{1\leqslant k\leqslant h-1}\exp(b_{\underline{J}-k})+\lambda^{(0)},\mu^{(0)}_{h-1}). \end{array}\right] f(\varepsilon_{\bar{J}+1})d\varepsilon_{\bar{J}+1},
\end{aligned}
$$

in which we have set $\underline{J}-1-j=\bar{J}+1$ i.e. $j=\underline{J}-2-\bar{J}=h-1$ and $h\geqslant 2$. Note that $\underline{J}-h=\bar{J}+1$. Then:

$$
\begin{aligned}
&G^*(\varepsilon_0)\\
=&\sum_{p=1}^{h-1}\sigma_{\{p=1\}}\left(\begin{array}{c} G(\varepsilon_0+\bar{\eta}-b_{\bar{J}+1},1+\lambda^{(\varepsilon)}_{p,h-1},\frac{\mu^{(\varepsilon)}_{p,h-1}}{1+\lambda^{(\varepsilon)}_{p,h-1}})\\ -G(\varepsilon_0-b_{\bar{J}+1},1+\lambda^{(\varepsilon)}_{p,h-1},\frac{\mu^{(\varepsilon)}_{p,h-1}}{1+\lambda^{(\varepsilon)}_{p,h-1}}) \end{array}\right)G(\varepsilon_0,\sum_{1\leqslant k\leqslant p-1}\exp(b_{\underline{J}-k})+\lambda^{(0)},\mu^{(0)}_{p-1})\\
&-\left(\begin{array}{c} G(\varepsilon_0+\bar{\eta}-b_{\bar{J}+1},1,1)\\ -G(\varepsilon_0-b_{\bar{J}+1},1,1) \end{array}\right)G(\varepsilon_0,\sum_{1\leqslant k\leqslant h-1}\exp(b_{\underline{J}-k})+\lambda^{(0)},\mu^{(0)}_{h-1})\\
=&\sum_{p=1}^{h-1}\sigma_{\{p=1\}}\left(\begin{array}{c} G(\varepsilon_0,\exp(b_{\bar{J}+1}-\bar{\eta})(1+\lambda^{(\varepsilon)}_{p,h-1})+\sum_{1\leqslant k\leqslant p-1}\exp(b_{\underline{J}-k})+\lambda^{(0)},\mu^{(\varepsilon)}_{p,h}\mu^{(0)}_{p-1})\\ -G(\varepsilon_0,\exp(b_{\bar{J}+1})(1+\lambda^{(\varepsilon)}_{p,h-1})+\sum_{1\leqslant k\leqslant p-1}\exp(b_{\underline{J}-k})+\lambda^{(0)},\mu^{(\varepsilon)}_{p,h}\mu^{(0)}_{p-1}) \end{array}\right)\\
&-\left(\begin{array}{c} G(\varepsilon_0,\exp(b_{\bar{J}+1}-\bar{\eta})+\sum_{1\leqslant k\leqslant h-1}\exp(b_{\underline{J}-k})+\lambda^{(0)},\mu^{(0)}_{h-1})\\ -G(\varepsilon_0,\exp(b_{\bar{J}+1})+\sum_{1\leqslant k\leqslant h-1}\exp(b_{\underline{J}-k})+\lambda^{(0)},\mu^{(0)}_{h-1}) \end{array}\right)
\end{aligned}
$$

By the same principle as above:

$$
\exp(b_{\bar{J}+1})(1+\lambda^{(\varepsilon)}_{p,h-1})=\frac{\exp(b_{\bar{J}+1})}{C(b_{\bar{J}+1};(b_{\underline{J}-k})_{p\leqslant k\leqslant h})}=\sum_{p\leqslant k\leqslant h}\exp(b_{\underline{J}-k}),
$$

so that the second and fourth terms can be combined to obtain:

$$
\begin{aligned}
&G(\varepsilon_0,\sum_{1\leqslant k\leqslant h}\exp(b_{\underline{J}-k})+\lambda^{(0)},\sum_{p=1}^{h-1}\sigma_{\{p=1\}}\mu^{(\varepsilon)}_{p,h}\mu^{(0)}_{p-1})-G(\varepsilon_0,\sum_{1\leqslant k\leqslant h}\exp(b_{\underline{J}-k})+\lambda^{(0)},\mu^{(0)}_{h-1})\\
=&\ G(\varepsilon_0,\sum_{1\leqslant k\leqslant h}\exp(b_{\underline{J}-k})+\lambda^{(0)},\sum_{p=1}^{h-1}\sigma_{\{p=1\}}\mu^{(\varepsilon)}_{p,h}\mu^{(0)}_{p-1}-\mu^{(0)}_{h-1})\\
=&\ G(\varepsilon_0,\sum_{1\leqslant k\leqslant h}\exp(b_{\underline{J}-k})+\lambda^{(0)},\mu^{(0)}_{h}),
\end{aligned}
$$

where we use the definition of $\mu^{(0)}_h$ from equation (7).

Because:

$$\exp(b_{\bar{J}+1}-\bar{\eta})(1+\lambda^{(\varepsilon)}_{p,h-1}) = \exp(-\bar{\eta})\frac{\exp(b_{\bar{J}+1})}{C(b_{\bar{J}+1};(b_{\underline{J}-k})_{p\leqslant k\leqslant h})} = \exp(-\bar{\eta})\sum_{p\leqslant k\leqslant h}\exp(b_{\underline{J}-k}),$$

we can then write:

$$\begin{aligned} G^*(\varepsilon_0) &= \sum_{p=1}^{h-1}\sigma_{\{p=1\}}G(\varepsilon_0,\exp(-\bar{\eta})\sum_{p\leqslant k\leqslant h}\exp(b_{\underline{J}-k})+\sum_{1\leqslant k\leqslant p-1}\exp(b_{\underline{J}-k})+\lambda^{(0)},\mu^{(\varepsilon)}_{p,h}\mu^{(0)}_{p-1}) \\ &\quad -G(\varepsilon_0,\sum_{1\leqslant k\leqslant h}\exp(b_{\underline{J}-k})+\lambda^{(0)},\mu^{(0)}_{h}) \\ &\quad -G(\varepsilon_0,\exp(b_{\bar{J}+1}-\bar{\eta})+\sum_{1\leqslant k\leqslant h-1}\exp(b_{\underline{J}-k})+\lambda^{(0)},\mu^{(0)}_{h-1}) \end{aligned}$$

Final Derivation We end up with considering that:

$$\Pr(v_{\bar{J}}\geqslant\bar{v}_0\geqslant v_{\bar{J}+1}\geqslant ...\geqslant v_{\underline{J}-1}\geqslant\underline{v}_0\geqslant v_{\underline{J}}\geqslant ...\geqslant v_J) = \mathbb{E}_{\bar{J}}\left(H^{(h)}(\varepsilon_{\bar{J}})\right),$$

in which

$$H^{(h)}(\varepsilon_{\bar{J}}) = \int_{-\infty}^{\varepsilon_{\bar{J}}+b_{\bar{J}}-\bar{\eta}} f(\varepsilon_0)G^*(\varepsilon_0)d\varepsilon_0$$

and therefore:

$$H^{(h)}(\varepsilon_{\bar{J}}) = \sum_{p=1}^{h+1}\sigma_{\{p=1\}}G(\varepsilon_{\bar{J}},\bar{\lambda}^{(0)}_p,\bar{\mu}^{(0)}_p),$$

in which:

$$\bar{\lambda}^{(0)}_p = \exp(\bar{\eta}-b_{\bar{J}})(1+\exp(-\bar{\eta})\sum_{p\leqslant k\leqslant h}\exp(b_{\underline{J}-k})+\sum_{1\leqslant k\leqslant p-1}\exp(b_{\underline{J}-k})+\lambda^{(0)})$$

because $\underline{J}-h=\bar{J}+1$. We can then write:

$$\bar{\lambda}^{(0)}_p = \exp(-b_{\bar{J}})(\exp(\bar{\eta})+\sum_{p\leqslant k\leqslant h}\exp(b_{\underline{J}-k})+\sum_{1\leqslant k\leqslant p-1}\exp(b_{\underline{J}-k}+\bar{\eta})+\exp(\bar{\eta})\lambda^{(0)})$$

and:

$$1+\bar{\lambda}^{(0)}_p = \frac{1}{C((b_{\bar{J}};\bar{\eta},(b_{\underline{J}-k})_{p\leqslant k\leqslant h}),(b_{\underline{J}-k}+\bar{\eta})_{1\leqslant k\leqslant p-1}),(b_k+\bar{\eta})_{k\geqslant\underline{J}})}.$$

We also have for $1\leqslant p<h$:

$$\begin{aligned} \bar{\mu}^{(0)}_p &= \frac{1}{(1+\exp(-\bar{\eta})\sum_{p\leqslant k\leqslant h}\exp(b_{\underline{J}-k})+\sum_{1\leqslant k\leqslant p-1}\exp(b_{\underline{J}-k})+\lambda^{(0)})}\mu^{(\varepsilon)}_{p,h}\mu^{(0)}_{p-1} \\ &= C(\bar{\eta};\bar{\eta},(b_{\underline{J}-k})_{p\leqslant k\leqslant h},(b_{\underline{J}-k}+\bar{\eta})_{1\leqslant k\leqslant p-1},(b_k+\bar{\eta})_{k\geqslant\underline{J}}) \\ &\quad\times\prod_{p<m\leqslant h}C(b_{\underline{J}-m};(b_{\underline{J}-k})_{p\leqslant k\leqslant m})\mu^{(0)}_{p-1}. \end{aligned}$$

which is also valid for $p = h$ because:

$$\begin{aligned}\bar{\mu}_h^{(0)} &= \frac{1}{1+\exp(b_{\bar{J}+1}-\bar{\eta})+\sum_{1\leqslant k\leqslant h-1}\exp(b_{\underline{J}-k})+\lambda^{(0)}}\mu_{h-1}^{(0)}, \\ &= C(\bar{\eta};\bar{\eta},(b_{\underline{J}-k})_{p\leqslant k\leqslant h},(b_{\underline{J}-k}+\bar{\eta})_{1\leqslant k\leqslant p-1},(b_k+\bar{\eta})_{k\geqslant \underline{J}})\mu_{h-1}^{(0)}.\end{aligned}$$

When $p = h + 1$, we have:

$$\bar{\mu}_{h+1}^{(0)} = \frac{1}{1+\sum_{1\leqslant k\leqslant h}\exp(b_{\underline{J}-k})+\lambda^{(0)}}\mu_h^{(0)} = C(0;0,(b_k)_{k\geqslant \bar{J}+1})\mu_h^{(0)}.$$

To derive the likelihood $\Pr(v_1 \geqslant ... \geqslant v_{\bar{J}} \geqslant \overline{v}_0 \geqslant v_{\bar{J}+1} \geqslant ... \geqslant v_{\underline{J}-1} \geqslant \underline{v}_0 \geqslant v_{\underline{J}} \geqslant ... \geqslant v_J)$, we reconsider the backward induction as in the previous sections:

$$\bar{\lambda}_{j,p} = \exp(-(b_j - b_{j+1}))(1+\bar{\lambda}_{j+1,p}), \bar{\mu}_{j,p} = \frac{\bar{\mu}_{j+1,p}}{1+\bar{\lambda}_{j+1,p}},$$

so that:

$$1+\bar{\lambda}_{j,p} = \frac{1}{C((b_j;(b_k)_{j\leqslant k\leqslant \bar{J}},\bar{\eta},(b_{\underline{J}-k})_{p\leqslant k\leqslant h},(b_{\underline{J}-k}+\bar{\eta})_{1\leqslant k\leqslant p-1},(b_k+\bar{\eta})_{j\geqslant \underline{J}})},$$

and therefore if $p \leqslant h$:

$$\begin{aligned}\bar{\mu}_{0,p} &= \left[\prod_{1\leqslant m\leqslant \bar{J}} C(b_m;((b_k)_{m\leqslant k\leqslant \bar{J}}),\bar{\eta},(b_{\underline{J}-k})_{p\leqslant k\leqslant h},(b_{\underline{J}-k}+\bar{\eta})_{1\leqslant k\leqslant p-1},(b_k+\bar{\eta})_{k\geqslant \underline{J}})\right] \\ &\quad \times C(\bar{\eta};\bar{\eta},(b_{\underline{J}-k})_{p\leqslant k\leqslant h},(b_{\underline{J}-k}+\bar{\eta})_{1\leqslant k\leqslant p-1},(b_k+\bar{\eta})_{k\geqslant \underline{J}}) \\ &\quad \times \prod_{p<m\leqslant h} C(b_{\underline{J}-m};(b_{\underline{J}-k})_{p\leqslant k\leqslant m})\mu_{p-1}^{(0)},\end{aligned}$$

and if $p = h + 1$:

$$\bar{\mu}_{0,h+1} = \left[\prod_{1\leqslant m\leqslant \bar{J}} C(b_m;((b_k)_{m\leqslant k\leqslant \bar{J}}),\bar{\eta},(b_{\underline{J}-k}+\bar{\eta})_{1\leqslant k\leqslant h},(b_k+\bar{\eta})_{k\geqslant \underline{J}})\right] C(0;0,(b_k)_{k\geqslant \bar{J}+1})\mu_h^{(0)},$$

Finally:

$$\Pr(v_1 \geqslant ... \geqslant v_{\bar{J}} \geqslant \overline{v}_0 \geqslant v_{\bar{J}+1} \geqslant ... \geqslant v_{\underline{J}-1} \geqslant \underline{v}_0 \geqslant v_{\underline{J}} \geqslant ... \geqslant v_J) = \sum_{p=1}^{h+1} \sigma_{\{p=1\}}\bar{\mu}_{0,p}.$$

Possible mistakes are checked by summing over all possible cases under different scenarios of variables and coefficients and assessing that the sum of probabilities is equal to one. In an additional note which is available upon request, cases for $h = 0, 1$ and 2 are written directly, and we have checked that the formula above applies as well. Cases

$h=0$ and $h=1$ are derived below.

h=0 We have by application of the above a single term $p = h + 1 = 1$:

$$\Pr(v_1 \geqslant ... \geqslant v_{\bar{J}} \geqslant \overline{v}_0 \geqslant \underline{v}_0 \geqslant v_{\underline{J}} \geqslant ... \geqslant v_J) = \bar{\mu}_{0,1},$$

in which:

$$\bar{\mu}_{0,1} = \left[\prod_{1\leqslant m\leqslant \bar{J}} C(b_m; ((b_k)_{m\leqslant k\leqslant \bar{J}}), \bar{\eta}, (b_k+\bar{\eta})_{k\geqslant \underline{J}})\right] C(0;0,(b_k)_{k\geqslant \bar{J}+1}) \prod_{m\geqslant \underline{J}} C(b_m;(b_k)_{k\geqslant m}),$$

because

$$\mu_0^{(0)} = \mu^{(0)} = \prod_{m\geqslant \underline{J}} C(b_m;(b_k)_{k\geqslant m}).$$

h=1 We have by application of the above two terms $p=1$ and $p = h+1 = 2$:

$$\Pr(v_1 \geqslant ... \geqslant v_{\bar{J}} \geqslant \overline{v}_0 \geqslant v_{\bar{J}+1} \geqslant \underline{v}_0 \geqslant v_{\underline{J}} \geqslant ... \geqslant v_J) = \bar{\mu}_{0,1} - \bar{\mu}_{0,2},$$

in which:

$$\begin{aligned}
\bar{\mu}_{0,1} &= \left[\prod_{1\leqslant m\leqslant \bar{J}} C(b_m; ((b_k)_{m\leqslant k\leqslant \bar{J}}), \bar{\eta}, b_{\bar{J}+1}, (b_k+\bar{\eta})_{k\geqslant \underline{J}})\right] \\
&\quad \times C(\bar{\eta}; \bar{\eta}, b_{\bar{J}+1}, (b_k+\bar{\eta})_{k\geqslant \underline{J}}) \prod_{m\geqslant \underline{J}} C(b_m;(b_k)_{k\geqslant m}), \\
\bar{\mu}_{0,2} &= \left[\prod_{1\leqslant m\leqslant \bar{J}} C(b_m; ((b_k)_{m\leqslant k\leqslant \bar{J}}), \bar{\eta}, (b_k+\bar{\eta})_{k\geqslant \bar{J}+1})\right] \\
&\quad \times C(0;0,(b_k)_{k\geqslant \bar{J}+1}) \prod_{m\geqslant \underline{J}} C(b_m;(b_k)_{k\geqslant m}),
\end{aligned}$$

because $\mu_1^{(0)} = \mu^{(0)} = \prod_{m\geqslant \underline{J}} C(b_m;(b_k)_{k\geqslant m})$.

Appendix D Counterfactual Analysis

D.1 Solving Equilibrium

This appendix specifies the algorithm to numerically solve an equilibrium under DA-T of different degrees or DA-C with various marginal costs.

As explained in Section 5.5, we begin with drawing $M(= 4000)$ samples of size I (number of applicants) in which random shocks to applicant preferences ($\varepsilon_i^{(m)}$), to applicants' final grades ($\nu_i^{(m)}$), and to program preferences ($\xi_{i,j}^{(m)}$) are generated. Using these simulated shocks as well as estimated coefficients and observed characteristics, we compute applicant and program preferences, both cardinal and ordinal; moreover, program preferences are translated into rankings over applicants from 1 to I denoted by $r_{i,j}^{(m)}$, with larger $r_{i,j}^{(m)}$ indicating being more preferred.

D.1.1 Solving Bayesian Nash Equilibrium under DA-T

For DA-T of degree $K \in \{1, 2, \ldots, 6\}$, we adopt an iterative process defined by index t $(= 0, 1, \ldots)$:

- *Initialization*: We first initialize the cutoff distribution, Φ_0. For each simulation sample m:

  (i) We let the matching outcome $\mu_{(m,0)}$ (artificially) be $\hat{\mu}_{DA}$, where $\hat{\mu}_{(DA)}$ is the applicant-optimal stable matching observed in our real data (i.e., the outcome from DA as calculated in Section 4);

  (ii) We then calculate the distribution of cutoffs Φ_0 by setting the cutoff of each program in each simulation sample,

$$\kappa_j^{(m,0)} = \begin{cases} \min_{i \in \mu_{(m,0)}^{-1}(j)} r_{i,j}^{(m)} & \text{if } |\mu_{(m,0)}^{-1}(j)| \geqslant q_j, \\ 0 & \text{if } |\mu_{(m,0)}^{-1}(j)| < q_j. \end{cases}$$

  Because $\mu_{(m,0)} = \hat{\mu}_{(DA)}$ for all m, $\kappa_j^{(m,0)}$ is constant across all simulation samples. This leads to a degenerate distribution, Φ_0.

  We then calculate the simulated preferences of applicants and programs:

  (i) We compute applicants' cardinal and therefore ordinal preferences by inputting the simulated shocks $\varepsilon_i^{(m)}$, estimated coefficients, and observed characteristics into the utility functions specified in equations (1) and (2). Note that shocks to applicant preferences ($\varepsilon_i^{(m)}$) are drawn **unconditionally** on ordinal preferences observed in the sample.

(ii) We also identify all ROLs $\{L_{i,n}^{(m)}\}_{n=1,\ldots,N_i}$ for all i such that (a) $L_{i,n}^{(m)}$ ranks no more than K programs; (b) $L_{i,n}^{(m)}$ is a partial order of the true preference order, i.e., the programs included in $L_{i,n}^{(m)}$ are ranked according to the true ordinal preferences; and (c) $L_{i,n}^{(m)}$ does not include any unacceptable or unqualified program. We only need to focus on these ROLs to find an optimal ROL because all other ROLs are strictly dominated (see Section 2).

(iii) We attribute to each program the same capacity as in the experiment and compute its "cardinal" preferences over applicants by plugging the simulated shocks ($\nu_i^{(m)}$ and $\xi_{i,j}^{(m)}$), estimated coefficients, and observed first-semester grades into the grade function (equation 5) and the latent score function (equation 4). These preferences are then translated into rankings over applicants in increasing order of preference from 1 to I, denoted by $r_{i,j}^{(m)}$.

- *Iteration*: For $t \in \{1, 2, \ldots\}$, given $\mu_{m,t-1}$ and $(\kappa_j^{(m,t-1)})_{j\in\mathcal{J}}$ for all m (i.e., Φ_{t-1}), iteration t goes through the following steps to find the optimal ROL for each applicant:

  (i) Define $A_i^{(m,t)} = \{j \in \mathcal{J} : r_{i,j}^{(m)} \geqslant \kappa_j^{(m,t-1)}\}$ for each i and each m. That is, $A_i^{(m,t)}$ is the set of programs that have lower cutoffs in iteration $t-1$ than i's ranks in sample m.

  (ii) Compute for each $L_{i,n}^{(m)}$ the expected utility:[21]

$$W_{i,n}^{(m,t)} = \frac{1}{M}\sum_{l=1}^{M}\left[\sum_{j=1}^{J}\mathbb{1}_{\{j \text{ is } L_{i,n}^{(m)}\text{-preferred in } A_i^{(l,t)}\}}E_\eta \max\{x_i\beta_j + \varepsilon_{i,j}^{(m)}, \eta + \varepsilon_{i,j}^{(0)})\}\right], \tag{8}$$

  in which the expectation is taken with respect to the ex-post shock, η, on the outside option and "j is $L_{i,n}^{(m)}$-preferred in $A_i^{(l,t)}$" is defined such that j is ranked higher than j' for all $j' \in L_{i,n}^{(m)} \cap A_i^{(l,t)}$.

  (iii) Find the optimal ROL $L_{i,*}^{(m,t)} = \arg\max_{n=1,\ldots,N_i} W_{i,n}^{(m,t)}$. If there are ties, we choose the unique longest list among them. Given Assumption 3, there is always a positive marginal benefit to apply to more programs, as long as the total number applications is not more than K. Tie occurs due to the discrete number of simulations.

We then calculate the matching outcome in each sample and update the cutoff distribution:

[21] This calculation amounts to assuming that i plays against a distribution of others' actions (which is approximated by their actions in iteration $t-1$ in all samples). Notice that i is present in every sample, although ideally one could take i out. Besides, we use sample m also notwithstanding the introduction of spurious correlation. These two simplifications speed up the computation, while the error introduced is either $O(1/n)$ or $O(1/M)$ and is therefore negligible.

(i) With $L_{i,*}^{(m,t)}$ and $r_{i,j}^{(m)}$ for all applicants and programs, we run the DA algorithm to find the new matching $\mu_{(m,t)}$ in each simulation sample.

(ii) We calculate the new cutoffs $\kappa_j^{(m,t)}$ for all j based on $\mu^{(m,t)}$ and the difference between two distributions of cutoffs:

$$d^{(t)} = \frac{1}{M} \sum_{j,m} \left(\kappa_j^{(m,t)} - \kappa_j^{(m,t-1)} \right)^2$$

- *Stopping rule*: Stop when $d^{(t)}$ is small enough.

Suppose that we stop at $t = t_f$. This leads to a joint distribution of cutoffs across simulation samples, $\{\kappa_j^{(m,t_f)}\}$ for all j and m, which is defined as Φ_e in the main text and "describes" an equilibrium under DA-T-K.

Counterfactual analysis of welfare. In counterfactual analysis, we use a **different set of simulation samples** and also need to find best response for each applicant with given cardinal preferences. Recall that ordinal preferences of applicants and programs are fixed at the value observed in the experimental data (cf. Table X). However, across the simulation samples for counterfactual analysis, applicant cardinal preferences, as simulated in Appendix D.2, can differ.

Given the simulated cardinal preferences in a simulation sample ($x_i\beta_j + \varepsilon_{i,j}$ for all j), i's best response under DA-T-K is found through the following steps:

(i) Given the equilibrium cutoff distribution solved above, $\{\kappa_j^{(m,t_f)}\}$ for all j and m, we define $A_i^{(m,t_f)} = \{j \in \mathcal{J} : \ r_{i,j}^{(m)} \geqslant \kappa_j^{(m,t_f)}\}$ for each i.

(ii) We identify all ROLs $\{L_{i,n}^{(m)}\}_{n=1,\dots,N_i}$ for all i such that (a) $L_{i,n}^{(m)}$ ranks no more than K programs; (b) $L_{i,n}^{(m)}$ is a partial order of the true preference order; and (c) $L_{i,n}^{(m)}$ does not include any unacceptable or unqualified program.

(iii) Compute for each $L_{i,n}^m$ the expected utility:

$$W_{i,n}^{(t_f)} = \frac{1}{M} \sum_{m=1}^{M} \left[\sum_{j=1}^{J} \mathbb{1}_{\{j \text{ is } L_{i,n}^m\text{-preferred in } A_i^{(m,t_f)}\}} E_\eta \max\{x_i\beta_j + \varepsilon_{i,j}^{(m)}, \eta + \varepsilon_{i,j}^{(0)})\} \right],$$

where "j is $L_{i,n}^m$-preferred in $A_i^{(m,t_f)}$" is defined such that j is ranked higher than j' for all $j' \in L_{i,n}^m \cap A_i^{(m,t_f)}$.

(iv) i's best response is the optimal ROL $L_{i,*}^m = \arg\max_{n=1,\dots,N_i} W_{i,n}^{(t_f)}$. If there are ties, we choose the unique longest list.

D.1.2 Solving Bayes-Nash equilibrium under DA-C

Similar to DA-T, equilibrium under DA-C also has to be solved as a fixed point. The application cost has the following form in the counterfactual analysis:

$$C(L) = \begin{cases} 0 \text{ if } |L| \leqslant 1, \\ c|L| \text{ if } |L| > 1; \end{cases}$$

in which c is the constant marginal cost for second and later choices; and c can take one of the 11 values in $\{10^{-5}, 3.1 \times 10^{-5}, 10^{-4}, 3.1 \times 10^{-4}, ..., 0.01, 0.31, 1\}$.

For a given c, we adopt an iterative process almost the same as the one for DA-T and we do not repeat here except that the expected utility, equation (8), is replaced by:

$$\tilde{W}_{i,n}^{(m,t)} = W_{i,n}^{(m,t)} - \frac{1}{M} \sum_{m=1}^{M} c \times (|L_{i,n}^{(m)}| - 1),$$

which takes into account the application cost associated with $L_{i,n}^{(m)}$.

D.2 Simulating Preference Shocks: Conditional Drawings

Having solved equilibrium strategy numerically, we conduct counterfactual welfare analysis. Our simulation experiment is to answer the following question: *Had a counterfactual market design been implemented at TSE in 2013, what is our best prediction of the counterfactual outcome, based on all the information that is observed by the researcher?*

Therefore, we take the observed applicant and program ordinal preferences as given, although their cardinal preferences can vary while being compatible with the ordinal ones.

To simplify notations, we define $v_j \equiv x\beta_j + \varepsilon_j \equiv b_j + \varepsilon_j$ for $j = 1, 2, \ldots, J$, omitting the index for applicants. Recall that $v_0 = \varepsilon_0$. For a given applicant, write $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_J)$; L^f is an **extended** rank-order list (i.e., an applicant's submitted ROL under DA combined with program acceptability). Define $\varepsilon_{-j} = (\varepsilon_0, \ldots, \varepsilon_{j-1}, \varepsilon_{j+1}, \ldots, \varepsilon_J)$ and also the following set:

$$S(L^f) = \{\varepsilon \in \mathbb{R}^{J+1} : \varepsilon \text{ is compatible with } L^f \text{ given } (b_j)_{j \in \mathcal{J}}\}.$$

This means that we draw the vector ε into the density function given by:

$$\frac{\mathbb{1}\{\varepsilon \in S(L^f)\} \Pi_{j=0}^{J} f(\varepsilon_j)}{\Pr(L^f)}.$$

Note that the conditional distribution of ε_j given ε_{-j} is given by:

$$\frac{f(\varepsilon_j)\mathbb{1}\{(\varepsilon_j, \varepsilon_{-j}) \in S(L^f)\}}{\Pr((\varepsilon_j, \varepsilon_{-j}) \in S(L^f) \mid \varepsilon_{-j})}.$$

For $j \neq 0$, the set $(\varepsilon_j, \varepsilon_{-j}) \in S(L^f)$ given ε_{-j} is an interval which can be open or not on the left or on the right according to the following cases:

(i) Program j is ranked first and therefore if the second ranked is l, the only constraint that $S(L^f)$ imposes on ε_j is :

$$b_j + \varepsilon_j > b_l + \varepsilon_l.$$

(ii) Program j is ranked last and therefore if the before-the-last ranked is u, the only constraint that $S(L^f)$ imposes on ε_j is :

$$b_j + \varepsilon_j < b_u + \varepsilon_u.$$

(iii) If program j has two neighbors, from below, l and from above u, then the only constraint is:

$$b_l + \varepsilon_l < b_j + \varepsilon_j < b_u + \varepsilon_u.$$

For $j = 0$, the set $(\varepsilon_0, \varepsilon_{-0}) \in S(L^f)$ is slightly more complicated but follows the same principles. Suppose the ranking of programs is given by $(1, \ldots, \overline{J}, \overline{O}, \overline{J}{+}1, \ldots, \underline{O}, \underline{J}, \ldots, J)$. It translates into the set of inequalities on ε_0:

$$\begin{aligned} b_{\overline{J}} + \varepsilon_{\overline{J}} > \bar{\eta} + \varepsilon_0 > b_{\overline{J}+1} + \varepsilon_{\overline{J}+1}, \\ b_{\underline{J}-1} + \varepsilon_{\underline{J}-1} > \varepsilon_0 > b_{\underline{J}} + \varepsilon_{\underline{J}}, \end{aligned}$$

using the conventions that $b_0 = +\infty$ and $b_{J+1} = -\infty$. This means that:

$$\varepsilon_0 \in \left[\max(b_{\overline{J}+1} + \varepsilon_{\overline{J}+1} - \bar{\eta}, b_{\underline{J}} + \varepsilon_{\underline{J}}), \min(b_{\overline{J}} + \varepsilon_{\overline{J}} - \bar{\eta}, b_{\underline{J}-1} + \varepsilon_{\underline{J}-1})\right],$$

and the interval is not empty since $\varepsilon \in S(L^f)$.

Defining all these domain intervals as $I_j(\varepsilon_{-j}, L^f)$, the conditional distribution of ε_j given ε_{-j} and L^f is given by:

$$\frac{f(\varepsilon_j)\mathbb{1}\{\varepsilon_j \in I_j(\varepsilon_{-j}, L^f)\}}{\Pr(\varepsilon_j \in I_j(\varepsilon_{-j}, L^f) \mid \varepsilon_{-j})}.$$

It is easy to simulate since $F(\varepsilon_j) = \exp(-\exp(-\varepsilon_j))$ and therefore for $\delta \in (0,1)$:

$$F^{-1}(\delta) = -\log(-\log(\delta)).$$

Assume that $I_j(\varepsilon_{-j}, L^f) = [a_j(\varepsilon_{-j}), b_j(\varepsilon_{-j)}]$ and draw $u \rightsquigarrow \mathcal{U}_{[0,1]}$ then:

$$\varepsilon_j^* = -\log(-\log(F(a_j(\varepsilon_{-j})) + u(F(b_j(\varepsilon_{-j})) - F(a_j(\varepsilon_{-j}))))) \tag{9}$$

is distributed as ε_j given ε_{-j} and L^f.

For Gibbs sampling, the Markov chain that we use is the following:

(i) Draw $\varepsilon_0^{(0)}$ in the type-I extreme value distribution.

 (a) Above $\overline{O}$, draw $\varepsilon_j^{(0)}$ sequentially in the reverse order of $1, \ldots, \overline{J}$ imposing the constraints sequentially:

$$b_{\overline{J}} + \varepsilon_{\overline{J}} > \bar{\eta} + \varepsilon_0, b_{\overline{J}-1} + \varepsilon_{\overline{J}-1} > b_{\overline{J}} + \varepsilon_{\overline{J}}, \text{ etc.}$$

 (b) Between $\overline{O}$ and $\underline{O}$, draw $\varepsilon_j^{(0)}$ sequentially in the order of $\overline{J}+1, \ldots, \underline{J}-1$ imposing the constraints sequentially:

$$\bar{\eta} + \varepsilon_0 > b_{\overline{J}+1} + \varepsilon_{\overline{J}+1} > \varepsilon_0, b_{\overline{J}+1} + \varepsilon_{\overline{J}+1} > b_{\overline{J}+2} + \varepsilon_{\overline{J}+2} > \varepsilon_0, \text{ etc.}$$

 (c) Below $\underline{O}$, draw $\varepsilon_j^{(0)}$ sequentially in the order of $\underline{J}, \ldots, J$ imposing the constraints sequentially:

$$\varepsilon_0 > b_{\underline{J}} + \varepsilon_{\underline{J}}, b_{\underline{J}} + \varepsilon_{\underline{J}} > b_{\underline{J}+1} + \varepsilon_{\underline{J}+1}, \text{ etc.}$$

 In this way, $\varepsilon^{(0)} \in S(L^f)$.

(ii) For $t = \{1, 2, \ldots\}$, repeat the steps in which $\varepsilon_0^{(t)}$ is drawn according to equation (9) given $\varepsilon_{-0}^{(t-1)}$, and then follow the same order for drawing $\varepsilon_j^{(t)}$ as in steps (ia) to (ic) that is the order of drawings given by:

$$\overline{J}, \ldots, 1, \overline{J}+1, \ldots, J;$$

in the meantime, impose the exact intervals $I_j(\varepsilon_{-j}, L^f)$. Note that at every step, the ε of interest belongs to $S(L^f)$ since we draw in the correct intervals.

(iii) If the chain is long enough, then the distribution of ε is the distribution of interest and expectations of all ε_j can be approximated.