The consumption Euler equation or the Keynesian consumption function?

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February 16, 2018

Abstract

We examine aggregate consumer behaviour, both before and after the financial crisis hit the Norwegian economy. To this end, we rely on a cointegrated vector autoregressive (CVAR) model that nests both a Keynesian consumption function and a class of consumption Euler equations. Using likelihood based methods, we find existence of cointegration between consumption, income and wealth once a structural break around the financial crisis is allowed for. That consumption cointegrates with both income and wealth and not only with income points to the empirical irrelevance of an Euler equation. Moreover, we find that consumption equilibrium corrects to changes in income and wealth and not that income equilibrium corrects to changes in consumption, which would be the case if an Euler equation is true. We also find that most of the parameters stemming from the class of Euler equations are not corroborated by the data when considering conditional expectations of future consumption and income in CVAR models. Only habit formation, typically included in DSGE models, seems important in explaining the Norwegian consumer behaviour. We therefore end up with a dynamic Keynesian consumption function with an average marginal propensity to consume close to 30 per cent well in line with the findings in recent literature.

Keywords: Consumption Euler equation, Keynesian consumption function, financial crisis, structural break, conditional expectations

JEL classification: C51, C52, E21

Acknowledgements: We are grateful to seminar participants at Statistics Norway, Jennifer Castle and John Muellbauer in particular, for helpful discussions, and to Thomas von Brasch, Håvard Hungnes and Terje Skjerpen for comments and suggestions on earlier drafts. The usual disclaimer applies.

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1 Introduction

Economists have for a long time been concerned with how households react to changes in fiscal policy. The financial crises in 2008 led to renewed interest in how household asset composition, liquidity and credit market conditions may affect consumption. As a response to the financial crises many government used expansionary fiscal policies, but at the expense of increasing public and private debt levels in later periods. Thus, expansionary fiscal policies were followed by contractionary policies in the wake of the financial crisis in many countries. The effects of such fiscal policies depend on the marginal propensity to consume (MPC) out of shocks to income. In the economics literature there was until recently no consensus regarding the size of the MPC and the role of fiscal policy in stabilizing the economy was controversial. That said, there seems now to be a consensus emerging on the size of the MPC that is much higher than what was the standard choice in most DSGE models before the financial crisis, see for instance Carroll *et al.* (2017) and the references cited therein.

In contrast to the Keynesian consumption function, which says that increased household income causes consumption to rise, both the permanent income hypothesis by Friedman (1957) and the life-cycle hypothesis by Ando and Modigliani (1963) imply that consumption depends on unanticipated and not on anticipated income shocks with a stronger response to permanent than transitory shocks. Recent microeconometric studies, however, find that households react much more to transitory income shocks than what the standard forward-looking theory of consumption predicts. For instance, Jappelli and Pistaferri (2014) estimate an average MPC of 48 per cent using Italian data and Fagereng *et al.* (2016) find an MPC of 35 per cent using Norwegian data. Also, studies of anticipated tax cuts find larger responses to consumption (excess sensitivity) than what is expected from the forward-looking theory of consumption, see for instance Parker *et al.* (2013).

Extended versions of the standard theory that allow for precautionary savings, liquidity constraints and habit formation can explain some of the empirical results found in the literature. Campbell and Mankiw (1991) among others account for precautionary savings and liquidity constraints in a model for aggregate consumption assuming constant relative risk aversion (CRRA) utility preferences and some of the households being current income consumers. Carroll (1992), on the other hand, assumes that consumers who face income uncertainty and are both impatient and prudent behave according to the so-called buffer-stock theory of saving. According to this theory unemployment expectations may explain parts of household behaviour as unpredictable fluctuations in income caused by spells of unemployment are an important source of uncertainty facing many households even in countries where replacement ratios are quite high.[1] Deaton (1991) presents another version of the buffer-stock model based on income uncertainty and liquidity constraints where households use liquid assets to buffer against temporary income shocks. In a recent study by Kaplan and Violante (2014) trading costs are introduced to explain evidence of hand to mouth consumers even for those who are asset-rich due to illiquid assets and credit constraints. The consumption model by Smets and Wouters (2003), which DSGE models typically are based upon, includes habit formation in that current consumption is

[1]For instance, the replacement ratio for average wage earners in Norway is 62 per cent.

proportional to past consumption.

In this paper, we study aggregated models of Norwegian consumer behaviour, both before and after the financial crises, building on Jansen (2013) who demonstrates the empirical relevance of a Keynesian type consumption function with household income and wealth as the main determinants. We contribute to the empirical literature by estimating and testing a general cointegrated vector autoregressive (CVAR) model that nests both a Keynesian consumption function and a class of consumption Euler equations. These include the martingale hypothesis by Hall (1978) and the equations of precautionary savings and liquidity constraints as in Campbell and Mankiw (1991) and of habit formation as in Smets and Wouters (2003). Using likelihood methods, we test the properties of *cointegration* between consumption and income only and of *equilibrium correction* in the nested CVAR. Drawing upon Eitrheim *et al.* (2002), the former property represents the common ground for a Keynesian consumption function and an Euler equation and the latter represents the discriminating feature between them. We also contribute to the literature by considering conditional expectations of future consumption and income in CVAR models within the context of Johansen and Swensen (1999, 2004, 2008). As such, we test the role of forward-looking behaviour in consumption in much the same way as what has been done in the new Keynesian literature on pricing behaviour, see Boug *et al.* (2010, 2017).

Based on a VAR in levels with seasonally unadjusted data that span the period 1970 to 2014, we find existence of one cointegrating relationship between consumption, income and wealth once a structural break around the financial crises in 2008 is allowed for. That consumption cointegrates with both income and wealth and not only with income is evidence against an Euler equation of consumer behaviour. Likelihood ratio tests further support that consumption equilibrium corrects to changes in income and wealth and not that income equilibrium corrects to changes in consumption, as would be the case when an Euler equation is true. We also find that most of the parameters stemming from the class of Euler equations are not corroborated by the data when considering conditional expectations of future consumption and income in CVAR models. Only habit formation in accordance with Smets and Wouters (2003) model seems to play an important role in explaining the Norwegian consumer behaviour. We therefore end up with a dynamic Keynesian type consumption function that has an average MPC of around 30 per cent well in line with the findings in the recent literature.

The rest of the paper is structured as follows: Section 2 discusses the theoretical background for the different models studied. Section 3 presents the data used in the empirical analysis. Section 4 reports findings from the cointegration analysis. Section 5 presents results from considering conditional expectations of consumption and income in CVAR models. Section 6 provides a conclusion.

2 Theory

As a useful benchmark for the empirical analysis, we begin this section by outlining the martingale hypothesis derived by Hall (1978). Then, we present the often used consumption Euler equations with precautionary savings, liquidity constraints and habit formation based on CRRA utility preferences. Finally, we outline a general CVAR, inspired by Jansen (2013), that nests the various hypotheses from the set of Euler equations.

2.1 The martingale hypothesis

Since the seminal paper by Hall (1978) the martingale hypothesis, saying that no other variable than consumption at time t should help predict consumption at time $t+1$, has been subject to extensive empirical investigation, see for instance Flavin (1981), Campbell and Deaton (1989), Muellbauer and Lattimore (1995), Palumbo *et al.* (2006) and Muellbauer (2010).

The main idea behind the martingale hypothesis, which builds on the permanent income hypothesis by Friedman (1957) and the life cycle hypothesis by Ando and Modigliani (1963), is that the representative consumer bases the choice between consumption and saving on both current income and prospect of future income. Formally, the consumer maximises the following intertemporal optimisation problem under uncertainty:[2]

$$\max E_t \sum_{i=0}^{\infty} (1+\theta)^{-i} U(C_{t+i})^{-t} \tag{1}$$

subject to

$$W_{t+1} = (1+R_t)(W_t + YL_t - C_t) \tag{2}$$

and

$$\lim_{i \to \infty} E_t \left[W_{t+i}/(1+R_t)^i \right] = 0, \tag{3}$$

where E_t, θ, $U(\cdot)$ and C_{t+i} in (1) denote expectations conditional on information at time t, the subjective discount rate, assumed constant, the utility function, assumed additive over time, and consumption at time $t+i$, respectively. Thus, the consumer maximises the present discounted value of expected utility conditional on information at time t subject to the budget constraint in (2) and the No-Ponzi Game condition in (3), where W_t denotes financial wealth at time t, YL_t is labour income at time t and R_t denotes the riskless rate of real return at time t.

The well known first order condition or the Euler equation for this optimisation problem takes the form

$$U'(C_t) = (1+R_t)(1+\theta)^{-1} E_t U'(C_{t+1}). \tag{4}$$

Assuming that the utility function is quadratic and that the riskless rate of real return is constant and equal to the subjective discount rate, as in Hall (1978), (4) becomes

$$E_t C_{t+1} = C_t, \tag{5}$$

[2] We follow Blanchard and Fischer (1989, p. 279) here. Our exposition differs slightly, however, in that we use an infinite time approach and assume that savings by the consumer can only be invested in riskless assets, which essentially are bank deposits in practice. Although the riskless rate of real return may vary over time, we further assume that it can be treated as non-stochastic at time t.

or $\Delta C_{t+1} = \varepsilon_{t+1}$, where $E_t \varepsilon_{t+1} = 0$ is an unforecastable innovation in permanent income. Hence, consumption follows a martingale, which means that the consumer never plans to change the consumption level from one period to the next. The change in consumption is thus unforecastable. Also, (5) implies the familiar property of certainty equivalence such that no precautionary savings is undertaken by the consumer. Using the result in (5) together with the constraints in (2) and (3), we obtain

$$C_t = R(1+R)^{-1} W_t + R(1+R)^{-1} \sum_{i=0}^{\infty} (1+R)^{-i} E_t Y L_{t+i} \equiv Y P_t, \tag{6}$$

which says that optimal consumption equals the sum of the proceeds from financial wealth and the expected present value of future labour income, defined to equal the permanent income YP_t. Finally, (6) and (2) imply that

$$\Delta C_t = R(1+R)^{-1} \sum_{i=0}^{\infty} (1+R)^{-i} (E_t - E_{t-1}) Y L_{t+i} \equiv \Delta Y P_t. \tag{7}$$

Any change in consumption, ΔC_t, is equal to the annuity value of the revisions in expectations from the last period to the present one about current and future labor income, defined to equal the change in permanent income, $\Delta Y P_t$. This implication is consistent with (5) as any change in consumption must be founded in new and unexpected information about how much the consumer can afford. No other factors change consumption over time. If we for instance assume that labour income follows a stationary first order autoregressive process with coefficient ρ, (7) becomes $\Delta C_t = R(1 + R - \rho)^{-1} \epsilon_t$, where ϵ_t represents an unexpected change or innovation in labour income from period $t-1$ to t. Accordingly, the marginal propensity to consume in response to an unexpected change in labour income is given by $R(1 + R - \rho)^{-1}$, which is less than unity. Consumption is thus smoother than transitory changes in labour income. When ρ approaches unity, however, the marginal propensity to consume also approaches unity and unexpected labour income is not smoothed.

A useful alternative formulation of the forward looking theory of consumption, suggested by Campbell (1987), is the so called “saving for a rainy day” hypothesis, which says that

$$S_t = -\sum_{i=1}^{\infty} (1+R)^{-i} E_t \Delta Y L_{t+i}, \tag{8}$$

where $S_t \equiv Y_t - C_t$ and $Y_t \equiv RW_t + YL_t$. Hence, savings equal the expected discounted value of future declines in labour income. That is, the consumer "saves for a rainy day". Note also that if labour income is integrated of the first order, savings are stationary and income and consumption are cointegrated.

We have seen that quadratic preferences lead to certainty equivalence with the consequence that an increase in uncertainty faced by the consumer has no effect on consumption and saving. Moreover, the forward looking model above relies heavily on the assumption of perfect capital markets and does not include habit formation. To allow for precautionary savings, liquidity constraints and habit formation, we now turn to consumption Euler equations with CRRA preferences.

2.2 Euler equations with CRRA preferences

Whereas Blundell and Stoker (2005) consider heterogeneity across consumers with CRRA preferences, we simplify matters following Campbell and Mankiw (1991) and Smets and Wouters (2003) among others and assume that all consumers are identical with respect to marginal utility and willingness to move consumption from one period to another. Our point of departure, as in Campbell and Mankiw (1991), is a CRRA utility function of the form[3]

$$U(C_t) = (1-\delta)^{-1} C_t^{1-\delta} \text{ for } 1 \neq \delta > 0, \tag{9}$$

where δ is the inverse of the intertemporal elasticity of substitution σ. The Euler equation now becomes

$$E_t C_{t+1}^{-\delta} = (1+\theta)(1+R_t)^{-1} C_t^{-\delta}, \tag{10}$$

or $E_t[\exp(-\delta \ln C_{t+1})] = (1+\theta)(1+R_t)^{-1}\exp(-\delta \ln C_t)$. Unlike Campbell and Mankiw (1991), who allow for ex ante real interest rates to vary over time, we simplify matters further by considering ex post real interest rates in (10). Assuming that the logarithms of consumption is normally distributed with mean $E_t \ln C_{t+1}$ and variance η_{t+1}^2, and making use of the approximation $\ln[(1+\theta)(1+R_t)^{-1}] \cong \theta - R_t$, we may write the Euler equation as

$$E_t \Delta c_{t+1} = \frac{\eta_{t+1}^2}{2\sigma} - \sigma\theta + \sigma R_t, \tag{11}$$

or $\Delta c_{t+1} = \frac{\eta_{t+1}^2}{2\sigma} - \sigma\theta + \sigma R_t + \varepsilon_{t+1}$, where a lower case letter here and below denotes the logarithms of a variable and $E_t \varepsilon_{t+1} = 0$ is as before an innovation term in permanent income. Clearly, if the consumer faces more uncertainty, that is the larger η_{t+1}^2, the more consumption is expected to increase from this period to the next. Thus, the consumer reduces consumption now in response to increased uncertainty to have a larger safety buffer, that is precautionary savings, for more consumption in the next period. As pointed out by Blundell and Stoker (2005), consumption growth with precautionary savings generally depends on the conditional variance of the uninsurable components of innovations to income. It may be convenient for estimation purposes to assume that the variance is constant, η^2, such that (11) simplifies to

$$E_t \Delta c_{t+1} = \phi + \sigma R_t, \tag{12}$$

or $\Delta c_{t+1} = \phi + \sigma R_t + \varepsilon_{t+1}$, where the constant term, $\phi = \frac{\eta^2}{2\sigma} - \sigma\theta$, partly reflects precautionary savings. According to (12), savings by the consumer is also associated with intertemporal substitution in consumption. An increase in the real interest rate makes savings due to relatively costly consumption today more profitable, hence consumption is expected to increase from this period to the next. We note that (12) collapses to $E_t \Delta c_{t+1} = \varphi$, where $\varphi = \phi + \sigma R$, in the special case when the real interest rate is constant.

[3] $U(C_t) = \ln C_t$ for $\delta = 1$.

The underlying assumption that the consumer has access to perfect capital markets in the sense of no liquidity constraints, permits consumption to move freely in accordance with (12). In practice, however, the consumer may be credit rationed by lending criteria based on payment-to-income ratios, which prevents the consumer from acting in accordance with the forward-looking hypothesis. To account for liquidity constraints in a simple way, Campbell and Mankiw (1991) assume that aggregate consumption is equal to a weighted average with weights μ and $1-\mu$ reflecting the proportions of households being rule of thumb consumers and permanent income consumers. Campbell and Mankiw (1991) further assume that the rule of thumb consumers determine consumption growth as a weighted average of current and one period lag of income growth with weights λ and $1-\lambda$.[4] We can then formulate an augmented version of (12) as

$$E_t \Delta c_{t+1} = (1-\mu)\phi + \mu[\lambda E_t \Delta y_{t+1} + (1-\lambda)\Delta y_t] + (1-\mu)\sigma R_t, \tag{13}$$

or $\Delta c_{t+1} = (1-\mu)\phi + \mu[\lambda \Delta y_{t+1} + (1-\lambda)\Delta y_t] + (1-\mu)\sigma R_t + (1-\mu)\varepsilon_{t+1}$, where Δy_{t+1} and Δy_t is disposable income growth at time $t+1$ and t. When $\mu = 0$, the augmented model collapses to (12). As emphasised by Campbell and Mankiw (1991), (13) can only serve as an approximation to a model in which liquidity constraints are explicitly modelled. A fully worked out model with liquidity constraints involves more complicated consumer behaviour, see for instance Deaton (1992, p. 194-213) and Blundell and Stoker (2005).

The consumption Euler equation by Smets and Wouters (2003), typically included in DSGE models, is also based on CRRA preferences appearing in a utility function separable in consumption and labour (leisure). However, the marginal utility of consumption at time t now equals $\epsilon_t^b (C_t - hC_{t-1})^{-\delta}$, where ϵ_t^b and hC_{t-1} denote a shock to the subjective discount rate that affects the intertemporal substitution and a habit formation that is proportional to past consumption, respectively.[5] Hence, Smets and Wouters (2003) extend the Euler equation in (10) by taking into account the possibility of habit formation. To obtain a tractable empirical model, Smets and Wouters (2003) log-linearize the Euler equation around a non-stochastic steady state such that consumption obeys

$$c_t = (1-\omega_1)c_{t-1} + \omega_1 E_t c_{t+1} - \omega_2 \hat{r}_t, \tag{14}$$

where $\omega_1 = (1+h)^{-1}$, $\omega_2 = \frac{(1-h)}{(1+h)\delta}$ and $\hat{r}_t$ is the log deviation of the ex ante real interest rate from its non-stochastic steady state.[6] Consumption thus depends on a weighted average of past and expected future consumption and the ex ante real interest rate. The higher the degree of habit formation, the smaller is the impact of the real interest rate on consumption for a given elasticity of substitution. We note that (14) collapses to $E_t \Delta c_{t+1} = \sigma \hat{r}_t$ when $h = 0$, which essentially is the same as (12). Adding and subtracting

[4] Campbell and Deaton (1989) argue that consumption is smooth because it responds with a lag to changes in income. As pointed out by Campbell and Mankiw (1991), (13) with lagged income growth is also in the spirit of Flavin (1981).

[5] Note that $\delta = \sigma_c$ in Smets and Wouters (2003).

[6] We simplicity matters by disregarding shocks from ε_t^b in (14). Note also that the log deviation of the consumption level from its non-stochastic steady state, $\ln(C/\bar{C})$, and the homogeneity restriction between the past and the future consumption levels imply that $\ln \bar{C}$ cancels throughout in (14).

c_{t-1} and $\omega_1 E_t c_t$ on the right hand side of (14) and rearranging, we can write expected consumption growth when $h \neq 0$ as

$$E_t \Delta c_{t+1} = \varpi_1 \Delta c_t + \varpi_2 \hat{r}_t, \tag{15}$$

where $\varpi_1 = \frac{1-\omega_1}{\omega_1}$ and $\varpi_2 = \frac{\omega_2}{\omega_1}$. We add as a final remark that it is also common in DSGE models, in the spirit of Campbell and Mankiw (1991), to incorporate rule of thumb consumers into the model, see for instance Amato and Laubach (2003) and Di Bartolomeo and Rossi (2007) and the references cited therein.

As shown by Campbell and Deaton (1989) and used by Palumbo *et al.* (2006) among others, the "saving for a rainy day" hypothesis in (8) has a very similar form in logarithms. By approximating the saving ratio, $\frac{S_t}{Y_t}$, with the logarithms of the income to consumption ratio, $y_t - c_t$, we may write a logarithmic version of (8) as

$$y_t - c_t \approx - \sum_{i=1}^{\infty} \rho^i E_t \Delta y_{t+i} + \kappa, \tag{16}$$

where ρ and κ denote a discount factor and a constant, respectively, cf. Campbell and Deaton (1989, equation 8). Equation (16) says that the saving ratio and the expected future income growth is negatively related so that when savings are increasing today the consumer expects income to decline tomorrow. Again, when income is non-stationary, the saving ratio is stationary and income and consumption are cointegrated with a coefficient equal to unity.

Thus far we have focused on various consumption models based on Euler equations. There exist, however, a huge empirical literature initiated by Davidson *et al.* (1978) based on a rather different theoretical framework, which goes back to Keynes (1936), saying that current aggregate income is the main determinant of current aggregate consumption. The consumption model by Jansen (2013) belongs to this literature.

2.3 A nested CVAR

As previously noted, a general CVAR nests all the Euler equations considered above. To show this, we will draw upon the analysis by Eitrheim *et al.* (2002) saying that both the Keynesian consumption function approach and the Euler equation approach are consistent with cointegration between consumption and income and that the discriminating feature is their implications for the direction of equilibrium correction (weak exogeneity) in a CVAR.

We start out, building on Jansen (2013), with a CVAR representation of a p-dimensional VAR of order k written as

$$\Delta X_t = \Pi X_{t-1}^* + \sum_{j=1}^{k-1} \Gamma_j \Delta X_{t-j} + \psi R_{t-1} + \vartheta + \Phi D_t + \epsilon_t, \tag{17}$$

where $X_t = (c_t, y_t, w_t)'$ comprises real consumption, c_t, real disposable income, y_t, and real household wealth, w_t, as the modelled variables, $X_t^* = (c_t, y_t, w_t, AGE_t, R_t, t)'$ comprises X_t along with an age composition variable, AGE_t, and the ex post real after tax interest rate, R_t, as the non-modelled variables, in addition to a deterministic trend, t, Γ_j are matrices of coefficients, ϑ is a vector of constants, D_t is a vector of centered seasonal dummies, and ϵ_t are identically distributed, independent random variables with expectation zero and covariance matrix Ω.[7] The initial observations $X_1, \ldots, X_k$ are considered as given. The impact matrix Π has rank $0 \leq r \leq p$, and therefore can be written $\Pi = \alpha\beta'$ where α and β are $p \times r$ matrices of full rank r.

When $r = 1$ and the cointegration vector is normalised with respect to consumption, we can write out the CVAR in (17) as

$$
\begin{aligned}
(18) \quad \begin{pmatrix} \Delta c_t \\ \Delta y_t \\ \Delta w_t \end{pmatrix} &= \begin{pmatrix} \alpha_c \\ \alpha_y \\ \alpha_w \end{pmatrix} [c_{t-1} - \beta_y y_{t-1} - \beta_w w_{t-1} - \beta_{AGE} AGE_t + \beta_R R_t + \gamma t] \\
&+ \sum_{j=1}^{k-1} \begin{pmatrix} \gamma_{j,11} & \gamma_{j,12} & \gamma_{j,13} \\ \gamma_{j,21} & \gamma_{j,22} & \gamma_{j,23} \\ \gamma_{j,31} & \gamma_{j,32} & \gamma_{j,33} \end{pmatrix} \begin{pmatrix} \Delta c_{t-j} \\ \Delta y_{t-j} \\ \Delta w_{t-j} \end{pmatrix} \\
&+ \psi R_{t-1} + \vartheta + \Phi D_t + \epsilon_t,
\end{aligned}
$$

where β_y, β_w, β_{AGE} and β_R are the cointegration coefficients for income, wealth, the age composition variable and the real after tax interest rate, and α_c, α_y and α_w are the adjustment coefficients for consumption, income and wealth, respectively. It follows from the Keynesian consumption function approach that consumption is equilibrium correcting, that is when $0 < -\alpha_c < 1$, and that two possibilities exist in the case of both income and wealth. If $0 < \alpha_y, \alpha_w < 1$, income and wealth are also equilibrium correcting. When $\alpha_y = \alpha_w = 0$, on the other hand, income and wealth are weakly exogenous with respect to β and the Keynesian consumption function from (18) is represented by

$$
\begin{aligned}
(19) \quad \Delta c_t &= \alpha_c [c_{t-1} - \beta_y y_{t-1} - \beta_w w_{t-1} - \beta_{AGE} AGE_t + \beta_R R_t + \gamma t] \\
&+ \sum_{j=1}^{k-1} \begin{pmatrix} \gamma_{j,11} & \gamma_{j,12} & \gamma_{j,13} \end{pmatrix} \begin{pmatrix} \Delta c_{t-j} \\ \Delta y_{t-j} \\ \Delta w_{t-j} \end{pmatrix} \\
&+ \psi_c R_{t-1} + \vartheta_c + \Phi_c D_t + \epsilon_{ct}.
\end{aligned}
$$

The Euler equation approach, however, implies that consumption and wealth are *not* equilibrium correcting, that is $\alpha_c = \alpha_w = 0$ in (18), and that income alone, in line with the "savings for a rainy day" hypothesis in (16), is equilibrium correcting, that is $0 < \alpha_y < 1$. The theoretical prediction that income is equilibrium correcting still carries over to situations with some rule of thumb consumers as long as they respond to income with some delay, that is when $\lambda = 0$ in (13). Hence, the Euler equation approach also

[7] Jansen (2013) included the age composition variable in X_t^* because, as documented in Erlandsen and Nymoen (2008), aggregate consumption may rise and savings decrease when the share of elderly persons increases in the population. Following the recommendations in Harbo *et al.* (1998) for partial VAR, Jansen (2013) also included the deterministic trend in X_t^*. We also note that Jansen (2013) included an income policy dummy ($CPSTOP$) in D_t, in line with Brodin and Nymoen (1992), to catch up the inflationary pressure that build up during the wage and price freeze in 1978.

implies that $\gamma_{1,12} > 0$ in (18) in addition to $\gamma_{j,12} = 0 \;\; \forall \; j \neq 1$, $\gamma_{j,13} = \gamma_{j,21} = \gamma_{j,22} = \gamma_{j,23} = \gamma_{j,31} = \gamma_{j,32} = \gamma_{j,33} = 0 \; \forall \; j$, $\beta_y = 1$ and $\beta_w = \beta_{AGE} = \beta_R = \gamma = 0$. Then, leading the CVAR one period, taking conditional expectations of Δc_{t+1}, $E_t \Delta c_{t+1}$, and multiplying through by the matrix $c' = (1, 0, 0)$, the consumption Euler equation from (18) can be expressed as

$$E_t \Delta c_{t+1} = \sum_{j=1}^{k-1} \gamma_{j,11} \Delta c_{t+1-j} + \gamma_{1,12} \Delta y_t + \psi_c R_t + \vartheta_c + \Phi_c D_{t+1}. \tag{20}$$

We can now see how (18) nests all the various hypotheses considered in Subsections 2.1 and 2.2. First, a modified version of the martingale hypothesis by Hall (1978), $E_t \Delta c_{t+1} = 0$,[8] implies no significant terms on the right hand side of (20). Second, precautionary savings in response to uncertainty are reflected in the constant term, ϑ_c. Third, a significant ψ_c can be interpreted as the intertemporal elasticity of substitution in consumption. Fourth, any significant $\gamma_{j,11}$ points to habit formation in consumption. Finally, a significant $\gamma_{1,12}$ indicates a substantial portion of rule of thumb consumers responding only to one period lag in income growth.

If the rule of thumb consumers, on the other hand, determine consumption growth by reference to current income growth only, that is when $\lambda = 1$ in (13), then the equilibrium correction property of income contradicts a consumption Euler equation interpretation from (18). We can verify this by once again leading the CVAR one period, but now taking conditional expectations of both consumption and income and multiplying through by the matrix $c' = (1, -\mu, 0)$ to obtain

$$\begin{aligned} E_t \Delta c_{t+1} - \mu E_t \Delta y_{t+1} &= -\mu \alpha_y [c_{t-1} - y_{t-1}] + c' \sum_{j=1}^{k-1} \Gamma_j \begin{pmatrix} \Delta c_{t-j} \\ \Delta y_{t-j} \\ \Delta w_{t-j} \end{pmatrix} \\ &\quad + c' \psi R_t + c' \vartheta + c' \Phi D_{t+1}, \end{aligned} \tag{21}$$

where $c'\psi = \psi_c - \mu \psi_y = (1 - \mu)\sigma$ and $c'\vartheta = \vartheta_c - \mu \vartheta_y = (1 - \mu)\phi$ in (13). The fact that $0 < \mu, \alpha_y < 1$ implies that consumption is equilibrium correcting in (21), which cannot be the case if an Euler equation is true. Consequently, the CVAR is not able to distinguish between permanent income consumers and rule of thumb consumers, as assumed in (13), once the latter respond to current income in some way, that is when $0 < \lambda \leq 1$. Hence, in order for the CVAR to be a model nesting all the Euler equations in Subsections 2.1 and 2.2 we must have $\lambda = 0$.

We have seen that cointegration in (18) represents the common ground between the Keynesian consumption function approach and the Euler equation approach and that the theoretical predictions from the two approaches put different restrictions with respect to exogeneity on consumption and income. In the empirical analysis, we shall therefore consider hypotheses of cointegration and equilibrium correction as restrictions on $\Pi = \alpha\beta'$, both before and after the financial crises, to discriminate between the two approaches. Because CVAR models considering conditional expectations of future consumption and

[8] Hall (1978) used levels rather than logarithms of the variables in his regressions.

income may corroborate parameters stemming from the class of Euler equations, we shall also examine the empirical relevance of such models within the context of Johansen and Swensen (1999, 2004, 2008). Having established a nested CVAR in theory, we now turn to the data underlying the empirical analysis.

3 Data

As previously mentioned, Jansen (2013) analysed cointegration between consumption, income and wealth conditioning on the two exogenous variables, the real after tax interest rate and the age composition variable, for the sample period $1970q3$ to $2008q2$. Because the national accounts data have been revised several times, we extend the original data set in Jansen (2013) by using quarterly growth rates for the period $2008q2-2014q2$, keeping $2008q2$ fixed.

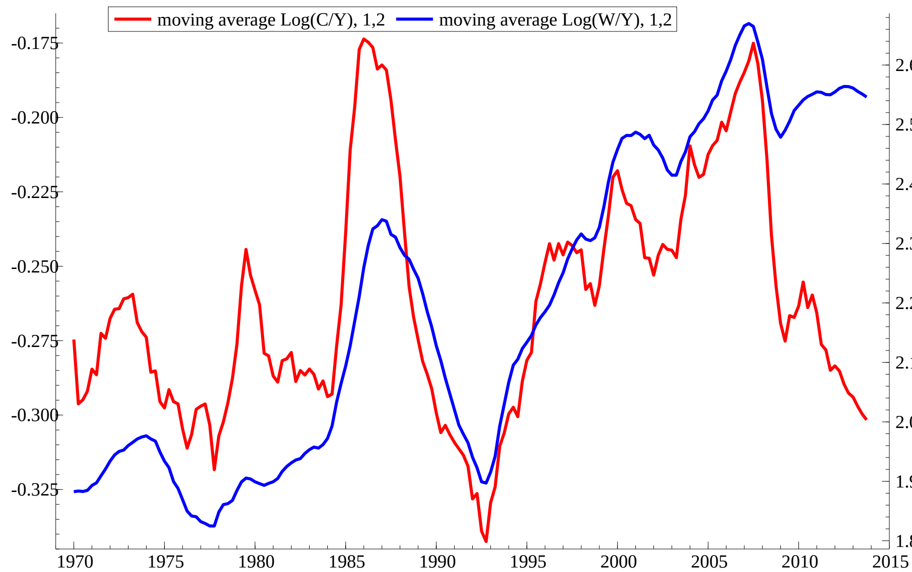
The consumption variable is defined as real consumption excluding expenditures on health services and housing. Expenditures on health services are excluded from the consumption variable as almost all of these are refunded by the government. Likewise, the imputed housing consumption is closely related to the imputed value of housing income by construction in the national accounts. Thus it does not make sense to include this component in the consumption variable when one purpose of our study is to estimate the MPC. Since we want to test the theory implications of the permanent income hypothesis, it could be argued that also durable goods should be excluded from the consumption variable under study, see for instance Deaton (1992, p. 99-103). However, data inspection reveals that the ratio between consumption of durables and our consumption variable fluctuates around a constant level, which suggests long run constancy. Taking logarithms means that the difference between the two consumption variables will be captured adequately by the constant term in the consumption models under study.

The income variable is real disposable income excluding equity income. The latter is left out because of episodes where tax increases on equity incomes were announced for the coming year leading to substantial tax motivated fluctuations in this income component, bearing in mind that equity incomes are likely to be less motivating for consumption than other incomes. Likewise, the wealth variable is measured in real terms net of household debt and consists of the value of housing as well as financial wealth. These entities differ widely in terms of liquidity and availability for the purpose of consumption of goods and services. We have nontheless maintained the aggregated wealth measure in the sequel.

Following Erlandsen and Nymoen (2008), the age composition variable is defined as the ratio between the number of persons between 50 and 66 years old and the persons between 20 and 49 years old plus the number of persons over 66 years old. Finally, the real after tax interest rate is defined as the average nominal interest rates on bank loans faced by households net of marginal income tax and adjusted for inflation. Again, following Erlandsen and Nymoen (2008), the real after tax interest rate interacts with a step dummy which takes on the value one as of $1984q1$ when direct regulations on the interest rates was lifted, zero otherwise. In Appendix 1, we give more precise definitions of all the variables entering the empirical models in Sections 4 and 5.

Figure 1 shows the consumption to income and the wealth to income ratios for the

Figure 1: Consumption to income and wealth to income ratios



Notes: Sample period: $1970q3-2014q2$. Moving averages of the ratios, with two quarters lags and one quarter lead, in logarithms. Consumption to income ratio (left axes), wealth to income ratio (right axes).

sample period $1970q3-2014q2$. We observe a strong comovement between the two ratios in the sample period of Jansen (2013) and this is *prima facie* evidence for cointegration between the three variables. However, a break in the cointegration relationship seems evident in the subsequent period − shortly after the financial crisis hit the Norwegian economy in the fall of 2008 − as the two ratios then diverge and move in opposite directions. These features of the data are the premises for the cointegration analysis, which we now turn to.

4 Cointegration analysis[9]

In this section, we first make a reappraisal of the cointegration results in Jansen (2013) in light of (17) by maintaining the data set from that study *as is*. Then, we conduct a cointegration analysis on the extended sample period without and with a structural break around the financial crises in 2008, applying the models in Johansen *et al.* (2000).

4.1 A reappraisal of Jansen (2013)

While unit root tests suggest that consumption and income are $I(1)$ variables and that wealth is an $I(1)$ variable with a deterministic trend, the tests for the conditioning vari-

[9]The econometric modelling in this section is carried out by PcGive 14, see Doornik and Hendry (2013).

Table 1: Trace test for cointegration*

Eigenvalue: λ_i	H_0	H_A	λ_{trace}	5%**
0.16	$r = 0$	$r \geq 1$	59.23	57.32
0.12	$r \leq 1$	$r \geq 2$	33.09	35.96
0.09	$r \leq 2$	$r \geq 3$	14.37	18.16

Diagnostics	Test statistic	Value[p-value]
Vector AR 1-5 test:	F(45,333)	1.24 [0.15]
Vector Normality test:	$\chi^2(6)$	4.20 [0.65]
Vector Hetero test:	F(246,633)	1.01 [0.46]

Notes: Sample period: $1970q3-2008q2$, VAR of order 5, modelled variables: c_t, y_t and w_t, non-modelled variables: AGE_t (restricted) and R_t (unrestricted), deterministic variables: trend (restricted), constant (unrestricted), income policy dummy ($CPSTOP$) (unrestricted) and centered seasonals (unrestricted), * see Doornik and Hendry (2013), ** critical values are obtained from Table 13 in Doornik (2003) - with two non-modelled variables.

ables, the age composition variable and the real interest rate, are ambiguous with respect to time series properties. That said, the underlying CVAR in Jansen (2013) can be formulated as

$$\Delta X_t = \alpha\beta' X_{t-1}^* + \Gamma_1 \Delta X_{t-1} + \cdots + \Gamma_{k-1} \Delta X_{t-(k-1)} + \Phi D_t + \epsilon_t, \tag{22}$$

where X_t and X_t^* are defined below (17), D_t now contains the constants, the centered seasonals and the income policy dummy, $CPSTOP$, and both AGE_t and R_t are treated as $I(1)$ variables entering the cointegration space only. Because (22) assumes that $\psi = 0$ at the outset, we shall make the reappraisal of the cointegration tests in Jansen (2013) by letting R_t enter the CVAR *unrestrictedly* in accordance with (17).

A VAR with five lags in c_t, y_t and w_t and one lag in AGE_t and R_t in addition to the deterministic terms specified above yields a well-specified model according to the diagnostic tests reported in Table 1. The trace tests for the rank of the VAR support a hypothesis of only one cointegrating vector between c_t, y_t, w_t, AGE_t and R_t. These tests are almost identical to the ones in Jansen (2013) for the case where R_t is restricted to lie in the cointegration space.[10] To test the hypothesis that $\psi = 0$, we compare the log likelihood value of (17) against the log likelihood value of (22) when the rank of the impact matrix equals unity. Restricting R_t to lie in the cointegrating space reduces the log likelihood value with only 0.25 ($1166.50 - 1166.25$), which gives a p-value of 0.97 to the corresponding likelihood ratio test. Since the two models are nested, this suggests that the restriction $\psi = 0$ is indeed supported by the data and that the intertemporal elasticity of substitution is not significantly different from zero. We may thus already at this stage of the analysis claim the empirical irrelevance of a consumption Euler equation like (12).

To discriminate between a consumption Euler equation and a Keynesian consumption function in more detail, we consider restrictions on $\Pi = \alpha\beta'$ based on (22). Table 2

[10]See Table 2 in Jansen (2013).

Table 2: Tests of restrictions on $\Pi = \alpha\beta'$*

Model (i): $\beta_c = 1$

$$c_t - \beta_y y_t - \beta_w w_t - \beta_{AGE} AGE_t + \beta_R R_t + \gamma t$$

$\log L = 1166.25$

Model (ii): $\beta_c = 1$, $\gamma = 0$

$$c_t - \underset{(0.05)}{0.79} y_t - \underset{(0.03)}{0.18} w_t + \underset{(0.12)}{0.07} AGE_t + \underset{(0.30)}{0.70} R_t,\ \alpha_c = -\underset{(0.10)}{0.48},\ \alpha_y = \underset{(0.10)}{0.02},\ \alpha_w = -\underset{(0.13)}{0.18}$$

$\log L = 1165.67$

$\chi^2(1) = 1.15[0.28]$

Model (iii): $\beta_c = 1$, $\gamma = 0$, $\beta_{AGE} = 0$

$$c_t - \underset{(0.05)}{0.78} y_t - \underset{(0.03)}{0.18} w_t + \underset{(0.30)}{0.58} R_t,\ \alpha_c = -\underset{(0.10)}{0.48},\ \alpha_y = \underset{(0.09)}{0.01},\ \alpha_w = -\underset{(0.12)}{0.16}$$

$\log L = 1165.53$

$\chi^2(2) = 1.43[0.49]$, $\chi^2(1) = 0.28[0.59]$

Model (iv): $\beta_c = 1$, $\gamma = 0$, $\beta_{AGE} = 0$, $\alpha_y = 0$, $\alpha_w = 0$

$$c_t - \underset{(0.05)}{0.74} y_t - \underset{(0.03)}{0.20} w_t + \underset{(0.20)}{0.49} R_t,\ \alpha_c = -\underset{(0.10)}{0.47}$$

$\log L = 1164.93$

$\chi^2(4) = 2.62[0.62]$, $\chi^2(2) = 1.20[0.55]$

Model (v): $\beta_c = 1$, $\gamma = 0$, $\beta_{AGE} = 0$, $\alpha_y = 0$, $\alpha_w = 0$, $\beta_y + \beta_w = 1$

$$c_t - \underset{(-)}{0.85} y_t - \underset{(0.02)}{0.15} w_t + \underset{(0.22)}{0.71} R_t,\ \alpha_c = -\underset{(0.08)}{0.38}$$

$\log L = 1163.37$

$\chi^2(5) = 5.75[0.33]$, $\chi^2(1) = 3.13[0.08]$

Notes: Sample period: $1970q3-2008q2$, VAR of order 5, rank = 1, modelled variables: c_t, y_t and w_t, non-modelled variables: AGE_t (restricted) and R_t (restricted), deterministic variables: trend (restricted), constant (unrestricted), income policy dummy ($CPSTOP$) (unrestricted) and centered seasonals (unrestricted), standard errors in parenthesis, p-values in square brackets, * see Doornik and Hendry (2013).

summarises the tests of such restrictions as documented in Jansen (2013, Table 3) conditioning on the rank being one. We see that the trend is insignificant (p-value = 0.28) as is the age composition variable (p-value = 0.59). Moreover, both y_t and w_t have significant coefficients, which contradict cointegration between consumption and income alone, as predicted by the consumption Euler equation. We also see that the loadings of the cointegrating vector are insignificant in the equations for y_t and w_t in the CVAR. This entails that both variables can be considered weakly exogenous with respect to the long run parameters in the cointegrating vector (p-value = 0.55 for the joint test). Hence, it is consumption, not income or wealth, that equilibrium corrects, and it follows that Δc_t is a Keynesian consumption function similar to (19) when $\beta_{AGE} = \gamma = 0$, $\alpha_y = \alpha_w = 0$ and $\psi = 0$. Finally, data support homogeneity (albeit marginally) in income and wealth (p-value = 0.08), yielding a long run relationship which can be written as

$$\widehat{eqcm}_{1,t} = c_t - 0.85y_t - 0.15w_t + 0.71R_t. \tag{23}$$

Recursive estimates demonstrate that the long run coefficients for wealth, and hence for income, are stable.[11] FIML estimation, conditioning on weak exogeneity of income and wealth and on homogeneity in the long run, yields a well specified CVAR and the following consumption equation with standard errors in parenthesis:[12]

$$\begin{aligned}
\widehat{\Delta c_t} \;=\; & -\underset{(0.10)}{0.19}\Delta c_{t-1} - \underset{(0.09)}{0.13}\Delta c_{t-2} - \underset{(0.09)}{0.07}\Delta c_{t-3} + \underset{(0.08)}{0.27}\Delta c_{t-4} \\
& -\underset{(0.12)}{0.26}\Delta y_{t-1} - \underset{(0.12)}{0.22}\Delta y_{t-2} + \underset{(0.11)}{0.01}\Delta y_{t-3} + \underset{(0.09)}{0.10}\Delta y_{t-4} \\
& +\underset{(0.07)}{0.29}\Delta w_{t-1} + \underset{(0.07)}{0.05}\Delta w_{t-2} - \underset{(0.07)}{0.02}\Delta w_{t-3} - \underset{(0.07)}{0.05}\Delta w_{t-4} \\
& -\underset{(0.08)}{0.38}\widehat{eqcm}_{1,t-1} - \underset{(0.05)}{0.21}const. + \underset{(0.09)}{0.14}CPSTOP_t \\
& +seasonals \\
& \\
& \hat{\sigma} = 0.01968
\end{aligned} \tag{24}$$

Two observations stand out in (24). First, the equilibrium correction term, $eqcm_{1,t-1}$, is highly significant, which corroborates the finding that the consumption Euler equation is not supported by the data. Second, the significant coefficients for Δy_{t-1}, Δy_{t-2} and Δw_{t-1} also contradict the Euler equation interpretation of the consumption equation. We end this subsection by noting that Jansen (2013) reports a conditional consumption function, based on a general to specific modelling approach, that has an average MPC of around 30 per cent well in line with the findings in the recent literature on consumer behaviour.[13]

[11] Note that (23) is not specified in per capita terms by the population, N_t, because $C_t/N_t = (Y_t/N_t)^{(1-\beta_w)} \cdot (W_t/N_t)^{\beta_w}$ is equivalent to $c_t = (1-\beta_w)y_t + \beta_w w_t$ due to the homogeneity restriction between consumption, income and wealth. The empirical results here and below are thus not affected in any substantive way by the non-per capita formulation.

[12] The full system of equations are reported in Appendix 2 along with system diagnostics.

[13] See Model A1 in Table 4 in Jansen (2013).

Table 3: Trace test for cointegration without a structural break*

Eigenvalue: λ_i	H_0	H_A	λ_{trace}	5%**
0.16	$r = 0$	$r \geq 1$	56.32	57.32
0.12	$r \leq 1$	$r \geq 2$	34.88	35.96
0.09	$r \leq 2$	$r \geq 3$	16.46	18.16

Diagnostics	Test statistic	Value[p-value]
Vector AR 1-5 test:	F(45,404)	1.62 [0.009]
Vector Normality test:	$\chi^2(6)$	11.21 [0.08]
Vector Hetero test:	F(246,775)	1.13 [0.11]

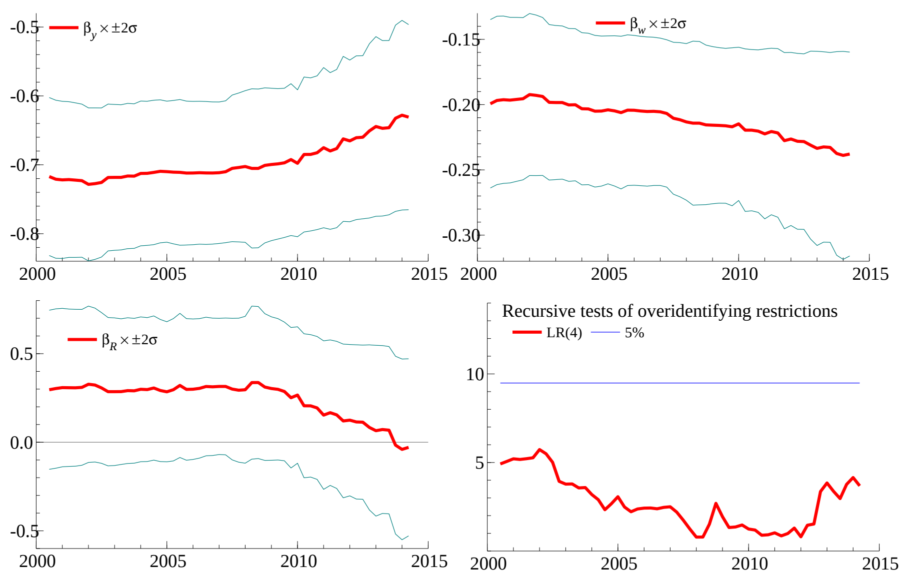
Notes: Sample period: 1970q3$-$2014q2, VAR of order 5, modelled variables: c_t, y_t and w_t, non-modelled variables: AGE_t (restricted) and R_t (restricted), deterministic variables: trend (restricted), constant (unrestricted), income policy dummy ($CPSTOP$) (unrestricted) and centered seasonals (unrestricted), * see Doornik and Hendry (2013), ** critical values are obtained from Table 13 in Doornik (2003) - with two non-modelled variables.

4.2 The extended sample period

When extending the sample period with 24 additional quarters, up to and including 2014q2, we build on (22) and continue to formulate a VAR with 5 lags in c_t, y_t and w_t and one lag in AGE_t and R_t in addition to the deterministic terms described above. Table 3 shows that the trace tests for cointegration now indicate zero rank. Hence, there is no longer support for cointegration between c_t, y_t, w_t, AGE_t and R_t. We also note that the test for autoregression suggests that the VAR is misspecified. If we, despite these findings, condition on the rank being equal to one, all the restrictions on $\Pi = \alpha\beta'$ considered in Subsection 4.1 are still supported by the data except long run homogeneity in income and wealth based on a partial likelihood ratio test. Figure 2 shows, however, that recursive likelihood ratio tests support the joint hypothesis of $\beta_{AGE} = 0$, $\beta_y + \beta_w = 1$, $\alpha_y = 0$ and $\alpha_w = 0$. The recursive estimates of the long run coefficients are now unstable and reveal a structural break around the financial crisis. The break affects both the income and the wealth elasticity, which increases and decreases, respectively, towards the end of the extended sample period. The break is, however, particularly clear for the semielasticity of the real interest rate, which is stable up to the second half of 2008 around a value of -0.35 and then drops to around zero by the first half of 2014.

Following Johansen *et al.* (2000), a structural break in the long run relationship can be captured by a model which takes into account the possibility of separate trends in the two periods $1, \ldots, T_1$ and $T_1+1, \ldots, T$. The idea is to allow for two VAR models where the k first observations are conditioned upon, but where the stochastic components are the same for both models and where the deterministic parameters may differ corresponding to the broken trend. Formally, let $T_0 = 0$ and $T_2 = T$. If $ID_{j,t} = 1$ for $t = T_{j-1}$ and $ID_{j,t} = 0$ else so that $ID_{j,t-i}$ is the indicator for the ith observation in the jth period, $j = 1, 2$, it follows that $SD_{j,t} = \Sigma_{i=k+1}^{T_j - T_{j-1}} ID_{j,t-i} = 1$ for $t = T_{j-1} + k + 1, \ldots, T_j$ and $SD_{j,t} = 0$ else.

Figure 2: Recursive estimates of long run coefficients



Notes: Sample period: $1970q3-2014q2$.

The CVAR in (22) is then formulated for $t = k+1, \ldots, T$ as

$$\Delta X_t = \alpha \begin{pmatrix} \beta \\ \gamma \end{pmatrix}' \begin{pmatrix} X^*_{t-1} \\ tSD_t \end{pmatrix} + \mu SD_t + \Gamma_1 \Delta X_{t-1} + \cdots + \Gamma_{k-1} \Delta X_{t-(k-1)} + \Phi D_t + \kappa_{2,1} ID_{2,t-1} + \cdots + \kappa_{2,k} ID_{2,t-k} + \epsilon_t, \tag{25}$$

where $SD_t = (SD_{1,t}, SD_{2,t})'$, $\gamma = (\gamma_1', \gamma_2')'$ and $\mu = (\mu_1, \mu_2)$. In our case we assume, referring to Figure 1, that the break occurs in 2008q3. This means that we augment the 5th order VAR above by defining $SD_{1,t}$ as a step dummy which is unity in the period $1970q3-2008q3$, $SD_{2,t}$ as a step dummy which is unity in the period $2010q1-2014q2$ and $ID_{2,t}$ as an impulse indicator which is unity for $t = 2008q4,...,2009q4$.

We repeat the cointegration analysis with the augmented VAR, letting the deterministic variables SD_t and $ID_{2,t}$ enter the VAR unrestrictedly, whereas tSD_t is restricted to lie in the cointegration space. The resulting trace tests are shown in Table 4. Critical values are not available for this case, but the magnitude of the test statistics suggests that there are at least one cointegrating vector. Table 5 reports test of restrictions on $\Pi = \alpha\beta'$ assuming one and only one cointegrating vector. Eliminating the age composition variable and imposing weak exogeneity as well as homogeneity are, as before, accepted by the data. We find that the trend is shifting equilibrium consumption downward both before and after the financial crisis. However, the shift is much larger after 2008q3, with a factor of 12 according to model (iii) – which corresponds to the modelling assumption in Section 5 – and a factor of 7.5 according to model (iv). When imposing homogeneity in income and wealth and weak exogeneity in accordance with model (iv), the long run relationship

Table 4: Trace test for cointegration* with a structural break**

Eigenvalue: λ_i	H_0	H_A	λ_{trace}	5%***
0.16	$r = 0$	$r \geq 1$	68.19	–
0.12	$r \leq 1$	$r \geq 2$	37.74	–
0.09	$r \leq 2$	$r \geq 3$	16.84	–

Diagnostics	Test statistic	Value[p-value]
Vector AR 1-5 test:	F(45,381)	1.51 [0.02]
Vector Normality test:	$\chi^2(6)$	8.00 [0.24]
Vector Hetero test:	F(246,729)	1.03 [0.38]

Notes: Sample period: $1970q3-2014q2$, VAR of order 5, modelled variables: c_t, y_t and w_t, non-modelled variables: AGE_t (restricted) and R_t (restricted), deterministic variables: tSD_t (restricted), SD_t (unrestricted), $ID_{2,t}$ (unrestricted), constant (unrestricted), income policy dummy ($CPSTOP$) (unrestricted) and centered seasonals (unrestricted), * see Doornik and Hendry (2013), ** see Johansen *et al.* (2000), *** critical values are not available.

can now be written

$$\widehat{eqcm}_{2,t} = c_t - 0.79y_t - 0.21w_t + 0.32R_t + 0.00045tSD_{1,t} + 0.0031tSD_{2,t}. \tag{26}$$

Figure 3 shows that recursive estimates of the coefficients for w_t and R_t, and hence also for y_t, are stable before and after the financial crises once the structural break is allowed for. FIML estimation of (25), based on the findings in Table 5, yields the following consumption equation with standard errors in parenthesis:[14]

$$\begin{aligned}
\widehat{\Delta c_t} = & -\underset{(0.09)}{0.40}\Delta c_{t-1} - \underset{(0.09)}{0.29}\Delta c_{t-2} - \underset{(0.09)}{0.18}\Delta c_{t-3} + \underset{(0.08)}{0.26}\Delta c_{t-4} \\
& -\underset{(0.10)}{0.04}\Delta y_{t-1} - \underset{(0.11)}{0.04}\Delta y_{t-2} + \underset{(0.11)}{0.11}\Delta y_{t-3} + \underset{(0.09)}{0.14}\Delta y_{t-4} \\
& +\underset{(0.07)}{0.34}\Delta w_{t-1} + \underset{(0.07)}{0.03}\Delta w_{t-2} - \underset{(0.07)}{0.03}\Delta w_{t-3} - \underset{(0.07)}{0.04}\Delta w_{t-4} \\
& -\underset{(0.03)}{0.10}\widehat{eqcm}_{2,t-1} - \underset{(0.03)}{0.06}SD_{1,t} - \underset{(0.01)}{0.02}SD_{2,t} + \underset{(0.10)}{0.13}CPSTOP_t \\
& -\underset{(0.03)}{0.11}D08q4 - \underset{(0.03)}{0.08}D09q1 - \underset{(0.03)}{0.07}D09q2 - \underset{(0.03)}{0.06}D09q3 \\
& -\underset{(0.03)}{0.09}D09q4 + seasonals \\
& \hat{\sigma} = 0.02049
\end{aligned} \tag{27}$$

The estimated CVAR has some signs of misspecification as normality is rejected by the data. We note that the equilibrium correction term is smaller in (27) than in (24), but still significant in the consumption equation. The significant coefficient for Δw_{t-1} is also at odds with the Euler equation interpretation of the consumption relationship. The

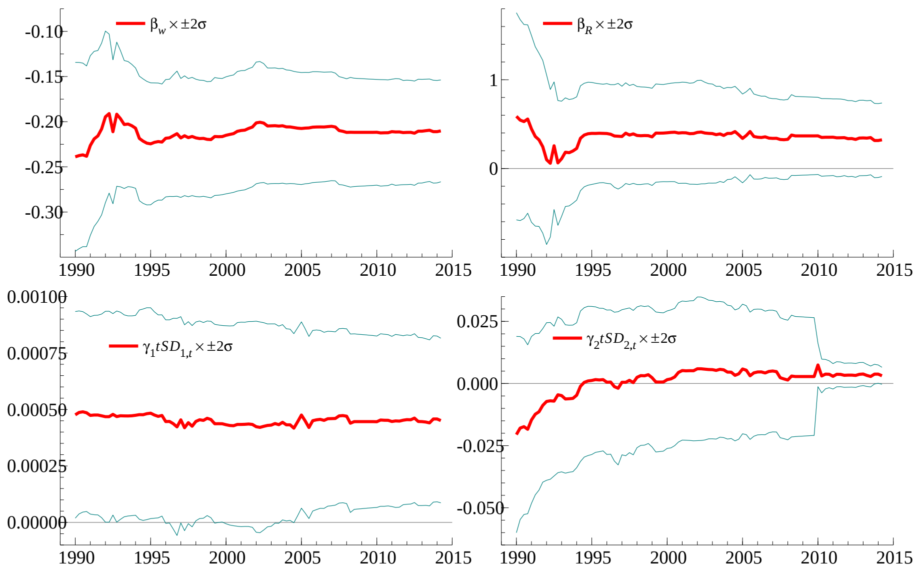
[14]The full system of equations are reported in Appendix 2 along with system diagnostics.

Table 5: Tests of restrictions on $\Pi = \alpha\beta'$* with a structural break**

Model (i): $\beta_c = 1$
$c_t - \beta_y y_t - \beta_w w_t + \beta_{AGE} AGE_t + \beta_R R_t + \gamma_1 tSD_{1,t} + \gamma_2 tSD_{2,t}$
$\log L = 1370.11$
Model (ii): $\beta_c = 1$, $\beta_{AGE} = 0$
$c_t - \underset{(0.11)}{0.85} y_t - \underset{(0.03)}{0.18} w_t + \underset{(0.26)}{0.36} R_t + \underset{(0.00081)}{0.00056} tSD_{1,t} + \underset{(0.002)}{0.0030} tSD_{2,t}$,
$\alpha_c = \underset{(0.10)}{-0.48}$, $\alpha_y = \underset{(0.09)}{0.01}$, $\alpha_w = \underset{(0.12)}{-0.16}$
$\log L = 1369.29$
$\chi^2(1) = 1.63[0.20]$
Model (iii): $\beta_c = 1$, $\beta_{AGE} = 0$, $\beta_y + \beta_w = 1$
$c_t - \underset{(-)}{0.82} y_t - \underset{(0.03)}{0.18} w_t + \underset{(0.22)}{0.43} R_t + \underset{(0.00019)}{0.00027} tSD_{1,t} + \underset{(0.0018)}{0.0034} tSD_{2,t}$,
$\alpha_c = \underset{(0.10)}{-0.48}$, $\alpha_y = \underset{(0.09)}{0.03}$, $\alpha_w = \underset{(0.11)}{-0.15}$
$\log L = 1369.25$
$\chi^2(2) = 1.72[0.42]$, $\chi^2(1) = 0.08[0.77]$
Model (iv): $\beta_c = 1$, $\beta_{AGE} = 0$, $\beta_y + \beta_w = 1$, $\alpha_y = 0$, $\alpha_w = 0$
$c_t - \underset{(-)}{0.79} y_t - \underset{(0.03)}{0.21} w_t + \underset{(0.21)}{0.32} R_t + \underset{(0.00018)}{0.00045} tSD_{1,t} + \underset{(0.0017)}{0.0031} tSD_{2,t}$,
$\alpha_c = \underset{(0.10)}{-0.50}$
$\log L = 1368.85$
$\chi^2(4) = 2.53[0.64]$, $\chi^2(2) = 0.81[0.67]$

Notes: Sample period: $1970q3-2014q2$, VAR of order 5 with a structural break in 2008q3, rank = 1, modelled variables: c_t, y_t and w_t, non-modelled variables: AGE_t (restricted) and R_t (restricted), deterministic variables: tSD_t (restricted), SD_t (unrestricted), $ID_{2,t}$ (unrestricted), constant (unrestricted), income policy dummy ($CPSTOP$) (unrestricted) and centered seasonals (unrestricted), standard errors in parenthesis, p-values in square brackets, * see Doornik and Hendry (2013), ** see Johansen *et al.* (2000).

Figure 3: Recursive estimates of long run coefficients



Notes: Sample period: $1970q3{-}2014q2$.

short term effects of the structural break are mainly picked up by the impulse indicators for the five quarters $2008q4{-}2009q4$, which have larger coefficients than the step dummies $SD_{j,t}$ in the consumption equation. The total short term effects of the impulse indicators and the step dummies are much smaller for the income and wealth equations compared to the consumption equation.

To facilitate a comparison of the magnitude of the MPC before and after the financial crisis, we estimate a conditional consumption function, general to specific, starting with Δc_{t-1}, Δc_{t-2}, Δc_{t-3}, Δc_{t-4}, Δy_t and four lags, Δw_t and four lags, ΔR_t and four lags in addition to $\widehat{eqcm}_{2,t-1}$ and all the dummies for the seasonals, the income policy and the structural break around the financial crisis as regressors. This general model is fully in accordance with the fitted CVAR underlying (27), both in terms of the number of lags and the weak exogeneity status of income and wealth. The autometrics procedure available in PcGive, see Doornik and Hendry (2013), picks the following conditional model together with diagnostic tests and standard errors in parenthesis[15]:

$$(28)\quad \widehat{\Delta c_t} = \underset{(0.07)}{-0.43}\Delta c_{t-1} - \underset{(0.07)}{0.30}\Delta c_{t-2} - \underset{(0.07)}{0.27}\Delta c_{t-3} + \underset{(0.005)}{0.23}\Delta c_{t-4} + \underset{(0.07)}{0.29}\Delta y_t$$
$$+\underset{(0.07)}{0.19}\Delta y_{t-3} + \underset{(0.06)}{0.19}\Delta w_t + \underset{(0.06)}{0.30}\Delta w_{t-1} - \underset{(0.03)}{0.12}\widehat{eqcm}_{2,t-1} + dummies$$

$OLS, T = 176\ (1970q3 - 2014q2), \hat{\sigma} = 0.01932$
AR_{1-5}: $F(5, 151) = 0.91\ [0.48]$, $ARCH_{1-4}$: $F(4, 168) = 0.30\ [0.88]$,
$NORM$: $\chi^2(2) = 2.22\ [0.33]$, HET: $F(24, 146) = 1.00\ [0.47]$.

Interestingly, the conditional consumption function for the extended sample has an average MPC of around 30 per cent similar to Model A1 in Jansen (2013). These findings are in line with the argument in Doornik and Hendry (1997) that the main source of forecast failure is deterministic shifts in equilibrium means, e.g. the equilibrium saving ratio, and not shifts in the derivative coefficients, e.g. the marginal propensity to consume, that are of primary interest for policy analysis.

Based on the findings from testing restrictions on $\Pi = \alpha\beta'$ in the CVAR, we may conclude that data support a Keynesian consumption function and not a consumption Euler equation. Left unanswered, however, is whether conditional expectations of consumption and income play a role in explaining the consumer behaviour.

5 Conditional expectations[16]

We recall from (21) in Subsection 2.3 that the CVAR is not able to distinguish between permanent income consumers and rule of thumb consumers once the latter respond to current and not only to lagged income in some way. Hence, we are motivated to examine the empirical relevance of CVAR models considering conditional expectations of future consumption and income within the context of Johansen and Swensen (1999, 2004, 2008), building on the findings in Section 4. That is, we still let the rank order of the impact matrix to be unity, but do not restrict income and wealth to be weakly exogenous, which matches the underlying assumptions of the methods used in this section. As such, we test the role of forward-looking behaviour in consumption in much the same way as what has been done in the new Keynesian literature on pricing behaviour, see Boug *et al.* (2010, 2017). We shall throughout the analysis simplify matters by specifying CVAR models in their *exact* form and not introduce a stochastic error term. As discussed in Boug *et al.* (2017), the numerical treatment of *inexact* models is complicated to handle using likelihood based methods when a trivariate VAR is the underlying model. First, we outline the testing procedure, paying particular attention to the conditional expectations

[15] AR_{1-5} is a test for until 5th order residual autocorrelation; $ARCH_{1-4}$ is a test for until 4th order autoregressive conditional heteroskedasticity in the residuals; $NORM$ is a joint test for residual normality (no skewness and excess kurtosis) and HET is a test for residual heteroskedasticity, see Doornik and Hendry (2013). The numbers in square brackets are p-values.

[16] The estimation and testing in this section are performed by the statistical package R, see http://www.r-project.org/.

restrictions on the stochastic part of the CVAR. Then, we estimate CVAR models with conditional expectations and examine whether data can corroborate parameters stemming from the class of Euler equations, both before and after the financial crisis.

5.1 Outline of the testing procedure

Our main reference is the CVAR in (22), which we repeat here for convenience.

$$\Delta X_t = \alpha\beta' X^*_{t-1} + \Gamma_1 \Delta X_{t-1} + \cdots + \Gamma_{k-1}\Delta X_{t-(k-1)} + \Phi D_t + \epsilon_t, \tag{29}$$

where $X_t = (c_t, y_t, w_t)'$, $X^*_t = (c_t, y_t, w_t, R_t, AGE_t, t)'$ and D_t contains both the constants, the centered seasonals and the income policy dummy ($CPSTOP$). The Euler equations involving expectations of future variables can be expressed as $c'E_t\Delta X_{t+1} = d'X_t$, which implies restrictions on the coefficients in (29). For instance, a bivariate system where the variables satisfies a martingale hypothesis can be written $(1,0)E_t(X_{1,t+1}, X_{2,t+1})' = (1,0)(X_{1,t}, X_{2,t})'$ or $(1,0)E_t\Delta(X_{1,t+1}, X_{2,t+1})' = 0$. Often it is convenient to have a more flexible specification of the form

$$c'E_t\Delta X_{t+1} - d'X^*_t + d'_{-1}\Delta X_t + \cdots + d'_{-k+1}\Delta X_{t-k+2} + \Phi_0 D_{t+1} = 0 \tag{30}$$

where $c, d, d_{-1}, \cdots, d_{-k+1}$ and Φ_0 are known matrices.

A surprisingly versatile formulation arises by assuming that the $p \times q$ matrix c is known and allowing $d, d_{-1}, \ldots, d_{k-1}$ and Φ_0 to be treated as parameters. If they are allowed to vary freely (30) does not imply any constraints. It is only a statement that the conditional expectation of the next observation can be expressed by the past and present values. The more general formulation can be exploited by testing whether any of the matrices $d_{-1}, \ldots, d_{-k+1}$ and Φ_0 in (30) are equal to zero or any specified value, that is to investigate whether a simplification of the conditional expectation relation is possible.

The restrictions implied by (30) can be taken into account as follows: Leading (29) one period, the conditional expectations of ΔX_{t+1}, $E_t\Delta X_{t+1}$, given the present and previous values of X^*_t, can be expressed as

$$E_t\Delta X_{t+1} - \alpha\beta' X^*_t - \Gamma_1\Delta X_t - \cdots - \Gamma_{k-1}\Delta X_{t-(k-2)} - \Phi D_{t+1} = 0 \tag{31}$$

By equating coefficients restrictions of the form (30) are equivalent to

$$c'\alpha\beta' = d, c'\Gamma_1 = -d'_{-1}, \cdots, c'\Gamma_{k-1} = -d'_{-k+1}, c'\Phi = -\Phi_0. \tag{32}$$

These restrictions consist of two separate parts. One set of restrictions on the random variables and another one on the deterministic variables. In the following, we concentrate on the stochastic part and leave the coefficient of the deterministic terms unrestricted.

Using the methods described in Johansen and Swensen (1999, 2004, 2008) the value of the concentrated likelihood $L_c(d, d_{-1}, \ldots, d_{k-1}, \Phi_0)$, where the restrictions (30) are imposed, can be computed. Further maximization over $d, d_{-1}, \ldots, d_{-k+1}, \Phi_0$ yields a value $\max L_c(d, d_{-1}, \ldots, d_{-k+1}, \Phi_0)$ which is equal to the maximal value of the likelihood for (29), denoted as L_{max}. The likelihood ratio for a test of a particular hypothesis, for instance $d_{-k+1} = d^0_{-k+1}$, can then be found as

$$\frac{\max_{d,d_{-1},\ldots,d_{-k+1},\Phi_0} L_c(d,d_{-1},\ldots,d^0_{-k+1},\Phi_0)}{\max_{d,d_{-1},\ldots,d_{-k+2},d_{-k+1},\Phi_0} L_c(d,d_{-1},\ldots,d_{-k+1},\Phi_0)} = \frac{\max_{d,d_{-1},\ldots,d_{-k+2},\Phi_0} L_c(d,d_{-1},\ldots,d^0_{-k+1},\Phi_0)}{L_{max}}.$$

By considering the details of the methods described in Johansen and Swensen (1999, 2004, 2008) we can see that the maximization with respect to $d_{-1},\ldots,d_{-k+2},\Phi_0$ can be performed by ordinary least squares (OLS) and reduced rank regression, while maximizing with respect to d must be carried out using numerical optimization. A more detailed explanation can be found in Appendix 3.

5.2 Estimation with no break in trend

The testing procedure outlined above relies on a general to specific approach akin to the backward selection process common in linear models. The alternative specific to general approach starts from a simple model and adds additional lags until the fit is satisfactory. When considering the conditional expectations of consumption in the next period, such that $c=(1,0,0)'$, (29) takes the form

$$(33) \qquad E_t\Delta c_{t+1} = \alpha_c(1,\beta_y,\ldots,\beta_\gamma)\begin{pmatrix} c_t \\ y_t \\ w_t \\ R_t \\ AGE_t \\ t \end{pmatrix} + (\gamma_{1,11},\gamma_{1,12},\gamma_{1,13})\begin{pmatrix} \Delta c_t \\ \Delta y_t \\ \Delta w_t \end{pmatrix} + \cdots + (\gamma_{4,11},\gamma_{4,12},\gamma_{4,13})\begin{pmatrix} \Delta c_{t-3} \\ \Delta y_{t-3} \\ \Delta w_{t-3} \end{pmatrix} + \Phi D_{t+1},$$

where the first row of the matrix Γ_j is denoted as $(\gamma_{j,11},\gamma_{j,12},\gamma_{j,13})$ for $j=1,\ldots,4$.

Table 6 reports likelihood ratio tests for simplifying restrictions on the coefficients of (33) with no break in trend during the sample period $1970q3-2008q2$. When there are no restrictions on the coefficients of the differences, no restrictions on the coefficients of the dummies and the likelihood is maximized over the parameters in β, no restrictions are imposed, and the value of the maximum of the likelihood is the same as for (29). This is displayed in the first line of Table 6. The second and third lines show the likelihood ratio tests when imposing homogeneity in income and wealth and zero coefficient on the trend, and zero coefficient on the age composition variable, respectively. The next four lines show the tests of setting the coefficients on the fourth lag of the differences equal to zero. The coefficient on the fourth lag of consumption growth, $\gamma_{4,11}$, is significant at all reasonable levels, whereas the coefficients on the fourth lag of income and wealth, $\gamma_{4,12}$ and $\gamma_{4,13}$, are not significant, both individually and jointly. Table 6 also provides tests for further simplifying restrictions on the coefficients of lagged income and wealth. Model 9 indicates that the coefficients of the third and fourth lag of income growth and of the second, third and fourth lag of wealth growth are not significant. Further simplifying restrictions on the coefficients of (33) are, however, not justified by the data.

Table 6: Likelihood ratio tests for simplifying restrictions* on (33)**

Model	Restrictions	$\log L_i$	i-j	$-2\log\frac{L_j}{L_i}$	df	p-value
1	-	1166.25	-	-	-	
2	$\beta_y + \beta_y = 1,\ \gamma = 0$	1165.36	1-2	1.78	2	0.42
3	$2, \beta_{AGE} = 0$	1165.22	2-3	0.28	1	0.60
4	$\gamma_{4,11} = 0$	1158.82	3-4	12.80	1	0.0003
5	$\gamma_{4,12} = 0$	1164.66	3-5	1.12	1	0.27
6	$\gamma_{4,13} = 0$	1165.04	3-6	0.36	1	0.55
7	$\gamma_{4,12} = \gamma_{4,13} = 0$	1164.36	3-7	1.72	2	0.42
8	$6, \gamma_{3,12} = \gamma_{3,13} = 0$	1164.08	7-8	0.56	2	0.76
9	$8, \gamma_{2,13} = 0$	1163.73	8-9	0.70	1	0.40
10	$9, \gamma_{2,12} = 0$	1158.44	9-10	10.58	1	0.001
11	$9, \gamma_{1,13} = 0$	1155.23	9-11	17.00	1	0.0004

Notes: Sample period: $1970q3-2008q2$, * see Johansen and Swensen (1999, 2004, 2008), ** model with no break in trend.

For model 9, the estimated version of the right hand side of (33) with standard errors in parenthesis, apart from the estimated coefficient of the dummies, is

$$\widehat{E_t\Delta c_{t+1}} = -0.41(1.0, -0.85, -0.15, 0.68, 0.0, 0.0)\begin{pmatrix} c_t \\ y_t \\ w_t \\ R_t \\ AGE_t \\ t \end{pmatrix} \tag{34}$$
$$-\underset{(0.07)}{0.10}\Delta c_t - \underset{(0.07)}{0.10}\Delta c_{t-1} - \underset{(0.07)}{0.07}\Delta c_{t-2} + \underset{(0.07)}{0.27}\Delta c_{t-3}$$
$$-\underset{(0.08)}{0.33}\Delta y_t - \underset{(0.08)}{0.26}\Delta y_{t-1} + \underset{(0.06)}{0.27}\Delta w_t + \hat{\Phi}\hat{D}_{t+1}.$$

In order to interpret the estimates it is useful to consider the parameters as consisting of two parts: the first part, the parameters of interest, are the parameters stemming from the Euler equations described in Subsections 2.1 and 2.2 and the second part, the nuisance parameters, are the parameters necessary to ensure empirically well specified models. There are several interesting consequences of (34). The first is a clear rejection of the hypothesis that log consumption is a martingale, $E_t[\Delta c_{t+1}] = 0$, which is a variant of the hypothesis of Hall (1978) that the level of consumption is a martingale. The result found in Jansen (2013) is therefore confirmed. The second is a comparison with the relation

$$E_t\Delta c_{t+1} = \sigma R_t, \tag{35}$$

which, as outlined in Subsection 2.2, is derived using CRRA utility preferences. The restriction in (35) is clearly rejected since it is nested in the formulation (34) which cannot be reduced further. The third is evidence of strong habit formation through lags of consumption growth in line with the Smets and Wouters (2003) model in (15). Finally, we may rewrite (33) as

$$(36)\quad \Delta c_t = \frac{-\alpha_c}{\gamma_{1,11}}(1, \beta_y, \ldots, \beta_\gamma) \begin{pmatrix} c_t \\ y_t \\ w_t \\ R_t \\ AGE_t \\ t \end{pmatrix} - \left(\frac{-1}{\gamma_{1,11}}, \frac{\gamma_{1,12}}{\gamma_{1,11}}, \frac{\gamma_{1,13}}{\gamma_{1,11}}\right) \begin{pmatrix} E_t \Delta c_{t+1} \\ \Delta y_t \\ \Delta w_t \end{pmatrix}$$

$$- \cdots - \left(\frac{\gamma_{4,11}}{\gamma_{1,11}}, \frac{\gamma_{4,12}}{\gamma_{1,11}}, \frac{\gamma_{4,13}}{\gamma_{1,11}}\right) \begin{pmatrix} \Delta c_{t-3} \\ \Delta y_{t-3} \\ \Delta w_{t-3} \end{pmatrix} - \frac{\Phi}{\gamma_{1,11}} D_{t+1}.$$

This formulation can be considered as a consumption function involving expected future consumption. The estimate $1/\gamma_{1,11}$ of the coefficient of $E_t \Delta c_{t+1}$ is of particular interest and equals $-1/0.10 = -10$ with estimated standard error of $(1/0.10)^2 = 100$. Thus, even if the estimate is outside the region where the parameter has an economically meaningful interpretation, the estimated standard error is so large that for all reasonable coefficients a confidence interval will cover such regions.

Thus far, we have only considered conditional expectations of future consumption in the CVAR. We now consider a more general formulation of the model involving conditional expectations of both future consumption and income to shed light on the magnitude of the proportion of rule of thumb consumers taking current income, and not future income, into account when determining the amount of consumption. Specifically, we consider a more general formulation of (33) expressed as

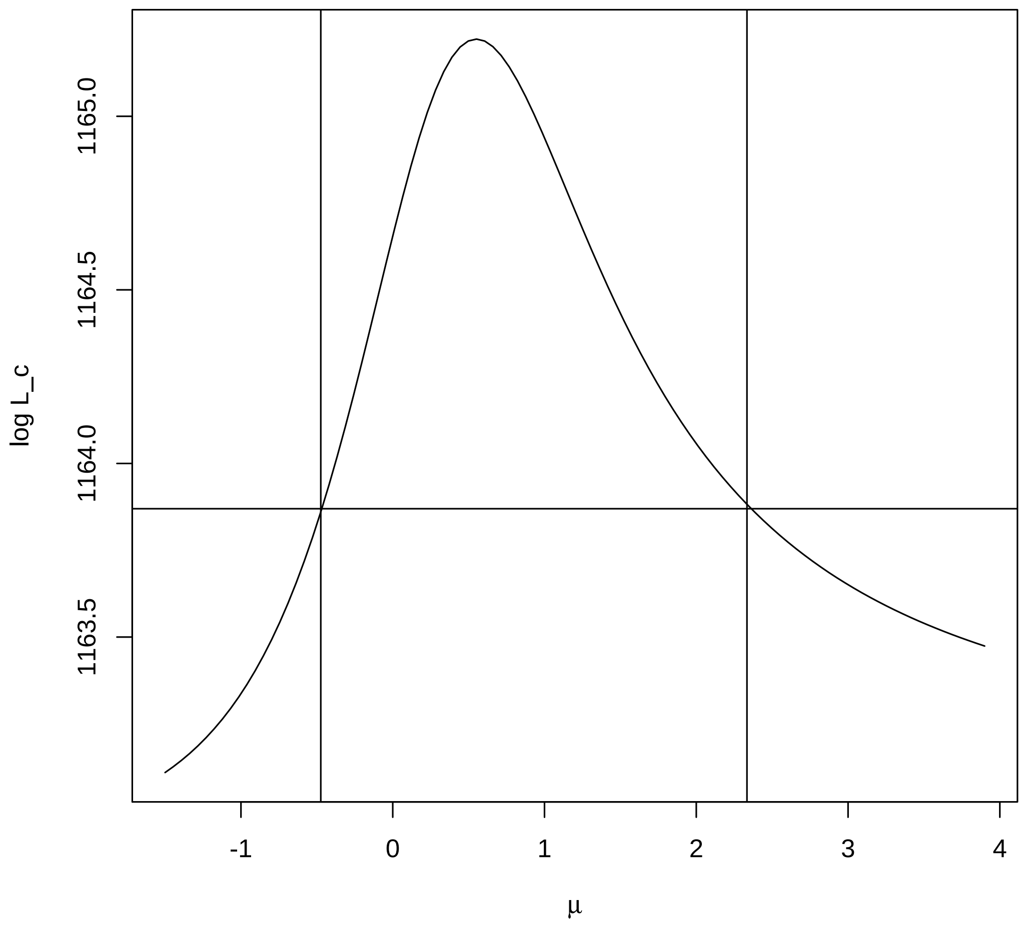
$$(37) E_t(\Delta c_{t+1} - \mu \Delta y_{t+1}) = \alpha_c(1, \beta_y, \ldots, \beta_\gamma) \begin{pmatrix} c_t \\ y_t \\ w_t \\ R_t \\ AGE_t \\ t \end{pmatrix} + (\gamma_{1,11}, \gamma_{1,12}, \gamma_{1,13}) \begin{pmatrix} \Delta c_t \\ \Delta y_t \\ \Delta w_t \end{pmatrix}$$

$$+ \cdots + (\gamma_{4,11}, \gamma_{4,12}, \gamma_{4,13}) \begin{pmatrix} \Delta c_{t-3} \\ \Delta y_{t-3} \\ \Delta w_{t-3} \end{pmatrix} + \Phi D_{t+1},$$

where a special case is (13) when $\lambda = 1$.

After having simplified (37) with regards to lags to obtain a satisfactory model we can estimate the likelihood function for fixed various values of μ and plot the maximal values. The maximal value will correspond to the maximum likelihood estimator. Figure 4 shows the concentrated log likelihood for μ with a 90 per cent confidence interval. We see that the maximal value of the likelihood is 1165.22, which corresponds to model 3 from Table 6, and that the maximal likelihood estimator of μ is around 0.5. However, the indicated 90 per cent confidence interval has the limits -0.4 and 2.3. Thus a large part of the confidence interval covers values for μ, which in light of (13), have no reasonable economic interpretation.

Figure 4: Concentrated log likelihood for the parameter μ in (37)* with a 90 per cent confidence interval



Notes: Sample period: 1970q3−2008q2. *Model without a break in trend

5.3 Estimation with a break in trend

As shown in Subsection 4.2, one cointegrating relationship between consumption, income and wealth seems evident once a structural break around the financial crises in 2008 is allowed for by the augmented CVAR in (25). Therefore, the empirical relevance of the augmented model with conditional expectations of future consumption and income will also be examined in the context of Johansen and Swensen (1999, 2004, 2008). We repeat (25) for convenience.

$$
\begin{aligned}
\text{(38)} \qquad \Delta X_t &= \alpha \begin{pmatrix} \beta \\ \gamma \end{pmatrix}' \begin{pmatrix} X_{t-1}^* \\ tSD_t \end{pmatrix} + \mu SD_t + \Gamma_1 \Delta X_{t-1} + \cdots + \Gamma_{k-1} \Delta X_{t-(k-1)} \\
&\quad + \Phi D_t + \kappa_{2,1} ID_{2,t-1} + \cdots + \kappa_{2,k} ID_{2,t-k} + \epsilon_t,
\end{aligned}
$$

where $SD_t = (SD_{1,t}, SD_{2,t})'$, $\gamma = (\gamma_1', \gamma_2')'$ and $\mu = (\mu_1, \mu_2)$. We now have that

$$
\begin{aligned}
E_t \Delta X_{t+1} &= \alpha\beta' X_t^* + \alpha\gamma' tSD_{t+1} + (\alpha\gamma' + \mu) SD_{t+1} \\
&+ \Gamma_1 \Delta X_t + \cdots + \Gamma_{k-1} \Delta X_{t-k} \\
&+ \Phi D_{t+1} + + \kappa_{2,1} ID_{2,t} + \cdots + \kappa_{2,k} ID_{2,t-k+1}
\end{aligned}
$$

and

$$
\begin{aligned}
c' E_{t+1} \Delta X_{t+1} &= c'\alpha\beta' X_t^* + c'\alpha\gamma' tSD_{t+1} + c'(\alpha\gamma' + \mu) SD_{t+1} \\
&+ c'\Gamma_1 \Delta X_t + \cdots + c'\Gamma_{k-1} \Delta X_{t-k} \\
&+ c'\Phi D_{t+1} + c'\kappa_{2,1} ID_{2,t} + \cdots + c'\kappa_{2,k} ID_{2,t-k+1}.
\end{aligned}
$$

Inserting the restrictions from (30) for the non-deterministic terms yields

$$
\begin{aligned}
c' E_t \Delta X_{t+1} &= d' X_t^* + c'\alpha\gamma' (tSD_{t+1} + SD_{t+1}) + c'\mu SD_{t+1} \\
&- d_{-1}' \Delta X_t + \cdots - d_{-k+1}' \Delta X_{t-k} \\
&+ c'\Phi D_{t+1} + c'\kappa_{2,1} ID_{2,t} + \cdots + c'\kappa_{2,k} ID_{2,t-k+1}.
\end{aligned}
$$

Remark that the coefficient matrix $c'\alpha\gamma'$ of $tSD_{t+1} + SD_{t+1}$ has reduced rank. It is now possible to proceed as before to see if any of the matrices $d_{-1}', \cdots, d_{-k+1}'$ can be deleted, by expanding X_t^* to also include $tSD_{t+1} + SD_{t+1}$.

Table 7 shows likelihood ratio tests for simplifying restrictions on the coefficients of the augmented CVAR considering conditional expectations of consumption only over the sample period $1970q3-2014q2$ and allowing for a break in trend at $2008q3$. The best model is model 7, which is of the same type as for the case without a broken trend: four lags of consumption growth, two lags of income growth and one lag of wealth growth. The estimates corresponding to model 7 are displayed in (39) with standard errors in parenthesis.

Table 7: Likelihood ratio tests for simplifying restrictions* on the augmented CVAR**

Model	Restrictions	$\log L_i$	i-j	$-2\log\frac{L_j}{L_i}$	df	p-value
1	-	1370.11	-	-	-	
2	$\beta_y + \beta_y = 1$, $\beta_{AGE} = 0$	1369.25	1-2	1.72	2	0.42
3	2, $SD_{1,t} = 0$	1368.93	2-3	0.64	1	0.42
4	3, $\gamma_{4,11} = 0$	1361.86	3-4	14.14	1	0.0002
5	3, $\gamma_{4,12} = \gamma_{4,13} = 0$	1368.74	3-5	0.38	2	0.83
6	5, $\gamma_{3,12} = \gamma_{3,13} = 0$	1367.99	5-6	1.50	2	0.47
7	6, $\gamma_{2,13} = 0$	1367.23	6-7	1.52	1	0.22
8	6, $\gamma_{2,12} = 0$	1364.18	6-8	7.62	1	0.006
9	6, $\gamma_{1,13} = 0$	1356.38	6-9	23.22	1	4E-6

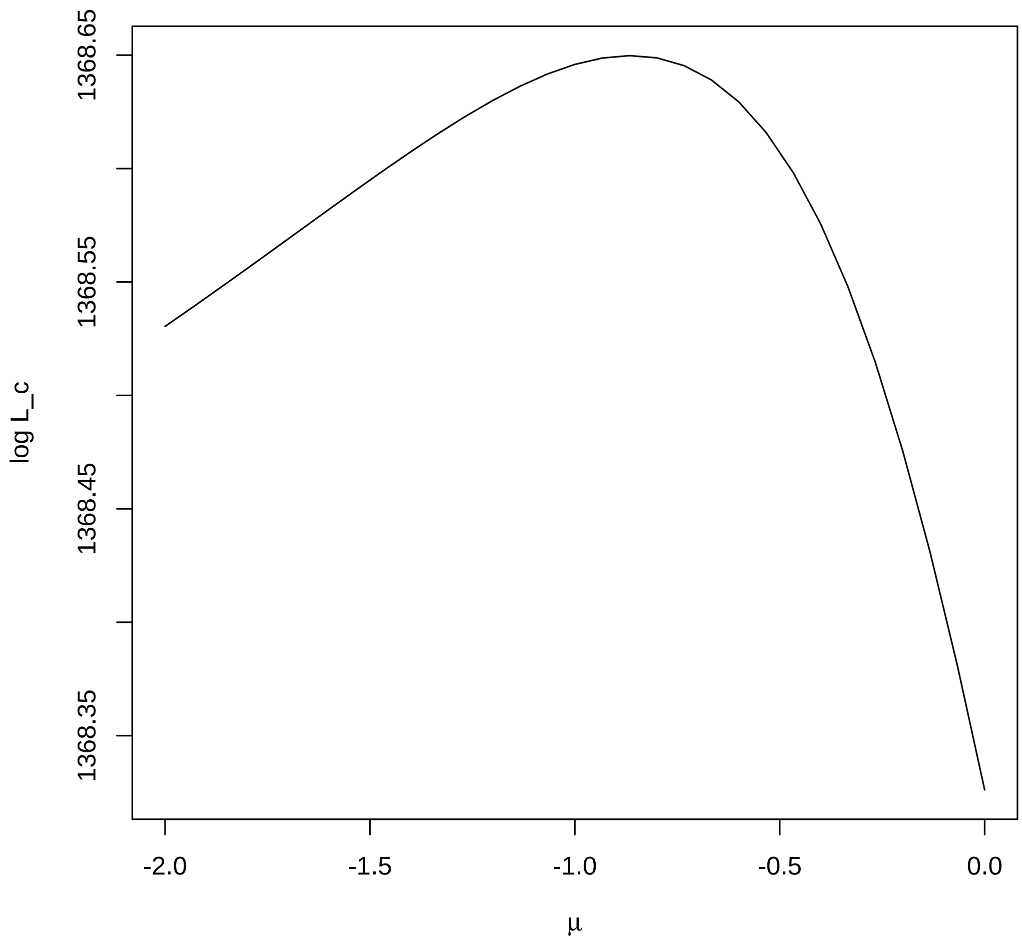
Notes: Sample period: 1970q3−2014q2, * see Johansen and Swensen (1999, 2004, 2008), ** model with a break in trend, see Johansen *et al.* (2000).

$$\widehat{E_t \Delta c_{t+1}} = -0.40(1.0, -0.87, -0.13, 0.64, 0.0, 0.0, 0.002)\begin{pmatrix} c_t \\ y_t \\ w_t \\ R_t \\ AGE_t \\ tSD_{1,t+1} \\ tSD_{2,t+1} \end{pmatrix} \tag{39}$$
$$-\underset{(0.06)}{0.14}\Delta c_t - \underset{(0.07)}{0.15}\Delta c_{t-1} - \underset{(0.07)}{0.10}\Delta c_{t-2} + \underset{(0.06)}{0.28}\Delta c_{t-3}$$
$$-\underset{(0.07)}{0.29}\Delta y_t - \underset{(0.08)}{0.23}\Delta y_{t-1} + \underset{(0.06)}{0.29}\Delta w_t + \hat{\Phi}\hat{D}_{t+1}.$$

We see that the coefficients are close to those obtained for the model without a break in trend. The coefficient $1/\gamma_{1,11}$ is now estimated to $-1/0.14 = -7.1$ with a standard error of 51.0. Thus the conclusions from the case of no break in trend are maintained. Finally, Figure 5 plots the maximal value of the concentrated log likelihood for various values of μ based on a simplification of (37) allowing for a break in the trend. We see that the maximal value of the likelihood is around 1368 with a corresponding $\mu \approx -0.8$, which does not make sense economically. Moreover, the variation in the value of the likelihood over the interval $(-2, 0)$ is small, which means that a reasonable parameter estimate for μ will not be contained by the confidence interval.

We conclude from all the findings in this section that most of the parameters stemming from the class of Euler equations are not supported by the data when considering conditional expectations of consumption and income in CVAR models. Only habit formation in line with Smets and Wouters (2003) model seems to play an important role in explaining the Norwegian consumer behaviour.

Figure 5: Concentrated log likelihood for the parameter μ in (37)* with a 90 per cent confidence interval



Notes: Sample period: $1970q3-2014q2$. *Model with a break in trend.

6 Conclusions

In this paper, we have examined aggregated Norwegian consumer behaviour, both before and after the financial crisis hit the economy. Using likelihood based methods, we have estimated and tested a general CVAR that nests both a Keynesian consumption function and a class of consumption Euler equations. These include the martingale hypothesis and the often used equations of precautionary savings, liquidity constraints and habit formation in consumption.

We found evidence of one cointegrating vector between consumption, income and wealth once a structural break around the financial crises is accounted for. That consumption cointegrates with both income and wealth and not only with income demonstrates the empirical irrelevance of an Euler type equation. We also found that consumption equilibrium corrects to changes in income and wealth and not that income equilibrium corrects to changes in consumption, as would be the case when an Euler equation is true. Finally, we found that most of the parameters stemming from the class of Euler equations are not corroborated by the data when considering conditional expectations of future consumption and income in CVAR models. Only habit formation, typically included in DSGE models, seems to be important in explaining the Norwegian consumer behaviour. We therefore end up with a dynamic Keynesian type consumption function with an average MPC of around 30 per cent well in line with the findings in the recent literature.

We have relied on a CVAR in which a structural break in the cointegration relationship between consumption, income and wealth around the event of the financial crisis has been accounted for by a broken trend. One possibility is that this broken trend picks up some important effects of omitted variables necessary to explain the consumer behaviour after the financial crises. For instance, we have not included a variable for the credit shifts faced by households and not disaggregated the wealth variable into separate variables for liquid assets, illiquid assets, debt and housing. Such variables may be important in a CVAR to adequately pick up effects of the household financial accelerator on consumption in the wake of the financial crises. We leave this issue for future work.

References

[1] Amato, J.D. and T. Laubach (2003): Rule-of-thumb behaviour and monetary policy, *European Economic Review*, 47(5), 791-831.

[2] Ando, A. and F. Modigliano (1963): The life cycle hypothesis of saving: Aggregate implications and tests, *American Economic Review*, 53, 55-84.

[3] Bartolomeo, G.D. and L. Rossi (2007): Effectiveness of monetary policy and limited asset market participation: Neoclassical versus Keynesian effects, *International Journal of Economic Theory*, 3(3), 213-218.

[4] Blanchard, O.J. and S. Fischer (1989): Lectures on Macroeconomics, Cambridge, Massachusetts, MIT Press.

[5] Blundell, R. and T.M. Stoker (2005): Heterogeneity and aggregation, *Journal of Economic Literature*, XLIII, 347-391.

[6] Boug, P., Å. Cappelen and A.R. Swensen (2017): Inflation dynamics in a small open economy, *Scandinavian Journal of Economics*, 119(4), 1010-1039.

[7] Boug, P., Å. Cappelen and A.R. Swensen (2010): The new Keynesian Phillips curve revisited, *Journal of Economic Dynamics & Control*, 34, 858-874.

[8] Brodin, P.A. and R. Nymoen (1992): Wealth effects and exogeneity. The Norwegian consumption function 1966(1)-1989(4), *Oxford Bulletin of Economics and Statistics*, 54, 431-454.

[9] Campbell, J.Y. (1987): Does saving anticipate declining labor income? An alternative test of the permanent income hypothesis, *Econometrica*, 55, 1249-1273.

[10] Campbell, J.Y. and A.S. Deaton (1989): Why is consumption so smooth? *Review of Economic Studies*, 56, 357-374.

[11] Campbell, J.Y. and N.G. Mankiw (1991): The response of consumption to income: A cross-country investigation, *European Economic Review*, 35, 723-767.

[12] Carroll, C., J. Slacalek, K. Tokuoka and M.N. White (2017): The distribution of wealth and the marginal propensity to consume, *Quantitative Economics*, 8, 977-1020.

[13] Carroll, C.D. (1992): The buffer-stock theory of saving: Some macroeconomic evidence, *Brookings Papers on Economic Activity*, 1992(2), 61-156

[14] Davidson, J.E.H, D.F. Hendry, F. Srba and S. Yeo (1978): Econometric modelling of the aggregate time-series relationship between consumers' expenditure and income in the United Kingdom, *Economic Journal*, 88, 661-692.

[15] Deaton, A.S. (1992): Understanding consumption, Clarendon Press, Oxford.

[16] Deaton, A.S. (1991): Saving and liquidity constraints, *Econometrica*, 59, 1221-1248.

[17] Doornik, J.A. (2003): Asymptotic tables for cointegration tests based on the Gamma-distribution approximation, http://www.doornik.com/research/coigamma_tables.pdf

[18] Doornik, J.A. and Hendry, D.F. (2013). Modelling Dynamic Systems Using PcGive 14: Volume II, London: Timberlake Consultants Press.

[19] Doornik, J.A. and D.F. Hendry (1997): The implications for econometric modelling of forecast failure, *Scottish Journal of Political Economy*, 44, 437-461.

[20] Eitrheim, Ø., E.S. Jansen and R. Nymoen (2002): Progress from forecast failure − the Norwegian consumption function, *Econometrics Journal*, 5, 40-64.

[21] Erlandsen, S. and R. Nymoen (2008): Consumption and the population age structure, *Journal of Population Economics*, 21, 505-520.

[22] Fagereng, A., M.B. Holm and G.J. Natvik (2016): MPC Heterogeneity and Household Balance Sheets, Discussion Papers 852, Statistisk sentralbyr.

[23] Flavin, M.A. (1981): The adjustment of consumption to changing expectations about future income, *Journal of Political Economy*, 89, 974-1009.

[24] Friedman, M. (1957): A theory of the consumption function, Princeton University Press, Princeton.

[25] Hall, R.E. (1978): Stochastic implications of the life cycle-permanent income hypothesis: Theory and evidence. *Journal of Political Economy*, 86, 971-987.

[26] Harbo, I., S. Johansen, B. Nielsen and A. Rahbek (1998): Asymptotic inference on cointegration rank in partial systems, *Journal of Business & Economic Statistics*, 16, 388-399.

[27] Jansen, E.S. (2013): Wealth effects on consumption in financial crises: The case of Norway, *Empirical Economics*, 45, 873-904.

[28] Jappelli, T. and L. Pistaferri (2014): Fiscal policy and MPC Heterogeneity: *AEJ Macroeconomics*, 6(4), 107-136.

[29] Johansen, S., R. Mosconi and B. Nielsen (2000): Cointegration analysis in the presence of structural breaks in the deterministic trend, *Econometrics Journal*, 3, 216-249.

[30] Johansen, S. and Swensen, A.R. (2008): Exact rational expectations, cointegration and reduced rank regression, *Journal of Statistical Planning and Inference*, 138, 2738-2748.

[31] Johansen, S. and Swensen, A.R. (2004): More on testing exact rational expectations in cointegrated vector autoregressive models: Restricted constant and linear term, *Econometrics Journal*, 7(2), 389-397.

[32] Johansen, S. and Swensen, A.R. (1999): Testing exact rational expectations in cointegrated vector autoregressive models, *Journal of Econometrics*, 93, 73-91.

[33] Kaplan, G. and G.L. Violante (2014): A model of the consumption response to fiscal stimulus payments, *Econometrica*, 82(4) 1199-1239.

[34] Keynes, J.M. (1936): The general theory of employment, interest and money, Macmillan, London.

[35] Muellbauer, J. (2010): Household decisions, credit markets and the macroeconomy: Implications for the design of central bank models. Bank for International Settlements Discussion Paper 306, http://ideas.repec.org/p/bis/biswps/.

[36] Muellbauer, J. and R. Lattimore (1995): The Consumption function: A theoretical and empirical overview, in Handbook of Applied Econometrics, edited by H. Pesaran and M. Wickens. Oxford: Blackwell.

[37] Palumbo, M., J. Rudd and K. Whelan (2006): On the relationships between real consumption, income and wealth, *Journal of Business & Economic Statistics*, 24(1), 1-11.

[38] Parker, J.A., N.S. Souleles, D.S. Johnson and R. McClelland (2013): Consumer spending and the economic stimulus payments of 2008, *American Economic Review*, 103(6), 2530-2553.

[39] Smets, F. and R. Wouters (2003): An estimated dynamic stochastic general equilibrium model of the euro area, *Journal of the European Economic Association*, 1, 1123-1175.

Appendix 1. Data definitions and sources

The original data set used in Jansen (2013) as well as the extended data set are collected from Statistics Norway and Norges Bank, unless otherwise noted.

C: Consumption expenditures in households and ideal organisations excluding expenditures on heath services and housing, fixed 2006 prices. Source: Statistics Norway.

Y: Households real disposable income, excluding equity income, defined as nominal income deflated by PC. Source: Statistics Norway.

W: Real household wealth defined as nominal household wealth (NW), the sum of financial and housing wealth, deflated by PC.

$NW_t = (L_{t-1}+ML_{t-1}+NL_{t-1}-CR_{t-1} +(PH/PC)_t a_t K_{t-1} PC_t$, where L_t is household liquid assets (money stock and deposits), ML_t is household medium liquid assets (equity and bonds), NL_t is household non-liquid assets (insurance claims), CR_t is household debt to banks and other financial institutions, PH_t is housing price index (2006=1), a_t is fraction of residential housing stock owned by households and K_t is real values of residential housing stock (fixed 2006 prices). The residential housing stock is updated each quarter by adding the gross investments in housing capital (in real terms deflated by PH) and deducting 0.4 per cent depreciation per quarter. Source: Statistics Norway. Data for the period 1970q1 to 1992q3 for nominal household wealth are from Erlandsen and Nymoen (2008). These data are chained in 1992q3 with data from Statistics Norway.

AGE: $(P5066)/(P2049+P67\text{up})$, where $P5066$ is the population $50-66$ years old, $P2049$ is population $20-49$ years old and $P67\text{up}$ is population more than 66 years old. Source: Statistics Norway.

R: Real after tax interest rate for households. 1970q1$-$1983q4, zero. 1984q1$-$2009q4, $RLB(1-\tau)-\Delta_4\log CPI$, where τ is marginal income tax rate faced by households. Source: Statistics Norway and Norges Bank.

RLB: Average interest rate on households bank loans. Source: Norges Bank.

CPI: Headline consumer price index (2006=1). Source: Statistics Norway.

PC: Price deflator for total consumption expenditures in households and ideal organisations excluding expenditures on heath services and housing (2006=1). Source: Statistics Norway.

$CPSTOP$: Income policy dummy constructed to catch up the inflationary pressure that build up during the wage and price freeze in 1978. It takes non-zero values from 1979q1 to 1980q1 and zero elsewhere. See Brodin and Nymoen (1992) for details.

Appendix 2. Full systems and diagnostics

The full system and diagnostics underlying (24) in the main text are

$$\begin{aligned}
\widehat{\Delta c_t} = & -\underset{(0.10)}{0.19}\Delta c_{t-1} - \underset{(0.09)}{0.13}\Delta c_{t-2} - \underset{(0.09)}{0.07}\Delta c_{t-3} + \underset{(0.08)}{0.27}\Delta c_{t-4} \\
& -\underset{(0.12)}{0.26}\Delta y_{t-1} - \underset{(0.12)}{0.22}\Delta y_{t-2} + \underset{(0.11)}{0.01}\Delta y_{t-3} + \underset{(0.09)}{0.10}\Delta y_{t-4} \\
& +\underset{(0.07)}{0.29}\Delta w_{t-1} + \underset{(0.07)}{0.05}\Delta w_{t-2} - \underset{(0.07)}{0.02}\Delta w_{t-3} - \underset{(0.07)}{0.05}\Delta w_{t-4} \\
& -\underset{(0.08)}{0.38}\widehat{eqcm}_{1,t-1} - \underset{(0.05)}{0.21}const. + \underset{(0.09)}{0.14}CPSTOP_t \\
& +seasonals
\end{aligned}$$

$$\hat{\sigma} = 0.01968$$

$$\begin{aligned}
\widehat{\Delta y_t} = & +\underset{(0.07)}{0.28}\Delta c_{t-1} + \underset{(0.08)}{0.13}\Delta c_{t-2} + \underset{(0.08)}{0.09}\Delta c_{t-3} + \underset{(0.07)}{0.06}\Delta c_{t-4} \\
& -\underset{(0.09)}{0.58}\Delta y_{t-1} - \underset{(0.10)}{0.43}\Delta y_{t-2} - \underset{(0.10)}{0.33}\Delta y_{t-3} + \underset{(0.09)}{0.17}\Delta y_{t-4} \\
& -\underset{(0.06)}{0.08}\Delta w_{t-1} - \underset{(0.07)}{0.05}\Delta w_{t-2} - \underset{(0.07)}{0.02}\Delta w_{t-3} - \underset{(0.06)}{0.03}\Delta w_{t-4} \\
& + \underset{(0.003)}{0.01}\, const. - \underset{(0.09)}{0.13}CPSTOP_t \\
& +seasonals
\end{aligned}$$

$$\hat{\sigma} = 0.01821$$

$$\begin{aligned}
\widehat{\Delta w_t} = & +\underset{(0.10)}{0.07}\Delta c_{t-1} + \underset{(0.11)}{0.04}\Delta c_{t-2} + \underset{(0.10)}{0.28}\Delta c_{t-3} + \underset{(0.09)}{0.04}\Delta c_{t-4} \\
& -\underset{(0.12)}{0.04}\Delta y_{t-1} - \underset{(0.13)}{0.05}\Delta y_{t-2} - \underset{(0.13)}{0.26}\Delta y_{t-3} - \underset{(0.11)}{0.13}\Delta y_{t-4} \\
& +\underset{(0.08)}{0.16}\Delta w_{t-1} + \underset{(0.09)}{0.09}\Delta w_{t-2} - \underset{(0.09)}{0.04}\Delta w_{t-3} + \underset{(0.08)}{0.30}\Delta w_{t-4} \\
& +\underset{(0.004)}{0.006}const. - \underset{(0.11)}{0.08}CPSTOP_t \\
& +seasonals
\end{aligned}$$

$$\hat{\sigma} = 0.02385$$

Diagnostics:
Vector AR 1-5 test: F(45,351) = 0.99 [0.49]
Vector Normalty test: $\chi^2(6) = 5.98$ [0.43]
Vector Heteroscedasticity test: $F(234, 644) = 1.10[0.19]$

The full system and diagnostics underlying (27) in the main text are

$$
\begin{aligned}
\widehat{\Delta c_t} = & -\underset{(0.09)}{0.40}\Delta c_{t-1} - \underset{(0.09)}{0.29}\Delta c_{t-2} - \underset{(0.09)}{0.18}\Delta c_{t-3} + \underset{(0.08)}{0.26}\Delta c_{t-4} \\
& -\underset{(0.10)}{0.04}\Delta y_{t-1} - \underset{(0.11)}{0.04}\Delta y_{t-2} + \underset{(0.11)}{0.11}\Delta y_{t-3} + \underset{(0.09)}{0.14}\Delta y_{t-4} \\
& +\underset{(0.07)}{0.34}\Delta w_{t-1} + \underset{(0.07)}{0.03}\Delta w_{t-2} - \underset{(0.07)}{0.03}\Delta w_{t-3} - \underset{(0.07)}{0.04}\Delta w_{t-4} \\
& -\underset{(0.03)}{0.10}\widehat{eqcm}_{2,t-1} - \underset{(0.03)}{0.06}SD_{1,t} - \underset{(0.01)}{0.02}SD_{2,t} + \underset{(0.10)}{0.13}CPSTOP_t \\
& -\underset{(0.03)}{0.11}D08q4 - \underset{(0.03)}{0.08}D09q1 - \underset{(0.03)}{0.07}D09q2 - \underset{(0.03)}{0.06}D09q3 \\
& -\underset{(0.03)}{0.09}D09q4 + seasonals \\
& \hat{\sigma} = 0.02049
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\Delta y_t} = & +\underset{(0.07)}{0.25}\Delta c_{t-1} + \underset{(0.07)}{0.12}\Delta c_{t-2} + \underset{(0.07)}{0.10}\Delta c_{t-3} + \underset{(0.07)}{0.07}\Delta c_{t-4} \\
& -\underset{(0.08)}{0.57}\Delta y_{t-1} - \underset{(0.09)}{0.43}\Delta y_{t-2} - \underset{(0.09)}{0.35}\Delta y_{t-3} + \underset{(0.08)}{0.16}\Delta y_{t-4} \\
& -\underset{(0.06)}{0.08}\Delta w_{t-1} - \underset{(0.06)}{0.03}\Delta w_{t-2} - \underset{(0.06)}{0.02}\Delta w_{t-3} - \underset{(0.06)}{0.05}\Delta w_{t-4} \\
& -\underset{(0.002)}{0.01}SD_{1,t} - \underset{(0.005)}{0.02}SD_{2,t} + \underset{(0.08)}{0.13}CPSTOP_t \\
& +\underset{(0.02)}{0.03}D08q4 + \underset{(0.02)}{0.03}D09q1 + \underset{(0.02)}{0.03}D09q2 + \underset{(0.02)}{0.03}D09q3 \\
& -\underset{(0.02)}{0.04}D09q4 + seasonals \\
& \hat{\sigma} = 0.01751
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\Delta w_t} = & +\underset{(0.09)}{0.09}\Delta c_{t-1} + \underset{(0.10)}{0.02}\Delta c_{t-2} + \underset{(0.10)}{0.25}\Delta c_{t-3} + \underset{(0.09)}{0.07}\Delta c_{t-4} \\
& -\underset{(0.11)}{0.06}\Delta y_{t-1} - \underset{(0.12)}{0.02}\Delta y_{t-2} - \underset{(0.12)}{0.03}\Delta y_{t-3} - \underset{(0.10)}{0.17}\Delta y_{t-4} \\
& +\underset{(0.08)}{0.15}\Delta w_{t-1} + \underset{(0.08)}{0.09}\Delta w_{t-2} - \underset{(0.08)}{0.02}\Delta w_{t-3} + \underset{(0.08)}{0.30}\Delta w_{t-4} \\
& +\underset{(0.003)}{0.006}SD_{1,t} + \underset{(0.006)}{0.004}SD_{2,t} - \underset{(0.11)}{0.07}CPSTOP_t \\
& -\underset{(0.02)}{0.08}D08q4 + \underset{(0.02)}{0.04}D09q1 + \underset{(0.03)}{0.06}D09q2 + \underset{(0.03)}{0.06}D09q3 \\
& +\underset{(0.03)}{0.04}D09q4 + seasonals \\
& \hat{\sigma} = 0.02298
\end{aligned}
$$

Diagnostics:
Vector AR 1-5 test: F(45,404) = 1.15 [0.23]
Vector Normalty test: $\chi^2(6)$ = 15.46 [0.02]
Vector Heteroscedasticity test: $F(288, 706) = 1.17[0.05]$

Appendix 3. Details about the testing procedure

Assume that the variables are Gaussian and consider first restrictions of the form (30) dropping the restrictions on the nonstochastic terms and where c, d, $d_{-1}, \ldots, d_{-k+1}$ are fixed. For simplicity let $X_t = X_t^*$ in (29).

From Johansen and Swensen (1999) it follows that to find the maximum likelihood estimators one has to consider a conditional equation and a marginal equation. The likelihood is of the form $L_{1\cdot 2,max}(d, d_{-1}, \ldots, d_{-k+1}) L_{2,max}(d, d_{-1}, \ldots, d_{-k+1})$.

The marginal equation takes the form

$$c'\Delta X_t = d'X_{t-1} - d'_{-1}\Delta X_{t-1} - \cdots - d'_{-k+2}\Delta X_{t-k+2} - d_{-k+1}\prime\Delta X_{t-k+1} - c'\Phi D_t + c'\epsilon_t.$$

The maximal value of the marginal likelihood can therefore be found by regressing $c'\Delta X_t - d'X_{t-1} + d'_{-1}\Delta X_{t-1} + \cdots + d'_{-k+1}\Delta X_{t-k+1}$ on D_t so $L_{2,max}(d, d_{-1}, \ldots, d_{-k+1})$ has a closed form. The conditional equation takes the form

$$\begin{aligned} c'_{\perp}\Delta X_t &= \eta\xi'\bar{d}'_{\perp}X_{t-1} \\ &- \rho(c'\Delta X_{t-1} - d'X_{t-1} + d'_{-1}\Delta X_{t-1} + \cdots + d'_{-k+1}\Delta X_{t-k+1} + c'\Phi D_t) \\ &+ \Theta(d'd)^{-1}d'X_{t-1} + c'_{\perp}\Gamma_1\Delta X_{t-1} + \cdots + c'_{\perp}\Gamma_{k-1}\Delta X_{t-k+1} + c'_{\perp}\Phi D_t + u_t. \end{aligned}$$

where $u_t = (c'_{\perp} - \rho c')\epsilon_t$. For d and $d_{-1}, \ldots, d_{-k+1}$ fixed the maximal values of the likelihood can be computed by reduced rank regression. The matrices η, ξ, $\rho = c'_{\perp}\Omega c(c'\Omega c)^{-1}$ and θ have dimensions $(p-q)\times(r-q)$, $(p-q)\times(r-q)$, $(p-q)\times q$ and $(p-q)\times q$ respectively.

Now we want to consider maximization over d, $d_{-1}, \ldots, d_{-k+1}$ and Φ_0. Since these quantities occur in both the marginal and conditional equations the product $L_{1\cdot 2,max}(d, d_{-1}, \ldots, d_{-k+1}) L_{2,max}(d, d_{-1}, \ldots, d_{-k+1})$ must be considered. Using a generic numerical optimization procedure is an option, but the number of parameters quickly gets large. We therefore propose another procedure. If d is fixed, new values for $d_{-1}, \ldots, d_{-k+1}$ can be found by regressing $c'\Delta X_t - d'X_{t-1}$ on $\Delta X_{t-1}, \ldots, \cdots, \Delta X_{t-k+1}$ and D_{t-1}

Because $d_{-1}, \ldots, d_{-k+1}$ also occur in the conditional equation it may be that also restrictions arising from this part must be taken into account. Reformulating the conditional equation as

$$\begin{aligned} c'_{\perp}\Delta X_t &= \eta\xi'\bar{d}'_{\perp}X_{t-1} \\ &- \rho(c'\Delta X_{t-1} - d'X_{t-1}) + \Theta(d'd)^{-1}d'X_{t-1} \\ &+ (c'_{\perp}\Gamma_1 - \rho d'_{-1})\Delta X_{t-1} - \cdots + (c'_{\perp}\Gamma_{k-1} - \rho d'_{-k+1})\Delta X_{t-k+1} \\ &+ c'_{\perp}\Gamma_{k-1}\Delta X_{t-k+1} + (c'_{\perp} - \rho c')\Phi D_t + u_t \end{aligned}$$

one can se that there are no such constraints since $(c'_{\perp}\Gamma_1 - \rho d'_{-1}), \ldots, (c'_{\perp}\Gamma_{k-1} - \rho d'_{-k+1})$ and $(c'_{\perp} - \rho c')\Phi$ vary freely.

Thus $L_{1\cdot 2,max}(d, d_{-1}, \ldots, d_{-k+1}) L_{2,max}(d, d_{-1}, \ldots, d_{-k+1})$ is concentrated and depends only on the values in d and the maximum value can be found using a general optimization procedure. As there are no restrictions on $d, d_1, \ldots, d_{k-1}$ this maximum will be the same as for the reduced rank VAR-model.

Restrictions on $d_{-1}, \ldots, d_{-k+1}$, e.g. $d_{-k+1} = d^0_{-k+1}$ can be treated as above, but regressing $c'\Delta X_t - d'X_{t-1} + d^0_{k-1}\Delta X_{t-k+1}$ on $d'_{-1}\Delta X_{t-1}, \cdots, d'_{-k+2}\Delta X_{t-k+2}$. The conditional equation takes the form

$$\begin{aligned} c'_\perp \Delta X_t &= \eta\xi'\bar{d}'_\perp X_{t-1} \\ &- \rho(c'\Delta X_{t-1} - d'X_{t-1} + d^0_{k-1}\Delta X_{t-k+1}) + \Theta(d'd)^{-1}d'X_{t-1} \\ &+ (c'_\perp\Gamma_1 - \rho d'_{-1})\Delta X_{t-1} + \cdots + (c'_\perp\Gamma_{k-2} - \rho d'_{-k+2})\Delta X_{t-k+2} \\ &+ c'_\perp\Gamma_{k-1}\Delta X_{t-k+1} + (c'_\perp - \rho c')\Phi D_t + u_t \end{aligned}$$

where the parameters can be found by reduced rank regression.