

Weak beta, strong beta: multi-factor pricing and rank restrictions.¹

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ABSTRACT

A plethora of competing factors appear to explain the cross-section of expected returns, as recent methodological works in asset pricing call for raising statistical standards. This paper contributes to this literature via an identification-robust methodology to assess factor pricing, regardless of whether factor betas are jointly informative or not, or heterogenous enough to identify risk premiums. We provide a formal definition of non-informative factors in this regard, and document the extent to which inferences on Fama-French-Carhart factors are affected by taking into account such source of uncertainty. Confidence sets are proposed that invert popular asset pricing tests, and analytical and tractable solutions are derived. An extensive simulation study is reported in addition to our empirical assessment of the standard as well as the recent profitability and investment factors. Results suggest that the standard three factors are not statistically insignificant, yet resulting betas are not convincingly priced. Evidence of pricing weakens post 1970s, as the three factor model more broadly does not perform well. We do not find evidence favouring size or book-to-market risk over the market risk. The Carhart and recent Fama-French factors are not utterly redundant, yet heterogeneity of betas is not sufficient to distinguish a priced momentum anomaly from profitability or investment risk. All models struggle when tested with all portfolios jointly and identification of risk premium worsens globally post 2000s.

1 Introduction

Size, book-to-market and momentum factors [Fama and French (1992), Fama and French (1993), Carhart (1997), Fama and French (2015)] have emerged as powerful benchmarks to explain the cross-section of returns. Their adoption as the backbone of many prominent models for asset pricing is a clear reflection of their success. In practice, a plethora of alternative factors as well as conditioning variables seem to also pass available conditional or unconditional asset pricing tests. On methodological advances in this regard and resulting empirical findings, see for example, Lewellen and Nagel (2006), Boguth, Carlson, Fisher and Simutin (2011), Shanken and Zhou (2007), Lewellen, Nagel and Shanken (2010), Penaranda and Sentana (2015) and the emerging literature that recognizes the adverse effects of weak and/or many factors [for example, Kleibergen (2009), Beaulieu, Dufour and Khalaf (2010), Kan, Robotti and Shanken (2013), Beaulieu, Dufour and Khalaf (2013), Gospodinov, Kan and Robotti (2014), Kleibergen and Zhan (2015) and Harvey, Liu and Zhu (2015)].¹

Broadly, this literature: (i) concludes that the habitually used statistical standards set misleadingly low hurdles, and (ii) poses the question of whether researchers should re-assess widespread and popular factors. While momentum effects and consumption-based factors, among others, have been challenged via such arguments², Fama and French's factors are generally upheld in applied work, which reflects, perhaps, the common presumption that size and book-to-market sensitivities are well identified.

This paper makes three main contributions to this literature. First, we propose a set of analytically tractable tools for confidence set based inference on factor price, regardless of whether factor betas are jointly informative or not, or heterogenous enough to identify risk premiums. In particular, these confidence sets serve to robustly assess whether candidate factors are priced risk-drivers. This approach extends the single-benchmark robust method proposed by Beaulieu et al. (2013) to multivariate beta-pricing models. Secondly, we offer a more comprehensive analysis on the factors' information content than can be gleaned from statistical insignificance tests (aside from raising cut-offs). Third, we document, empirically and via extensive simulations, the extent to which inferences on Fama-French-Carhart factors are affected by taking into account such sources of uncertainty; the standard size and book-to-market factors are studied, in addition to the recent investment and profitability factors [Fama and French (2015)]. This analysis yields useful practical suggestions for factor evaluation more broadly.

¹These works, as well as the bulk of this literature, typically rely on test portfolios. We maintain the same approach in the present paper; see also Gagliardini, Ossola and Scaillet (2016) for a recent big data perspective based on test assets instead.

²Refer to the above-cited works.

The proposed methods combine traditional time series and cross-sectional asset pricing strategies, whether factors are traded portfolios, non-return factors or a mixture of both. As in Lewellen et al. (2010), we focus on empirical pricing equations encompassing a rather limited number of given benchmarks. With reference to *e.g.* Harvey et al. (2015) who consider a very large number of factors, we empirically document problems with a given number (four to five) well known factors.

We first produce simultaneous confidence intervals for (in turn), the unrestricted components of the time-series regression *alphas*, and of each of the *betas*. The goal of our simultaneous approach is to guarantee joint coverage, that is, to ensure that the joint probability that each of the intervals covers true values, up to the desired level (for example, 95%). Next, we construct confidence intervals for the components of the cross-sectional regression coefficients, that is, the zero-beta rate and the risk price vector, again viewed jointly and using traditional first-pass estimates of betas, yet accounting for estimation errors and for uncertainty surrounding the factors' information content.

This is achieved building on, although modifying, the strategy developed by Lewellen et al. (2010). One of their suggestions (Prescription 6) is to recover, from popular test statistics, a confidence interval for a model's unexplained generalized Sharpe ratio which measures how far are maximally-correlated mimicking portfolios from the efficient frontier; see also Campbell, Lo and MacKinlay (1997, Chapter 6, Section 6). We follow a different although related rationale, using these same statistics to recover the range of factor loadings and risk premium compatible with the model and data.

The statistics considered include: (i) the joint regression intercept test statistic by Gibbons, Ross and Shanken (1989) and its counterparts pertaining to each factor [see, *e.g.* Dufour and Khalaf (2002) and Beaulieu et al. (2010)], and (ii) the likelihood based statistics discussed by Shanken and Zhou (2007) and reinterpreted as a cross-sectional test. When underlying parameters are fixed, all these statistics are of the Hotelling form [Hotelling (1947)]. We thus screen through each parameter space, testing all possible values and assembling non-rejected ones, at a given level. This implies *inverting* the considered tests, which relates to the graphical procedure of Lewellen et al. (2010), as follows.

Recall the duality between: (i) a $1 - \alpha$ level confidence interval of the usual Wald-type form $CI(\delta; \alpha) = [\hat{\delta} - z_{\alpha/2}SEE, \hat{\delta} + z_{\alpha/2}SEE]$ where $\hat{\delta}$ is the estimate of a scalar parameter δ with standard error SEE and $z_{\alpha/2}$ is a justified critical point, and (ii) the two-sided t -test associated with $H(\delta_0) : \delta - \delta_0 = 0$ based on $\mathbf{t}_D(\delta_0) = (\hat{\delta} - \delta_0)/SEE$, δ_0 known. $CI(\delta; \alpha)$ assembles all δ_0 values that are not rejected via $\mathbf{t}_D(\delta_0)$ which algebraically results from solving, over δ_0 , inequality

$$|\hat{\delta} - \delta_0| \leq z_{\alpha/2} (SEE). \quad (1)$$

We implement this same definition with non-scalar parameters, but the considered statistics follow a

different form. Instead of solving for the scalar unexplained generalized Sharpe ratio [as in Lewellen et al. (2010)], we solve for the loading or the risk premium vectors.

We show that the resulting inversion requires, in contrast to inequalities of the form (1), multi-dimensional quadratic inequalities. We provide a unified and tractable analytical solution to these inequalities and supportive finite sample and simulation assessments in non *i.i.d.* and non Gaussian settings, all of which are new to both asset pricing and econometric literatures. Analytical computations rely on the mathematics of *quadrics* [Dufour and Taamouti (2005)].³

Several features of our methodology are worth emphasizing, all of which are illustrated via an extensive simulation study. One is joint treatment of factors viewed simultaneously rather than individual proxies. Formally, our methodology makes explicit the joint hypothesis on loadings and risk premiums that defines models, yet preserving traditional cross-sectional interpretations of models and tests. The second feature is our reliance on interval rather than just point estimates for parameters of interest. Furthermore, confidence intervals embed traded factor restrictions and can be empty or unbounded, reflecting mis-specified information or lack thereof. This fulfills several key prescriptions from Lewellen et al. (2010).

The third notable feature is our analytical solution to both point and set estimates. While analytical point estimates of risk premia are available that avoid documented numerical problems [see Shanken and Zhou (2007) and Kleibergen (2009)], relevant analytical identification-robust interval estimation methods are lacking. Our solutions cover the joint numerical inference method introduced by Kleibergen (2009) which extends its practical usefulness since precise multivariate numerical searches can be prohibitive. Importantly, our methodology allows one to estimate the zero-beta rate and to control for factors that are traded portfolios, while both problems are not considered by Kleibergen (2009). Lastly, the statistics we consider are not *t*-type measures, in contrast to *e.g.* Kan et al. (2013) and Gospodinov et al. (2014). This fulfills the key prescription from Dufour (1997), which we summarize here as it relates to the problem at hand.

When some parameters of a given model are weakly identified, for confidence sets to be valid, they should allow for the possibility of being unbounded. That is, if objective functions can be almost flat in a parameter, all its corresponding values should be equally admissible as estimates, the collection of which should result in an unbounded confidence region. Any method that, by construction, leads to a confidence interval with bounded limits, should thus undercover true values. The confidence sets we produce can be unbounded, which will reflect the model's identification status. Wald-type intervals by

³For further quadric based solutions in different contexts, see Bolduc, Khalaf and Yelou (2010) for inference on multiple ratios, and Khalaf and Urga (2014) for inference on cointegration vectors.

construction also cannot be empty and must be supplemented with model tests if fit is to be assessed. In contrast, our method can produce empty tests which suggest lack-of-fit. Our intervals will thus signal problems when present so pre-tests are avoided altogether. Our procedure thus summarizes statistical information in one single size-controlled step.

Our main empirical finding concerns the weak identification of size and book-to-market risk premium. Using industry and size sorted equally or value weighted return with NYSE data from 1961-2010, we find that although Fama and French factors are not statistically insignificant (using our new and well known metrics), the resulting betas are not necessarily priced. In fact, the Fama-French three factors are confirmed to be priced concurrently only before 1970, with equally weighted industry portfolios, and only in the 1970s, with size sorted portfolios. Evidence of pricing weakens thereafter, as the model more broadly does not perform well (is either rejected or weakly identified, depending on intercept restrictions or test portfolios). Interestingly, we do not find evidence favouring size and book-to-market risk over the market risk. For instance, with size portfolios, in all subperiods except the 1980s and 2000s in which our confidence sets on the market risk are uninformative, it is significantly priced.

The Carhart and the recent Fama and French (2015) factors do not fare convincingly either. Though not utterly redundant, heterogeneity of betas is not sufficient to distinguish a priced momentum anomaly from profitability or investment risk. Finally, all considered models do not fare well when test assets are used jointly, and data is generally less informative post-2000s.

The paper is organized as follows. Section 2 sets the framework. Section 3 provides our inference methodology. Our data-based analysis is reported in section 4, with simulation-based evidence in sub-section 4.1 and empirical results in sub-section 4.2. Section 5 concludes the paper. A technical appendix presents proofs and some empirical results reported for completion.

2 Model and Reduced Rank Restrictions

Let $r_i, i = 1, \dots, n$, be a vector of T returns on n securities or portfolios for the period $t = 1, \dots, T$, and $\mathcal{R} = [\mathcal{R}_1 \dots \mathcal{R}_q]$ a $T \times q$ matrix of observations on q factors or benchmarks. Arbitrage Pricing Theory (APT) based models rely on regressions such as

$$r_i = a_i \iota_T + \mathcal{R}b_i + u_i, \quad i = 1, \dots, n, \quad (2)$$

with non-linear restrictions embedding zero-beta rates and risk premia.

To introduce the latter as model parameters, suppose that \mathcal{R}_1 corresponds to a vector of returns

on a tradable factor, for example some market benchmark. Partition \mathcal{R} into \mathcal{R}_1 and \mathcal{F} so the latter $T \times (q - 1)$ submatrix contains the observations on the $(q - 1)$ factors $[\mathcal{R}_2 \dots \mathcal{R}_q]$ and partition b_i conformably into the scalar b_{i1} and the $(q - 1)$ dimensional vector $b_{i\mathcal{F}}$.⁴ Equilibrium analysis, summarized *e.g.* in Campbell et al. (1997, Chapter 6), Shanken and Zhou (2007) or Penaranda and Sentana (2015) and the references therein, yields zero-intercept constraints in the regression of returns in excess of the zero-beta rate, denoted γ_c , on the factors in excess of a conformable risk price measure:

$$r_i - \iota_T \gamma_c = (\mathcal{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathcal{F} - \iota_T \gamma'_{\mathcal{F}}) b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n \quad (3)$$

where γ_0 denotes the risk price associated with \mathcal{R}_1 , the restriction $\gamma_0 = \gamma_c$ is imposed or relaxed, and the $(q - 1)$ dimensional vector $\gamma_{\mathcal{F}}$ refers to the price of the remaining factors.

Here, γ_c , γ_0 and $\gamma_{\mathcal{F}}$ are unknown parameters (possibly imposing $\gamma_0 = \gamma_c$) in addition to the factor loadings b_{i1} and $b_{i\mathcal{F}}$, which leads back to (2) with non-linear constraints:

$$r_i = a_i \iota_T + \mathcal{R}_1 b_{i1} + \mathcal{F} b_{i\mathcal{F}} + u_i, \quad (4)$$

$$a_i = \gamma_0(1 - b_{i1}) - \gamma'_{\mathcal{F}} b_{i\mathcal{F}} \text{ [restricted]} \quad (5)$$

$$a_i = \gamma_c - \gamma_0 b_{i1} - \gamma'_{\mathcal{F}} b_{i\mathcal{F}} \text{ [unrestricted]} \quad (6)$$

For further reference, define the q -dimensional parameter

$$\theta = (\gamma_0, \gamma'_{\mathcal{F}}). \quad (7)$$

From (5) or (6), it is clear that the components of θ cannot be identified if *e.g.* the corresponding components of b_i do not differ enough over i (that is, in cross-section), and in particular, are jointly close to one or to zero. Possibly non-informative factors and reliance on portfolios which tends to equalize betas implies that identification cannot be taken for granted for the problem at hand.

2.1 Asset Pricing Context

Typically, θ is analyzed relative to factor pricing in cross-section. The APT assumes that the unconditional expectation of returns (the n -dimensional vector $\boldsymbol{\mu}_R$) is linear in *betas*:

$$\boldsymbol{\mu}_R = \iota_n \gamma_c + b' \Gamma \quad (8)$$

where the scalar γ_c is the above defined return on the zero-beta portfolio, $b = [b_1 \dots b_n]$ is the $q \times n$ matrix of factor betas and $\Gamma = (\Gamma_1, \dots, \Gamma_q)'$ denotes the risk premium. Estimating the latter

⁴Extensions to multiple tradable factors follow straightforwardly; our main empirical analysis considers this restriction for the market benchmark only, hence this notation.

parameter of interest confronts enduring hurdles since b is unobserved and the choice of factors remains subject to wide debate.

Given a set of q benchmarks \mathcal{R} , which (as in Lewellen et al. (2010)) is the problem we focus on, (8) yields the restricted regression:

$$r_i = \iota_T (\gamma_c + b_i' \Gamma) + (\mathcal{R} - \iota_T \boldsymbol{\mu}'_F) b_i + u_i, \quad i = 1, \dots, n \quad (9)$$

where $\boldsymbol{\mu}_F$ is the q -dimensional vector of factor means [see Shanken and Zhou (2007) and references therein]. Taking the time series average of (9) with usual notation yields

$$\bar{r}_i = \gamma_c + b_i' (\Gamma + \bar{\mathcal{R}} - \boldsymbol{\mu}_F) + \bar{u}_i, \quad i = 1, \dots, n. \quad (10)$$

The coefficient of b_i' in the resulting cross-sectional regression (which motivates the popular two-pass estimation and testing methods) is a random parameter defined since Shanken (1985) as the ex-post risk price to set it apart from Γ .

It seems clear enough that (9) relates to (3). Take the time series average of, say the unrestricted (4), leading (again with obvious notation) to

$$\bar{r}_i = \gamma_c + b_i' (\bar{\mathcal{R}} - \theta) + \bar{u}_i, \quad i = 1, \dots, n. \quad (11)$$

Clearly, the latter corresponds to (10) with

$$\Gamma = \boldsymbol{\mu}_F - \theta. \quad (12)$$

Consequently, testing (4) provides a test of the APT [see *e.g.* Campbell et al. (1997, Chapters 5 & 6)] even though $\boldsymbol{\mu}_F$ is unknown. The issue is relevant however for estimation purposes: as a matter of fact, the only sample information we have on $\boldsymbol{\mu}_F$ is $\bar{\mathcal{R}}$.⁵ In the context of (4)-(11), this suggests a random parametrization via $\bar{\mathcal{R}} - \theta$.

Confidence intervals on the latter can be built nevertheless conditioning on the factors. We thus proceed conformably, to exploit the reduced rank statistical properties of (4) in line with pioneering works in this literature [reviewed *e.g.* in Campbell et al. (1997, Chapters 5 & 6) and more recently by Shanken and Zhou (2007) and Lewellen et al. (2010)]. Formally, we derive a joint confidence region for θ conditioning on the factors (imposing or relaxing $\gamma_0 = \gamma_c$) and project this region to obtain

⁵"An average return carries no information about the mean of the factor that is not already observed in the sample mean of the factor." [Cochrane (2005, p. 245)]. See also Shanken and Zhou (2007) or Penaranda and Sentana (2015), on including moment conditions for factors with GMM.

simultaneous confidence sets for each of the components of θ . We next assess pricing reflecting zero-restrictions on the components of $\bar{\mathcal{R}} - \theta$: each factor is considered not priced if its empirical mean is not covered by the confidence set associated with the corresponding component of θ . Our confidence intervals are simultaneous, which implies that decisions on pricing will also be simultaneous.

The zero-beta rate γ_c also holds important information and so does the restriction $\gamma_0 = \gamma_c$, as emphasized by Lewellen et al. (2010). Our method will produce a confidence interval for γ_c in addition to each component of $(\gamma_0, \gamma'_{\mathcal{F}})$. Testing $\gamma_c = \gamma_0$ can be conducted in our context via the following simple reparametrization:

$$\gamma_c - \gamma_0 b_{i1} - \gamma'_{\mathcal{F}} b_{i\mathcal{F}} = \gamma_c^* - \gamma_0 (b_{i1} - 1) - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}, \quad (13)$$

$$\gamma_c^* = \gamma_c - \gamma_0. \quad (14)$$

The intercept can be partialled-out if we rewrite the regressions in deviation from one of them, leading to $n - 1$ equations, in which case we will obtain another confidence interval for each component of $(\gamma_0, \gamma'_{\mathcal{F}})$, at the expense of foregoing information on γ_c . The statistic we consider is a monotonic transformation of the LR-based criterion [the so-called FAR test] introduced by Kleibergen (2009). Interpretations on pricing are unchanged, and as in Kleibergen (2009), the statistic is invariant to the equation chosen as the deviation basis. We formally assess the pros and cons of evacuating γ_c , for estimation and fit purposes.

For further reference, the three frameworks we consider are categorized as follows: model (4)-(5) is denoted **RAPT** where **R** stands for "restricted" which refers to traded factor constraints; model (4)-(6) which relaxes the latter constraints is denoted **UAPT** where **U** stands for "unrestricted", in which case we refer to partialling γ_c out as the **PAPT** approach, where **P** stands for "partialling-out". We will also refer to the hypothesis

$$H_c^* : \gamma_c^* \equiv \gamma_c - \gamma_0 = 0 \quad (15)$$

that can be tested by checking whether the confidence set for γ_c^* in (13) covers zero.

2.2 Statistical Context

The above equilibrium models can be defined via rank restrictions on a multivariate regression of the form:

$$Y = XB + U, \quad U = WJ' \Leftrightarrow Y_t = B'X_t + U_t, \quad U_t = JW_t, \quad t = 1, \dots, T \quad (16)$$

where Y is a $T \times n$ matrix of observations on n endogenous variables, X is a $T \times k$ full-column rank matrix of exogenous variables, Y'_t and X'_t are, respectively, the t -th row of Y and X so that Y_t and X_t

provide the t -th observation on the dependant variables and regressors, J is unknown, non-singular and possibly random, U'_t is the t -th row of U , W is a $T \times n$ matrix of random errors, W'_t is the t -th row of W , and the joint distribution of W_1, \dots, W_T is either fully specified, or specified up to a nuisance parameter μ . Finite sample results in this paper assume we can condition on X for statistical analysis.

Throughout the paper, we maintain the following assumptions and notation. $\mathcal{D}(d_1, \dots, d_m)$ refers to an m -dimensional diagonal matrix with diagonal elements d_1, \dots, d_m . ι_j refers to a j -dimensional vector of ones. The number of factors is $q = k - 1$. $\text{DIAG}(A)$ refers to a column vector from the diagonal of a matrix A . For any $N \times K$ matrix A , $\text{vec}(A)$ returns an $NK \times 1$ vector, with the columns of A stacked on top of each other; $\mathcal{M}[A] = I - A(A'A)^{-1}A'$ for any full column rank matrix A . We refer to a $1 - \alpha$ level CS for a parameter as $\text{CS}_\alpha(\cdot)$. For presentation ease, we partition the coefficient matrix as follows:

$$B = \begin{bmatrix} a' \\ b \end{bmatrix}, \quad a = (a_1, \dots, a_n)', \quad b = [b_1 \quad \dots \quad b_n] = \begin{bmatrix} \beta'_2 \\ \vdots \\ \beta'_k \end{bmatrix} \quad (17)$$

where a is the vector of n intercepts, and b is $q \times n$. Conformably, we partition \hat{B} as follows

$$\hat{B} = \begin{bmatrix} \hat{a}' \\ \hat{b} \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} \hat{\beta}'_2 \\ \vdots \\ \hat{\beta}'_k \end{bmatrix}. \quad (18)$$

We also use the following matrix partitions:

$$(X'X)^{-1} = \begin{bmatrix} x^{11} & x^{12} \\ x^{21} & x^{22} \end{bmatrix} \quad (19)$$

where x^{11} is a scalar, $x^{21} = x^{12'}$ is $q \times 1$ and x^{22} is $q \times q$.

The rank restrictions in question can be written as:

$$H_{\text{RAPT}} : (1, \theta')B = 0, \quad \text{for some unknown vector } \theta, \quad (20)$$

$$H_{\text{UAPT}} : (1, \theta')B = \phi \iota'_n, \quad \text{for some unknown vector } (\theta', \phi)', \quad (21)$$

where θ is $q \times 1$ and ϕ is an unknown scalar.⁶ Indeed, rewriting (4)-(5) with left-hand side returns in deviation from \mathcal{R}_1 yields, for $i = 1, \dots, n$:

$$r_i - \mathcal{R}_1 = (\mathcal{R}_1 - \iota_T \gamma_0) [b_{i1} - 1] + (\mathcal{F} - \iota_T \gamma'_{\mathcal{F}}) b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n$$

⁶Typically, (20) and (21) assume that $k \leq n$.

or alternatively

$$\begin{aligned} r_i - \mathcal{R}_1 &= a_i \iota_T + \mathcal{R}_1 d_i + \mathcal{F} b_{i\mathcal{F}} + u_i, & i = 1, \dots, n \\ a_i &= -\gamma_0 d_i - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}, & d_i = b_{i1} - 1 \end{aligned} \quad (22)$$

which is a special case of (16) where Y stacks the matrix of returns in deviation from the tradable benchmark, imposing (20) with

$$\theta = (\gamma_0, \gamma'_{\mathcal{F}})'. \quad (23)$$

The non-tradable case (4)-(6) is the regression

$$\begin{aligned} r_i &= a_i \iota_T + \mathcal{R}_1 b_{i1} + \mathcal{F} b_{i\mathcal{F}} + u_i, & i = 1, \dots, n \\ a_i &= \gamma_c - \gamma_0 b_{i1} - \gamma'_{\mathcal{F}} b_{i\mathcal{F}} \end{aligned}$$

which again is a special case of (16) where Y stacks the matrix of returns, imposing (21) with

$$\theta = (\gamma_0, \gamma'_{\mathcal{F}})', \text{ and } \phi = \gamma_c.$$

Regression (4)-(6) can also be re-expressed as

$$\begin{aligned} r_i - \mathcal{R}_1 &= a_i \iota_T + \mathcal{R}_1 d_i + \mathcal{F} b_{i\mathcal{F}} + u_i, & i = 1, \dots, n, \\ a_i &= \gamma_c^* - \gamma_0 d_i - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}, & d_i = b_{i1} - 1, \quad \gamma_c^* = \gamma_c - \gamma_0, \end{aligned} \quad (24)$$

in which imposing (21) provides with $\phi = \gamma_c^*$ provides a test of (15). Finally, it is also straightforward to see that the model in deviation from one of the returns yields a system of $n-1$ equations conformable with (20).

This paper focuses on estimating and testing θ and ϕ . Furthermore, we provide cross-equation simultaneous confidence sets for the parameters of the unrestricted asset pricing regression. Formally, we invert the test that fixes each row of B , in turn, to a fixed vector; the associated hypotheses takes the form

$$H_j : s_k[j]' B = \bar{\beta}'_j, \quad j \in \{1, \dots, k\}, \quad \bar{\beta}_j \text{ known} \quad (25)$$

where $s_k[j]$ denotes a k -dimensional selection vector with all elements equal to zero except for the j -th element which equals 1. To interpret H_j , recall that the classical zero restriction hypothesis underlying the Hotelling statistic which is viewed as the multivariate extension of the Student-t based significance test corresponds to

$$H_{0j} : s_k[j]' B = 0, \quad j \in \{1, \dots, k\} \quad (26)$$

so for example using $s_k[1]$ provides inference on the unrestricted regression intercept, and in the context of an unrestricted regression in deviation from the tradable factor [(22) above, ignoring the constraints], $s_k[2]$ allows one to assess the *betas* on the tradable factor in deviation from one. Assembling the $\bar{\beta}_j$ vectors that are not rejected at a given level yields a joint confidence set for the corresponding row of B which contain, in turn for $j = 1, \dots, k$, the n -dimensional vector of intercepts, and the n -dimensional vector of betas (possibly in deviation from one) on each factor over all considered assets.

In addition to useful information on underlying assets, the unrestricted regression intercepts and *betas* underlie identification of the above defined risk premia. Formally, for θ to be recoverable with no further data and information (in particular in the absence of other instruments), the *betas* per factor need to vary enough across equation. Concrete identification failure problems discussed in Beaulieu et al. (2013) (and the reference therein) relate to benchmark *betas* jointly [across i] equal to one. Kleibergen (2009) discusses the case of small *betas* in the sense of jointly [across i] equal to zero that may be traced back to Kan and Zhang (1999). Regardless of the source, identification of θ is driven by the joint cross-equation nature of the information conveyed by each factor. Our simultaneous approach for inference on θ as well as for the underlying reduced form *betas* thus zooms in on the core of the financial problem. Concretely, using portfolios rather than individual assets as test assets (that is for r_i in our notation) tends to equalize *betas* across equations; whether moving away from portfolios to individual assets which calls for alternative information reduction technique is an answer to this problem remains an open question which is beyond the scope of the bulk of the this literature as well as the present paper which requires $T - k - n > 0$. Our methodology is presented in the next section for the general (16) regression.

3 Confidence Sets for Factor Loadings and Risk Premia

Following Beaulieu et al. (2013) and Kleibergen (2009), we focus on inverting identification-robust statistics, that is, statistics whose null distributions are provably invariant to whether identification holds or not. We focus on Hotelling-type statistics

$$\Lambda(\theta, \phi) = \frac{\left((1, \theta') \hat{B} - \phi \iota'_n \right) \hat{S}^{-1} \left(\hat{B}'(1, \theta')' - \phi \iota_n \right)}{(1, \theta')(X'X)^{-1}(1, \theta)'} \quad (27)$$

$$\Lambda(\theta) = \frac{(1, \theta') \hat{B} \hat{S}^{-1} \hat{B}'(1, \theta)'}{(1, \theta')(X'X)^{-1}(1, \theta)'} \quad (28)$$

where θ' and ϕ are given and

$$\hat{B} = (X'X)^{-1}X'Y, \quad \hat{S} = \hat{U}'\hat{U}, \quad \hat{U} = Y - X\hat{B}. \quad (29)$$

These statistics serve to assess the special cases of $H_{\mathbf{RAPT}}$ and $H_{\mathbf{UAPT}}$ [(20)) and (21)] respectively

$$H_{\mathbf{R}} : (1, \theta')B = 0, \quad \theta \text{ known.} \quad (30)$$

$$H_{\mathbf{U}} : (1, \theta')B = \phi u'_n, \quad (\theta', \phi)' \text{ known,} \quad (31)$$

In addition, we invert the series of statistics associated with each of the H_j (25):

$$\Lambda(\bar{\beta}_j) = \frac{(\hat{\beta}_j - \bar{\beta}_j)' \hat{S}^{-1} (\hat{\beta}_j - \bar{\beta}_j) \tau_n}{s_k[j]' (X'X)^{-1} s_k[j]} \frac{\tau_n}{n} \quad (32)$$

where and $\hat{\beta}'_j$ is the j th row of \hat{B} . These statistics are also of the Hotelling form [see e.g. Dufour and Khalaf (2002)]; note that the classical Hotelling statistics to assess each of H_{0j} (26) are

$$\Lambda_{0j} = \frac{s_k[i]' \hat{B} \hat{S}^{-1} \hat{B}' s_k[i]}{s_k[i]' (X'X)^{-1} s_k[i]}. \quad (33)$$

When errors are normal then

$$\Lambda(\theta) \frac{\tau_n}{n} \sim F(n, \tau_n), \quad \Lambda(\theta, \phi) \frac{\tau_n}{n} \sim F(n, \tau_n), \quad \Lambda(\bar{\beta}_j) \frac{\tau_n}{n} \sim F(n, \tau_n) \quad (34)$$

where $\tau_n = T - k - n + 1$. The latter distributional results do not require any identification restriction.⁷ Λ_{0j} also follow the same null distribution. Underlying finite sample theory is discussed in section 3.3. Simulations reported in 4.1 show that for the problem under consideration corresponding to 5 or 10 year subsamples, the normal cut-off controls size whether errors are multivariate Student-t. Prior to these analyses, we discuss in the next section how inverting the proposed tests can be performed analytically.

3.1 A Unified Analytical Solution

Inverting the above tests requires solving in turn, over (θ, ϕ) , θ and $\bar{\beta}_j$ respectively, the inequalities

$$\Lambda(\theta, \phi) \frac{\tau_n}{n} \leq f_{n, \tau_n, \alpha}, \quad \Lambda(\theta) \frac{\tau_n}{n} \leq f_{n, \tau_n, \alpha}, \quad \Lambda(\bar{\beta}_j) \frac{\tau_n}{n} \leq f_{n, \tau_n, \alpha}, \quad (35)$$

where $f_{n, \tau_n, \alpha}$ denotes the α -level cut off point from the $F(n, \tau_n)$ distribution. The following unified analysis using the mathematics of Quadric (see Dufour and Taamouti (2005, 2007)) generalizes the Beaulieu et al. (2013) solution to: (ii) the multi-factor context, and (iii) the estimation of factor loadings and Jensen-type *alphas*.

Each inequation in (35) is rewritten as

$$(1, \zeta') A(1, \zeta')' \leq 0 \quad (36)$$

⁷Other than the usual Least Squares assumptions on $X'X$ and \hat{S} of course.

where ζ is the $m \times 1$ vector of unknown parameters and A is an $(m + 1) \times (m + 1)$ data dependent matrix. Next, inequality (36) is re-expressed as

$$\zeta' A_{22} \zeta + 2A_{12} \zeta + A_{11} \leq 0 \quad (37)$$

which leads to the set-up of Dufour and Taamouti (2005, 2007) so projections based CSs for any linear transformation of ζ of the form $\omega' \zeta$ can be obtained as described in these papers. The solution is reproduced in the Appendix for completion.

Moving from (36) to (37) requires partitioning A as follows

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad (38)$$

where A_{11} is a scalar, A_{22} is $m \times m$, and $A_{12} = A'_{21}$ is $1 \times m$. Simple algebraic manipulations suffice to show that for test defined by (28), we have:

$$A = \hat{B} \hat{S}^{-1} \hat{B}' - (X'X)^{-1} f_{n,\tau_n,\alpha}(n/\tau_n) \quad (39)$$

setting $\zeta = \theta$. Using the partitionings (18) and (19), $A_{11} = \hat{a}' \hat{S}^{-1} \hat{a} - (f_{n,\tau_n,\alpha}(n/\tau_n)) x^{11}$, $A_{12} = A'_{21} = \hat{a}' \hat{S}^{-1} \hat{b}' - (f_{n,\tau_n,\alpha}(n/\tau_n)) x^{12}$ and

$$A_{22} = \hat{b} \hat{S}^{-1} \hat{b}' - (f_{n,\tau_n,\alpha}(n/\tau_n)) x^{22}. \quad (40)$$

In the case of (27), we have $\zeta = (\theta', \phi)'$ and

$$A = \begin{bmatrix} \hat{B} \hat{S}^{-1} \hat{B}' - (X'X)^{-1} f_{n,\tau_n,\alpha} \frac{n}{\tau_n} & -\hat{B} \hat{S}^{-1} t_n \\ -t'_n \hat{S}^{-1} \hat{B}' & t'_n \hat{S}^{-1} t_n \end{bmatrix}. \quad (41)$$

Finally, inverting (32) yields $\zeta = \bar{\beta}'_i$ and the quadric form (37) with

$$A_{22} = \hat{S}^{-1}, \quad A_{12} = -\hat{\beta}'_i \hat{S}^{-1}, \quad A_{11} = -n (s_k[i]' (X'X)^{-1} s_k[i]) / (\tau_n). \quad (42)$$

The outcome of resulting projections can be empty, bounded, or the union of two unbounded disjoint sets. Dufour and Taamouti (2005, 2007) discuss such outcomes depending importantly on the A_{22} matrix. In particular, the confidence set is unbounded if A_{22} is not positive definite. It is thus clear that inverting (32) produces bounded sets as \hat{S} is assumed invertible. The following Theorem further shows that if any of the Hotelling tests based on Λ_{0j} , $j = 2, \dots, k$ is not significant then the A_{22} matrix will not be positive definite and the confidence set will be unbounded. That is, if any of the factors is redundant from a joint significance perspective, then information on risk prices *for all factors* is compromised.

Theorem 1 *In the context of (16), if*

$$(\tau_n/n) \Lambda_j < f_{n,\tau_n,\alpha}, \quad j \in \{2, \dots, k\} \quad (43)$$

where Λ_i are the Hotelling statistics defined in (32), then the CS for θ as defined in (20) which inverts the statistic (28) at the α -level is unbounded.

The above condition is sufficient but not necessary. It follows that although Hotelling tests on each factor are useful, they remain insufficient, and perhaps more importantly, are embedded in our methodology without compounding type-I errors. This characterization also holds when inverting the test defined in (27) and (34).

3.2 Empty Confidence Sets and Minimum Distance Statistics

The confidence set for factor loadings cannot be empty. Indeed, Dufour and Taamouti (2005, 2007) show that in the context of (37) and a positive definite A_{22} , the confidence set is empty if $\tilde{D} = A_{12}A_{22}^{-1}A_{12} - A_{11} < 0$. Here, from (42) we have

$$\tilde{D} = \hat{\beta}_i' \hat{S}^{-1} \hat{\beta}_i + n (s_k[i]' (X'X)^{-1} s_k[i]) / (\tau_n) \geq 0.$$

Moving on to the case of $\Lambda(\theta)$, we proceed by generalizing the single-beta results in Beaulieu et al. (2013). Because the cut-off point underlying test inversion denoted $f_{n,\tau_n,\alpha}$ above is the same for all θ values, an empty set would result when $\min_{\theta} \Lambda(\theta) \geq f_{n,\tau_n,\alpha}$. It can be shown that minimizing $\Lambda(\theta)$ produces the Gaussian-LR statistic to test the non-linear restriction which defines θ , namely (20); general derivation are available in *e.g.* Gouriéroux, Monfort and Renault (1996).

Theorem 2 *In the context of (16) and the non-linear hypothesis (20) the confidence set estimate for θ which inverts the statistic $\Lambda(\theta)$ defined in (28) at the α -level is empty if and only if*

$$\Lambda_{\mathbf{RAPT}} = \min_{\theta} \Lambda(\theta) = \Lambda(\hat{\theta}_{\mathbf{RAPT}}) = \hat{g}/(1 - \hat{g}) \equiv \hat{\rho} \geq f_{n,\tau_n,\alpha} \quad (44)$$

where \hat{g} is the minimum non-zero root of $C(X, Y) = (X'X)^{-1}X'Y(Y'Y)^{-1}Y'X$ and

$$\hat{\theta}_{\mathbf{RAPT}} = - \left(\hat{b} \hat{S}^{-1} \hat{b}' - \hat{\gamma} x^{22} \right)^{-1} \left(\hat{b} \hat{S}^{-1} \hat{a} - \hat{\gamma} x^{21} \right). \quad (45)$$

In other words, the confidence set estimate for θ is empty if and only if the minimum distance LR-based Hotelling statistic associated with (20) is significant when referred to the $f_{n,\tau_n,\alpha}$ cut-off

which can be viewed as a finite-sample bound cut-off point for this test. Observe that $\hat{\rho}$ coincides with the minimum root of both determinantal equations

$$\left| \hat{B} \hat{S}^{-1} \hat{B}' - \rho (X'X)^{-1} \right| = 0 \quad (46)$$

$$\left| \hat{B}' (X'X) \hat{B} - \rho \hat{S} \right| = 0. \quad (47)$$

Since the underlying eigen vector solution is not unique, we provide a proof for (45) in the Appendix which easily extends to the less restricted (21) definition and will allow us to link our results to existing related works namely Kandel (1984, 1986) and more recently to Kleibergen (2009). Our approach is general and relates to important applications of reduced rank regression based inference in econometrics; these include limited information simultaneous equation or cointegration models; see Dhrymes (1974, Chapter 7), Davidson and MacKinnon (2004, Chapter 12), or Johansen (1995). We rely on the following matrix algebra result pertaining to an equation of the form

$$\Sigma(1, \zeta)' = 0, \quad |\Sigma| = 0, \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \quad (48)$$

where ζ is the m -dimensional unknown given an $(m+1) \times (m+1)$ matrix Σ , and Σ_{11} is a scalar, $\Sigma_{12} = \Sigma'_{21}$ is $1 \times m$ and Σ_{22} is $m \times m$ and is invertible. As summarized in the Appendix,

$$\hat{\zeta} = -\Sigma_{22}^{-1} \Sigma_{21} \quad (49)$$

provides a unique solution to this system.⁸

Theorem 3 *In the context of (16) and the non-linear hypothesis (21), the minimum distance estimators associated with the criterion $\Lambda(\theta, \phi)$ defined in (27) can be derived as*

$$\Lambda_{\mathbf{UAPT}} = \min_{\theta, \phi} \Lambda(\theta, \phi) = \Lambda(\hat{\theta}_{\mathbf{UAPT}}, \hat{\phi}_{\mathbf{UAPT}}) = \hat{\nu} \quad (50)$$

where $\hat{\nu}$ is the minimum root of

$$\left| \hat{B} \left(\hat{S}^{-1} - \hat{S}^{-1} \iota_n \left(\iota_n' \hat{S}^{-1} \iota_n \right)^{-1} \iota_n' \hat{S}^{-1} \right) \hat{B}' - \nu (X'X)^{-1} \right| = 0 \quad (51)$$

and

$$\hat{\theta}_{\mathbf{UAPT}} = - \left(\hat{b} \left(\hat{S}^{-1} - \hat{S}^{-1} \iota_n \left(\iota_n' \hat{S}^{-1} \iota_n \right)^{-1} \iota_n' \hat{S}^{-1} \right) \hat{b}' - \hat{\nu} x^{22} \right)^{-1} \quad (52)$$

$$\left(\hat{b} \left(\hat{S}^{-1} - \hat{S}^{-1} \iota_n \left(\iota_n' \hat{S}^{-1} \iota_n \right)^{-1} \iota_n' \hat{S}^{-1} \right) \hat{a} - \hat{\nu} x^{21} \right) \quad (53)$$

$$\hat{\phi}_{\mathbf{UAPT}} = \frac{(1, \hat{\theta}'_{\mathbf{UAPT}}) \hat{B} \hat{S}^{-1} \iota_n}{\iota_n' \hat{S}^{-1} \iota_n}. \quad (54)$$

⁸Such solutions have been used in econometrics in *e.g.* limited information simultaneous equations or cointegration models; see Dhrymes (1974, Chapter 7), Davidson and MacKinnon (2004, Chapter 12), or Johansen (1995).

The formulas in (52)-(54) coincide with the solution obtained (using another method of proof) by Shanken and Zhou (2007).

3.3 Underlying Finite Sample Distributional Results

For some though not all inferential problems considered, we will assume the following mixture distributional setting:

$$W = VZ \quad (55)$$

where V is $T \times T$, unknown and possibly random (in which case it is independent of Z), and Z is a $T \times n$ matrix of *i.i.d.* n -dimensional standard normal variables *i.e.* if we denote the t -th row of Z as Z'_t , then

$$Z_t \stackrel{i.i.d.}{\sim} N(0, I_n). \quad (56)$$

Assumption (55) is sufficiently general and includes various n -dimensional elliptically contoured distributions and skew-elliptical distributions. Special cases of (55) include the normal distribution

$$W_t = Z_t \stackrel{i.i.d.}{\sim} N(0, I_n) \quad (57)$$

and the multivariate Student- t distribution with μ degrees-of-freedom [denoted as $t(\mu)$].

The hypotheses associated with all statistics we aim to invert as introduced in the previous section fall within the uniform linear class [see Dufour and Khalaf (2002), Beaulieu, Dufour and Khalaf (2007) and the references therein] of the form:

$$\tilde{H}(C, G, D) : CBG = D \text{ for known } C, G \text{ and } D \quad (58)$$

where C is $c \times k$ with rank c , $0 \leq c \leq k$, and G is $n \times g$, with rank g . The restricted estimators in this case are:

$$\tilde{B}(C, G, D) = \hat{B} - (X'X)^{-1} C' [C(X'X)^{-1} C']^{-1} (C\hat{B}G - D) (G'\hat{S}G)^{-1} G'\hat{S}, \quad (59)$$

$$\tilde{S}(C, G, D) = \tilde{U}'(C, G, D) \tilde{U}(C, G, D) \quad (60)$$

$$\tilde{U}(C, G, D) = Y - X\tilde{B}(C, G, D), \quad (61)$$

with $\tilde{S}(C, G, D) = \hat{S} + \hat{S}G (G'\hat{S}G)^{-1} (C\hat{B}G - D)' [C(X'X)^{-1} C']^{-1} (C\hat{B}G - D) (G'\hat{S}G)^{-1} G'\hat{S}$.

Commonly used statistics including the LR and Wald criteria [see Berndt and Savin (1977), Gouriéroux, Monfort and Renault (1995), Dufour and Khalaf (2002) and the references therein] to

test $\tilde{H}(C, G, D)$ can be expressed as

$$\mathcal{L}(C, G, D) = T \ln (|\tilde{S}(C, G, D)|/|\hat{S}|) = -T \sum_{i=1}^l \ln(1 - \lambda_i(C, G, D)) \quad (62)$$

$$\mathcal{W}(C, G, D) = T \operatorname{tr} (\hat{S}^{-1}[\tilde{S}(C, G, D) - \hat{S}]) = T \sum_{i=1}^l \frac{\lambda_i(C, G, D)}{1 - \lambda_i(C, G, D)} \quad (63)$$

where $l = \min\{c, g\}$ and $\lambda_1(C, G, D) \geq \dots \geq \lambda_n(C, G, D)$ are the eigenvalues of $\tilde{S}(C, G, D)^{-1}[\tilde{S}(C, G, D) - \hat{S}]$. Clearly, $\lambda_i(C, G, D)$, $i = 1, \dots, l$ coincide with the roots of $S^{-1}(C, G, D)[S(C, G, D) - G'\hat{S}G]$ where

$$S(C, G, D) = G'\hat{S}G + (C\hat{B}G - D)' [C(X'X)^{-1}C']^{-1} (C\hat{B}G - D). \quad (64)$$

Solving for eigenvalues in question thus requires considering the determinantal equation

$$\left| (S(C, G, D) - G'\hat{S}G) - \lambda S(C, G, D) \right| = 0. \quad (65)$$

Theorem 4 *In the context of (16) and the null hypothesis $\tilde{H}(C, I_n, D)$, the LR criterion simplifies to following form $\mathcal{L}(C, I_n, D) = T \ln (|I_c + \tilde{\Lambda}(C, I_n, D)|)$ where*

$$\tilde{\Lambda}(C, I_n, D) = [C(X'X)^{-1}C']^{-1}(C\hat{B} - D)\hat{S}^{-1}(C\hat{B} - D)'. \quad (66)$$

The latter Theorem covers the linear hypotheses relevant to the APT introduced above since

$$H_{\mathbf{R}} \equiv \tilde{H}((1, \theta'), I_n, 0), \quad H_{\mathbf{U}} \equiv \tilde{H}((1, \theta'), I_n, \phi_n'), \quad H_j \equiv \tilde{H}(s_k[j]', I_n, \bar{\beta}_j), \quad H_{0j} \equiv \tilde{H}(s_k[j]', I_n, 0)$$

leading to

$$\begin{aligned} \Lambda(\theta, \phi) &= \tilde{\Lambda}((1, \theta'), I_n, \phi_n'), & \Lambda(\theta) &= \tilde{\Lambda}((1, \theta'), I_n, 0), \\ \Lambda(\bar{\beta}_j) &= \tilde{\Lambda}(s_k[j]', I_n, \bar{\beta}_j), & \Lambda_{0j} &= \tilde{\Lambda}(s_k[j]', I_n, 0). \end{aligned}$$

Furthermore, the **PAPT** approach amounts to a convenient selection of the postmultiplying matrix G in (58). Via this matrix, the analysis in sections 2.1 and 3 above extends to any non-redundant combination of returns one may wish to consider. The following Theorem establishes the finite sample distribution for the above eigen-value based statistics with emphasis on the role of G , which was not discussed in (Dufour and Khalaf (2002)).

Theorem 5 *In the context of (16) and under the null hypothesis $\tilde{H}(C, G, D)$ in (58), the vector of the roots of (65) is distributed like the vector of the roots of*

$$|\mathcal{G}'W'(\mathcal{M}_0[X, C])W\mathcal{G} - \lambda\mathcal{G}'W'(\mathcal{M}[X] + \mathcal{M}_0[X, C])W\mathcal{G}| = 0 \quad (67)$$

where G is the orthogonal $n \times g$ matrix which includes the eigenvectors associated with the non-zero eigenvalues of $J'GG'J$ and

$$\mathcal{M}_0[X, C] = X(X'X)^{-1}C'[C(X'X)^{-1}C']^{-1}C(X'X)^{-1}X'. \quad (68)$$

Furthermore, under assumption (55), the distribution in question follows that of the roots of

$$|\mathcal{Z}'V'(\mathcal{M}_0[X, C])V\mathcal{Z} - \lambda\mathcal{Z}'V'(\mathcal{M}[X] + \mathcal{M}_0[X, C])V\mathcal{Z}| = 0 \quad (69)$$

where Z is a $T \times g$ matrix of *i.i.d.* g -dimensional standard normal variables, and is thus invariant to B and J . For the special case where $G = I_n$, *i.e.* hypothesis $\tilde{H}(C, I_n, D)$, the distribution in question follows that of the roots of

$$|W'(\mathcal{M}_0[X, C])W - \lambda W'(\mathcal{M}[X] + \mathcal{M}_0[X, C])W| = 0 \quad (70)$$

so invariance to B and J holds imposing or ignoring assumption (55).

Two results emerging from Theorem 5 deserve discussion for the problem under consideration.

1. The pivotal characterization (69) may be used to obtain finite sample p -values using the Monte Carlo test method [see e.g. Dufour and Khalaf (2002) and Dufour (2006)] if the variates underlying V are simulable.
2. Under assumption (55), the distribution of the roots will depend on C but not on D , and depends on G only through its rank.

So for the family of mixture distributions (55), the fact that null distributions depend on G only through its rank underlies and generalizes (beyond the deviation form) the invariance property noted by Kleibergen (2009). The fact that null distributions do not depend on D imply that the null distribution of $\Lambda(\bar{\beta}_j)$ does not depend on $\bar{\beta}_j$, so extending the above defined test inversion beyond normality preserves its quadrics-based analytics. Indeed, it suffices to obtain a simulation-based cut-off point depending on the assumed disturbance distributions which will be the same for all $\bar{\beta}_j$. Dependence on θ will not be evacuated in the same way, since θ intervenes in the null distributions in questions via $\mathcal{M}_0[X, C]$. These distributions do not depend on ϕ , which again supports partialling this parameter out at least from a statistical perspective. Recall however that ϕ and whether it differs empirically from the hypothesized zero-beta rate γ_0 [which in our notation is the first component of θ] is an important empirical question in finance. Simultaneous inference on θ and ϕ remains relevant. Our empirical analysis sheds more light on this matter using a well-known prototypical data set.

To conclude, we review two useful approximations to the above finite sample distributions. Given normal errors and if $\min(c, g) \leq 2$ [see Rao (1973, Chapter 8)] then

$$\left(\frac{\varkappa_1 \varkappa_3 - 2\varkappa_2}{cg} \right) \frac{1 - \left(|\hat{S}| / |\tilde{S}(C, G, D)| \right)^{1/\varkappa_3}}{\left(|\hat{S}| / |\tilde{S}(C, G, D)| \right)^{1/\varkappa_3}} \sim F(cg, \varkappa_1 \varkappa_3 - 2\varkappa_2) \quad (71)$$

$$\varkappa_1 = T - k - ((g - c + 1) / 2), \quad \varkappa_2 = (cg - 2) / 4, \quad (72)$$

$$\varkappa_3 = \begin{cases} [(c^2 g^2 - 4) / (c^2 + g^2 - 5)]^{1/2}, & \text{if } c^2 + g^2 - 5 > 0 \\ 1, & \text{otherwise} \end{cases} \quad (73)$$

The latter result holds as a reliable [relative to the standard χ^2] approximation when $\min(c, g) > 2$. The following approximation is also recommended [see McKeon (1974)]:

$$\text{tr}(\hat{S}^{-1}[\tilde{S}(C, G, D) - \hat{S}]) / \varkappa_6 \sim F(cg, \varkappa_5)$$

$$\varkappa_4 = (T - k + c - g - 1)(T - k - 1) / [(T - k - g - 3)(T - k - g)]$$

$$\varkappa_5 = 4 + (gc + 2) / (\varkappa_4 - 1), \quad \varkappa_6 = (cg)(\varkappa_5 - 2) / [\varkappa_5(T - k - g - 1)]$$

and is exact [exactly coincides with (71)] for $\min(c, g) = 1$. The cut-offs we use to invert the statistics considered in section 3 follow from these approximations. We also verify that deviations from the *i.i.d.* or normal errors assumption are not distortive, in empirically relevant multifactor simulation designs.

4 Empirical Analysis: Fama-French and Momentum Factors

In our empirical analysis of a multifactor asset pricing model, we conduct: (i) a simulation study calibrated to observed returns and factors, and (ii) a data-based assessment of factor pricing.

We first produce results for industry portfolios for the US, as in Beaulieu et al. (2013). We use monthly returns from 1961 to 2010, obtained from the University of Chicago's Center for Research in Security Prices (CRSP), on standard 12 portfolios of New York Stock Exchange (NYSE) firms grouped by standard two-digit industrial classification (SIC).⁹ For each month the industry portfolios include the firms for which the return, price per common share and number of shares outstanding are recorded by CRSP. Equally and value-weighted portfolios are analyzed.

Following recommendations of Lewellen et al. (2010), we also produce results for size portfolios, based on Fama and French's data base. We consider monthly returns of 25 value weighted and equally weighted portfolios from 1961 to 2010. The portfolios which are constructed at the end of June are

⁹The sectors studied include: (1) petroleum; (2) finance and real estate; (3) consumer durables; (4) basic industries; (5) food and tobacco; (6) construction; (7) capital goods; (8) transportation; (9) utilities; (10) textile and trade; (11) services; (12) leisure.

the intersections of five portfolios formed on size (market equity) and five portfolios formed on the ratio of book equity to market equity. The size breakpoints for year s are the NYSE market equity quintiles at the end of June of year s . The ratio of book equity to market equity for June of year s is the book equity for the last fiscal year end in $s - 1$ divided by market equity for December of year $s - 1$. The ratio of book equity to market equity is NYSE quintiles. The portfolios for July of year s to June of year $s + 1$ include all NYSE, AMEX, NASDAQ stocks for which market equity data is available for December of year $s - 1$ and June of year s , and (positive) book equity data for $s - 1$.

The benchmark factors are: 1) **MKT**, the excess return on the market defined as the value-weighted return on all NYSE, AMEX, and NASDAQ stocks (from CRSP) minus the one-month Treasury bill rate (from Ibbotson Associates); 2) **SMB** (small minus big) defined as the average return on three small portfolios minus the average return on three big portfolios; 3) **HML** (high minus low) defined as the average return on two value portfolios minus the average return on two growth portfolios; 4) **MOM** (momentum), the average return on the two high prior return portfolios minus the average return on the two low prior return portfolios; 5) **RMW** (robust minus weak) operating profitability; and 6) **CMA** (conservative minus aggressive) investment, constructed from conservative minus aggressive growth of assets for the fiscal year.

Fama and French benchmark factors, SMB, HML, RMW and CMA are constructed from benchmark portfolios that do not include hold ranges and do not incur transaction costs. The portfolios for these factors are rebalanced quarterly using two independent sorts, on size (market equity, ME), book-to-market (the ratio of book equity to market equity, BE/ME), profitability (annual revenues minus cost of goods sold, interest expense, and selling, general, and administrative expenses, all divided by book equity at the end of fiscal year $s - 1$) and investment (the growth of total assets for the fiscal year ending in $s-1$ divided by total assets at the end of $s - 1$). The profitability and investment factors, RMW and CMA, are constructed in the same way as HML except the second sort is either on operating profitability (robust minus weak) or investment (conservative minus aggressive). As HML, RMW and CMA can be interpreted as averages of profitability and investment factors for small and big stocks.

For the construction of the MOM factor, six value-weighted portfolios formed on size and prior (2–12) returns are used. The portfolios, which are formed monthly, are the intersections of two portfolios formed on size (market equity, ME) and three portfolios formed on prior (2–12) return. The size breakpoint (which determines the buy range for the small and big portfolios) is the median NYSE market equity. The BE/ME breakpoints are the 30th and 70th NYSE percentiles. The monthly prior

(2–12) return breakpoints are also the 30th and 70th NYSE percentiles.

4.1 Simulation Evidence

The experiment reported in this section is designed to assess three issues. First, the above proposed CSs can be conservative when k is large, so we aim to document their coverage properties. Second, we study the performance of the inverted proposed tests when errors are fat-tailed. Third, we assess the implications of imposing and relaxing tradability restrictions; see Lewellen et al. (2010) and Penaranda and Sentana (2015) for theoretical and practical discussions in this regard. To the best of our knowledge, an identification-robust assessment of this important equilibrium-based restriction is as yet unavailable.

We consider an empirically relevant design based on the above data set and the Fama-French three factor model [with MKT, SMB and HML] in which returns are generated by (3) and/or (?). We derive confidence intervals for risk price based on inverting the various above defined tests. We calibrate designs using observed data on value-weighted portfolios covering the full sample [$n = 12$ and $T = 624$], and the last sub-period [$n = 12, T = 120$]. The full sample exercise allows us to analyze our results relative to the literature, while the shorter sample documents performance as it applies to a standard sub-period analysis.

The factors are drawn as normal with means and variance/covariance calibrated to match the considered sample. We set the factor loadings and the variance/covariance of disturbances to their OLS estimated counterparts for the observed sample. J is obtained as the Cholesky root of this variance/covariance matrix. For conformity, we also compute the cross-sectional two-pass OLS estimates of the zero-beta rate and risk price [denoted θ_{*0}], and their companion standard errors [denoted $SE(\theta_{*0})$]. We use these cross-section estimates to calculate θ as in Shanken and Zhou (2007) and from there on, to initialize the simulations underlying the size study. To assess power, we set the parameter under the alternative as

$$\theta_* = \theta_{*0} + \text{step} \times SE(\theta_{*0}) \tag{74}$$

where **step** measures departure from the null hypotheses; the intercept term γ_c is calibrated in the same way. Of course, for our empirical analysis, we do not compute confidence intervals using OLS estimates nor Wald-based MLEs for that matter. All reported intervals invert the Hotelling-tests we proposed above. The cross-sectional OLS estimates from the training samples, despite their imperfections, are used as well-understood prototypical metric to initialize our data generating processes. For space considerations, the simulation values are not reported here but are available from the authors upon

request.

These settings are maintained for all analyzed tests, except in one case in which we provoke under-identification by fixing the MKT betas jointly to zero or one. We report test sizes as a worst scenario check to confirm that no over-rejections occur despite identification failure.

The disturbances W_t are generated, in turn, as *i.i.d.* normal, multivariate Student(t) with 5 degrees of freedom, and multivariate GARCH using in this case the data generating process

$$W_t = \mathbf{G}_t^{1/2} Z_t, \quad \mathbf{G}_t = (1 - \mu_1 - \mu_2)I_n + \mu_1 W_{t-1} W_{t-1}' + \mu_2 \mathbf{G}_{t-1} \quad (75)$$

where Z_t are uncorrelated n -dimensional standard normal variables, so in this case the conditional variance of JW_t is given by Σ_t with

$$\begin{aligned} \Sigma_t &= J\mathbf{G}_t J' = (1 - \mu_1 - \mu_2)JJ' + \mu_1 JW_{t-1} W_{t-1}' J' + \mu_2 J\mathbf{G}_{t-1} J' \\ &= (1 - \mu_1 - \mu_2)JJ' + \mu_1 JW_{t-1} W_{t-1}' J' + \mu_2 \Sigma_{t-1} \end{aligned} \quad (76)$$

which corresponds to a special case proposed by Engle and Kroner (1995). We use $(\mu_1, \mu_2) = (.15, .80)$. The considered inference methods are *not* corrected for departures from the *i.i.d.* assumption nor from normality. Tests and confidence intervals in what follows are at the 5% and 95% level. We report empirical rejections over 10000 replications for each parameter. In the results below, \mathcal{R}_1 is the MKT factor.

Design and results are reported in two tables and 7 figures. Table 1 reports the proportion of empty joint confidence sets in the experiments underlying each figure. Table 2 reports the unidentified experiment results. Figures 1-8 summarize our simulation findings. In each figure, we report: (i) under the heading "True Model", the specification used to generate data, and (ii) under the heading "Empirical Model", the specification that was considered for estimation and inference. The true and empirical specifications differ only regarding the treatment of \mathcal{R}_1 , as summarized in table 3. Figure 7 replicates the design in 1 with a smaller sample size; we do not replicate all designs for space considerations, so figure 7 aims to broadly illustrates sample size issues. The design underlying figure 8 differs from the rest in that it assesses our proposed tradability test; further discussions below will clarify the design and its implications. In all figures, the parameters corresponding to the null hypothesis are identified via a dashed vertical line. Results can be summarized as follows.

1. *Deviations from normality.*

In all designs including the under-identified case, deviations from normality are not distortive in the following sense: no over-rejections occur under the null hypothesis when the *i.i.d.* normal assumption

Table 1: Proportion of Empty Confidence Sets in Reported Figures

Figure	Errors	Step								
		0	-4	-3	-2	-1	1	2	3	4
1	Normal	.0110	.0099	.0098	.0093	.0102	.0112	.0113	.0115	.0121
	Student t(5)	.0103	.0110	.0112	.0101	.0103	.0098	.0094	.0095	.0098
	GARCH	.0124	.0124	.0120	.0123	.0127	.0130	.0126	.0122	.0120
2	Normal	.0055	.0050	.0049	.0042	.0044	.0060	.0065	.0068	.0070
	Student t(5)	.0051	.0053	.0052	.0055	.0052	.0052	.0052	.0047	.0049
	GARCH	.0068	.0054	.0060	.0066	.0067	.0062	.0066	.0059	.0064
3	Normal	.0100	.0086	.0089	.0091	.0097	.0106	.0105	.0108	.0119
	Student t(5)	.0084	.0090	.0093	.0086	.0084	.0085	.0084	.0082	.0084
	GARCH	.0108	.0098	.0103	.0106	.0109	.0110	.0106	.0104	.0106
4	Normal	.0055	.0050	.0049	.0042	.0044	.0060	.0065	.0068	.0070
	Student t(5)	.0053	.0051	.0052	.0055	.0052	.0052	.0052	.0047	.0049
	GARCH	.0068	.0054	.0060	.0066	.0067	.0062	.0066	.0059	.0064
5	Normal	.0100	.0086	.0089	.0091	.0097	.0106	.0105	.0108	.0119
	Student t(5)	.0084	.0090	.0093	.0086	.0084	.0085	.0084	.0082	.0084
	GARCH	.0108	.0098	.0103	.0106	.0109	.0110	.0106	.0104	.0106
6	Normal	.6683	1.0	1.0	1.0	.9700	.9954	1.0	1.0	1.0
	Student t(5)	.5854	1.0	1.0	1.0	.9553	.9930	1.0	1.0	1.0
	GARCH	.7690	1.0	1.0	1.0	.9811	.9973	1.0	1.0	1.0
7	Normal	.0095	.0096	.0091	.0091	.0094	.0076	.0073	.0069	.0069
	Student t(5)	.0051	.0056	.0055	.0054	.0054	.0039	.0037	.0032	.0039
	GARCH	.0111	.0107	.0106	.0107	.0104	.0093	.0083	.0074	.0084
8	Normal	.0055	.0049	.0053	.0050	.0055	.0054	.0060	.0062	.0056
	Student t(5)	.0051	.0055	.0058	.0056	.0052	.0046	.0047	.0047	.0049
	GARCH	.0068	.0061	.0070	.0069	.0073	.0066	.0062	.0058	.0055

Note – Numbers reported are the proportion of empty confidence sets which correspond to tests that reject the specification.

Table 2: Monte Carlo Study: Tests in Unidentified Models

$n = 12$ $T = 624$	$r_i = a_i \iota_T + \mathcal{R}_1 b_{i1} + \mathcal{F} b_{i\mathcal{F}} + u_i$					
<i>Model</i>	$a_i = [\gamma_0(1 - b_{i1}) - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}]$ $\gamma_0 = \theta_{\text{MKT}}, \gamma_{\mathcal{F}} = (\theta_{\text{SMB}}, \theta_{\text{HML}})'$					
Inverted Statistic	$\Lambda(\theta), \theta = (\gamma_0, \gamma'_{\mathcal{F}})'$					
	MKT betas = zero			MKT betas = one		
Errors	Normal	Student $t(5)$	GARCH	Normal	Student $t(5)$	GARCH
θ_{MKT}	.0199	.0185	.0207	.0199	.0185	.0207
θ_{SMB}	.0190	.0198	.0209	.0001	.0004	.0003
θ_{HML}	.0203	.0191	.0218	.0003	.0005	.0003
Rejected	.0115	.0101	.0124	.0000	.0002	.0002
<i>Model</i>	$a_i = [\gamma_c - \gamma_0 b_{i1} - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}]$ $\gamma_0 = \theta_{\text{MKT}}, \gamma_{\mathcal{F}} = (\theta_{\text{SMB}}, \theta_{\text{HML}})'$					
	MKT betas = zero					
Inverted Statistic	$\Lambda(\theta, \phi)$ $\theta = (\gamma_0, \gamma'_{\mathcal{F}})', \phi = \gamma_c$			$\Lambda(\theta)$ on $r_i - r_n$ $\theta = (\gamma_0, \gamma'_{\mathcal{F}})'$		
Errors	Normal	Student $t(5)$	GARCH	Normal	Student $t(5)$	GARCH
θ_{MKT}	.0114	.0103	.0124	.0189	.0163	.0201
θ_{SMB}	.0000	.0001	.0000	.0000	.0002	.0001
θ_{HML}	.0000	.0002	.0000	.0001	.0003	.0002
ϕ	.0000	.0002	.0002	-	-	-
Rejected	.0000	.0000	.0000	.0000	.0001	.0001

Note – Numbers reported are test sizes, for 5% tests, given models in which identification problems are provoked by setting MKT betas jointly to zeros or ones. All other model parameters and inverted test are kept as in the original designs. For this design, (unreported) power curves remain below 5%.

Table 3: Designs Underlying Reported Figures

Empirical Model	True Data Generating Process					
	R_1 Tradable			R_1 Non-Tradable		
	Inverted Statistic			Inverted Statistic		
R_1 Tradable	RAPT	PAPT	UAPT	RAPT	PAPT	UAPT
	$\Lambda(\theta)$	$\Lambda_D(\theta)$	$\Lambda(\theta, \phi)$	$\Lambda(\theta)$	$\Lambda_D(\theta)$	$\Lambda(\theta, \phi)$
	Fig. 1 & 7	NA	NA	Fig. 6	NA	NA
	Inverted Statistic			Inverted Statistic		
R_1 Non-Tradable	RAPT	PAPT	UAPT	RAPT	PAPT	UAPT
	$\Lambda(\theta)$	$\Lambda_D(\theta)$	$\Lambda(\theta, \phi)$	$\Lambda(\theta)$	$\Lambda_D(\theta)$	$\Lambda(\theta, \phi)$
	NA	Fig. 3	Fig. 2 & 8	NA	Fig. 5	Fig. 4

Note – This table summarizes designs and methods in reported figures below. The **RAPT** $\Lambda(\theta)$ [in (28)] and **UAPT** $\Lambda(\theta, \phi)$ [in (27)] statistics, where **R** and **U** stands for "restricted" and "restricted", impose and relax the assumption that R_1 is tradable, respectively. The **PAPT** $\Lambda_D(\theta)$ statistic, where **P** stands for "partialling-out", denotes $\Lambda(\theta)$ applied to a system on $n - 1$ returns in deviation from r_n . In this case, all factors are assumed non-tradable in estimating and testing the model but the resulting unrestricted constant is evacuated from the statistical objective function as it is applied to $r_i - r_n, i = 1, \dots, n - 1$.

is violated. Recall that tests rely on the above defined F critical points regardless of the distributions we use to draw simulated samples. This result is noteworthy given the prevalence of multivariate GARCH or Student-t type assumptions on disturbances in theoretical and empirical asset pricing work.

Although power studies are typically design dependent, we find that power results with GARCH-based designs dominate the Gaussian based ones which in turn dominate the Student-t case. On balance, a maximum of around 10% difference in power is observed between power curves, respectively. Power costs resulting from Student- t errors are expected, since our tests rely on least-squares. Power results with GARCH deserve discussion. Recall that test sizes are controlled even though GARCH was not accounted for. We do not advocate hasty conclusions suggesting that GARCH enhances test performance. Instead, we find that the GARCH case underscores the practical usefulness of our tests in realistic settings: most likely, GARCH adjusts the scale of left-hand side simulated variates relative to the model's covariates, to better match the considered initializing parameters which rely on observed data here. For further insight on single equation least-squares based inference in the presence of ARCH, see (Hamilton (2010)). Both Student- t and GARCH errors do not seem to affect power ranking relative to factor informativeness.

2. *Size of tests, and identification*

Empirical rejections under the null hypothesis do not exceed 5% in all cases, including the under-identified case. Reported results on designs in which we provoke under-identification [see table 2] are restricted to size, since power is expected not to exceed size in this case, a fact we verified. These results confirm that even when identification fails, the size problems documented in this literature with standard methods are solved via our proposed tests.

To quantify identification in the baseline design, refer to table 4 [reported in our empirical section below] which summarizes joint factor significance tests¹⁰ and confidence sets for factor loadings, using the last sub-period of our dataset. This is relevant because these loadings drive the simulated design for our short sample (with $T = 120$) which we view as a "stress test" for our methodology. The main point we aim to underscore from this table as it relates to our simulation design, is the following. While the (unrestricted) intercept is not significant at 5% using the Hotelling test, all factors are significant at 5%. Nevertheless, all confidence intervals for the SMB beta cover zero, while a small proportion of the intervals for the MKT or HML loadings do not cover zero. Thus factors are not necessarily

¹⁰Industry portfolios are used although (unreported) results with size factors convey qualitatively similar information. With reference to market betas, we assess joint deviations from one since the market factor is assumed tradable in this design, in which case bunching up at one is more relevant to gauge identification.

redundant, yet joint information may not be strong across portfolios with shorter samples. This turns out to matter as we will see below.

Although not obvious on first sight, figure 8 contains a size study on our unrestricted test, when applied in deviation from \mathcal{R}_1 . In this case, the intercept will measure deviations between the risk price of \mathcal{R}_1 [$\gamma_0 = \theta_{\text{MKT}}$] and the zero beta rate [γ_c], which will assess the tradable factor assumption. In this design, the null hypothesis corresponds to $\gamma_0 = \theta_{\text{MKT}} = \gamma_c$ that is, \mathcal{R}_1 is tradable. Departures from the null hypotheses vary γ_c keeping γ_0 constant. Since $\gamma_0 = \theta_{\text{MKT}}$ is kept constant throughout, the curves corresponding to θ_{MKT} describe size and not power. We thus see that size is well controlled again, a point worth emphasizing since this test is new to the (identification-robust) literature.

3. *Power of tests, general findings*

Except with the under-identified case [see table 2], all tests display good and empirically relevant power. Recall that a joint three (or four) parameter test is inverted here, which confirms that the benefits of simultaneous inference is not offset by power losses unless identification fails completely.

Comparing figures 1 and 7, we see that tests are powerful on all parameters even when the sample size drops from 624 to 120. Strikingly, power curves do not differ much between the large and small samples. In particular, and though information on other factors suffers to some extent, power on the HML price with $T = 120$ almost matches the $T = 624$ experiment. This leads us to analyze with further detail how power differs across considered factors; refer to point 4, below.

The unrestricted and partialled-out tests perform exactly the same for inference on risk price, a result which lines up with the above theory [refer to (38)-(41)]. The main advantage of our test compared to the partialled-out one which relates to an asymptotic test proposed by (Kleibergen (2009)) is the information we provide on the model's intercept. See in particular figure 4: while partialling-out evacuates this parameter, our test provides tremendous power on this fundamental coefficient without sacrificing any information on the model's risk price.

Perhaps equally important here is our finding in table 1: inverting the test that imposes tradability when it does not hold produces a very large proportion of empty sets, which implies it successfully detects the false assumptions. Taken collectively, results reinforce the prescription in Lewellen et al. (2010) regarding tradable factors particularly because we provide a method to validate this assumption. Further discussion of restricted versus unrestricted testing is discussed in point 5, below.

4. *Power of tests, across factors*

Broadly, tests are more informative on one of the three factors relative to the others. Given the historical debate on MKT beta, comparing figures 4 and 5 to figures 1 and 7 is particularly enlightening.

The former imply that the MKT risk is harder to test than the remaining factors. In contrast, tests on the zero-beta rate as depicted in the latter seem more powerful than those on SMB and HML.

In all four figures true and empirical assumptions on R_1 coincide and conformable tests are applied. Thus, both of these alternative findings may initially appear plausible. However, while figures 4 and 5 relax tradability of \mathcal{R}_1 , figures 1 and 7 replicate empirical consensus on \mathcal{R}_1 , *i.e.* that it is a tradable factor. This underscores the usefulness of our restricted test which is depicted in figures 1 and 7, which in turn leads us to further analyze the effects of restricting versus assessing the tradable factor assumption; refer to point 5 below.

Figures 1 and 7 also suggest that SMB is the least informative factor. The above comments warning that confidence sets on the SMB loading all cover zero apply in this case. Relative power ranking are thus clearly driven by the relative identification strength of SMB in this design, which we emphasize is based on observable factors.

5. *Power of tests, restricted versus unrestricted*

Figure 8 (with the exception of results on θ_{MKT} as noted above) depicts the power of our proposed test which assesses that \mathcal{R}_1 is tradable, or formally whether $\gamma_0 = \theta_{\text{MKT}} = \gamma_c$. Departures from this null hypotheses vary γ_c keeping γ_0 constant. The power on non-market risk prices is unaffected relative to the previous designs, while we find very good power on $\gamma_c - \gamma_0$. Though ex-ante decisions regarding some factors is possible, most cross-sectional based works in asset pricing tend to leave the intercept unrestricted. Our approach provides an identification-robust assessment of tradable factor restrictions, which, to the best of our knowledge, is a new and useful contribution.

Figure 2 documents the consequences of neglecting this assumption when it holds. Comparing power curves between figure 1 and figure 2, we find that the risk price of \mathcal{R}_1 bares all the cost as power is much weaker in figure 2 than in figure 1, which results of course from disregarding a relevant restriction. Our design suggests that resulting power losses are sizable: the risk price of a tradable factor is harder to pin-down even in identified contexts when an unrestricted cross-sectional constant is maintained, which quantifies the consequences of an important "pitfall" raised in particular by Lewellen et al. (2010) and more recently though in a GMM context by Penaranda and Sentana (2015). In contrast to the traditional literature, our findings are based on methods that are robust to the identification of all factors which provides new insights into the historical debate surrounding the role of the MKT factor in multi-factor models.

To further interpret the evidence on intercept tests, note that in the design underlying figure 2, as γ_c is taken away from its value under the null hypothesis, γ_0 follows conformably since the true

model throughout imposes $\gamma_0 = \gamma_c$ (hence the need for figure 8, to assess the test for inference on this discrepancy). So results for $\gamma_0 = \gamma_c$ can be interpreted as size [despite the misspecification], while tests on γ_c confirms the excellent power properties we noted while commenting on figure 4. Figure 3 illustrates the limitations of partialled-out tests: power on all coefficients, again, coincides exactly with that of our unrestricted test as depicted in figure 2, which we noted to be way lower than in figure 1 for inference regarding \mathcal{R}_1 . As an added major cost, partialling-out takes away all sources of information on the validity of tradability assumptions, whose usefulness we quantified via figure 8.

Figure 6 illustrates the consequences of imposing the traded factor assumption when it does not hold. Here, what is indicated as a parameter value under the null is in fact a false null, since the model falsely imposes a restriction that does not hold. An important contrast with figure 2 in which case we found that inference regarding \mathcal{R}_1 is only affected, here results show that spurious inference *on all* model parameters results, with notable size distortions as empirical rejections exceed 60%. A important cautionary remark about this figure: as may be seen from table 1, the rejections we depict actually correspond to empty confidence sets in almost all simulations. This means that our companion model checks are conveying evidence of *misspecification*, with very good power. This finding leads to clear prescriptions for empirical work: despite the importance of imposing traded factor assumption, their empirical validation remains a must as serious distortions will result otherwise. This reinforced the usefulness of our proposed intercept test, and more importantly, the usefulness of our built-in specification checks which will return empty sets when the model deviates importantly from asset pricing equilibrium relations.

6. Size and power, and empirical results

Taken collectively, our simulation results suggest that the unbounded confidence sets we observe empirically as reported in section 4.2 are most likely driven by weak factors. Inference problems are thus highly likely even outside the high dimensional settings analyzed for example by Harvey et al. (2015) where k is by far larger than 5, the maximum number we consider. Our results with a small number of commonly used factors also suggest that our confidence sets in which an identification check is "hard-wired" are extremely valuable in practice, since they will allow the researcher to qualify non-rejections. An unbounded confidence set guards the researcher from misreading nonsignificant tests as evidence in favour of models on which data is not informative. Our analytical projection method thus provides an invaluable tool because it easily and surely confirms an unbounded solution, in contrast to *e.g.* numerical searches that are typically subject to precision, convergence and tractability constraints. On balance, results with both sample sizes illustrate the worth of our analytical F-based motivation

for relying on our proposed analytical test inversion formula.

4.2 Empirical Results

In the following discussion, significance refers to the 5% level and the restricted test refers to treating MKT as a tradable factor. Our empirical analysis builds on the prescriptions of Lewellen et al. (2010). From a general standpoint, our results can be summarized as follows.

Risk premiums are better identified with industry portfolios than with size portfolios. This result is not driven by the number of the portfolios in question. In fact, when the whole set of portfolios is used jointly following Lewellen et al. (2010, Prescription 1), all considered unconditional models are rejected. Although noteworthy and consistent with the discussion in Lewellen et al. (2010), the latter test may pose an unconventionally high hurdle for goodness of fit. We thus do not aim to overemphasize these rejections. Instead, we view these results as confirming the power of our tests as n increases relative to T .

An alternative and more fundamental argument is that stacking portfolios increases dispersion of factor sensitivities; in contrast, size sorted portfolios yield much more clustered betas than their industry counterparts, which ill-conditions the rank of the associated *beta* matrix thus compromising identification of risk price. Similar distortive clustering results with value weighted portfolios whether we use industry or size sorting, while size sorted value-weighted portfolios are the least informative in our considered tests. This is illustrated in table 4 which reports confidence sets for factor loading based on inverting $\Lambda(\bar{\beta}_j)$ in (35), as well as zero-parameter test based on Λ_{0j} (33) under the heading Hotelling, corresponding to the last sub-period of our dataset with industry value-weighted portfolios.¹¹ Though all factors seem relevant via significant Hotelling test, a small number of confidence intervals for MKT excludes one, one of the intervals for HML excludes zero and all intervals for SMB cover zero, which suggests severe clustering. For this purpose, we base the bulk of our analysis on equally weighted industry portfolios.

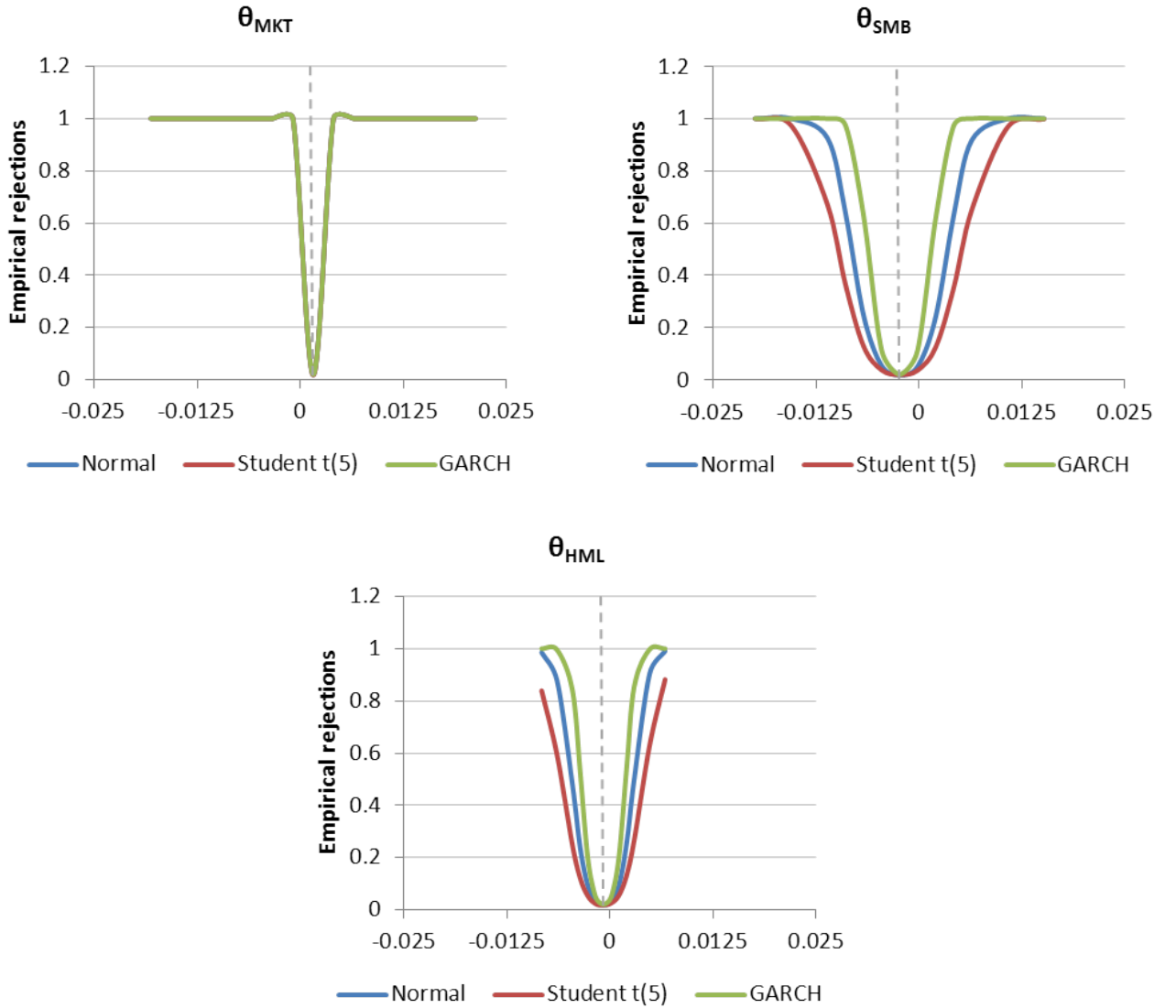
The Fama-French five factor model is severely under-identified even via our most informative checks, namely our restricted test, industry portfolios, and over the whole sample. Since Fama and French (2015) argue that HML in this model is unidentified, we repeat our analysis excluding this factor.

Analyzed results are reported in six tables; the appendix includes results with value weighted portfolios and further results with weaker identification evidence for completion.

¹¹We discussed these results above as they relate to our simulation design.

Figure 1: Monte Carlo Study: Tests imposing tradable market factor

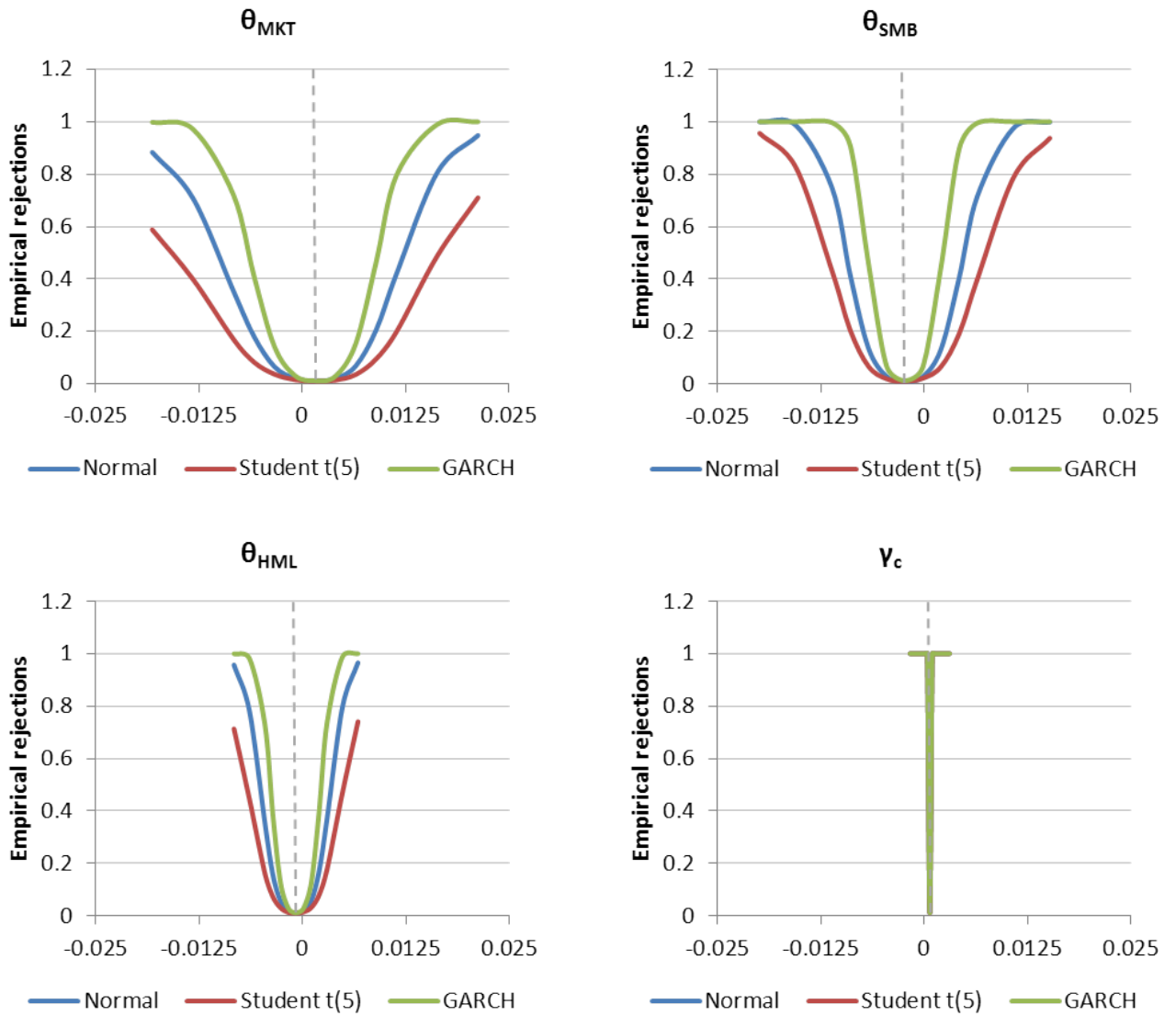
$n = 12$ $T = 624$	$r_i = a_i \iota_T + \mathcal{R}_1 b_{i1} + \mathcal{F} b_{i\mathcal{F}} + u_i$
<i>True Model</i>	$a_i = [\gamma_0(1 - b_{i1}) - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}], \quad \gamma_0 = \theta_{\text{MKT}}, \quad \gamma_{\mathcal{F}} = (\theta_{\text{SMB}}, \theta_{\text{HML}})'$
<i>Empirical Model</i>	$a_i = [\gamma_0(1 - b_{i1}) - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}]$
RAPT: Inverted Statistic $\Lambda(\theta)$, $\theta = (\gamma_0, \gamma'_{\mathcal{F}})'$	

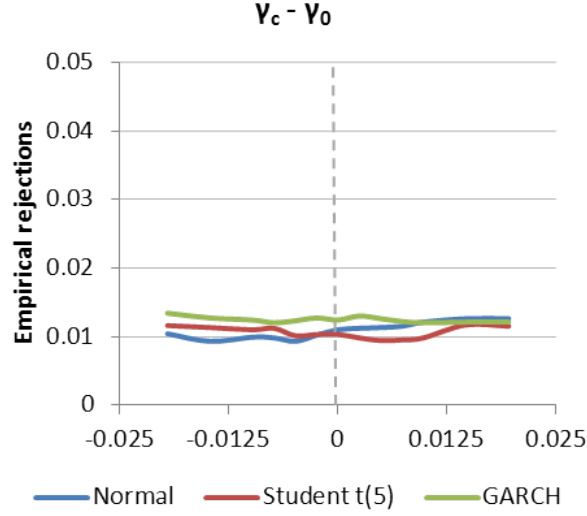


Note – Dashed vertical lines denotes null parameter values. Empirical rejections pertain to 5% tests associated with 95% confidence sets: if the 95% set does not cover the null value or is empty, the reported test is considered significant. Parameters are calibrated to OLS cross-sectional two-pass estimates from a training sample based on industry portfolios, Fama-French factors and monthly data, 1961-2010 hence $T = 624$. See table 3 for further details on design and inverted tests. When one uni-color curve is depicted while the legend refers to three cases, this implies that all visually coincide.

Figure 2: Monte Carlo Study: Joint tests, market factor tradable assumed non-tradable

$n = 12$ $T = 624$	$r_i = a_i \iota_T + \mathcal{R}_1 b_{i1} + \mathcal{F} b_{i\mathcal{F}} + u_i$
<i>True Model</i>	$a_i = [\gamma_0(1 - b_{i1}) - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}], \quad \gamma_0 = \theta_{\text{MKT}}, \quad \gamma_{\mathcal{F}} = (\theta_{\text{SMB}}, \theta_{\text{HML}})'$
<i>Empirical Model</i>	$a_i = [\gamma_c - \gamma_0 b_{i1} - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}]$
UAPT: Inverted Statistic $\Lambda(\theta, \phi)$, $\theta = (\gamma_0, \gamma'_{\mathcal{F}})'$, $\phi = \gamma_c$ or $(\gamma_c - \gamma_0)$	





Note – See notes to figure 1 and table 3. Results under the heading $\gamma_c - \gamma_0$ are obtained by executing $\Lambda(\theta, \phi)$ on returns in deviation from \mathcal{R}_1 in which case tests on risk θ are unchanged and tests on ϕ provide inference on $\gamma_c - \gamma_0$, which is zero throughout this design for all "steps" given the considered true model.

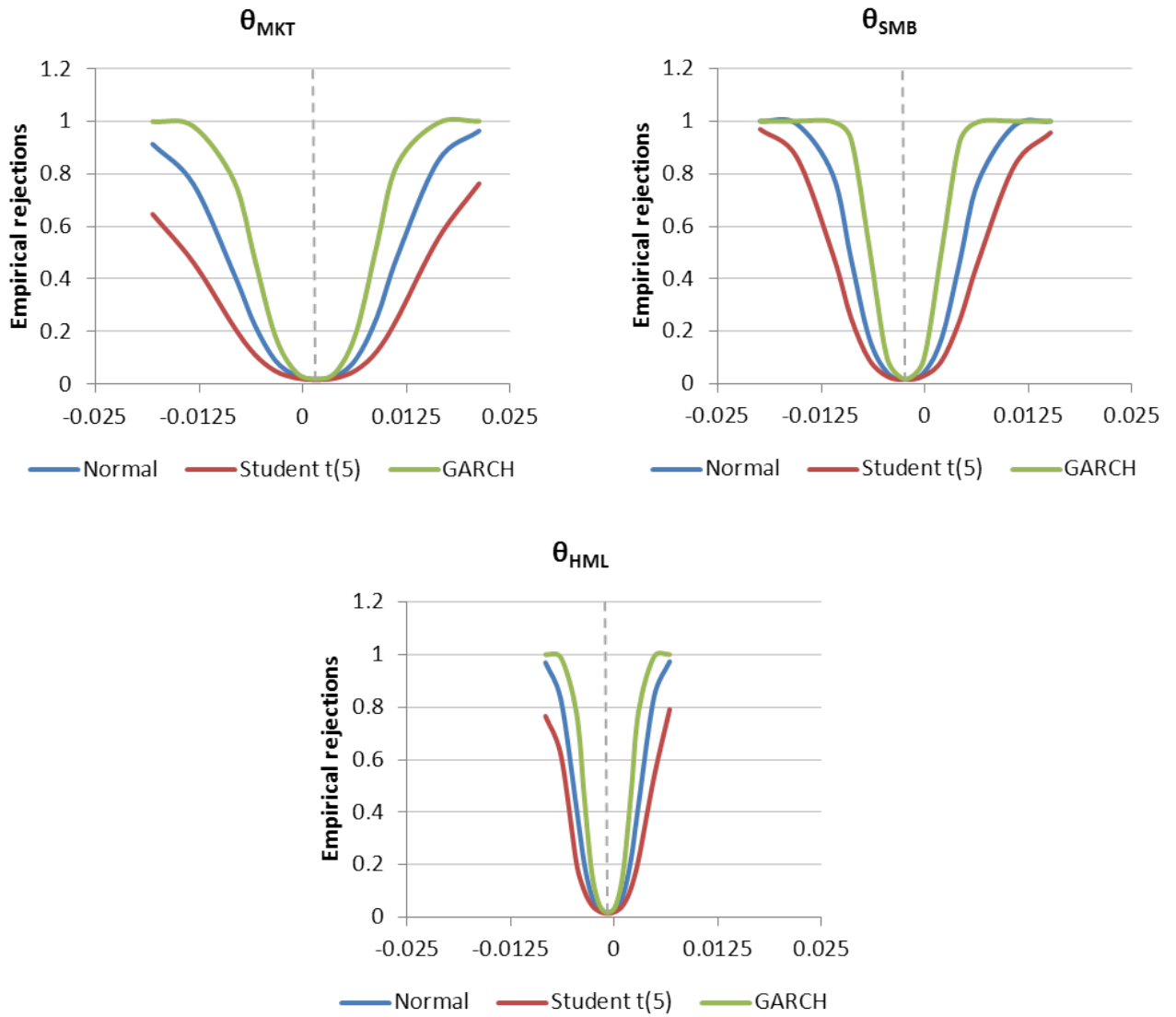
Table 4: Simultaneous confidence sets for factor loadings

2001-2010, Value-Weighted Industry Portfolios				
Eq.	Intercept	MKT -1	SMB	HML
1	[-.0062,.0129]	[-.5969,-.1868]	[-.4630,.2560]	[-.0498,.5577]
2	[-.0214,.0180]	[-.0563,.7905]	[-.2121,1.2723]	[-.0808,1.1737]
3	[-.0079,.0127]	[-.0405,.4033]	[-.1699,.6082]	[-.0701,.5874]
4	[-.0145,.0298]	[-.6664,.2849]	[-1.0358,.6319]	[-.4649,.9445]
5	[-.0063,.0147]	[-.4489,.0022]	[-.5510,.2397]	[-.0700,.5982]
6	[-.0095,.0113]	[.1698,.6156]	[-.0586,.7223]	[-1.2226, -.5623]
7	[-.0147,.0148]	[-.2125,.4201]	[-.9030,.2062]	[-.5811,.3563]
8	[-.0146,.0188]	[-.7735,-.0573]	[-.7852,.4705]	[-.2415,.8196]
9	[-.0102,.0113]	[-.4224,.0396]	[-.1606,.6493]	[-.2435,.4409]
10	[-.0118,.0122]	[-.5823,-.0659]	[-.8394,.0660]	[-0.3373,.4278]
11	[-.0156,.0062]	[-.1480,.3206]	[-.4751,.3463]	[.2860,.9802]
12	[-.0103,.0077]	[-.1198,.2679]	[-.3047,.3750]	[-.0969,.4776]
Hotelling	1.2131	16.9573	5.4253	23.2133
p-value	.284	.000	0.000	0.000

Note – See notes to table 5 for the definition of the considered sample. Intervals reported are the 95% joint (across equations) confidence sets for the coefficients (in turn) of each portfolio regression numbered 1-12. The inverted test in each case is $\Lambda(\bar{\beta}_j)$ defined in (32) to test H_j (25). The Classical Hotelling joint significance test with conforming p-value is reported at the bottom of each column to assess each of H_{0j} (26). $j = 1$ provides joint inference on the unrestricted regression intercepts, and as the unrestricted regression is in deviation from the tradable factor, here MKT, $j = 2$ provides joint inference on market betas in deviation from one, and $j = 3, 4$ provide inference, in turn, on SMB and HML betas. Confidence sets in bold are those that do not cover zero.

Figure 3: Monte Carlo Study: Partialled-out tests, market factor tradable assumed non-tradable

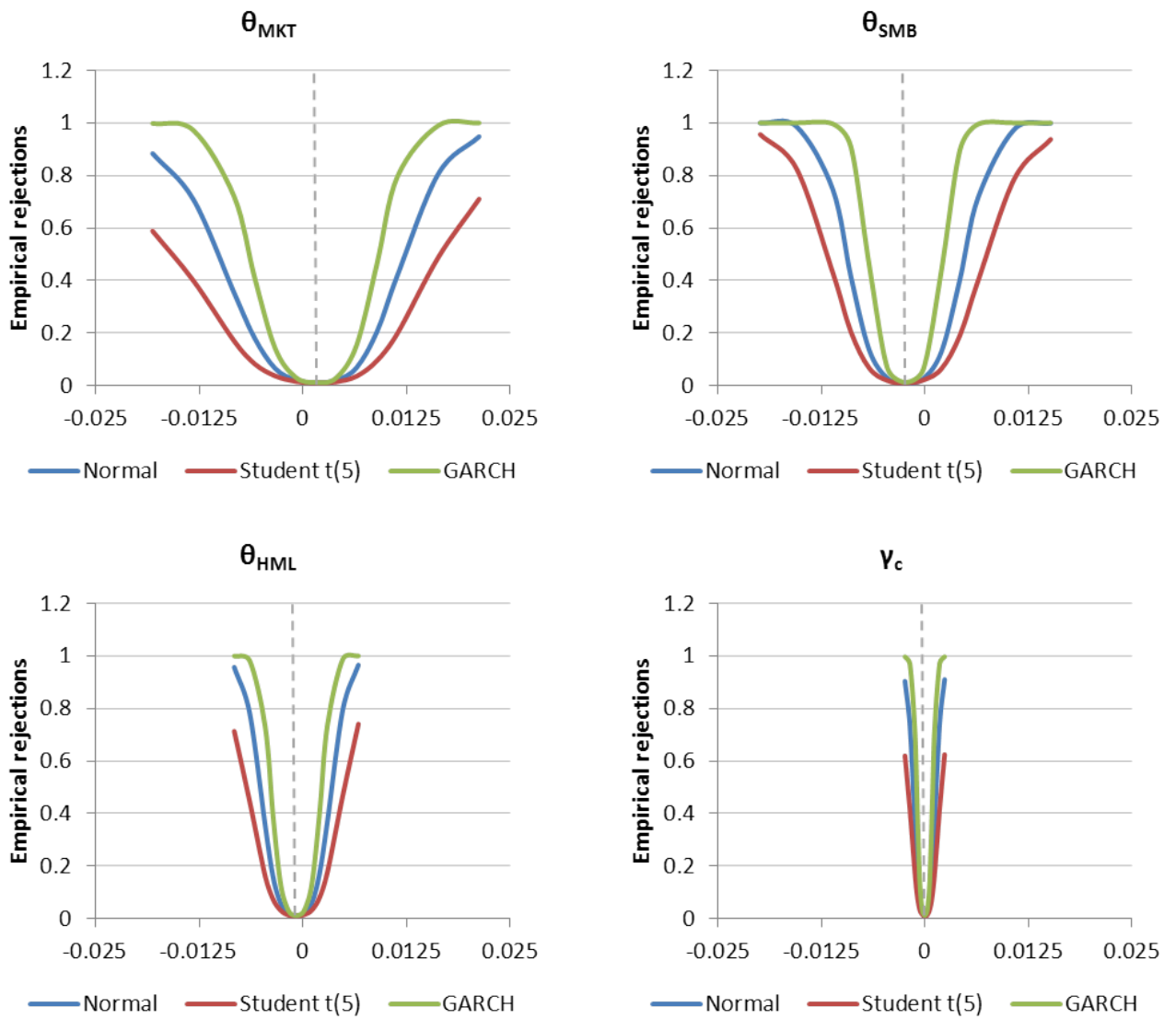
$n = 12$ $T = 624$	$r_i = a_i \nu_T + \mathcal{R}_1 b_{i1} + \mathcal{F} b_{i\mathcal{F}} + u_i$
<i>True Model</i>	$a_i = [\gamma_0(1 - b_{i1}) - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}], \quad \gamma_0 = \theta_{\text{MKT}}, \quad \gamma_{\mathcal{F}} = (\theta_{\text{SMB}}, \theta_{\text{HML}})'$
<i>Empirical Model</i>	$a_i = [\gamma_c - \gamma_0 b_{i1} - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}]$
PAPT: Inverted Statistic $\Lambda(\theta)$ on $r_i - r_n$ with $\theta = (\gamma_0, \gamma'_{\mathcal{F}})'$	



Note – See notes to figure 1 and table 3. The dashed vertical line denotes the value of the given parameter under the null.

Figure 4: Monte Carlo Study: Joints tests, relaxing tradable market factor

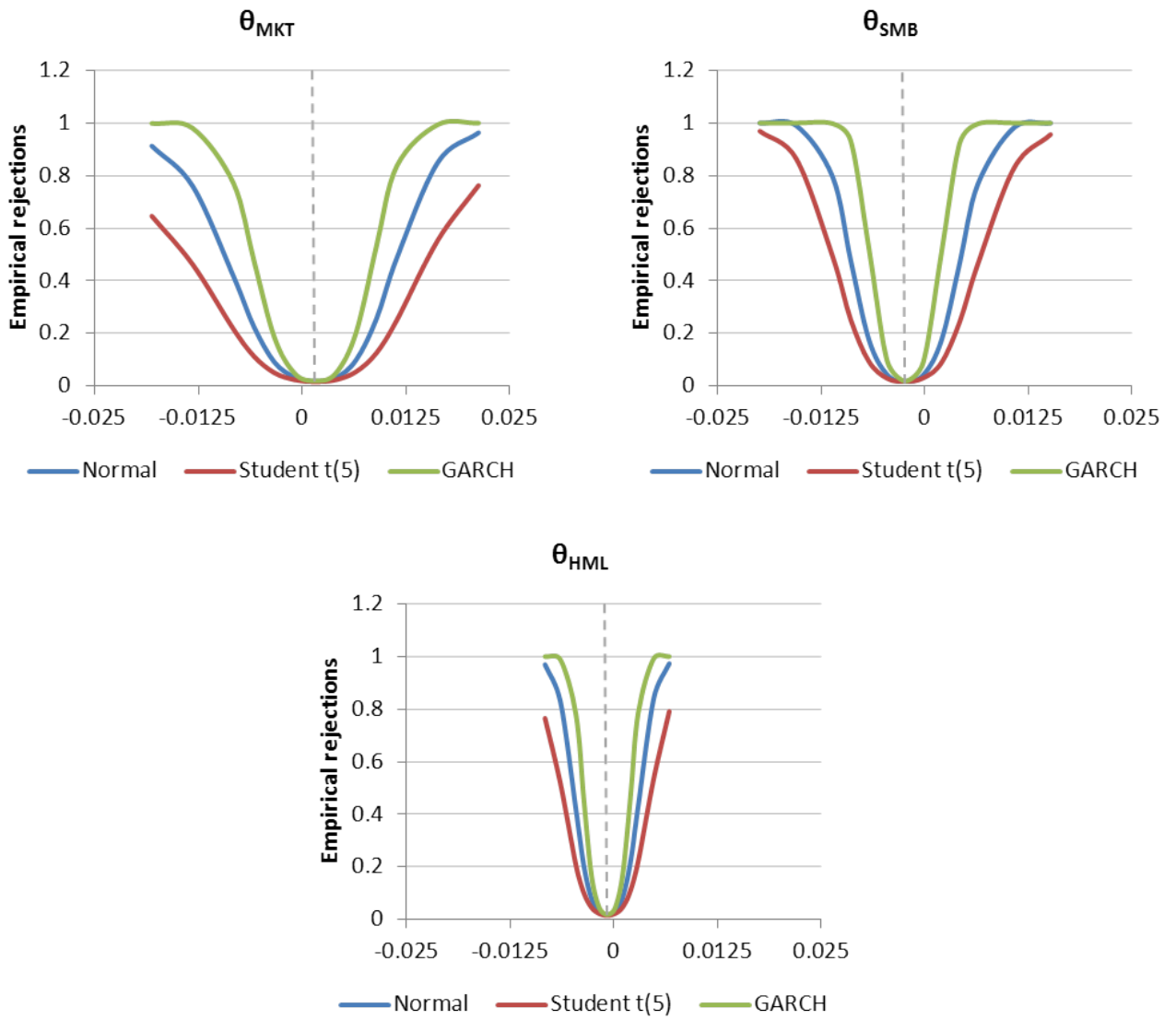
$n = 12$ $T = 624$	$r_i = a_i \iota_T + \mathcal{R}_1 b_{i1} + \mathcal{F} b_{i\mathcal{F}} + u_i$
<i>True Model</i>	$a_i = [\gamma_c - \gamma_0 b_{i1} - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}], \quad \gamma_0 = \theta_{\text{MKT}}, \quad \gamma_{\mathcal{F}} = (\theta_{\text{SMB}}, \theta_{\text{HML}})'$
<i>Empirical Model</i>	$a_i = [\gamma_c - \gamma_0 b_{i1} - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}]$
UAPT: Inverted Statistic $\Lambda(\theta, \phi), \quad \theta = (\gamma_0, \gamma'_{\mathcal{F}})', \quad \phi = \gamma_c$	



Note – See notes to figure 2.

Figure 5: Monte Carlo Study: Partialled-out tests, relaxing tradable market factor

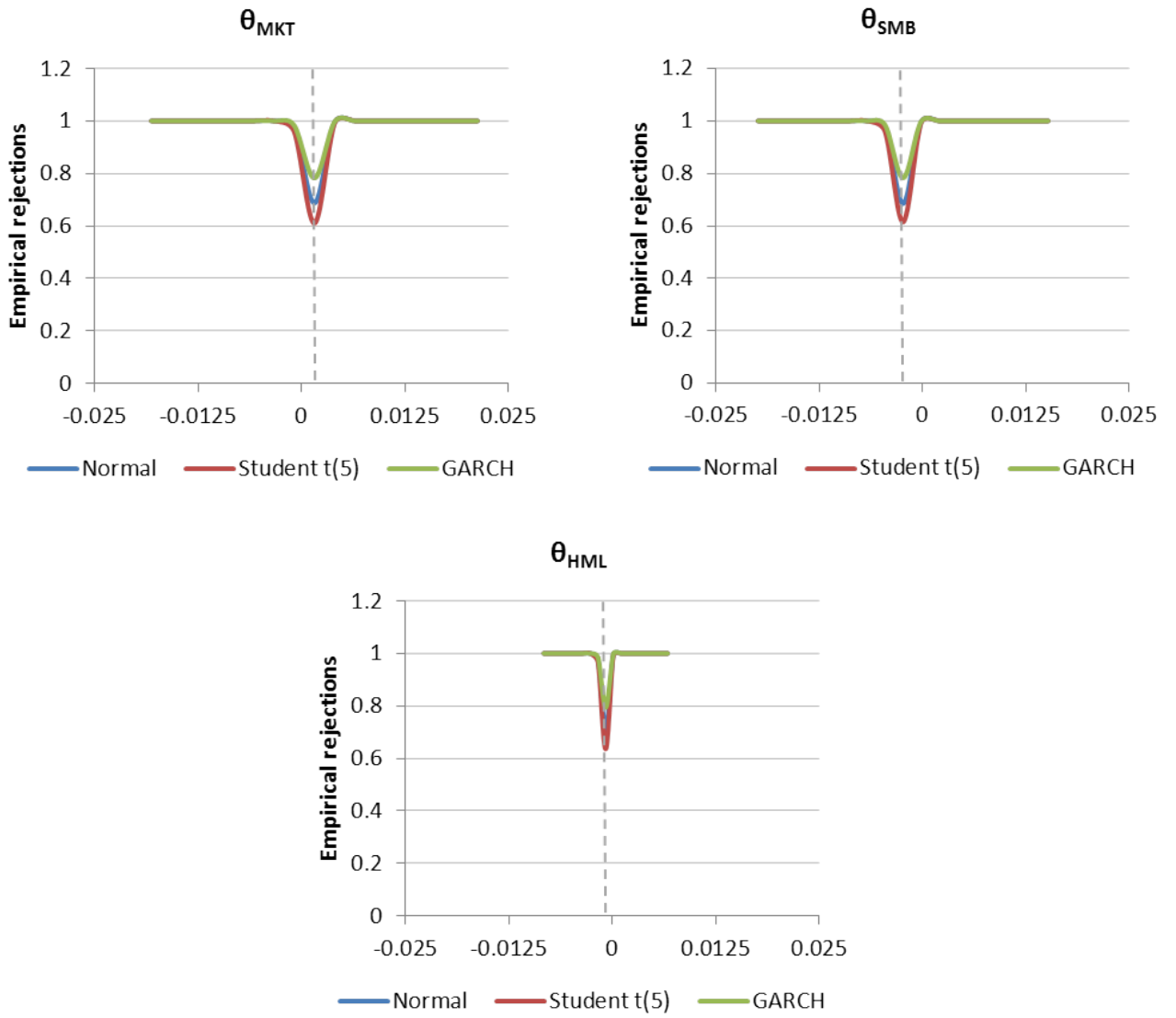
$n = 12$ $T = 624$	$r_i = a_i \iota_T + \mathcal{R}_1 b_{i1} + \mathcal{F} b_{i\mathcal{F}} + u_i$
<i>True Model</i>	$a_i = [\gamma_c - \gamma_0 b_{i1} - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}], \quad \gamma_0 = \theta_{\text{MKT}}, \quad \gamma_{\mathcal{F}} = (\theta_{\text{SMB}}, \theta_{\text{HML}})'$
<i>Empirical Model</i>	$a_i = [\gamma_c - \gamma_0 b_{i1} - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}]$
PAPT: Inverted Statistic $\Lambda(\theta)$ on $r_i - r_n$ where $\theta = (\gamma_0, \gamma'_{\mathcal{F}})'$	



Note – See notes to figure 3 and table 3.

Figure 6: Monte Carlo Study: Test imposing tradable market factor when non-tradable

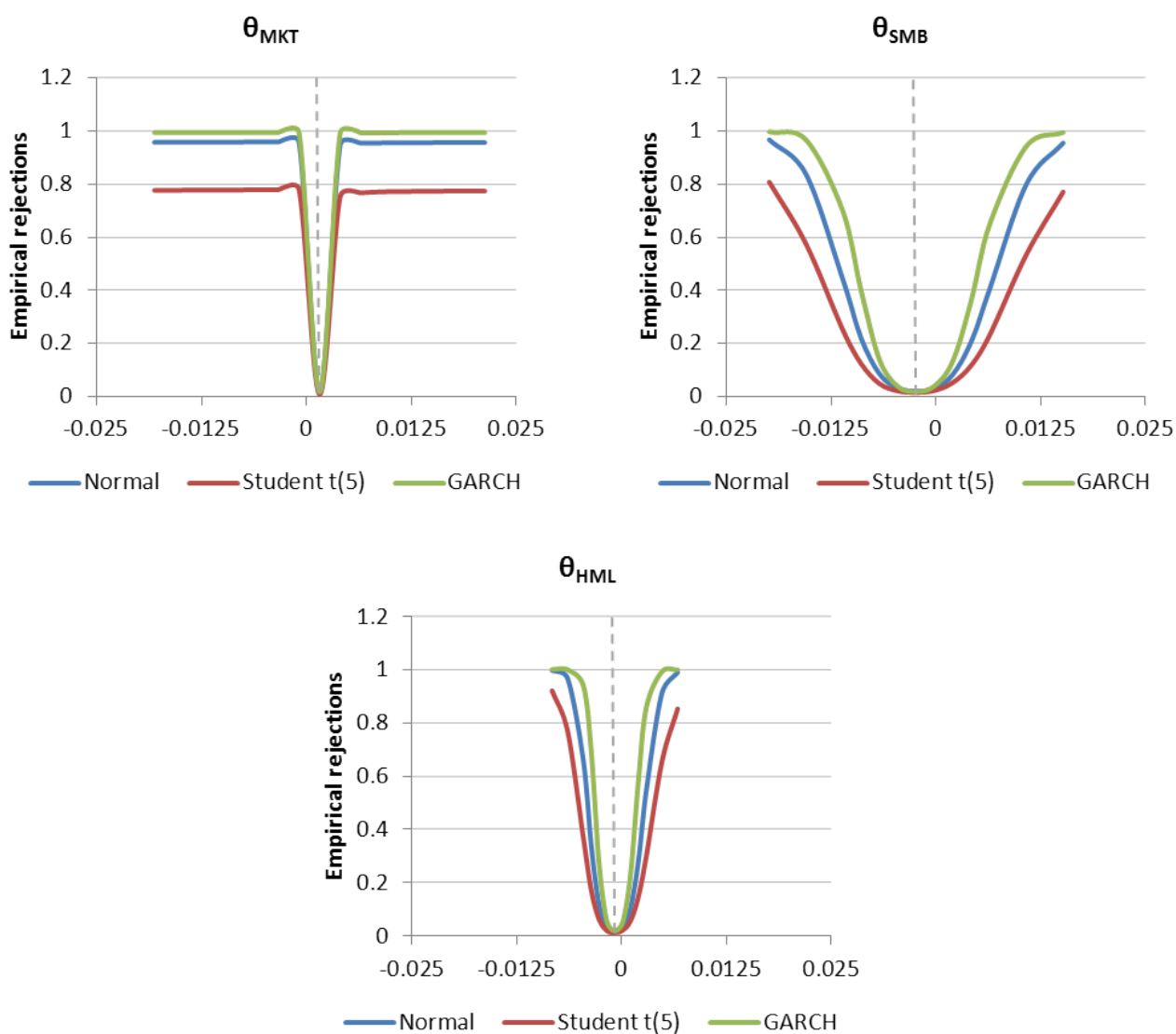
$n = 12$ $T = 624$	$r_i = a_i \iota_T + \mathcal{R}_1 b_{i1} + \mathcal{F} b_{i\mathcal{F}} + u_i$
<i>True Model</i>	$a_i = [\gamma_c - \gamma_0 b_{i1} - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}], \quad \gamma_0 = \theta_{\text{MKT}}, \quad \gamma_{\mathcal{F}} = (\theta_{\text{SMB}}, \theta_{\text{HML}})'$
<i>Empirical Model</i>	$a_i = [\gamma_0(1 - b_{i1}) - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}]$
RAPT: Inverted Statistic $\Lambda(\theta)$, $\theta = (\gamma_0, \gamma'_{\mathcal{F}})'$	



Note – See notes to figure 1.

Figure 7: Monte Carlo Study: Tests imposing tradable market factor, performance with 10 years of data

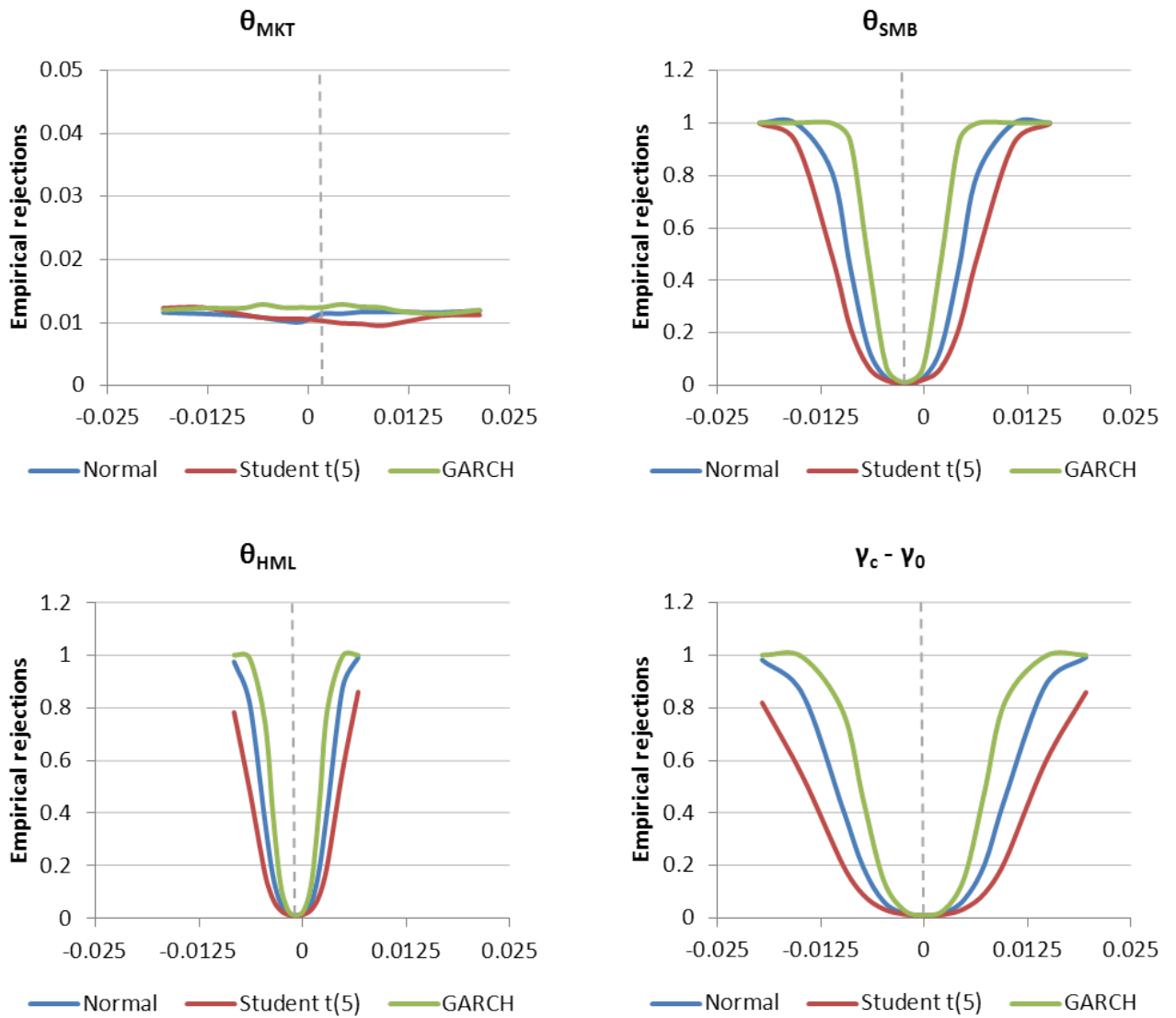
$n = 12$ $T = 120$	$r_i = a_i \iota_T + \mathcal{R}_1 b_{i1} + \mathcal{F} b_{i\mathcal{F}} + u_i$
<i>True Model</i>	$a_i = [\gamma_0(1 - b_{i1}) - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}], \quad \gamma_0 = \theta_{\text{MKT}}, \quad \gamma_{\mathcal{F}} = (\theta_{\text{SMB}}, \theta_{\text{HML}})'$
<i>Empirical Model</i>	$a_i = [\gamma_0(1 - b_{i1}) - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}]$
RAPT: Inverted Statistic $\Lambda(\theta)$, $\theta = (\gamma_0, \gamma'_{\mathcal{F}})'$	



Note – See notes to figure 1 and table 3. The training sample used to generate simulation parameters is restricted to the last 10 years of data.

Figure 8: Monte Carlo Study: Joint tests, market factor tradable assumed non-tradable

$n = 12$ $T = 624$	$r_i = a_i \iota_T + \mathcal{R}_1 b_{i1} + \mathcal{F} b_{i\mathcal{F}} + u_i$
<i>True Model</i>	$a_i = [\gamma_c - \gamma_0 b_{i1} - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}], \quad \gamma_0 = \theta_{\text{MKT}}, \quad \gamma_{\mathcal{F}} = (\theta_{\text{SMB}}, \theta_{\text{HML}})'$
<i>Empirical Model</i>	$a_i = [\gamma_c - \gamma_0 b_{i1} - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}]$
UAPT: Inverted Statistic $\Lambda(\theta, \phi)$, $\theta = (\gamma_0, \gamma'_{\mathcal{F}})'$, $\phi = (\gamma_c - \gamma_0)$	



Note – See notes to figure 1 and table 3. Results are obtained by executing $\Lambda(\theta, \phi)$ on returns in deviation from \mathcal{R}_1 . Under the null, we hold $\gamma_c = \gamma_0$, and under the alternative, γ_c moves away from the true value, while γ_0 does not.

4.2.1 Industry Portfolios

Results imposing tradable MKT differ importantly from their unrestricted counterparts. Compare for example Panel A to Panel B of table 5, and consider first the 1971-1980 and 1991-2000 subperiods in which we reject the three-factor model via our restricted test (of Panel A). For further reference, these subperiods will be denoted as the *atypical* ones, to underscore this rejection. In contrast to the restricted test, our unrestricted inference (in Panel B) fails to reject the model in these subperiods and confirms that: (i) MKT is priced in both cases, (2) HML is not priced while SMB is priced only in the 90s.

Does it seem reasonable to retain a model that ignores a key property of the market factor? With reference to Lewellen et al. (2010, Prescription 2), the unrestricted test seems a low hurdle to meet unless (among other explanations) the zero beta rate differs anomalously from the risk-free rate. Our inference on the cross-sectional intercept can inform in this regard, in contrast to the unrestricted test [adopted in particular by Kleibergen (2009)] that subtracts this intercept away. Let us thus refer to the upper Panel of table 7. For the subperiods in question, we find that despite notable estimation uncertain, the confidence set on γ_c^* does not cover zero which confirms that γ_0 significantly differs from γ_c . This discrepancy seems to be driving the rejection of the tradable MKT model. Interestingly, we are not able to refute a zero γ_c^* outside these subperiods.

Results with the Carhart model provide further insights on the above anomalous interpretation. Comparing the upper to the lower panels of table 6, we find that the unrestricted test is completely uninformative as the confidence sets are utterly wide. In contrast, with a tradable MKT and again, despite evidence of estimation uncertainty, we find that MOM is priced only in the 1971-1980 and 1991-2000 sub-periods and these are the only subperiods in which the restricted three-factors model is rejected while γ_0 significantly differs from γ_c . The MKT risk itself is no longer priced in these subperiods, which stands in sharp contrast with our unrestricted three-factor based evidence. In addition, SMB is priced in the presence of MOM in both subperiods, while it is priced in our unrestricted three-factor model only in the nineties.

Divergences between restricted and unrestricted inference also arise when the restricted three-factors model is not rejected. In particular, in the three factors model, the restricted tests confirm that SMB and HML are both priced in the 1960s while the unrestricted tests cover zero. Similarly, HML appears priced via the restricted test in 1981-90 and not priced using the unrestricted counterpart, and the same holds for SMB in 2000-2010. Overall, aside from the market and unless the restricted model is rejected, all factors that are priced via the restricted test are no longer priced when the

tradable MKT restriction is relaxed. Referring to the upper Panel of table 7 reveals no basis to refute $\gamma_0 = \gamma_c$ when the restricted model is not rejected, and as emphasized above, MOM is not priced in these subperiods as may be checked again from the upper Panel of table 6.

The above interpretation of the momentum effect may be qualified as we interpret results of the Fama-French model with SMB, RMW and CMA over and above the MKT factor. As with the Carhart model, the unrestricted test is completely uninformative yet the model passes our restricted test over all subperiods. The following discussion thus focuses on the restricted model outcomes in table 8. Interestingly, results for the atypical 70s and 90s stand in sharp contrast with those we obtained using Carhart’s model: MKT and one of the RMW and CMA factors are priced while SMB is not; instead, recall that SMB and MOM are jointly priced with the Carhart model, while the MKT factor was not.

The obvious question is, then, whether observed factors represent risks or anomalies. Yet any interpretation of our findings in this regard is hasty, given our focus thus far on the atypical subperiods. Outside these subperiods, interpretations in any direction are severely hampered by the more pernicious identification failures we observe therein. Indeed, over and above MKT and SMB, the addition of MOM and HML yields completely uninformative sets (the real line) prior to the 70s and so does the addition of RMW and CMA. In the 80s, the Carhart model is uninformative while MKT and RMW are priced despite unbounded sets. In contrast, post 2000, SMB is the only priced factor in the Carhart model while the four-factor Fama-French model is completely uninformative.

In sum, four key results are worth emphasizing. First, the three Fama-French factors are confirmed to be priced concurrently only before 1970. From there on, the factors are either: (i) jointly rejected in the sense that anomalies remain despite some evidence of pricing, or (ii) are weakly supported, in the following sense: we find no clear indication on which among the three is priced or not. Second, with regards to the historical debate on anomalies¹², we do not find convincing and uniform evidence favoring any factor relative to MKT. Third, MOM is not necessarily irrelevant despite its adverse effect on identification broadly, and may possibly proxy an outstanding anomaly relative to the three-factors model in the 70s and 90s atypical subperiods. Fourth, heterogeneity is not sufficient to distinguish a priced momentum anomaly from profitability or investment as presumably non-diversifiable risk drivers.

¹²See Campbell et al. (1997, Chapters 5 and 6), Fama and French (2004), Perold (2004), Campbell (2003), Sentana (2009) and the recent insight in Fama and French (2015).

Table 5: Confidence sets for risk price: Industry portfolios and three factor model

$r_i - \iota_T \gamma_0 = (\mathcal{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathcal{F} - \iota_T \gamma'_{\mathcal{F}}) b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n$						
PANEL A $\theta = (\gamma_0, \gamma'_{\mathcal{F}})' = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}})'$						
	MKT		SMB		HML	
$\times 10^{-4}$	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{HML}
61-70	38*	[-437, -55]	33*	[-17, 14]	53*	[-46, 41]
71-80	30	\emptyset	43	\emptyset	33	\emptyset
81-90	44*	$]-\infty, -194[$ $\cup [1156, \infty[$	-16	\mathbb{R}	56*	$]-\infty, -115[$ $\cup [611, \infty[$
91-00	103	\emptyset	4	\emptyset	29	\emptyset
00-10	14	[-79, 245]	57*	[-146, 3]	40	[-41, 178]
$r_i - \iota_T \gamma_c = (\mathcal{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathcal{F} - \iota_T \gamma'_{\mathcal{F}}) b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n$						
PANEL B $\theta = (\gamma_0, \gamma'_{\mathcal{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}), \quad \gamma_c$ partialled-out						
	MKT		SMB		HML	
$\times 10^{-4}$	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{HML}
61-70	38*	[-642, -14]	33	[-11, 83]	53	[-64, 74]
71-80	30*	[-621, -54]	43	[-26, 158]	33	[-137, 160]
81-90	44*	$]-\infty, -198[$ $\cup [807, \infty[$	-16	\mathbb{R}	56	\mathbb{R}
91-00	103*	$]-\infty, -432[$ $\cup [3087, \infty[$	4*	$]-\infty, -1499[$ $\cup [97, \infty[$	29	$]-\infty, -281[$ $\cup [-24, \infty[$
00-10	14	[-100, 1226]	57	[-1089, 66]	40	[-137, 192]

Note – Sample includes monthly observations from January 1991 to December 2010 on the US. Series include 12 equally weighted (EW) industry portfolios as well as US factors for market (MKT), size (SMB), book-to-market (HML). Confidence sets are at the 5% level. \bar{F} is the factor average over the considered time period; θ captures factor pricing as defined in (23). * denotes evidence of pricing at the 5% significance level interpreted as follows: given the reported confidence sets, each factor is priced if its average is not covered; see section 2.1. In Panel A, the inverted test is $\Lambda(\theta)$ defined in 28. This test follows our **RAPT** approach where **R** stands for "restricted" implying that tradable factor constraints are imposed, here on \mathcal{R}_1 , in estimating and testing the model. In Panel B, the inverted test $\Lambda(\theta)$ is applied on a system on $n - 1$ returns in deviation from r_n . This test follows our **PAPT** approach where **P** stands for "partialling-out" implying that all factors are assumed non-tradable but the resulting unrestricted constant is evacuated from the statistical objective function as it is based on $r_i - r_n, i = 1, \dots, n - 1$.

Table 6: Confidence sets for risk price: Industry portfolios and four factor model

$r_i - \iota_T \gamma_0 = (\mathcal{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathcal{F} - \iota_T \gamma'_{\mathcal{F}}) b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n$								
$\theta = (\gamma_0, \gamma'_{\mathcal{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{MOM}})$								
	MKT		SMB		HML		MOM	
$\times 10^{-4}$	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{HML}	\bar{F}	θ_{MOM}
61-70	38	\mathbb{R}	33	\mathbb{R}	53	\mathbb{R}	73	\mathbb{R}
71-80	30	[-300, 232]	43*	[-62, -17]	33	[-87, 111]	113*	[-333, 98]
81-90	44	\mathbb{R}	-16	\mathbb{R}	56	\mathbb{R}	66	\mathbb{R}
91-00	103	[-617, 123]	4*	[-387, -20]	29	[-267, 36]	112*	[-995, -45]
00-10	14	[-124, 258]	57*	[-212, 5]	40	[-52, 204]	-3	[-554, 129]
$r_i - \iota_T \gamma_c = (\mathcal{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathcal{F} - \iota_T \gamma'_{\mathcal{F}}) b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n$								
PANEL B $\theta = (\gamma_0, \gamma'_{\mathcal{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{MOM}}), \quad \gamma_c$ partialled-out								
	MKT		SMB		HML		MOM	
$\times 10^{-4}$	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{HML}	\bar{F}	θ_{MOM}
61-70	38	\mathbb{R}	33	\mathbb{R}	53	\mathbb{R}	73	\mathbb{R}
71-80	30	[-654, 316]	43	[-53, 166]	33	[-143, 230]	113	[-499, 286]
81-90	44	\mathbb{R}	-16	\mathbb{R}	56	\mathbb{R}	66	\mathbb{R}
91-00	103	$]-\infty, 112]$ $\cup [1959, \infty[$	4	$]-\infty, -764]$ $\cup [-344, \infty[$	29	\mathbb{R}	112	\mathbb{R}
00-10	14	[-148, 1552]	57	[-1502, 75]	40	[-172, 227]	-3	[-1547, 108]

Note – See notes to table 5. The considered model is the four factor case with market (MKT), size (SMB), book-to-market (HML) and momentum (MOM).

4.2.2 Size portfolios, and conditional specifications

Size portfolios preserve some of the above findings, though globally, evidence weakens as identification is visibly weaker. The latter finding reinforces the argument in Lewellen et al. (2010, Footnote 1), namely that size and book-to-market sorted betas on MKT are close to one, a fact that seems to empirically endure since Fama and French (1993).

Notwithstanding almost inevitable resulting under-identification, table 9 broadly underscores the following, relative to industry portfolios. First, the restricted three-factor model is no longer rejected in the 70s and 90s; in fact it passes our test overall. Interestingly, the three factors are jointly priced in the 70s, whereas in the 90s, the only priced factor is MKT. Second, in all subperiods except the 80s and 2000s in which our confidence sets on MKT risk are the real line, MKT is priced. Third, prior to the 90s, HML is always priced. Fourth, the data is not informative post-2000s, a result shared to some extent with industry sorts.

The addition of momentum provokes under-identification since almost all confidence sets for risk prices are the real line, and so does unrestricting the intercept with and without momentum (the latter results are not reported for space consideration). The same holds when adding RMW and CMA, with

Table 7: Industry portfolios, Testing Traded Factor Assumption

$r_i - \mathcal{R}_1 = a_i \iota_T + \mathcal{R}_1 d_i + \mathcal{F} b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n,$									
$a_i = \gamma_c^* - \gamma_0 d_i - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}, \quad d_i = b_{i1} - 1, \quad \gamma_c^* = \gamma_c - \gamma_0$									
$\theta = (\gamma_0, \gamma'_{\mathcal{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}})$									
CTE		MKT		SMB		HML		MOM	
$\times 10^{-4}$	γ_c^*	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{HML}	\bar{F}	θ_{MOM}
61-70	[-8, 101]	38*	[-777, 9]	33	[-17, 94]	53	[-81, 82]	-	-
71-80	[5, 217]	30*	[-715, -35]	43	[-34, 178]	33	[-179, 173]	-	-
81-90	\mathbb{R}	44*	$]-\infty, -164[$ $\cup [707, \infty[$	-16	\mathbb{R}	56	\mathbb{R}	-	-
91-00	$]-\infty, -898[$ $\cup [150, \infty[$	103*	$]-\infty, -365[$ $\cup [1950, \infty[$	4*	$]-\infty, -964[$ $\cup [58, \infty[$	29	\mathbb{R}	-	-
00-10	[-1876, 115]	14	[-121, 2768]	57	[-2461, 84]	40	[-254, 245]	-	-
$r_i - \mathcal{R}_1 = a_i \iota_T + \mathcal{R}_1 d_i + \mathcal{F} b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n,$									
$a_i = \gamma_c^* - \gamma_0 d_i - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}, \quad d_i = b_{i1} - 1, \quad \gamma_c^* = \gamma_c - \gamma_0$									
$\theta = (\gamma_0, \gamma'_{\mathcal{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{MOM}})$									
CTE		MKT		SMB		HML		MOM	
$\times 10^{-4}$	γ_c^*	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{HML}	\bar{F}	θ_{MOM}
61-70	\mathbb{R}	38	\mathbb{R}	33	\mathbb{R}	53	\mathbb{R}	73	\mathbb{R}
71-80	[-13, 224]	30	[-803, 430]	43	[-62, 182]	33	[-186, 260]	113	[-586, 375]
81-90	\mathbb{R}	44	\mathbb{R}	-16	\mathbb{R}	56	\mathbb{R}	66	\mathbb{R}
91-00	$]-\infty, -714[$ $\cup [-111, \infty[$	103	$]-\infty, 157[$ $\cup [1590, \infty[$	4	$]-\infty, -600[$ $\cup [-407, \infty[$	29	\mathbb{R}	112	\mathbb{R}
00-10	[-3548, 135]	14	[-170, 5023]	57	[-4838, 94]	40	[-398, 323]	-3	[-4568, 129]

Note – See notes to tables 5 and 6. The inverted test is $\Lambda(\theta, \phi)$ is defined in (27). This test follows our **UAPT** where **U** stands for "unrestricted" implies that factors are assumed non-tradable in estimating and testing the model. The test is applied on $r_i - \mathcal{R}_1$ so inference on ϕ allows to assess whether $\gamma_c = \gamma_0$: the hypothesis that \mathcal{R}_1 (here, MKT) is traded is rejected at the 5% level when the confidence set on ϕ excludes zero.

Table 8: Confidence sets for risk price: Industry portfolios and five-factor model, excluding HML

$r_i - \iota_T \gamma_0 = (\mathcal{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathcal{F} - \iota_T \gamma'_{\mathcal{F}}) b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n$								
PANEL A $\theta = (\gamma_0, \gamma'_{\mathcal{F}})' = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{RMW}}, \theta_{\text{CMA}})'$								
	MKT		SMB		RMW		CMA	
$\times 10^{-4}$	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{RMW}	\bar{F}	θ_{CMA}
63-70	28	\mathbb{R}	57	\mathbb{R}	2	\mathbb{R}	22	\mathbb{R}
71-80	30*	$]-\infty, -79]$ $\cup [34937, \infty[$	54	$]-\infty, -1511]$ $\cup [-79, \infty[$	5*	$]-\infty, -20138]$ $\cup [14, \infty[$	25	$]-\infty, 210]$ $\cup [14260, \infty[$
81-90	44*	$]-\infty, -78]$ $\cup [82, \infty[$	-20	$]-\infty, -1]$ $\cup [42, \infty[$	39*	$]-\infty, -75]$ $\cup [95, \infty[$	55	$]-\infty, -143]$ $\cup [-6, \infty[$
91-00	103*	$[-1568, 85]$	3	$[-277, 237]$	32	$[-118, 362]$	30*	$[-591, -8]$
00-10	14	\mathbb{R}	65	\mathbb{R}	44	\mathbb{R}	35	\mathbb{R}
$r_i - \iota_T \gamma_c = (\mathcal{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathcal{F} - \iota_T \gamma'_{\mathcal{F}}) b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n$								
PANEL B $\theta = (\gamma_0, \gamma'_{\mathcal{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{RMW}}, \theta_{\text{CMA}})$, γ_c partialled-out								
	MKT		SMB		RMW		CMA	
$\times 10^{-4}$	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{RMW}	\bar{F}	θ_{CMA}
63-70	28	\mathbb{R}	57	\mathbb{R}	2	\mathbb{R}	22	\mathbb{R}
71-80	30*	$]-\infty, -26]$ $\cup [1108, \infty[$	54	$]-\infty, 167]$ $\cup [282, \infty[$	5	$]-\infty, -1041]$ $\cup [-118, \infty[$	25	$]-\infty, 202]$ $\cup [664, \infty[$
81-90	44	\mathbb{R}	-20	\mathbb{R}	39	\mathbb{R}	55	\mathbb{R}
91-00	103	$]-\infty, 105]$ $\cup [1790, \infty[$	3	$]-\infty, -662]$ $\cup [-238, \infty[$	32	\mathbb{R}	30	\mathbb{R}
00-10	14	\mathbb{R}	65	\mathbb{R}	44	\mathbb{R}	35	\mathbb{R}

Note – Sample includes monthly observations from July 1963 to December 2010 on the US. Series include 12 equally weighted (EW) industry portfolios as well as US factors for market (MKT), size (SMB), profitability (RMW), and investment (CMA). Results in Panel A rely on our **RAPT** approach, and those in Panel B on its **PAPT** counterpart; see notes to notes to table 5 for further definitions.

Table 9: Confidence sets for risk price: Size portfolios

$r_i - \iota_T \gamma_0 = (\mathcal{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathcal{F} - \iota_T \gamma'_{\mathcal{F}}) b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n$							
$\theta = (\gamma_0, \gamma'_{\mathcal{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}})$							
	MKT		SMB		HML		MOM
$\times 10^{-4}$	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{HML}	\bar{F} θ_{MOM}
61-70	38*	$]-\infty, -256]$ $\cup [221, \infty[$	33	$]-\infty, -87]$ $\cup \{-71, \infty\}$	53*	$]-\infty, -128]$ $\cup [1008, \infty[$	- -
71-80	30*	$]-\infty, -229]$ $\cup [25121, \infty[$	43*	$]-\infty, -253]$ $\cup [9972, \infty[$	33*	$]-\infty, -30714]$ $\cup [639, \infty[$	- -
81-90	44	\mathbb{R}	-16	\mathbb{R}	56*	$]-\infty, -881]$ $\cup [975, \infty[$	- -
91-00	103*	$[-14492, 72]$	4	$[-732, 649]$	29	$[-3055, 886]$	- -
00-10	14	\mathbb{R}	57	\mathbb{R}	40	\mathbb{R}	- -
$r_i - \iota_T \gamma_0 = (\mathcal{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathcal{F} - \iota_T \gamma'_{\mathcal{F}}) b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n$							
$\theta = (\gamma_0, \gamma'_{\mathcal{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{MOM}})$							
	MKT		SMB		HML		MOM
$\times 10^{-4}$	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{HML}	\bar{F} θ_{MOM}
61-70	38	\mathbb{R}	33	\mathbb{R}	53	\mathbb{R}	73 \mathbb{R}
71-80	30	\mathbb{R}	43	\mathbb{R}	33	\mathbb{R}	113 \mathbb{R}
81-90	44	\mathbb{R}	-16	\mathbb{R}	56	\mathbb{R}	66 \mathbb{R}
91-00	103	\mathbb{R}	4	\mathbb{R}	29	\mathbb{R}	112 $]-\infty, 3029]$ $\cup [5573, \infty[$
00-10	14	\mathbb{R}	57	\mathbb{R}	40	\mathbb{R}	3 \mathbb{R}

Note – Sample includes monthly observations from January 1991 to December 2010 on the US. Series include 25 size sorted equally weighted (EW) and value-weighted (VW) portfolios as well as US factors for market (MKT), size (SMB), book-to-market (HML) and momentum (MOM). See notes to tables 5 and 6 for further definitions and applied inference methods.

and without HML; a sample of these results is reported in the Appendix.

As a final robustness check, we examine conditional models estimated over the full sub-period, using the full set of industry and size portfolios, and the (standard) conditioning variables as in Beaulieu et al. (2007). Assuming all betas are time varying returned real lines, the same holds when each set of portfolios was used on its own which is not surprising, given the number of regressors to add relative to the sample size.

We report a sample of results in the Appendix assuming that the MKT beta varies as a function of the difference between the one-month lagged returns of a three-month and a one month. This sample is representative in the following sense: as the conditioning information changes, confidence sets jump from empty to severely unbounded a result we observed even with a single benchmark conditional model.

Two points are worth emphasizing from tables 16 and 17. First, because industry and size sorted portfolios are used jointly, the advantage of equal or value weights no longer prevail. This reinforces

our earlier findings in this regard. Second, the conditional model in question does not fare well, in view of its rejection with value-weighted portfolios. Given the extensive instruments search we experimented with leading to these tables, we do not aim to over-emphasize these results, aside from the following broad yet empirically important message. The identification problems in this literature are not restricted to unconditional asset pricing, and are not solved by standard conditioning which seems instead to exacerbate complications.¹³ Our findings thus endorse identification-robust methods for assessing whether candidate factors are associated with risk premiums.

5 Conclusion

One of the key goals of asset pricing is to identify factors that drive asset returns and are associated with risk premiums. This paper contributes to this literature via an identification-robust methodology to assess pricing, regardless of whether betas are jointly informative or not, or heterogenous enough to identify risk price, that is, to identify factors that represent a non diversifiable source of risk rather than an idiosyncratic association with returns.

As with Lewellen et al. (2010), our methodology is applied to models with a given and relatively small number of popular factors. The motivation [see Lewellen et al. (2010, Footnote 3)] may be traced to Fama and French (1993) whose main message is that relevant risks can be summarized by a small number of factors. Since then, the literature does not necessarily dispute this fact, in the sense that more is not necessarily viewed as better. Instead of a consensus view on a common set of explanatory factors, a plethora of different although related candidate factors has been proposed, which raises enduring empirical puzzles, statistical concerns and ultimately, spurious pricing considerations [Harvey et al. (2015)].

The main message in both strands of the literature reflected by Lewellen et al. (2010) (on analyzing models given a small number of given factors), or Harvey et al. (2015) (on factor searches globally) is that more stringent practices are needed. Our methodology serves this purpose by robustifying inference on risk price, controlling for the quality of available betas. Whether practice moves towards more parsimonious ways of summarizing information on factors, or towards reliance on test assets instead of test portfolios, our message is that statistical inference on risk price should not take identification for granted.

¹³For other perspectives on conditioning complications, see *e.g.* Boguth et al. (2011) and Penaranda and Sentana (2016).

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Appendix

.1 Eigenvalue based equations and inequations

Equation (48) may be re-expressed as

$$\Sigma_{11} + \Sigma_{12}\zeta = 0, \quad (\text{A.1})$$

$$\Sigma_{21} + \Sigma_{22}\zeta = 0, \quad (\text{A.2})$$

and solving (A.2) for ζ leads to (49). Substituting $\hat{\zeta}$ into (A.1) yields $\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} = 0$. Assuming that Σ_{11} is non-singular, on recalling that $\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$ is a scalar and using the formulae for the determinant of partitioned matrices

$$|\Sigma| = |\Sigma_{22}| |\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}| = |\Sigma_{22}| (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$$

we thus see that if $\hat{\zeta}$ satisfies (A.2) then it satisfies (A.1).

We next summarize the solution of (37) from Dufour and Taamouti (2005). Projections based confidence sets for any linear transformation of ζ of the form $\omega'\zeta$ can be obtained as follows. Let $\tilde{A} = -A_{22}^{-1}A'_{12}$, $\tilde{D} = A_{12}A_{22}^{-1}A_{12} - A_{11}$. If all the eigenvalues of A_{22} [as defined in (38)] are positive so A_{22} is positive definite then:

$$\text{CS}_\alpha(\omega'\zeta) = \left[\omega'\tilde{A} - \sqrt{\tilde{D}(\omega'A_{22}^{-1}\omega)}, \omega'\tilde{A} + \sqrt{\tilde{D}(\omega'A_{22}^{-1}\omega)} \right], \quad \text{if } \tilde{D} \geq 0 \quad (\text{A.3})$$

$$\text{CS}_\alpha(\omega'\zeta) = \emptyset, \quad \text{if } \tilde{D} < 0. \quad (\text{A.4})$$

If A_{22} is non-singular and has one negative eigenvalue then: (i) if $\omega'A_{22}^{-1}\omega < 0$ and $\tilde{D} < 0$:

$$\text{CS}_\alpha(\omega'\zeta) = \left] -\infty, \omega'\tilde{A} - \sqrt{\tilde{D}(\omega'A_{22}^{-1}\omega)} \right] \cup \left[\omega'\tilde{A} + \sqrt{\tilde{D}(\omega'A_{22}^{-1}\omega)}, +\infty \right[; \quad (\text{A.5})$$

(ii) if $\omega'A_{22}^{-1}\omega > 0$ or if $\omega'A_{22}^{-1}\omega \leq 0$ and $\tilde{D} \geq 0$ then:

$$\text{CS}_\alpha(\omega'\zeta) = \mathbb{R}; \quad (\text{A.6})$$

(iii) if $\omega'A_{22}^{-1}\omega = 0$ and $\tilde{D} < 0$ then:

$$\text{CS}_\alpha(\omega'\zeta) = \mathbb{R} \setminus \left\{ \omega'\tilde{A} \right\}. \quad (\text{A.7})$$

The projection is given by (A.6) if A_{22} is non-singular and has at least two negative eigenvalues.

.2 Proofs of Theorems

Proof of Theorem 1

Equations (A.3) - - (A.7) applied with A as defined in (39) imply that an unbounded solution to the problem of inverting the test defined by (28) and (34) would occur if A_{22} [refer to the partitioning in (38) and (40)] is not positive definite. In this case, the diagonal term of A_{22} is given by $\text{DIAG}(A_{22}) = (F_2 \dots F_k)'$ where

$$F_i = s_k[i]' \hat{B} \hat{S}^{-1} \hat{B}' s_k[i] - s_k[i]' (X'X)^{-1} s_k[i] \frac{n f_{n, \tau_n, \alpha}}{\tau_n}.$$

Clearly, if any of the Hotelling tests based on Λ_i , $i \in \{2, \dots, k\}$ [as in (32) and using the distribution in (34)] is not significant at level α , then by the definition of Λ_i and F_i , $\Lambda_i(\tau_n)/n < f_{n, \tau_n, \alpha} \Leftrightarrow F_i < 0$, in which case A_{22} cannot be positive definite. On comparing (39) and (41) we see that $\Lambda_i(\tau_{n-1})/(n-1) \geq f_{n-1, \tau_{n-1}, \alpha}$, $i \in \{2, \dots, k\}$ holds for the problem of inverting the test defined by (27) and (34) as a necessary but not sufficient condition to obtain bounded CSs. ■

Proof of the minimum distance computations in Theorems 2 and 3

$\min_{\theta} \Lambda(\theta)$ is the minimum root [denoted $\hat{\rho}$] of the determinantal equation (46) so the minimization problem can be cast as an equation of the (48) where $\zeta = \theta$ and

$$\Sigma = \hat{B} \hat{S}^{-1} \hat{B}' - \hat{\rho} (X'X)^{-1}, \quad (\text{A.8})$$

$\Sigma_{11} = \hat{a}' \hat{S}^{-1} \hat{a} - \hat{\gamma} x^{11}$, $\Sigma_{12} = \Sigma'_{21} = \hat{a}' \hat{S}^{-1} \hat{b}' - \hat{\gamma} x^{12}$, $\Sigma_{22} = \hat{b} \hat{S}^{-1} \hat{b}' - \hat{\gamma} x^{22}$, using the partitioning (18). So $\hat{\theta}_{\text{RAFT}}$ obtains applying (49) leading to (45).

Turning to $\Lambda(\theta, \phi)$, we have

$$\frac{\partial \Lambda(\theta, \phi)}{\partial \phi} = \frac{-2(1, \theta') \hat{B} \hat{S}^{-1} \iota_n + 2\phi \iota_n' \hat{S}^{-1} \iota_n}{(1, \theta') (X'X)^{-1} (1, \theta)'} \quad (\text{A.9})$$

and the (non-zero) value of ϕ which sets the latter partial derivative to zero is

$$\phi(\theta) = \frac{(1, \theta') \hat{B} \hat{S}^{-1} \iota_n}{\iota_n' \hat{S}^{-1} \iota_n}. \quad (\text{A.10})$$

Substituting $\phi(\theta)$ in (A.9) leads to

$$\Lambda(\theta, \phi(\theta)) = \frac{(1, \theta') \hat{B} \left[\hat{S}^{-1} - \hat{S}^{-1} \iota_n \left(\iota_n' \hat{S}^{-1} \iota_n \right)^{-1} \iota_n' \hat{S}^{-1} \right] \hat{B}' (1, \theta)'}{(1, \theta') (X'X)^{-1} (1, \theta)'}$$

which proves 50. From there on, $\min_{\theta, \phi} \Lambda(\theta, \phi)$ requires to solve a system of the (48) form with $\zeta = \theta$, and

$$\begin{aligned}\Sigma &= \hat{B} \left[\hat{S}^{-1} - \hat{S}^{-1} \iota_n \left(\iota_n' \hat{S}^{-1} \iota_n \right)^{-1} \iota_n' \hat{S}^{-1} \right] \hat{B}' - \hat{\nu} (X'X)^{-1} \\ \Sigma_{11} &= \hat{a}' \left[\hat{S}^{-1} - \hat{S}^{-1} \iota_n \left(\iota_n' \hat{S}^{-1} \iota_n \right)^{-1} \iota_n' \hat{S}^{-1} \right] \hat{a} - \hat{\nu} x^{11} \\ \Sigma_{12} &= \Sigma'_{21} = \hat{a}' \left[\hat{S}^{-1} - \hat{S}^{-1} \iota_n \left(\iota_n' \hat{S}^{-1} \iota_n \right)^{-1} \iota_n' \hat{S}^{-1} \right] \hat{b}' - \hat{\nu} x^{12} \\ \Sigma_{22} &= \hat{b}' \left[\hat{S}^{-1} - \hat{S}^{-1} \iota_n \left(\iota_n' \hat{S}^{-1} \iota_n \right)^{-1} \iota_n' \hat{S}^{-1} \right] \hat{b}' - \hat{\nu} x^{22}\end{aligned}$$

using the partitionings (18) and (19). So a point estimate for θ [denoted $\hat{\theta}_{\mathbf{UAPT}}$] obtains applying (49) leading to (52) and an point estimate for Φ thus follows using (A.10) leading to (54).

Proof of Theorem 5

Consider the following decomposition of $G' \hat{S} G$ and \tilde{S}_0 :

$$\begin{aligned}G' \hat{S} G &= G' J W' \mathcal{M}[X] W J' G \\ C \hat{B} G - D &= C (X'X)^{-1} X' [X B + W J'] G - D = C B G - D + C (X'X)^{-1} X' W J' G\end{aligned}$$

so under the null hypothesis $C \hat{B} G - D = C (X'X)^{-1} X' W J' G$ and

$$\begin{aligned}S(C, G, D) - G' \hat{S} G &= G' J W' \mathcal{M}_0[X, C] W J' G, \\ S(C, G, D) &= G' J W' (\mathcal{M}[X] + \mathcal{M}_0[X, C]) W J' G\end{aligned}$$

which implies that (65) corresponds to:

$$|G' J W' (\mathcal{M}_0[X, C]) W J' G - \lambda G' J W' (\mathcal{M}[X] + \mathcal{M}_0[X, C]) W J' G| = 0. \quad (\text{A.11})$$

Since J is invertible and G has full column rank, the singular value decomposition of $J'G$ gives

$$J'G = \mathcal{G} \Delta^{1/2} \Xi$$

where Δ is a g -dimensional diagonal matrix which includes the non-zero eigenvalues of $J'GG'J$, \mathcal{G} is the $n \times g$ matrix which includes the corresponding eigenvectors so $\mathcal{G}'\mathcal{G} = I_g$ and Ξ is the g -dimensional matrix $\Xi = \Delta^{-1/2} \mathcal{G}' J' G$ so that $\Xi \Xi' = I_g$. Replacing the latter expressions in (A.11) leads to (67). In particular, under assumption (55), (67) reduces to (69) where $\mathcal{Z} = Z\mathcal{G}$ and in view of (57), the rows of $Z\mathcal{G}$ are $\overset{i.i.d.}{\sim} N(0, I_g)$. It follows that the null distribution of all test statistics which depend on the data via the roots of (65) are invariant to B and J . When $G = I_n$, (A.11) takes the form

$$|J W' (\mathcal{M}_0[X, C]) W J' - \lambda J W' (\mathcal{M}[X] + \mathcal{M}_0[X, C]) W J'| = 0 \quad (\text{A.12})$$

which leads to (67) so B and J are evacuated out. ■

Proof of Theorem 4

Given $\tilde{H}(C, I_n, D)$ the sum of squared error ratio simplifies to

$$\begin{aligned} |\tilde{S}(C, G, D)|/|\hat{S}| &= \left| I_n + \hat{S}^{-1}(C\hat{B} - D)'[C(X'X)^{-1}C']^{-1}(C\hat{B} - D) \right| \\ &= \left| I_c + [C(X'X)^{-1}C']^{-1}(C\hat{B} - D)\hat{S}^{-1}(C\hat{B} - D)' \right| \end{aligned}$$

using a well known result on determinants, which leads to (66).¹⁴

.3 Further empirical results

This appendix reports further empirical results, for completion. In particular, we provide results with value-weighted industry sorts, size sorts, the Fama-French five factor model and a representative set of results using conditioning information.

Confidence sets for risk price: Industry portfolios and three factor model

$r_i - \iota_T \gamma_0 = (\mathcal{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathcal{F} - \iota_T \gamma'_{\mathcal{F}}) b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n$						
PANEL A $\theta = (\gamma_0, \gamma'_{\mathcal{F}})' = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}})'$						
VW	MKT		SMB		HML	
$\times 10^{-4}$	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{HML}
61-70	38*	[-612,-77]	33	[10,158]	53	[-27,54]
71-80	30	[-842,286]	43	[-44,302]	33	[-244,133]
81-90	44*	[75,259]	-16	[-78,0]	56*	[-25,24]
91-00	103	[-248, 206]	4*	[83, 596]	29	[-8,159]
00-10	14	[-60,260]	57	[-276,105]	40	[-66,85]
$r_i - \iota_T \gamma_c = (\mathcal{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathcal{F} - \iota_T \gamma'_{\mathcal{F}}) b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n$						
PANEL B $\theta = (\gamma_0, \gamma'_{\mathcal{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}), \quad \gamma_c$ partialled-out						
VW	MKT		SMB		HML	
$\times 10^{-4}$	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{HML}
61-70	38	[-655, 137]	33	[-60, 172]	53	[-18, 166]
71-80	30	[-625, 413]	43	[-83, 244]	33	[-179, 219]
81-90	44	[-454, 204]	-16	[-56, 200]	56	[-11, 186]
91-00	103	[-230, 197]	4*	[38, 673]	29	[-19,156]
00-10	14	[-73, 284]	57	[-281, 135]	40	[-61, 124]

Note – See notes to table 5; VW refers to value-weighted portfolios.

¹⁴For any $n \times m$ matrix S and any $m \times n$ matrix U , $|I_n + SU| = |I_m + US|$; see e.g. Harville (1997, section 18.1, p. 416).

Table 10: Confidence sets for risk price: Industry portfolios and four factor model

$r_i - \iota_T \gamma_0 = (\mathcal{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathcal{F} - \iota_T \gamma'_{\mathcal{F}}) b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n$								
$\theta = (\gamma_0, \gamma'_{\mathcal{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{MOM}})$								
VW	MKT		SMB		HML		MOM	
$\times 10^{-4}$	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{HML}	\bar{F}	θ_{MOM}
61-70	38	\mathbb{R}	33	\mathbb{R}	53	\mathbb{R}	73	\mathbb{R}
71-80	30	[-1933, 657]	43	[-138, 595]	33	[-422, 171]	113	[-175, 594]
81-90	44	[-203, 554]	-16	[-258, 81]	56	[-199, 53]	66	[-1030, 81]
91-00	103	[-253, 209]	4*	[81, 606]	29	[-37, 174]	112	[-212, 443]
00-10	14	\mathbb{R}	57	\mathbb{R}	40	\mathbb{R}	-3	\mathbb{R}

$r_i - \iota_T \gamma_c = (\mathcal{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathcal{F} - \iota_T \gamma'_{\mathcal{F}}) b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n$								
PANEL B $\theta = (\gamma_0, \gamma'_{\mathcal{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{MOM}}), \quad \gamma_c$ partialled-out								
VW	MKT		SMB		HML		MOM	
$\times 10^{-4}$	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{HML}	\bar{F}	θ_{MOM}
61-70	38	\mathbb{R}	33	\mathbb{R}	53	\mathbb{R}	73	\mathbb{R}
71-80	30	[-945, 3459]	43	[-928, 324]	33	[-227, 945]	113	[-1267, 366]
81-90	44	[-3856, 581]	-16	[-240, 1333]	56	[-178, 956]	66	[-2194, 70]
91-00	103	\mathbb{R}	4	\mathbb{R}	29	\mathbb{R}	112	\mathbb{R}
00-10	14	$]-\infty, 496[$ $\cup [1223, \infty[$	57	\mathbb{R}	40	$]-\infty, 136[$ $\cup [273, \infty[$	-3	$]-\infty, 734[$ $\cup [2344, \infty[$

Note – See notes to table 6; VW refers to value-weighted portfolios.

Table 11: Industry portfolios, Testing Traded Factor Assumption

$r_i - \mathcal{R}_1 = a_i \iota_T + \mathcal{R}_1 d_i + \mathcal{F} b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n,$									
$a_i = \gamma_c^* - \gamma_0 d_i - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}, \quad d_i = b_{i1} - 1, \quad \gamma_c^* = \gamma_c - \gamma_0$									
$\theta = (\gamma_0, \gamma'_{\mathcal{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}})$									
VW	CTE	MKT		SMB		HML		MOM	
$\times 10^{-4}$	γ_c^*	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{HML}	\bar{F}	θ_{MOM}
61-70	[-2, 16]	38	[-765, 157]	33	[-61, 193]	53	[-28, 173]	-	-
71-80	[-7, 16]	30	[-843, 510]	43	[-104, 302]	33	[-251, 250]	-	-
81-90	[-4, 40]	44	[-707, 274]	-16	[-83, 298]	56	[-31, 255]	-	-
91-00	[-49, 35]	103	[-249, 213]	4*	[28, 726]	29	[-24, 163]	-	-
00-10	[-6, 21]	14	[-82, 307]	57	[-315, 146]	40	[-68, 129]	-	-

$r_i - \mathcal{R}_1 = a_i \iota_T + \mathcal{R}_1 d_i + \mathcal{F} b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n,$									
$a_i = \gamma_c^* - \gamma_0 d_i - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}, \quad d_i = b_{i1} - 1, \quad \gamma_c^* = \gamma_c - \gamma_0$									
$\theta = (\gamma_0, \gamma'_{\mathcal{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{MOM}})$									
VW	CTE	MKT		SMB		HML		MOM	
$\times 10^{-4}$	γ_c^*	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{HML}	\bar{F}	θ_{MOM}
61-70	\mathbb{R}	38	\mathbb{R}	33	\mathbb{R}	53	\mathbb{R}	73	\mathbb{R}
71-80	\mathbb{R}	30	\mathbb{R}	43	\mathbb{R}	33	\mathbb{R}	113	\mathbb{R}
81-90	[-20, 960]	44	[-19988, 683]	-16	[-283, 6831]	56	[-212, 4618]	66	[-12992, 82]
91-00	\mathbb{R}	103	\mathbb{R}	4	\mathbb{R}	29	\mathbb{R}	112	\mathbb{R}
00-10	\mathbb{R}	14	\mathbb{R}	57	\mathbb{R}	40	\mathbb{R}	-3	\mathbb{R}

Note – See notes to table 7; VW refers to value-weighted portfolios.

Table 12: Confidence sets for risk price: Size portfolios

$r_i - \iota_T \gamma_0 = (\mathcal{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathcal{F} - \iota_T \gamma'_{\mathcal{F}}) b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n$							
$\theta = (\gamma_0, \gamma'_{\mathcal{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}})$							
VW	MKT		SMB		HML		MOM
$\times 10^{-4}$	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{HML}	\bar{F} θ_{MOM}
61-70	38*	$]-\infty, -755]$ $\cup [1716, \infty[$	33	$]-\infty, -260]$ $\cup [-31, \infty[$	53*	$]-\infty, -731]$ $\cup [2599, \infty[$	- -
71-80	30	\mathbb{R}	43	\mathbb{R}	33	\mathbb{R}	- -
81-90	44	\mathbb{R}	-16	\mathbb{R}	56	$]-\infty, 707]$ $\cup [818, \infty[$	- -
91-00	103*	$[-11236, -1030]$	4	$[-568, 1375]$	29	$[-5394, 730]$	- -
00-10	14*	$]-\infty, -1788]$ $\cup [1895, \infty[$	57	\mathbb{R}	40	\mathbb{R}	- -
$r_i - \iota_T \gamma_0 = (\mathcal{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathcal{F} - \iota_T \gamma'_{\mathcal{F}}) b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n$							
$\theta = (\gamma_0, \gamma'_{\mathcal{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{MOM}})$							
VW	MKT		SMB		HML		MOM
$\times 10^{-4}$	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{HML}	\bar{F} θ_{MOM}
61-70	38	\mathbb{R}	33	\mathbb{R}	53	\mathbb{R}	73 \mathbb{R}
71-80	30	\mathbb{R}	43	\mathbb{R}	33	\mathbb{R}	113 \mathbb{R}
81-90	44	\mathbb{R}	-16	\mathbb{R}	56	\mathbb{R}	66 \mathbb{R}
91-00	103	\mathbb{R}	4	\mathbb{R}	29	$]-\infty, 969]$ $\cup [3955, \infty[$	112 $]-\infty, 3167]$ $\cup [5035, \infty[$
00-10	14*	$]-\infty, -1761]$ $\cup [1261, \infty[$	57	\mathbb{R}	40	\mathbb{R}	-3 \mathbb{R}

Note – See notes to table 9; VW refers to value-weighted portfolios.

Table 13: Confidence sets for risk price: Industry portfolios and five-factor model

$r_i - \iota_T \gamma_0 = (\mathcal{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathcal{F} - \iota_T \gamma'_{\mathcal{F}}) b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n$										
PANEL A $\theta = (\gamma_0, \gamma'_{\mathcal{F}})' = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{RMW}}, \theta_{\text{CMA}})'$										
EW	MKT		SMB		HML		RMW		CMA	
$\times 10^{-4}$	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{HML}	\bar{F}	θ_{RMW}	\bar{F}	θ_{CMA}
63-70	28	\mathbb{R}	57	\mathbb{R}	41	\mathbb{R}	2	\mathbb{R}	22	\mathbb{R}
71-80	30	\mathbb{R}	54	\mathbb{R}	33	\mathbb{R}	5	\mathbb{R}	25	\mathbb{R}
81-90	44	$]-\infty, -53]$ $\cup [9, \infty[$	-20	\mathbb{R}	57	$]-\infty, -101]$ $\cup [49, \infty[$	39*	$]-\infty, -67]$ $\cup [42, \infty[$	55	\mathbb{R}
91-00	103	$[-2014, 116]$	3	$[-389, 255]$	27	$[-567, 88]$	32	$[-298, 368]$	30*	$[-927, -3]$
00-10	14	\mathbb{R}	65	\mathbb{R}	41	\mathbb{R}	44	\mathbb{R}	35	\mathbb{R}
Full	45	$[-289, 134]$	30*	$[-54, 14]$	40	$[-132, 79]$	26	$[15, 326]$	34	$[-532, 87]$
VW	MKT		SMB		HML		RMW		CMA	
$\times 10^{-4}$	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{HML}	\bar{F}	θ_{RMW}	\bar{F}	θ_{CMA}
63-70	28	\mathbb{R}	57	\mathbb{R}	41	\mathbb{R}	2	\mathbb{R}	22	\mathbb{R}
71-80	30	\mathbb{R}	54	\mathbb{R}	33	\mathbb{R}	5	\mathbb{R}	25	\mathbb{R}
81-90	44	$[-149, 787]$	-20	$[-360, 91]$	57	$[-76, 123]$	39	$[-362, 154]$	55	$[-127, 217]$
91-00	103	$[-242, 1081]$	3*	$[76, 1641]$	27	$[-98, 167]$	32	$[-596, 48]$	30	$[4, 893]$
00-10	14	\mathbb{R}	65	\mathbb{R}	41	\mathbb{R}	44	\mathbb{R}	35	\mathbb{R}
Full	45	$[-92, 546]$	30	$[-13, 358]$	40	$[-25, 110]$	26	$[-446, 31]$	34	$[-31, 768]$
$r_i - \iota_T \gamma_c = (\mathcal{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathcal{F} - \iota_T \gamma'_{\mathcal{F}}) b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n$										
PANEL B $\theta = (\gamma_0, \gamma'_{\mathcal{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{RMW}}, \theta_{\text{CMA}})$, γ_c partialled-out										
EW	MKT		SMB		HML		RMW		CMA	
$\times 10^{-4}$	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{HML}	\bar{F}	θ_{RMW}	\bar{F}	θ_{CMA}
63-70	28	\mathbb{R}	57	\mathbb{R}	41	\mathbb{R}	2	\mathbb{R}	22	\mathbb{R}
71-80	30	\mathbb{R}	54	\mathbb{R}	33	\mathbb{R}	5	\mathbb{R}	25	\mathbb{R}
81-90	44	\mathbb{R}	-20	\mathbb{R}	57	\mathbb{R}	39	\mathbb{R}	55	\mathbb{R}
91-00	103	$]-\infty, 134]$ $\cup [1473, \infty[$	3	$]-\infty, -434]$ $\cup [-337, \infty[$	27	\mathbb{R}	32	\mathbb{R}	30	\mathbb{R}
00-10	14	\mathbb{R}	65	\mathbb{R}	41	\mathbb{R}	44	\mathbb{R}	35	\mathbb{R}
Full	45	$[-219, 136]$	30	$[-71, 138]$	40	$[-103, 243]$	26	$[-179, 264]$	34	$[-373, 369]$
VW	MKT		SMB		HML		RMW		CMA	
$\times 10^{-4}$	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{HML}	\bar{F}	θ_{RMW}	\bar{F}	θ_{CMA}
63-70	28	\mathbb{R}	57	\mathbb{R}	41	\mathbb{R}	2	\mathbb{R}	22	\mathbb{R}
71-80	30	\mathbb{R}	54	\mathbb{R}	33	\mathbb{R}	5	\mathbb{R}	25	\mathbb{R}
81-90	44	\mathbb{R}	-20	\mathbb{R}	57	\mathbb{R}	39	\mathbb{R}	55	\mathbb{R}
91-00	103	\mathbb{R}	3	\mathbb{R}	27	\mathbb{R}	32	\mathbb{R}	30	\mathbb{R}
00-10	14	\mathbb{R}	65	\mathbb{R}	41	\mathbb{R}	44	\mathbb{R}	35	\mathbb{R}
Full	45	\mathbb{R}	30	\mathbb{R}	40	\mathbb{R}	26	\mathbb{R}	34	\mathbb{R}

Note – Sample includes monthly observations from July 1963 to December 2010 on the US. Series include 12 equally weighted (EW) and value-weighted (VW) industry portfolios as well as US factors for market (MKT), size (SMB), book-to-market (HML), profitability (RMW), and investment (CMA). Confidence sets are at the 5% level. \bar{F} is the factor average over the considered time period; θ captures factor pricing. * denotes evidence of pricing at the 5% significance level interpreted as follows: given the reported confidence sets, each factor is priced if its average is not covered.

Table 14: Confidence sets for risk price: Industry portfolios and five-factor model, excluding HML

$r_i - \iota_T \gamma_0 = (\mathcal{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathcal{F} - \iota_T \gamma'_\mathcal{F}) b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n$								
PANEL A $\theta = (\gamma_0, \gamma'_\mathcal{F})' = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{RMW}}, \theta_{\text{CMA}})'$								
VW	MKT		SMB		RMW		CMA	
$\times 10^{-4}$	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{RMW}	\bar{F}	θ_{CMA}
63-70	28	[-126, 400]	57	[-113, 60]	2	[-101, 50]	22	[-50, 62]
71-80	30	\mathbb{R}	54	\mathbb{R}	5	\mathbb{R}	25	\mathbb{R}
81-90	44	[-141, 645]	-20	[-252, 75]	39	[-220, 126]	55	[-107, 122]
91-00	103	[-236, 594]	3*	[78, 1099]	32	[-370, 46]	30	[27, 477]
00-10	14	\mathbb{R}	65	\mathbb{R}	44	\mathbb{R}	35	\mathbb{R}
Full	45	[-91, 147]	30	[-13, 152]	26*	[-97, 20]	34	[12, 129]
$r_i - \iota_T \gamma_c = (\mathcal{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathcal{F} - \iota_T \gamma'_\mathcal{F}) b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n$								
PANEL B $\theta = (\gamma_0, \gamma'_\mathcal{F}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{RMW}}, \theta_{\text{CMA}}), \quad \gamma_c$ partialled-out								
VW	MKT		SMB		RMW		CMA	
$\times 10^{-4}$	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{RMW}	\bar{F}	θ_{CMA}
63-70	28	[-387, 572]	57	[-131, 285]	2	[-102, 229]	22	[-32, 407]
71-80	30	\mathbb{R}	54	\mathbb{R}	5	\mathbb{R}	25	\mathbb{R}
81-90	44	[-3897, 3713]	-20	[-1601, 1721]	39	[-1327, 1065]	55	[-1664, 1968]
91-00	103	[-221, 6116]	3*	[11, 12025]	32	[-7807, 151]	30	[13, 4639]
00-10	14	\mathbb{R}	65	\mathbb{R}	44	\mathbb{R}	35	\mathbb{R}
Full	45	[-104, 145]	30	[-18, 153]	26	[-93, 46]	34	[5, 127]

Note – See notes to table 8; VW refers to value-weighted portfolios.

Table 15: Industry portfolios, Five-factor model, Testing Traded Factor Assumption

$r_i - \mathcal{R}_1 = a_i \iota_T + \mathcal{R}_1 d_i + \mathcal{F} b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n,$							
$a_i = \gamma_c^* - \gamma_0 d_i - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}, \quad d_i = b_{i1} - 1, \quad \gamma_c^* = \gamma_c - \gamma_0$							
$\theta = (\gamma_0, \gamma'_{\mathcal{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{RMW}}, \theta_{\text{CMA}})$							
EW	CTE	MKT		SMB		RMW	CMA
$\times 10^{-4}$	γ_c^*	\bar{F}	θ_{MKT}	\bar{F}	θ_{SMB}	\bar{F}	θ_{CMA}
63-70	\mathbb{R}	28	\mathbb{R}	57	\mathbb{R}	2	\mathbb{R}
71-80	\mathbb{R}	30	\mathbb{R}	54	\mathbb{R}	5	\mathbb{R}
81-90	\mathbb{R}	44	\mathbb{R}	-20	\mathbb{R}	39	\mathbb{R}
91-00	$]-\infty, -486[$ $\cup]-168, \infty[$	103	$]-\infty, 165[$ $\cup]1266, \infty[$	3	\mathbb{R}	32	\mathbb{R}
00-10	\mathbb{R}	14	\mathbb{R}	65	\mathbb{R}	44	\mathbb{R}
VW	CTE	MKT		SMB		RMW	CMA
$\times 10^{-4}$	γ_c^*	θ_{MKT}		θ_{SMB}		θ_{RMW}	θ_{CMA}
63-70	\mathbb{R}	\mathbb{R}		\mathbb{R}		\mathbb{R}	\mathbb{R}
71-80	\mathbb{R}	\mathbb{R}		\mathbb{R}		\mathbb{R}	\mathbb{R}
81-90	\mathbb{R}	\mathbb{R}		\mathbb{R}		\mathbb{R}	\mathbb{R}
91-00	\mathbb{R}	\mathbb{R}		\mathbb{R}		\mathbb{R}	\mathbb{R}
00-10	\mathbb{R}	\mathbb{R}		\mathbb{R}		\mathbb{R}	\mathbb{R}
$r_i - \mathcal{R}_1 = a_i \iota_T + \mathcal{R}_1 d_i + \mathcal{F} b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n,$							
$a_i = \gamma_c^* - \gamma_0 d_i - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}, \quad d_i = b_{i1} - 1, \quad \gamma_c^* = \gamma_c - \gamma_0$							
$\theta = (\gamma_0, \gamma'_{\mathcal{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{RMW}}, \theta_{\text{CMA}})$							
EW	CTE	MKT		SMB		RMW	CMA
$\times 10^{-4}$	γ_c^*	θ_{MKT}		θ_{SMB}		θ_{RMW}	θ_{CMA}
63-70	\mathbb{R}	\mathbb{R}		\mathbb{R}		\mathbb{R}	\mathbb{R}
71-80	\mathbb{R}	$]-\infty, 29[$ *		\mathbb{R}		$]-\infty, -449[$ $\cup]-171, \infty[$	$]-\infty, 231[$ $\cup]335, \infty[$
81-90	\mathbb{R}	\mathbb{R}		\mathbb{R}		\mathbb{R}	\mathbb{R}
91-00	$]-\infty, -568[$ $\cup]-154, \infty[$	$]-\infty, 135[$ $\cup]1495, \infty[$		$]-\infty, -520[$ $\cup]-277, \infty[$		\mathbb{R}	\mathbb{R}
00-10	\mathbb{R}	\mathbb{R}		\mathbb{R}		\mathbb{R}	\mathbb{R}
VW	CTE	MKT		SMB		RMW	CMA
$\times 10^{-4}$	γ_c^*	θ_{MKT}		θ_{SMB}		θ_{RMW}	θ_{CMA}
63-70	$[-2, 62]$	$[-461, 1094]$		$[-152, 507]$		$[-118, 580]$	$[-50, 863]$
71-80	\mathbb{R}	\mathbb{R}		\mathbb{R}		\mathbb{R}	\mathbb{R}
81-90	\mathbb{R}	\mathbb{R}		\mathbb{R}		\mathbb{R}	\mathbb{R}
91-00	$]-\infty, 47[$ $\cup]465, \infty[$	$]-\infty, -2123[$ $\cup]-239, \infty[$		$]-\infty, -3915[$ $\cup]-10, \infty[$		$]-\infty, 168[$ $\cup]2722, \infty[$	$]-\infty, -1493[$ $\cup]4, \infty[$
00-10	\mathbb{R}	\mathbb{R}		\mathbb{R}		\mathbb{R}	\mathbb{R}

Note – The inverted test is $\Lambda(\theta, \phi)$ is defined in (27). This test follows our **UAPT** where **U** stands for "unrestricted" implies that factors are assumed non-tradable in estimating and testing the model. The test is applied on $r_i - \mathcal{R}_1$ so inference on ϕ allows to assess whether $\gamma_c = \gamma_0$: the hypothesis that \mathcal{R}_1 (here, MKT) is traded is rejected at the 5% level when the confidence set on ϕ excludes zero. In the upper Panel of this table, all confidence sets on the HML price are the real line; the lower Panel excludes HML.

Table 16: Confidence sets for risk price: Industry and size portfolios, Five-factor model, Instrumenting MKT with: LagTBill31

$r_i - \iota_T \gamma_0 = (\mathcal{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathcal{F} - \iota_T \gamma'_{\mathcal{F}}) b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n$						
PANEL A $\theta = (\gamma_0, \gamma'_{\mathcal{F}})' = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{RMW}}, \theta_{\text{CMA}})'$						
EW	MKT	SMB	HML	RMW	CMA	
$\times 10^{-4}$	θ_{MKT}	θ_{SMB}	θ_{HML}	θ_{RMW}	θ_{CMA}	
Full	[-1496, 232]	[-265, 29]*	[-577, 377]	[-354, 934]	[-2391, 197]	
VW	MKT	SMB	HML	RMW	CMA	
$\times 10^{-4}$	$\bar{F} \theta_{\text{MKT}}$	$\bar{F} \theta_{\text{SMB}}$	$\bar{F} \theta_{\text{HML}}$	$\bar{F} \theta_{\text{RMW}}$	$\bar{F} \theta_{\text{CMA}}$	
Full	45 \emptyset	30 \emptyset	40 \emptyset	26 \emptyset	34 \emptyset	
$r_i - \iota_T \gamma_c = (\mathcal{R}_1 - \iota_T \gamma_0) b_{i1} + (\mathcal{F} - \iota_T \gamma'_{\mathcal{F}}) b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n$						
PANEL B $\theta = (\gamma_0, \gamma'_{\mathcal{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{RMW}}, \theta_{\text{CMA}}), \quad \gamma_c$ partialled-out						
EW	MKT	SMB	HML	RMW	CMA	
$\times 10^{-4}$	θ_{MKT}	θ_{SMB}	θ_{HML}	θ_{RMW}	θ_{CMA}	
Full	$]-\infty, 268]$ $\cup [7594, \infty[$	$]-\infty, 30]$ $\cup [3219, \infty[$	$]-\infty, 337]$ $\cup [7422, \infty[$	$]-\infty, -10725]$ $\cup [-314, \infty[$	$]-\infty, 174]$ $\cup [23510, \infty[$	
VW	MKT	SMB	HML	RMW	CMA	
$\times 10^{-4}$	$\bar{F} \theta_{\text{MKT}}$	$\bar{F} \theta_{\text{SMB}}$	$\bar{F} \theta_{\text{HML}}$	$\bar{F} \theta_{\text{RMW}}$	$\bar{F} \theta_{\text{CMA}}$	
Full	45 \emptyset	30 \emptyset	40 \emptyset	26 \emptyset	34 \emptyset	

Note – Sample includes monthly observations from July 1963 to December 2010 on the US. Series include 37 equally weighted (EW) and value-weighted (VW) industry and size portfolios as well as US factors for market (MKT), size (SMB), book-to-market (HML), profitability (RMW), and investment (CMA).

Table 17: Industry and size portfolios, Five-factor model, Instrumenting MKT with: LagTBill31, Testing Traded Factor Assumption

$r_i - \mathcal{R}_1 = a_i \iota_T + \mathcal{R}_1 d_i + \mathcal{F} b_{i\mathcal{F}} + u_i, \quad i = 1, \dots, n,$						
$a_i = \gamma_c^* - \gamma_0 d_i - \gamma'_{\mathcal{F}} b_{i\mathcal{F}}, \quad d_i = b_{i1} - 1, \quad \gamma_c^* = \gamma_c - \gamma_0$						
$\theta = (\gamma_0, \gamma'_{\mathcal{F}}) = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{RMW}}, \theta_{\text{CMA}})$						
EW	CTE	MKT	SMB	HML	RMW	CMA
$\times 10^{-4}$	γ_c^*	θ_{MKT}	θ_{SMB}	θ_{HML}	θ_{RMW}	θ_{CMA}
Full	$]-\infty, 75]$ $\cup [762, \infty[$	$]-\infty, 411]$ $\cup [1775, \infty[$	$]-\infty, 37]$ $\cup [923, \infty[$	$]-\infty, 378]$ $\cup [2172, \infty[$	$]-\infty, -3201]$ $\cup [-355, \infty[$	$]-\infty, 225]$ $\cup [7036, \infty[$
VW	CTE	MKT	SMB	HML	RMW	CMA
$\times 10^{-4}$	γ_c^*	$\bar{F} \theta_{\text{MKT}}$	$\bar{F} \theta_{\text{SMB}}$	$\bar{F} \theta_{\text{HML}}$	$\bar{F} \theta_{\text{RMW}}$	$\bar{F} \theta_{\text{CMA}}$
Full	\emptyset	45 \emptyset	30 \emptyset	40 \emptyset	26 \emptyset	34 \emptyset

Note – The inverted test is $\Lambda(\theta, \phi)$ is defined in (27). This test follows our **UAPT** where **U** stands for "unrestricted" implies that factors are assumed non-tradable in estimating and testing the model. The test is applied on $r_i - \mathcal{R}_1$ so inference on ϕ allows to assess whether $\gamma_c = \gamma_0$: the hypothesis that \mathcal{R}_1 (here, MKT) is traded is rejected at the 5% level when the confidence set on ϕ excludes zero.