A Class of Time-Varying Parameter Structural VARs for Inference under Exact or Set Identification

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Structural Vector Autoregression

$$\underset{(1\times n)}{\mathbf{y}_t'}\underset{(n\times n)}{\mathbf{A}} = \mathbf{y}_{t-1}'\underset{(n\times n)}{\mathbf{F}_1} + \cdots + \mathbf{y}_{t-p}'\mathbf{F}_p + \underset{(1\times n)}{\mathbf{c}} + \underset{(1\times n)}{\boldsymbol{\varepsilon}_t'}, \quad \boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \mathbf{I}_n)$$

Define

$$\underset{(1\times m)}{\mathbf{x}_t'} \equiv [\mathbf{y}_{t-1}', \ldots, \mathbf{y}_{t-p}', 1] \quad \text{and} \quad \underset{(m\times n)}{\mathbf{F}} \equiv \begin{bmatrix} \mathbf{F}_1 \\ \vdots \\ \mathbf{F}_p \\ \mathbf{c} \end{bmatrix}$$

Write

$$\mathbf{y}_t'\mathbf{A} = \mathbf{x}_t'\mathbf{F} + \boldsymbol{\varepsilon}_t'$$

Goal is to infer $(\mathbf{A}, \mathbf{F})$ because they:

- represent equilibrium relationships between variables
- determine response of $\mathbf{y}_{t+h}$ to the "structural" shocks $\boldsymbol{\varepsilon}_t$

But $(\mathbf{A}, \mathbf{F})$ don't come for free.

The identification problem

Rewriting the SVAR

$$\mathbf{y}_t' = \mathbf{x}_t'\mathbf{F}\mathbf{A}^{-1} + \varepsilon_t'\mathbf{A}^{-1}, \quad \boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \mathbf{I}_n)$$

Likelihood for $\mathbf{y}_t$

$$p(\mathbf{y}_t|\mathbf{A}, \mathbf{F}, \mathbf{x}_t) = Npdf(\mathbf{y}_t|\mathbf{x}_t' \underbrace{\mathbf{F}\mathbf{A}^{-1}}_{\mathbf{B}}, (\underbrace{\mathbf{A}\mathbf{A}'}_{\mathbf{H}})^{-1})$$

Consider an alternative parameter point

$$(\widetilde{\mathbf{A}}, \widetilde{\mathbf{F}}) = (\mathbf{AQ}, \mathbf{FQ}) \quad \text{for } \mathbf{Q} \in \mathcal{O}_n$$

Key likelihood terms at the alternative parameter point:

$$\widetilde{\mathbf{B}} = \widetilde{\mathbf{F}}\widetilde{\mathbf{A}}^{-1} = \mathbf{FQ}(\mathbf{AQ})^{-1} = \mathbf{FQQ}^{-1}\mathbf{A}^{-1} = \mathbf{F}\mathbf{A}^{-1} = \mathbf{B}$$

$$\widetilde{\mathbf{H}} = \widetilde{\mathbf{A}}\widetilde{\mathbf{A}}' = (\mathbf{AQ})(\mathbf{AQ})' = \mathbf{AQQ}'\mathbf{A}' = \mathbf{A}\mathbf{A}' = \mathbf{H}$$

Hence, $p(\mathbf{y}_t|\mathbf{A}, \mathbf{F}, \mathbf{y}_{t-p:t-1}) = p(\mathbf{y}_t|\widetilde{\mathbf{A}}, \widetilde{\mathbf{F}}, \mathbf{y}_{t-p:t-1})$

Reduced-form VAR

Evidently **cannot** identify $(\mathbf{A}, \mathbf{F})$

Can identify

$$g(\mathbf{A}, \mathbf{F}) = (\mathbf{F}\mathbf{A}^{-1}, \mathbf{A}\mathbf{A}') = (\mathbf{B}, \boldsymbol{\Sigma})$$

$$\mathbf{y}_t' = \mathbf{x}_t\mathbf{B} + \mathbf{u}_t', \quad \mathbf{u}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma})$$

Key practical feature:

- Easy to estimate $(\mathbf{B}, \boldsymbol{\Sigma})$

Key drawback:

- $(\boldsymbol{\Sigma}, \mathbf{B})$ are not $(\mathbf{A}, \mathbf{F})$

SVAR approaches

Traditional approaches to inference in SVARs:

1. Estimate $(\boldsymbol{\Sigma}, \mathbf{B})$
2. Choose a 1-to-1 mapping between $(\boldsymbol{\Sigma}, \mathbf{B})$ and $(\mathbf{A}, \mathbf{F})$ e.g.
    - Ordering restrictions (aka short-run restrictions).
    - Long-run restrictions.

The literature since then:

1. Set identification (with **static** VAR parameters):
    - Canova and de Nicolo (2002)
    - Uhlig (2005)
2. Coefficients that change (with **exact** identification)
    - Cogley and Sargent (2005)
    - Primiceri (2005)
    - Sims and Zha (2006), Sims, Waggoner and Zha (2008)

Not obvious how to coherently combine 1 and 2.

Motivating example: Set ID and TVP

- Based on Baumeister and Peersman (2013, AEJ Macro)
- $\mathbf{y}_t = [\Delta p_t^{oil}, \Delta q_t^{oil}, \Delta GDP_t, \Delta p_t^{CPI}]'$
- Identify time-varying IRFs of oil supply shocks

Their method:

1. Estimate Primiceri (2005) VAR-TVP-SV
2. Reassemble into "reduced-form VAR" parameters t-by-t
3. Find structural parameters satisfying sign-restrictions

$$\varepsilon_t^{oil,s} < 0 \Rightarrow \Delta q_{t+h}^{oil} < 0 < \Delta p_{t+h}^{oil} \quad \text{for } h = 0, ..., 4$$

4. RRWZ "algorithm" applied to "reduced-form" parameters t-by-t.

"Reduced-form"

Primiceri (2005)

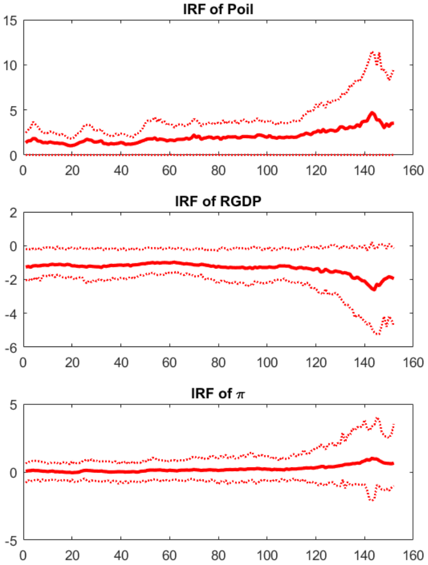
$$\mathbf{y}_t' = vec(\mathbf{B}_t)'(\mathbf{I}_n \otimes \mathbf{x}_t) + \varepsilon_t' \boldsymbol{\Xi}_t \boldsymbol{\Delta}_t^{-1}$$

where

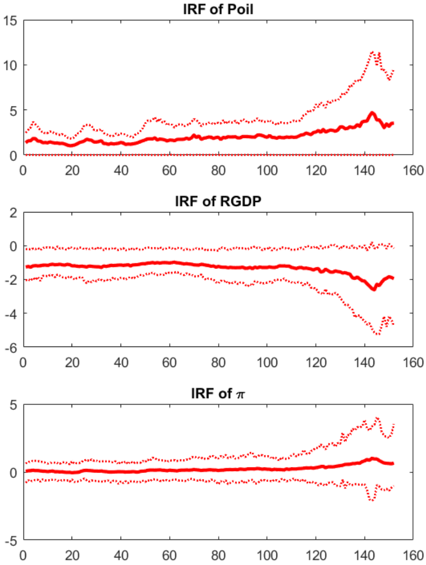
$$\boldsymbol{\Xi}_t = \begin{bmatrix} \xi_{1,t} & 0 & \cdots & 0 \\ 0 & \xi_{2,t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \xi_{n,t} \end{bmatrix}, \quad \boldsymbol{\Delta}_t = \begin{bmatrix} 1 & \delta_{12,t} & \cdots & \delta_{1n,t} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \delta_{n-1n,t} \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

and

$$\begin{aligned} \boldsymbol{\Xi}_t &= \boldsymbol{\Xi}_{t-1}\, diag(\exp(\boldsymbol{\eta}_t)), & \boldsymbol{\eta}_t &\sim N(\mathbf{0}_{n\times 1}, \boldsymbol{\Sigma}_{\boldsymbol{\eta}}) \\ \boldsymbol{\delta}_t &= \boldsymbol{\delta}_{t-1} + \boldsymbol{\zeta}_t, & \boldsymbol{\zeta}_t &\sim N(\mathbf{0}_{\frac{n(n-1)}{2}\times 1}, \boldsymbol{\Sigma}_{\boldsymbol{\zeta}}) \\ vec(\mathbf{B}_t) &= vec(\mathbf{B}_{t-1}) + \boldsymbol{\upsilon}_t, & \boldsymbol{\upsilon}_t &\sim N(\mathbf{0}_{mn\times 1}, \boldsymbol{\Sigma}_{\boldsymbol{\upsilon}}) \end{aligned}$$



- Supply shock causing $\Delta q^{oil} = -1\%$.
- "baseline" IRFs
- x-axis: time in quarters
- p_t^{oil} IRF: contemporaneous response at each t
- GDP_t and Δp_t IRFs: cumulative change over 4 quarters at each t



- Supply shock causing $\Delta q^{oil} = -1\%$.
- "baseline" IRFs
- Finding: oil demand has become increasingly inelastic

A Motivating Example

- Based on Baumeister and Peersman (2013, AEJ Macro)
- $\mathbf{y}_t = [\Delta p_t^{oil}, \Delta q_t^{oil}, \Delta GDP_t, \Delta p_t^{CPI}]'$
- Identify time-varying IRFs of oil supply shocks

The method:

- Estimate Primiceri (2005) VAR-TVP-SV
- Reassemble into "reduced-form VAR" parameters t-by-t
- Find structural parameters satisfying sign-restrictions

$$\varepsilon_t^{oil,s} < 0 \Rightarrow \Delta q_{t+h}^{oil} < 0 < \Delta p_{t+h}^{oil} \quad \text{for } h = 0, ..., 4$$

- RRWZ "algorithm"

A Motivating Example Revisited

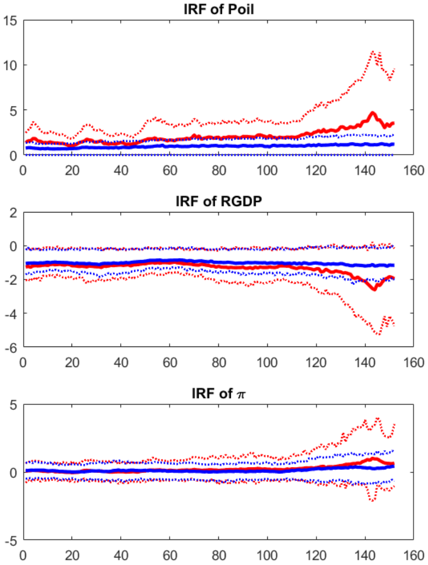
- Based on Baumeister and Peersman (2013, AEJ Macro)
- ~~$\mathbf{y}_t = [\Delta p_t^{oil}, \Delta q_t^{oil}, \Delta GDP_t, \Delta p_t^{CPI}]'$~~
  $\mathbf{y}_t = [\Delta p_t^{CPI}, \Delta GDP_t, \Delta q_t^{oil}, \Delta p_t^{oil}]'$
- Identify time-varying IRFs of oil supply shocks

The method:

- Estimate Primiceri (2005) VAR-TVP-SV
- Reassemble into "reduced-form VAR" parameters t-by-t
- Find structural parameters satisfying sign-restrictions

$$\varepsilon_t^{oil,s} < 0 \Rightarrow \Delta q_{t+h}^{oil} < 0 < \Delta p_{t+h}^{oil} \quad \text{for } h = 0, ..., 4$$

- RRWZ "algorithm"



- Supply shock causing $\Delta q^{oil} = -1\%$.
- "baseline" IRFs
- IRFs under alternative variable ordering

- **Time-variation in IRFs is gone!**
- **Would have been a different paper!**

Takeaway from the exercise

- **Not** that Baumeister Peersman are "wrong."
  (Indeed, I will find something similar them).

But

- Methodologically, the BP method is deeply problematic.

What goes wrong?

- "reduced-form" estimates sensitive to variable ordering.
- Spills over into any inference based on the "reduced-form"

Two key implications

1. Results can be driven as much by an unacknowledged modeling choice (variable ordering) as by the explicit identifying assumptions.
2. $n!$ different candidate reduced-forms.

Examining the posterior I

- Let $\mathbf{S}_t = (\mathbf{A}_t, \mathbf{F}_t)$ and $\mathbf{S}_t * \mathbf{Q}_t = (\mathbf{A}_t\mathbf{Q}_t, \mathbf{F}_t\mathbf{Q}_t)$

$$p(\phi, \mathbf{S}_{0:T}|\mathbf{y}_{1:T}) \propto \underbrace{p(\phi, \mathbf{S}_0)}_{\text{prior}} \underbrace{p(\mathbf{S}_{1:T}|\phi, \mathbf{S}_0)}_{\substack{\text{density of the } \mathbf{S}_{1:T} \\ \text{sequence under the} \\ \text{model's law of motion}}} \underbrace{p(\mathbf{y}_{1:T}|\phi, \mathbf{S}_0, \mathbf{S}_{1:T})}_{\text{data density given } \mathbf{S}_{0:T}}$$

where

$$\begin{aligned} p(\mathbf{y}_{1:T}|\phi, \mathbf{S}_0, \mathbf{S}_{1:T}) &= \prod_{t=1}^{T} p(\mathbf{y}_t|\mathbf{y}_{t-p:t-1}, \mathbf{S}_t) \\ &= \prod_{t=1}^{T} \underbrace{Npdf(\mathbf{y}_t|\mathbf{x}_t'\mathbf{F}_t\mathbf{A}_t^{-1},\ (\mathbf{A}_t\mathbf{A}_t')^{-1})}_{\substack{\text{invariant to multiplication by } \mathbf{Q}_t, \\ \text{just like the constant-parameter case}}} \\ &= \prod_{t=1}^{T} p(\mathbf{y}_t|\mathbf{y}_{t-p:t-1}, \mathbf{S}_t * \mathbf{Q}_t) \end{aligned}$$

Examining the posterior II

$$p(\phi, \mathbf{S}_{0:T} | \mathbf{y}_{1:T}) \propto \underbrace{p(\phi, \mathbf{S}_0)}_{\text{prior}} \underbrace{p(\mathbf{S}_{1:T} | \phi, \mathbf{S}_0)}_{\substack{\text{density of the } \mathbf{S}_{1:T} \\ \text{sequence under the} \\ \text{model's law of motion}}} \underbrace{p(\mathbf{y}_{1:T} | \phi, \mathbf{S}_0, \mathbf{S}_{1:T})}_{\text{data density given } \mathbf{S}_{0:T}}$$

where

$$p(\mathbf{S}_{1:T} | \phi, \mathbf{S}_0) = \prod_{t=1}^{T} p(\mathbf{S}_t | \phi, \mathbf{S}_{t-1})$$

(This is the tricky part.)

TVP density in Primiceri (2005)

Consider the 2×2 example with no lags.

$$\boldsymbol{\Sigma}_t = \boldsymbol{\Delta}_t^{-1'} \boldsymbol{\Xi}_t \boldsymbol{\Xi}_t \boldsymbol{\Delta}_t^{-1} = \begin{bmatrix} \xi_{1,t}^2 & -\delta_{1,t}\xi_{1,t}^2 \\ -\delta_{1,t}\xi_{1,t}^2 & \delta_{1,t}^2\xi_{1,t}^2 + \xi_{2,t}^2 \end{bmatrix}$$

Hence,

$$\boldsymbol{\Sigma}_{t,(1,1)} = \xi_{1,t}^2 = (\xi_{1,t-1} \underbrace{\exp(\boldsymbol{\eta}_{1,t})}_{\sim \text{Lognormal}})^2 \sim \text{Lognormal}$$

$$\begin{aligned} \boldsymbol{\Sigma}_{t,(2,2)} &= \delta_{1,t}^2\xi_{1,t}^2 + \xi_{2,t}^2 \\ &= \underbrace{(\delta_{1,t-1} + \boldsymbol{\zeta}_{1,t})^2}_{\sim \text{scaled noncentral } \chi^2} \underbrace{(\xi_{1,t-1}\exp(\boldsymbol{\eta}_{1,t}))^2}_{\sim \textit{Lognormal}} + \underbrace{(\xi_{2,t-1}\exp(\boldsymbol{\eta}_{2,t}))^2}_{\sim \textit{Lognormal}} \\ &\nsim \text{Lognormal} \end{aligned}$$

TVP density in Primiceri (2005)

- $\boldsymbol{\Sigma}_{t,(1,1)}$ and $\boldsymbol{\Sigma}_{t,(2,2)}$ in different distributional families.
- Suppose I flip the order of variables 1 and 2.
- Density of variable 1's marginal distribution of $\boldsymbol{\Sigma}_t$ necessarily changes when it becomes "variable 2"!

This paper

- Let's try something else.

This paper

- Let's try something else.
- I define a class of models with laws of motion for $\mathbf{S}_t$ such that:

1. whole sequences of $\mathbf{S}_{0:T}$ have densities invariant to orthogonal rotations
2. yield a shared reduced-form

Key benefits

- Time-varying parameter model amenable to identification driven by RRWZ conditions/algorithms.
- (Also, more straightforward to estimate.)

Outline

Extending the SVAR

$$\mathbf{y}_t'\mathbf{A}_t = \mathbf{x}_t'\mathbf{F}_t + \varepsilon_t', \quad \varepsilon_t \sim N(\mathbf{0}, \mathbf{I}_n)$$

Law of motion and stochastic processes for $(\mathbf{A}_t, \mathbf{F}_t)$:

$$(\mathbf{A}_t, \mathbf{F}_t) \sim p(\phi, \mathbf{A}_{t-1}, \mathbf{F}_{t-1})$$

Extending the SVAR

$$\mathbf{y}_t'\mathbf{A}_t = \mathbf{x}_t'\mathbf{F}_t + \varepsilon_t', \quad \varepsilon_t \sim N(\mathbf{0}, \mathbf{I}_n)$$

Law of motion for $(\mathbf{A}_t, \mathbf{F}_t)$:

$$\mathbf{A}_t = \beta^{-1/2}\,\mathbf{A}_{t-1}\mathbf{\Omega}_t$$
$$\mathbf{F}_t = \mathbf{F}_{t-1}\mathbf{A}_{t-1}^{-1}\mathbf{A}_t + \mathbf{\Theta}_t \ .$$

Shocks:

$$\mathbf{\Omega}_t = \mathbf{L}_t h(\mathbf{\Gamma}_t)\mathbf{R}_t \ , \qquad \mathbf{\Gamma}_t \sim B_n(\beta/(2(1-\beta)), 1/2)$$
$$\mathbf{\Theta}_t \sim MN_{m,n}(\mathbf{0}, \mathbf{W}, \mathbf{I}_n)$$

where

$$\beta \in [(n-1)/n, 1]$$
$$\mathbf{L}_t, \mathbf{R}_t \in \mathcal{O}_n$$

Detour: alternate form of SVAR

$$\mathbf{y}_t' = \mathbf{x}_t'\mathbf{B}_t + \varepsilon_t'\mathbf{Q}_t' h(\mathbf{H}_t)^{-1}, \quad \varepsilon_t \sim N(\mathbf{0}, \mathbf{I}_n)$$

Law of motion for $(\mathbf{B}_t, \mathbf{H}_t, \mathbf{Q}_t)$:

$$\overbrace{h(\mathbf{H}_t)\mathbf{Q}_t}^{\mathbf{A}_t} = \overbrace{h(\mathbf{H}_{t-1})\mathbf{Q}_{t-1}}^{\mathbf{A}_{t-1}}\mathbf{\Omega}_t$$

$$\mathbf{B}_t h(\mathbf{H}_t)\mathbf{Q}_t = \mathbf{B}_{t-1} h(\mathbf{H}_{t-1})\mathbf{Q}_t + \mathbf{\Theta}_t$$

$$\mathbf{Q}_t = p(\mathbf{Q}_t | \mathbf{B}_t, \mathbf{H}_t)$$

Shocks:

$$\mathbf{\Omega}_t = \beta^{-1/2} h(\mathbf{\Gamma}_t) \qquad \mathbf{\Gamma}_t \sim B_n(\beta/(2(1-\beta)), 1/2)$$

$$\mathbf{\Theta}_t \sim MN_{m,n}(\mathbf{0}, \mathbf{W}, \mathbf{I}_n)$$

where

$$\beta \in [(n-1)/n, 1]$$

Some notation

A Dynamic SVAR (call it DSVAR) denoted:

$$\mathcal{S}^{U}_{0:T}(\mathbf{L}_{1:T}, \mathbf{R}_{1:T})$$

and let

$$\boldsymbol{\phi} = (\beta, \mathbf{W})$$

Key result

Theorem (Theorem 1)

Let $\mathcal{S}^U_{0:T}(\mathbf{L}_{1:T}, \mathbf{R}_{1:T})$ *have prior* $p(\phi, \mathbf{S}_0)$ *for which* $p(\phi, \mathbf{S}_0) = p(\phi, \mathbf{S}_0 * \mathbf{P})$ *for any* $\mathbf{P} \in \mathcal{O}_n$.

For any $\mathbf{Q}_{0:T}$ *such that each* $\mathbf{Q}_t \in \mathcal{O}_n$, *the model* $\mathcal{S}^U_{0:T}(\widetilde{\mathbf{L}}_{1:T}, \widetilde{\mathbf{R}}_{1:T})$ *defined by* $(\widetilde{\mathbf{L}}_t, \widetilde{\mathbf{R}}_t) = (\mathbf{Q}'_{t-1}\mathbf{L}_t, \mathbf{R}_t\mathbf{Q}_t)$ *is such that, for every point* $\mathbf{S}_{0:T}$, *the point* $\widetilde{\mathbf{S}}_{0:T} = \mathbf{S}_{0:T} * \mathbf{Q}_{0:T}$ *satisfies*

$$p(\phi, \mathbf{S}_{0:T} | \mathbf{y}_{1:T}, \mathcal{S}^U_{0:T}(\mathbf{L}_{1:T}, \mathbf{R}_{1:T})) = p\left(\phi, \widetilde{\mathbf{S}}_{0:T} | \mathbf{y}_{1:T}, \mathcal{S}^U_{0:T}(\widetilde{\mathbf{L}}_{1:T}, \widetilde{\mathbf{R}}_{1:T})\right) .$$

Theorem 1: restatement and implications

For

1. any realization of the data,
2. any dynamic structural VAR,
3. and any $\mathbf{Q}_{1:T}$

there exists an alternative model with the "same posterior" as the original model, but with each point rotated by $\mathbf{Q}_{1:T}$.

- Set of equivalent models does not depend on $\mathbf{y}_{1:T}$
- $\Rightarrow$ All structural models in the class are observationally equivalent.

Outline

Reduced-form VAR with TVP-SV

Define $(\mathbf{H}_t, \mathbf{B}_t) = g(\mathbf{S}_t) = (\mathbf{A}_t \mathbf{A}_t', \mathbf{F}_t \mathbf{A}_t^{-1})$

$$\mathbf{y}_t' = \mathbf{x}_t' \mathbf{B}_t + \mathbf{u}_t', \quad \mathbf{u}_t \sim N(\mathbf{0}, \mathbf{H}_t^{-1})$$

Laws of motion for $(\mathbf{B}_t, \mathbf{H}_t)$:

$$\mathbf{H}_t = \frac{1}{\beta} h(\mathbf{H}_{t-1})' \, \mathbf{\Gamma}_t \, h(\mathbf{H}_{t-1})$$
$$\mathbf{B}_t = \mathbf{B}_{t-1} + \mathbf{V}_t$$

distributions of shocks $(\mathbf{\Gamma}_t, \mathbf{V}_t)$

$$\mathbf{\Gamma}_t \sim Beta_n(\beta/(2(1-\beta)), 1/2)$$
$$\mathbf{V}_t \sim MN_{m,n}(\mathbf{0}, \mathbf{W}, \mathbf{H}_t^{-1})$$

A short history of the reduced-form

The reduced-form model is "a known quantity."

- Uhlig (1994, 1997) – the stochastic volatility part
- Mike West and coauthors – "dynamic linear model with discounted Wishart stochastic volatility," (DLM-DWSV)

Why does this work?

Suppose I've estimated the reduced-form $\mathbf{H}_{0:T}$.
Shocks rationalizing movement from $\mathbf{A}_{t-1}$ to $\mathbf{A}_t$ satisfy,

$$\beta \mathbf{A}_{t-1}^{-1} \underbrace{\mathbf{H}_t}_{\mathbf{A}_t \mathbf{A}_t'} \mathbf{A}_{t-1}^{-1'} = \mathbf{\Gamma}_t$$

Suppose instead my identification scheme said that in $t-1$, $\widetilde{\mathbf{A}}_{t-1} = \mathbf{A}_{t-1} \mathbf{Q}_{t-1}$.
Shocks rationalizing movement to $\mathbf{H}_t$:

$$\beta \mathbf{A}_{t-1}^{-1} \mathbf{H}_t \mathbf{A}_{t-1}^{-1'} = \mathbf{Q}_{t-1} \mathbf{\Gamma}_t \mathbf{Q}_{t-1}' = \tilde{\mathbf{\Gamma}}_t$$

Critical thing: $\mathbf{\Gamma}_t$ and $\tilde{\mathbf{\Gamma}}_t$ have the same density!

A property of the multivariate Beta distribution:
Srivastava (2003) Corollary 4.1,

$$p(\mathbf{\Gamma}_t) = p(\mathbf{Q}_t \mathbf{\Gamma}_t \mathbf{Q}_t')$$

Estimation of reduced-form

Need to characterize

$$p(\beta, \mathbf{W}, \mathbf{B}_{0:T}, \mathbf{H}_{0:T} | \mathbf{y}_{1:T}) \,.$$

- Can't characterize it analytically.
- Can construct an MCMC algorithm.

Gibbs Sampler

- **Block 1.** $p(\mathbf{W} | \mathbf{y}_{1:T}, \beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T})$
- **Block 2.** $p(\beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T} | \mathbf{y}_{1:T}, \mathbf{W})$

Gibbs sampler: block 1

- **Block 1.** $p(\mathbf{W}|\mathbf{y}_{1:T}, \beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T})$
- **Block 2.** $p(\beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T}|\mathbf{y}_{1:T}, \mathbf{W})$

- Super easy.
- If prior is $\mathbf{W} \sim IW(\mathbf{\Psi}_0, \nu_0)$,

$$\mathbf{W}|\mathbf{y}_{1:T}, \beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T} \sim IW(\bar{\mathbf{\Psi}}, \bar{\nu})$$

where

$$\bar{\mathbf{\Psi}} = \mathbf{\Psi}(\mathbf{y}_{1:T}, \mathbf{B}_{0:T}, \mathbf{H}_{0:T}) + \mathbf{\Psi}_0$$
$$\bar{\nu} = Tn + \nu_0$$

Gibbs sampler: block 2

- **Block 1.** $p(\mathbf{W}|\mathbf{y}_{1:T}, \beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T})$
- **Block 2.** $p(\beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T}|\mathbf{y}_{1:T}, \mathbf{W})$

- Factor joint density as

$$p(\beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T}|\mathbf{y}_{1:T}, \mathbf{W}) = \underbrace{p(\beta|\mathbf{y}_{1:T}, \mathbf{W})}_{\text{Block 2a}} \cdot \underbrace{p(\mathbf{B}_{0:T}, \mathbf{H}_{0:T}|\mathbf{y}_{1:T}, \beta, \mathbf{W})}_{\text{Block 2b}}$$

Gibbs sampler: block 2

- **Block 1.** $p(\mathbf{W}|\mathbf{y}_{1:T}, \beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T})$
- **Block 2.** $p(\beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T}|\mathbf{y}_{1:T}, \mathbf{W})$
  - **2a.** $p(\beta|\mathbf{y}_{1:T}, \mathbf{W})$
  - **2b.** $p(\mathbf{B}_{0:T}, \mathbf{H}_{0:T}|\mathbf{y}_{1:T}, \beta, \mathbf{W})$

Gibbs sampler: block 2a

- **Block 1.** $p(\mathbf{W}|\mathbf{y}_{1:T}, \beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T})$
- **Block 2.** $p(\beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T}|\mathbf{y}_{1:T}, \mathbf{W})$
  - **2a.** $p(\beta|\mathbf{y}_{1:T}, \mathbf{W})$
  - **2b.** $p(\mathbf{B}_{0:T}, \mathbf{H}_{0:T}|\mathbf{y}_{1:T}, \beta, \mathbf{W})$

Random-walk Metropolis-Hastings,

- "Propose" a $\beta^* \sim q(\beta^*|\beta^{(i-1)}) = Npdf(\beta^{(i-1)}, \sigma^2_\beta)$
- Set $\beta^* = \beta^{(i)}$ with probability

$$\alpha(\beta^*|\mathbf{y}_{1:T}, \mathbf{W}) = \min\left\{ \frac{\overbrace{p\left(\beta^*, \mathbf{W}^{(i)}|\mathbf{y}_{1:T}\right)}^{\propto p(\beta^*, \mathbf{W}^{(i)}) \cdot p(\mathbf{y}_{1:T}|\beta^*, \mathbf{W}^{(i)})}}{p\left(\beta^{(i-1)}, \mathbf{W}^{(i)}|\mathbf{y}_{1:T}\right)},\ 1 \right\}$$

Gibbs sampler: block 2a

- **Block 1.** $p(\mathbf{W}|\mathbf{y}_{1:T}, \beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T})$
- **Block 2.** $p(\beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T}|\mathbf{y}_{1:T}, \mathbf{W})$
  - **2a.** $p(\beta|\mathbf{y}_{1:T}, \mathbf{W})$
  - **2b.** $p(\mathbf{B}_{0:T}, \mathbf{H}_{0:T}|\mathbf{y}_{1:T}, \beta, \mathbf{W})$

Evaluating $\alpha(\beta^*|\mathbf{y}_{1:T}, \mathbf{W})$ requires pointwise evaluation of

$$p(\mathbf{y}_{1:T}|\beta^*, \mathbf{W}^{(i)}) \\ = \int_{(\mathbf{H}_{0:T}, \mathbf{B}_{0:T})} p(\mathbf{y}_{1:T}|\beta^*, \mathbf{W}^{(i)}, \mathbf{H}_{0:T}, \mathbf{B}_{0:T}) p(\mathbf{H}_{0:T}, \mathbf{B}_{0:T}) d(\mathbf{H}_{0:T}, \mathbf{B}_{0:T})$$

Block 2a: evaluating $p(\mathbf{y}_{1:T}|\beta^*, \mathbf{W}^{(i)})$

Distribution of Interest	Distributional Family	Parameters and Supporting Computations
Step 1 – Prior for $\mathbf{D}_t$ given $\mathbf{y}_{1:t-1}$		
		$(d_{t-1\|t-1}, \mathbf{\Psi}_{t-1\|t-1}, \overline{\mathbf{B}}_{t-1\|t-1}, \mathbf{C}_{t-1\|t-1})$ given from iteration $t-1$
$(\mathbf{H}_t\|\mathbf{y}_{1:t-1}, \boldsymbol{\phi})$	$W\left(d_{t\|t-1}, \mathbf{\Psi}^{-1}_{t\|t-1}\right)$	$d_{t\|t-1} = \beta d_{t-1\|t-1}$ $\mathbf{\Psi}_{t\|t-1} = \beta \mathbf{\Psi}_{t-1\|t-1}$
$(\mathbf{B}_t\|\mathbf{y}_{1:t-1}, \boldsymbol{\phi}, \mathbf{H}_t)$	$N\left(\overline{\mathbf{B}}_{t\|t-1}, \mathbf{C}_{t\|t-1}, \mathbf{H}_t^{-1}\right)$	$\overline{\mathbf{B}}_{t\|t-1} = \mathbf{G}\overline{\mathbf{B}}_{t-1\|t-1}$ $\mathbf{C}_{t\|t-1} = \mathbf{G}\mathbf{C}_{t-1\|t-1}\mathbf{G}' + \mathbf{W}$
Step 1.5 – Forecast density of $\mathbf{y}_t$		
$(\mathbf{y}_t\|\mathbf{y}_{1:t-1}, \boldsymbol{\phi})$	$T_{\zeta_t}\left(\overline{\mathbf{y}}_{t\|t-1}, \mathbf{\Sigma}_{\mathbf{y}_t}\right)$	$\zeta_t = d_{t\|t-1} - n + 1$ $\overline{\mathbf{y}}_{t\|t-1} = \overline{\mathbf{B}}'_{t\|t-1}\mathbf{x}_t$ $q_t = \mathbf{x}'_t\mathbf{C}_{t\|t-1}\mathbf{x}_t + 1$ $\mathbf{\Sigma}_{\mathbf{y}_t} = (q_t/\zeta_t)\mathbf{\Psi}_{t\|t-1}$
Step 2 – Posterior for $\mathbf{D}_t$ after observing $\mathbf{y}_{1:t}$		
$(\mathbf{H}_t\|\mathbf{y}_{1:t}, \boldsymbol{\phi})$	$W\left(d_{t\|t}, \mathbf{\Psi}^{-1}_{t\|t}\right)$	$d_{t\|t} = d_{t\|t-1} + 1$ $\mathbf{e}_t = \mathbf{y}_t - \overline{\mathbf{y}}_{t\|t-1}$ $\mathbf{\Psi}_{t\|t} = \mathbf{\Psi}_{t\|t-1} + \frac{1}{q_t}\mathbf{e}_t\mathbf{e}'_t$
$(\mathbf{B}_t\|\mathbf{y}_{1:t}, \boldsymbol{\phi}, \mathbf{H}_t)$	$N\left(\overline{\mathbf{B}}_{t\|t}, \mathbf{C}_{t\|t}, \mathbf{H}_t^{-1}\right)$	$\mathbf{K}_t = \mathbf{C}_{t\|t-1}\mathbf{x}_t/q_t$ $\overline{\mathbf{B}}_{t\|t} = \overline{\mathbf{B}}_{t\|t-1} + \mathbf{K}_t\mathbf{e}'_t$ $\mathbf{C}_{t\|t} = \mathbf{C}_{t\|t-1} - \mathbf{K}_t\mathbf{K}'_t q_t$

Notes: The table summarizes results given in Prado and West (2010).

Block 2b: simulation smoother

- **Block 1.** $p(\mathbf{W}|\mathbf{y}_{1:T}, \beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T})$
- **Block 2.** $p(\beta, \mathbf{B}_{0:T}, \mathbf{H}_{0:T}|\mathbf{y}_{1:T}, \mathbf{W})$
  - **2a.** $p(\beta|\mathbf{y}_{1:T}, \mathbf{W})$
  - **2b.** $p(\mathbf{B}_{0:T}, \mathbf{H}_{0:T}|\mathbf{y}_{1:T}, \beta, \mathbf{W})$

Analogous to Kalman smoother.

Distribution to be sampled	Distributional Family	Parameters and Supporting Computations
		$(d_{t\|t}, \boldsymbol{\Psi}_{t\|t}, \overline{\mathbf{B}}_{t\|t}, \mathbf{C}_{t\|t}, \overline{\mathbf{B}}_{t+1\|t}, \mathbf{C}_{t+1\|t})$ given from forwards filter
$(\mathbf{H}_t\|Y_t, \boldsymbol{\phi}, \mathbf{H}_{t+1})$	$\mathbf{H}_t = \beta\mathbf{H}_{t+1} + \boldsymbol{\Upsilon}_t$ $\boldsymbol{\Upsilon}_t \sim W(d^*_{t\|t+1}, \boldsymbol{\Psi}^{-1}_{t\|t})$	$d^*_{t\|t+1} = (1-\beta)d_{t\|t}$
$(\mathbf{B}_t\|Y_t, \boldsymbol{\phi}, \mathbf{H}_{t+1}, \mathbf{B}_{t+1})$	$N(\overline{\mathbf{B}}_{t\|t+1}, \mathbf{C}_{t\|t+1}, \mathbf{H}_t^{-1})$	$\widetilde{\mathbf{K}}_t = \mathbf{C}_{t\|t}\mathbf{G}'\mathbf{C}^{-1}_{t+1\|t}$ $\overline{\mathbf{B}}_{t\|t+1} = \overline{\mathbf{B}}_{t\|t} + \widetilde{\mathbf{K}}_t(\mathbf{B}_{t+1} - \overline{\mathbf{B}}_{t+1\|t})$ $\mathbf{C}_{t\|t+1} = \mathbf{C}_{t\|t} - \widetilde{\mathbf{K}}_t\mathbf{C}_{t+1\|t}\widetilde{\mathbf{K}}'_t$

Note: the distribution of $\mathbf{B}_t$ corrects a typo in Prado and West (2010).

Outline

From reduced-form back to structural

Given

1. restriction regions $\mathcal{R}_t$ for each t
2. and posterior samples $\{\mathbf{H}_{0:T}^{(i)}, \mathbf{B}_{0:T}^{(i)}, \phi^{(i)}\}_{i=1}^{Nsim}$

one can

1. construct a sequence of arbitrary $(\mathbf{A}_{0:T}^{(i)}, \mathbf{F}_{0:T}^{(i)})$ consistent with $(\mathbf{H}_{0:T}^{(i)}, \mathbf{B}_{0:T}^{(i)})$ period-by-period
2. t-by-t, find $\mathbf{Q}_t^{(i)} \in \mathcal{O}_n$ such that $(\mathbf{A}_t^{(i)}\mathbf{Q}_t^{(i)}, \mathbf{F}_t^{(i)}\mathbf{Q}_t^{(i)}) \in \mathcal{R}_t$.
3. Set $(\widetilde{\mathbf{A}}_t^{(i)}, \widetilde{\mathbf{F}}_t^{(i)}) = (\mathbf{A}_t^{(i)}\mathbf{Q}_t^{(i)}, \mathbf{F}_t^{(i)}\mathbf{Q}_t^{(i)})$

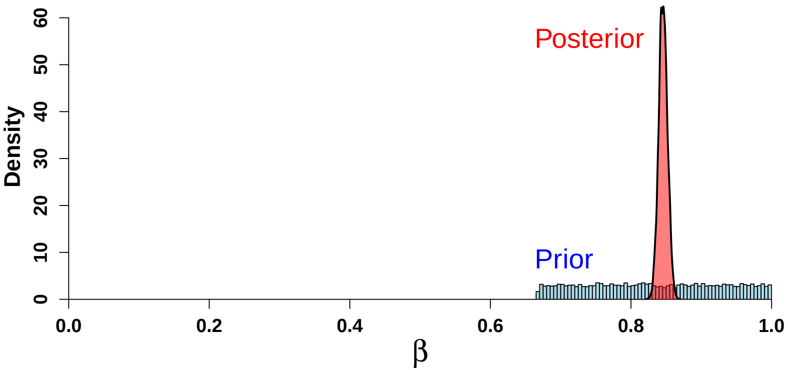
Note, $\mathbf{Q}_t^{(i)}$ can be constructed via:

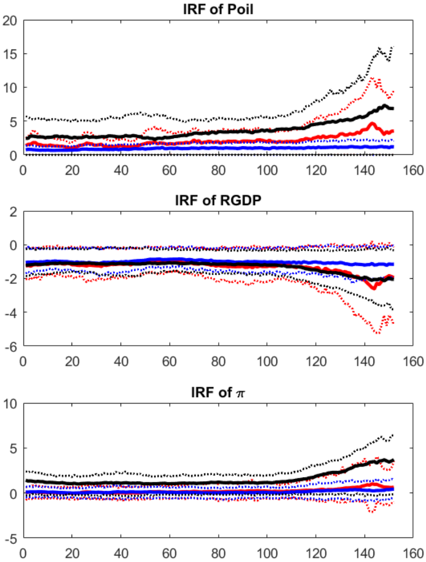
- Algorithm 1 of RRWZ (exact id), or
- Algorithm 2 of RRWZ (set id)

Outline

1. A new SVAR with dynamic parameters
2. Reduced-form Representation
3. Structural Inference Revisited
4. Revisiting the time-varying oil demand elasticity

Prior vs. Posterior: β





- Supply shock causing $\Delta q^{oil} = -1\%$.
- "baseline" IRFs
- IRFs under alternative variable ordering
- Results from **my model**.

Concluding Remarks

Main contributions:

1. Developed a new class of SVAR with time-varying parameters amenable to a variety of identification methods.
   - All models in the class have the same reduced-form representation.
2. Developed an MCMC algorithm for the fully-Bayesian estimation of the reduced-form model.
3. Applied to set identification of a time-varying object of interest about the effect of oil supply shocks.

Appendix