Shrinkage for Set-Identified Structural Vector Autoregressions[*†]

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Abstract

Set-identified SVARs, which relax exclusion restrictions and rely on weaker assumptions such as sign restrictions, are increasingly common. However, a known drawback is that the inference is rarely informative. This paper shows that robust restrictions on the Forecast Error Variance (FEV) decomposition may dramatically shrink the inference. Specifically, these restrictions are consistent with the implications of a variety of different DSGE models, with both real and nominal frictions, and with sufficiently wide ranges for their parameters. First, in a bivariate and trivariate setting, this paper analytically proves restrictions on the FEV decomposition are more informative than traditional sign restrictions. Second, sufficient conditions are provided to guarantee that the identified set is non-empty and convex. Finally, two applications are provided: using models of monetary policy and technology shocks, restrictions on the FEV decomposition tend to be highly informative, greatly shrink and even change the inference of models originally identified via traditional sign restrictions. Remarkably, shrinkage in inference is robust to the recent concerns over the unintended consequences of rotation matrix prior (Baumeister and Hamilton, 2015).

Keywords: Set Identification, Shrinkage, SVAR, Monetary Policy, Technology shock.

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1 Introduction and Related Literature

After Sims (1980), the structural vector autoregressive (SVAR) models are the common tool to study the dynamics caused by macroeconomic shocks. The early literature employs zero short-run, medium-run, or long-run restrictions on impulse response function (IRFs) for identification (Sims, 1980; Uhlig, 2004; Blanchard and Quah, 1989). However, recent contributions relax controversial restrictions and attempt to rely on weaker assumptions. Specifically, since Faust (1998), Canova and Nicolo (2002), and Uhlig (2005), it is increasingly common to identify structural shocks with sign restrictions on either the impulse response functions or the structural parameters. Such restrictions are usually weaker than classical identification schemes and, as a result, likely to be agreed upon by researchers. Additionally, because the structural parameters and IRFs are set-identified, or bounded, conclusions are robust across the set of structural models that satisfy the sign restrictions. But this minimalist, or agnostic, approach comes at a cost. Sign restrictions will usually deliver a set of structural parameters with very different implications for IRFs, elasticities, historical decompositions or forecasting error variance decompositions. On one hand, it will be extremely challenging to obtain informative inference and meaningful economic results. On the other hand, some of the admissible structural models may contain implausible implications. Specifically, Kilian and Murphy (2012) find that sign restrictions on IRFs of a SVAR for the oil market induce highly questionable implications for the price elasticity of oil supply to demand shocks. Arias, Caldara, and Rubio Ramírez (2016) show that identifying restrictions in Uhlig (2005) have counter-intuitive consequences for the systematic response of monetary policy to real output. Thus, the challenge is to come up with a small number of additional uncontroversial restrictions that help shrink the set of admissible structural parameters and allow us to reach clear economic conclusions.

I make the following contributions to the class of set-identified dynamic models. This paper shows that robust restrictions on the Forecast Error Variance (FEV) decomposition may dramatically shrink the inference. Specifically, these restrictions are consistent with the implications of a variety of different DSGE models, with both real and nominal frictions, and with sufficiently wide ranges for their parameters. First, in a bivariate and trivariate setting, this paper analytically proves restrictions on the FEV decomposition are more informative than traditional sign restrictions. Second, sufficient conditions are provided to guarantee that the identified set is non-empty and convex. Finally, two applications are provided: using models of monetary policy and technology shocks, restrictions on the FEV decomposition tend to be highly informative, greatly shrink and even change the inference of models originally identified via traditional sign restrictions. Remarkably, shrinkage in inference is robust to the recent concerns over the unintended consequences of rotation matrix prior (Baumeister and

Hamilton, 2015).

This paper shares with Antolin-Diaz and Rubio Ramírez (2017) and Amir-Ahmadi and Drautzburg (2017) the need to enrich traditional sign restrictions with additional information; however, the methodology greatly differs. First, Antolin-Diaz and Rubio Ramírez (2017) employ historical information to derive additional sign restrictions on the historical decomposition and truncate the likelihood distribution rather than the prior specification. Second, Amir-Ahmadi and Drautzburg (2017) propose a ranking of IRFs derived from micro data.

The paper is organised as follows. Section 2 provides the econometric framework for set-identified structural models. Section 3 analytically illustrates the shrinkage of identified set in a bivariate and trivariate setting. Section 4 establishes sufficient conditions for non-emptiness and convexity. Section 5 shows how to derive a set of restrictions on the FEV decomposition consistent with a variety of different theoretical models and shows the results for a monetary policy SVAR. Section 6 presents the second empirical application based on identification of technology shocks. Finally, Section 7 concludes.

2 The Econometric Framework

This section illustrates the SVAR. It then introduces the identification problem, the relationship between reduced-form and structural parameters and the class of equality, sign and FEV restrictions considered in this paper.

2.1 The Model

Consider a SVAR(p) model

$$\boldsymbol{A}_0\boldsymbol{y}_t = \boldsymbol{a} + \sum_{j=1}^{p} \boldsymbol{A}_j \boldsymbol{y}_{t-j} + \boldsymbol{\epsilon}_t \tag{2.1}$$

for $t = 1, \dots, T$, where $\boldsymbol{y}_t$ is an $n \times 1$ vector of endogenous variables, $\boldsymbol{\epsilon}_t$ an $n \times 1$ vector white noise process, normally distributed with mean zero and variance-covariance matrix $\boldsymbol{I}_n$, $\boldsymbol{A}_j$ for $j = 0, \dots, p$ is an $n \times n$ matrix of structural coefficient. As usual in literature, structural disturbances are assumed to be uncorrelated. The initial conditions $\boldsymbol{y}_1, \dots, \boldsymbol{y}_p$ are given. Let $\boldsymbol{\theta} = (\boldsymbol{A}_0, \boldsymbol{A}_+)$ collect the structural parameters, where $\boldsymbol{A}_+ = (\boldsymbol{a}, \boldsymbol{A}_j)$ for $j = 1, \dots, p$. The reduced-form VAR is as follows

$$\boldsymbol{y}_t = \boldsymbol{b} + \sum_{j=1}^{p} \boldsymbol{B}_j \boldsymbol{y}_{t-j} + \boldsymbol{u}_t, \tag{2.2}$$

where $\boldsymbol{b} = \boldsymbol{A}_0^{-1}\boldsymbol{c}$ is an $n \times 1$ vector of constants, $\boldsymbol{B}_j = \boldsymbol{A}_0^{-1}\boldsymbol{A}_j$, $\boldsymbol{u}_t = \boldsymbol{A}_0^{-1}\boldsymbol{\epsilon}_t$ denotes the $n \times 1$ vector of reduced-form errors. $var(\boldsymbol{u}_t) = E(\boldsymbol{u}_t\boldsymbol{u}_t') = \boldsymbol{\Sigma} = \boldsymbol{A}_0^{-1}(\boldsymbol{A}_0^{-1})'$ is the $n \times n$ variance-covariance matrix of reduced-form errors. Let $\boldsymbol{\phi} = (\boldsymbol{B}, \boldsymbol{\Sigma}) \in \boldsymbol{\Upsilon}$ collect the reduced-form parameters, where $\boldsymbol{B} \equiv [\boldsymbol{b}, \boldsymbol{B}_1, \ldots, \boldsymbol{B}_p]$. Note that $\boldsymbol{\Upsilon}$ is such that the VAR(p) is invertible into a VMA(∞), i.e., the model is stationary. Thus, the VMA(∞) representation of (2.2) is

$$\boldsymbol{y}_t = \boldsymbol{c} + \sum_{j=0}^{\infty} \boldsymbol{C}_j(\boldsymbol{B})\boldsymbol{A}_0^{-1}\boldsymbol{\epsilon}_{t-j}, \tag{2.3}$$

where $\boldsymbol{C}_j(\boldsymbol{B})$ is the j-th coefficient matrix of $(\boldsymbol{I}_n - \sum_{j=1}^{p} \boldsymbol{B}_j L^j)^{-1}$. Let the $n \times n$ matrix

$$\boldsymbol{IR}^h = \boldsymbol{C}_h(\boldsymbol{B})\boldsymbol{A}_0^{-1} \tag{2.4}$$

be the impulse response at h-th horizon, where its (i, j)-element denotes the effect on the i-th variable in $\boldsymbol{y}_{t+h}$ of a unit shock to the j-th element of $\boldsymbol{\epsilon}_t$ and $h = 0, 1, \ldots$.

2.2 The Identification Problem

Identifying restrictions are needed to point-identify structural parameters $\boldsymbol{A}_0$ and $\boldsymbol{A}_+$ from $\boldsymbol{\phi}$. Otherwise, reduced-form parameters $\boldsymbol{\phi}$ are not able to uniquely pin down the structural objects. In absence of any identifying restrictions, Uhlig (2005) shows that $\{\boldsymbol{A}_0 = \boldsymbol{Q}'\boldsymbol{\Sigma}_{tr}^{-1} : \boldsymbol{Q} \in \boldsymbol{\Theta}(n)\}$ is the set of observationally equivalent $\boldsymbol{A}_0$'s consistent with reduced-form parameters, where $\boldsymbol{\Sigma}$ relates to $\boldsymbol{A}_0$ by $\boldsymbol{\Sigma} = \boldsymbol{A}_0^{-1}(\boldsymbol{A}_0^{-1})'$, $\boldsymbol{\Sigma}_{tr}$ denotes the lower triangular Cholesky matrix with non-negative diagonal coefficients and $\boldsymbol{Q} \in \boldsymbol{\Theta}(n)$ is the $n \times n$ orthonormal matrix belonging to the space of $n \times n$ orthonormal matrices $\boldsymbol{\Theta}(n)$. The likelihood function depends on $\boldsymbol{\phi}$ and does not contain any information about $\boldsymbol{Q}$, leading to ambiguity in decomposing $\boldsymbol{\Sigma}$. Thus, in absence of point-identification, there is a multiplicity of $\boldsymbol{Q}$'s which deliver $\boldsymbol{A}_0$ given $\boldsymbol{\phi}$. Similarly, the rest of structural parameters $\boldsymbol{A}_+$ is a function of $\boldsymbol{Q}$ and Cholesky decomposition of reduced-form parameters. For simplicity, this section illustrates the identification problem relying on $\boldsymbol{A}_0$ only.

The set of $\boldsymbol{A}_0$ and $\boldsymbol{A}_+$ collapses to a singleton as long as identifying assumptions are able to deliver a unique $\boldsymbol{Q}$ which recovers structural parameter $\boldsymbol{A}_0$ and $\boldsymbol{A}_+$, i.e. point-identification. Rubio-Ramirez, Waggoner, and Zha (2010) establish sufficient conditions for point-identification: there must be at least $n - j$ equality restrictions on the j-th structural shock, for $1 \leq j \leq n$, and sign normalizations on the impulse responses to each structural shock.[1] This paper focuses on set-identification, so there will be fewer than $n - j$ equal-

[1] Rothenberg (1971) proves that necessary condition for point-identification require that the number of equality restrictions is greater than or equal to $n(n-1)/2$.

ity restrictions on the j-th structural shock. As a result, no matter how many sign restrictions are imposed, point-identification fails and will only be set-identification. Under set-identification, I follow Christiano, Eichenbaum, and Evans (1999) and assume that the diagonal elements of $\boldsymbol{A}_0$ are non-negative, i.e., a structural shock is a one standard-deviation positive shock to the related variable. Thus, the set of observationally equivalent $\boldsymbol{A}_0$'s becomes $\{\boldsymbol{A}_0 = \boldsymbol{Q}'\boldsymbol{\Sigma}_{tr}^{-1} : \boldsymbol{Q} \in \boldsymbol{\Theta}(n), diag(\boldsymbol{Q}'\boldsymbol{\Sigma}_{tr}^{-1}) \geq \boldsymbol{0}\}$, where $diag(\bullet) \geq \boldsymbol{0}$ implies that all diagonal elements of $\bullet$ are non-negative. Thus, in absence of any identifying restrictions, there is a multiplicity of $\boldsymbol{Q}$s consistent with $\boldsymbol{A}_0$, given the reduced-form parameters:

$$\boldsymbol{\mathcal{Q}}(\boldsymbol{\phi}) = \{\boldsymbol{Q} \in \boldsymbol{\Theta}(n) : diag(\boldsymbol{Q}'\boldsymbol{\Sigma}_{tr}^{-1}) \geq \boldsymbol{0}\}.$$

Without loss of generality, suppose one is interested in a specific (structural) impulse response - for instance, the $(i,j)-$th element of $\boldsymbol{IR}^h$ -:

$$g_{i,j}^h(\boldsymbol{\phi}, \boldsymbol{Q}) \equiv \boldsymbol{e}_i'\boldsymbol{C}_h(\boldsymbol{B})\boldsymbol{\Sigma}_{tr}\boldsymbol{Q}\boldsymbol{e}_j \equiv \boldsymbol{c}_{ih}'(\boldsymbol{\phi})\boldsymbol{q}_j,$$

where $g_{i,j}^h(\boldsymbol{\phi}, \boldsymbol{Q}) \in \mathcal{R}$, $\boldsymbol{e}_i$ is the i-th column vector of $\boldsymbol{I}_n$, $\boldsymbol{q}_j$ is the j-th column of $\boldsymbol{Q}$ and $\boldsymbol{c}_{ih}'(\boldsymbol{\phi})$ represents the i-th row vector of $\boldsymbol{C}_h(\boldsymbol{B})\boldsymbol{\Sigma}_{tr}$. Note that the analysis for the impulse responses can be easily extended to the structural parameters $\boldsymbol{A}_0$ and $\boldsymbol{A}_+$ since each structural parameter can be expressed by the inner product of a vector depending on $\boldsymbol{\phi}$ and a column vector of $\boldsymbol{Q}$.

2.2.1 Equality Restrictions

Typical equality restrictions include zero restrictions on off-diagonal elements of $\boldsymbol{A}_0^{-1}$[2] and zero restrictions on other components of the matrix.[3] Econometric framework here also allows to place zero restrictions on the lagged coefficients $\boldsymbol{A}_l : l = 1, \dots p$ and restrictions on the long-run impulse responses $\boldsymbol{IR}^\infty = (\boldsymbol{I}_n - \sum_{j=1}^p \boldsymbol{B}_j)^{-1}\boldsymbol{\Sigma}_{tr}\boldsymbol{Q}$. For simplicity and without loss of generality, this paper reduces the set of equality restrictions to zero restrictions only (in the short- or in the long-run). They can be considered as linear constraints on the columns of $\boldsymbol{Q}$ with coefficients depending on the reduced-form parameters $\boldsymbol{\phi}$. As a result, zero restrictions can be represented as follows:

$$\boldsymbol{F}(\boldsymbol{\phi}, \boldsymbol{Q}) \equiv \begin{pmatrix} \boldsymbol{F}_1(\boldsymbol{\phi})\boldsymbol{q}_1 \\ \vdots \\ \boldsymbol{F}_n(\boldsymbol{\phi})\boldsymbol{q}_n \end{pmatrix} = \boldsymbol{0}, \quad \boldsymbol{F}_i(\phi)\text{: } f_i \times n, \tag{2.5}$$

[2]This corresponds to a subset of the restrictions imposed by the classical recursive identification scheme that sets the upper-triangular elements of $\boldsymbol{A}_0^{-1}$ to zero.

[3]Zero restrictions on $\boldsymbol{A}_0^{-1}$ restrict the contemporaneous impulse responses.

where $f_i \times n$ matrix $\boldsymbol{F}_i(\boldsymbol{\phi})$ depends on $\boldsymbol{\phi}$. Each row vector in $\boldsymbol{F}_i(\boldsymbol{\phi})$ is the coefficient vector of a zero restriction that constrains the correspondent column of $\boldsymbol{Q}$. More generally, $\boldsymbol{F}_i(\boldsymbol{\phi})$ collects all the coefficient vectors that multiply $\boldsymbol{q}_i$ into a matrix and f_i denotes number of zero restrictions constraining $\boldsymbol{q}_i$.

2.2.2 Sign Restrictions

Assume that the researcher is interested in imposing some sign restrictions on the impulse response vector to the j-th structural shock and let $s_{h,j}$ denote the number of sign restrictions on impulse responses at horizon h. In this case, the impulse response is given by the j-th column vector of $\boldsymbol{IR}^h = \boldsymbol{C}_h(\boldsymbol{B})\boldsymbol{\Sigma}_{tr}\boldsymbol{Q}$ and the sign restrictions are

$$\boldsymbol{S}_{h,j}(\boldsymbol{\phi})\boldsymbol{q}_j \geq \boldsymbol{0},$$

where $\boldsymbol{S}(\boldsymbol{\phi})_{h,j} \equiv \boldsymbol{D}_{h,j}\boldsymbol{C}_h(\boldsymbol{B})\boldsymbol{\Sigma}_{tr}$ is a $s_{h,j} \times n$ matrix and $\boldsymbol{D}_{h,j}$ is the $s_{h,j} \times n$ selection matrix that selects the sign-restricted responses from the $n \times 1$ response vector $\boldsymbol{C}_h(\boldsymbol{B})\boldsymbol{\Sigma}_{tr}\boldsymbol{q}_j$. The nonzero elements of $\boldsymbol{D}_{h,j}$ can be equal to 1 or to -1 depending on the sign of restriction on the impulse response of interest. By considering multiple horizons, the whole set of sign restrictions is

$$\boldsymbol{S}_j(\boldsymbol{\phi})\boldsymbol{q}_j \geq \boldsymbol{0}. \tag{2.6}$$

Specifically, $\boldsymbol{S}_j$ is a $\left(\sum_{h=0}^{\bar{h}_j} s_{h,j}\right) \times n$ matrix defined by $\boldsymbol{S}_j(\boldsymbol{\phi}) = \left[\boldsymbol{S}_{0,j}(\boldsymbol{\phi})', \ldots, \boldsymbol{S}_{\bar{h}_j,j}(\boldsymbol{\phi})\right]'$. With abuse of notation, let $\boldsymbol{S}(\boldsymbol{\phi}, \boldsymbol{Q}) > 0$ collect the set of all sign restrictions $\boldsymbol{S}_j(\boldsymbol{\phi})\boldsymbol{q}_j > 0$ for any j.[4]

Sign restrictions above can be easily added to the zero restrictions; let $\boldsymbol{\mathcal{Q}}(\boldsymbol{\phi}|\boldsymbol{F}, \boldsymbol{S})$ be the set of $\boldsymbol{Q}$'s that satisfy sign normalizations, zero and sign restrictions, given $\boldsymbol{\phi}$:

$$\boldsymbol{\mathcal{Q}}(\boldsymbol{\phi}|\boldsymbol{F}, \boldsymbol{S}) = \{\boldsymbol{Q} \in \boldsymbol{\Theta}(n) : \boldsymbol{F}(\boldsymbol{\phi}, \boldsymbol{Q}) = \boldsymbol{0}, \boldsymbol{S}(\boldsymbol{\phi}, \boldsymbol{Q}) \geq \boldsymbol{0}, diag(\boldsymbol{Q}'\boldsymbol{\Sigma}_{tr}^{-1}) \geq \boldsymbol{0}\}.$$

The identified set for the object of interest is a set-valued map from $\boldsymbol{\phi}$ to a subset in $\mathcal{R}$ that delivers $g_{i,j}^h(\boldsymbol{\phi}, \boldsymbol{Q})$ when $\boldsymbol{Q}$ varies over $\boldsymbol{\mathcal{Q}}(\boldsymbol{Q}|\boldsymbol{F}, \boldsymbol{S})$:

$$IS_g(\boldsymbol{\phi}|\boldsymbol{F}, \boldsymbol{S}) = \{g_{i,j}^h(\boldsymbol{\phi}, \boldsymbol{Q}) : \boldsymbol{Q} \in \boldsymbol{\mathcal{Q}}(\boldsymbol{\phi}|\boldsymbol{F}, \boldsymbol{S})\}. \tag{2.7}$$

[4] Given the j-th shock, sign restrictions on $\boldsymbol{A}_0$ and $\boldsymbol{A}_+$ can be appended to equation (2.6) as they can be expressed as linear constraints on q_j.

2.2.3 Restrictions on the Forecast Error Variance

This Section illustrates how to impose restrictions on the Forecast Error Variance. First, the h-step ahead Forecast Error for a SVAR as in equation (2.1) is $\boldsymbol{FE}(h) \equiv \boldsymbol{y}_{t+h} - \boldsymbol{y}_{t+h|t} = \sum_{i=0}^{h-1} \boldsymbol{IR}^i \boldsymbol{\epsilon}_{t+h-i}$. Thus, the Forecast Error Variance at horizon h is

$$\boldsymbol{FEV}(h) \equiv E\left[(\boldsymbol{y}_{t+h} - \boldsymbol{y}_{t+h|t})(\boldsymbol{y}_{t+h} - \boldsymbol{y}_{t+h|t})'\right] = \sum_{i=0}^{h-1} \boldsymbol{IR}^i \boldsymbol{IR}^{i'}.$$

Thus, the contribution of shock j to Forecast Error Variance of variable z at horizon h is

$$CFEV_j^z(h) \equiv \frac{FEV_j^z(h)}{FEV^z(h)} = \frac{\sum_{i=0}^{h-1} \boldsymbol{IR}_{z,j}^{i2}}{\sum_{j=1}^{n} \sum_{i=0}^{h-1} \boldsymbol{IR}_{z,j}^{i2}}, \tag{2.8}$$

where $\boldsymbol{IR}_{z,j}^i$ is the (z,j)-th element of $\boldsymbol{IR}^i$. Equation (2.8) can be written as

$$CFEV_j^z(h) = \frac{\boldsymbol{q}_{\boldsymbol{j}}' \boldsymbol{S}^z(\boldsymbol{\phi}) \boldsymbol{q}_{\boldsymbol{j}}}{\sigma_z^2(\boldsymbol{\phi})}, \tag{2.9}$$

where $\boldsymbol{S}^z(\boldsymbol{\phi}) = \sum_{i=0}^{h-1} \boldsymbol{c}_{zi}(\boldsymbol{\phi}) \boldsymbol{c}_{zi}'(\boldsymbol{\phi})$ and $\sigma_z^2(\boldsymbol{\phi}) = \sum_{i=0}^{h-1} \boldsymbol{c}_{zi}'(\boldsymbol{\phi}) \boldsymbol{c}_{zi}(\boldsymbol{\phi})$ is the total Forecast Error Variance of variable z at horizon h.

Suppose that researcher believes the contribution of shock j to Forecast Error Variance of variable z at horizon h is not lower than contribution of shock $j*$ to Forecast Error Variance of variable $z*$. This implies that

$$\frac{\boldsymbol{q}_{\boldsymbol{j}}' \boldsymbol{S}^z(\boldsymbol{\phi}) \boldsymbol{q}_{\boldsymbol{j}}}{\sigma_z^2(\boldsymbol{\phi})} - \frac{\boldsymbol{q}_{\boldsymbol{j*}}' \boldsymbol{S}^{z*}(\boldsymbol{\phi}) \boldsymbol{q}_{\boldsymbol{j*}}}{\sigma_{z*}^2(\boldsymbol{\phi})} \geq 0. \tag{2.10}$$

For $j = j*$ and $z = z*$, inequality (2.10) is trivially always satisfied.

Let $I_{FEV} \subset \{1, 2, \ldots, n\}$ be the set of indices such that $j, j*, z, z* \in I_{FEV}$ if variables z and $z*$, subject to j- and $j*$-th shock, are FEV-restricted as in equation (2.10). The set of all the constraints on the FEV can be accordingly expressed by

$$\frac{\boldsymbol{q}_{\boldsymbol{j}}' \boldsymbol{S}^z(\boldsymbol{\phi}) \boldsymbol{q}_{\boldsymbol{j}}}{\sigma_z^2(\boldsymbol{\phi})} - \frac{\boldsymbol{q}_{\boldsymbol{j*}}' \boldsymbol{S}^{z*}(\boldsymbol{\phi}) \boldsymbol{q}_{\boldsymbol{j*}}}{\sigma_{z*}^2(\boldsymbol{\phi})} \geq \boldsymbol{0}, \text{for any } j, j*, z, z* \in I_{FEV}. \tag{2.11}$$

For simplicity, inequality (2.11) assumes that any restriction on the FEV has a common h; allowing different horizons is feasible at cost of making the notation dramatically heavier. As shorthand notation, let $\boldsymbol{\Gamma}(\boldsymbol{\phi}, \boldsymbol{Q}) \geq \boldsymbol{0}$ collect the whole set of rank restrictions on the FEV represented by (2.11). Note that standard sign assumptions in Section 2.2.2 sign-restrict the response functions at a given horizon and impose linear constraints on the columns of $\boldsymbol{Q}$. On the other hand, assumptions on the FEV are restricting the relative contribution of response functions in explaining the FEV of a target variable rather than constraining its sign. This implies that rank restriction on the FEV impose quadratic constraints on the columns of $\boldsymbol{Q}$ and can be used with/without standard sign restrictions.

The set of $\boldsymbol{Q}$'s that satisfy sign normalizations, zero restrictions, sign restrictions and restrictions on the FEV is

$$\mathcal{Q}(\boldsymbol{\phi}|\boldsymbol{F},\boldsymbol{S},\boldsymbol{\Gamma}) = \{\boldsymbol{Q} \in \boldsymbol{\Theta}(n) : \boldsymbol{F}(\boldsymbol{\phi},\boldsymbol{Q}) = \boldsymbol{0}, \boldsymbol{S}(\boldsymbol{\phi},\boldsymbol{Q}) \geq \boldsymbol{0}, \boldsymbol{\Gamma}(\boldsymbol{\phi},\boldsymbol{Q}) \geq \boldsymbol{0}, diag(\boldsymbol{Q'}\boldsymbol{\Sigma}_{tr}^{-1}) \geq \boldsymbol{0}\}.$$

The correspondent identified set for the object of interest is:

$$IS_g(\boldsymbol{\phi}|\boldsymbol{F},\boldsymbol{S},\boldsymbol{\Gamma}) = \{g_{i,j}^h(\boldsymbol{\phi},\boldsymbol{Q}) : \boldsymbol{Q} \in \mathcal{Q}(\boldsymbol{\phi}|\boldsymbol{F},\boldsymbol{S},\boldsymbol{\Gamma})\}. \tag{2.12}$$

Note that $\mathcal{Q}(\boldsymbol{\phi}|\boldsymbol{F},\boldsymbol{S},\boldsymbol{\Gamma})$ and $\mathcal{Q}(\boldsymbol{\phi}|\boldsymbol{F},\boldsymbol{S})$ can be empty sets depending on $\boldsymbol{\phi}$, unlike the case with zero restrictions only. If so, the correspondent identified set for $g_{i,j}^h$ is an empty set.

2.3 Estimation and Inference

The identification strategy is intuitive and relies on economic theory, but estimation and inference is not straightforward. While this paper does not take a stand over the exciting debate about the construction of priors and inference in set-identified SVARs, the current section briefly illustrates the methodologies used in this work and the main challenges.[5]

This paper follows the conventional Bayesian approach and places a Normal-inverse-Wishart prior on the reduced-form[6] and a uniform specification on $\boldsymbol{Q}$, which is uninformative in the Haar space. However, the implied priors for the structural impulse responses are clearly informative. This outcome derives from the fact that the responses are a weighted average of the elements of $\boldsymbol{Q}$, which are not flat. The resulting informative prior for the structural impulse responses does not rely on economic information; since the likelihood does not depend on $\boldsymbol{Q}$, data are unable to update this prior even asymptotically. A crucial practical question is to what extent the posterior distribution of the structural parameters of interest relies on the prior for $\boldsymbol{Q}$, as opposed to the data.

This paper employs the algorithm in Arias, Rubio-Ramirez, and Waggoner (2017). They propose to use an agnostic prior, in the sense that it does not imply further identifying restrictions beyond those explicitly imposed by the user. They define a prior over the structural representation $\boldsymbol{\theta}$ (or over the impulse response structural representation) to be agnostic if the prior density is invariant to $\boldsymbol{Q}$; they formally prove that a prior is agnostic if and only if it is equivalent to a prior over $(\boldsymbol{\phi},\boldsymbol{Q})$ that is flat over $\boldsymbol{Q}$. Thus, the conventional Bayesian approach

[5] Chapter 13 in Kilian and Lütkepohl (2017) provides an excellent survey.
[6] The hyperparameters are calibrated on a flat random walk.

discussed above is agnostic in this sense. As discussed, this does not imply that the prior is uninformative for the structural impulse responses.

In order to verify the potential effect of the prior for $\boldsymbol{Q}$ on the results, this paper uses the robust Bayesian approach in Giacomini and Kitagawa (2015) as additional check. They construct posterior bounds on the structural object of interest without taking a stand on the prior for $\boldsymbol{Q}$, which can be any specification as long as identifying restrictions are satisfied. Thus, their procedure only relies on a reduced-form prior and identifying restrictions. For practitioners, they provide an algorithm to construct a robustified credible region induced by posterior bounds. The comparison between this region and the credibility region implied by a specific prior for $\boldsymbol{Q}$ provides a diagnostic tool to evaluate how much the posterior inference relies on the prior for $\boldsymbol{Q}$, as opposed to the data. However, diagnostic in Giacomini and Kitagawa (2015) is a formal tool to detect unintended informative inference, but it does not provide any economic interpretation about its root.

Baumeister and Hamilton (2015) specify the prior directly on $\boldsymbol{\theta}$, draw in structural parametrization and generalize the approach in Sims and Zha (1998) instead of imposing priors on $\boldsymbol{\phi}$ and $\boldsymbol{Q}$. This is a very interesting and novel approach since the rest of the literature relies on the orthogonal reduced-form parametrization. Specifically, they place a prior, which can be explicitly uninformative or informative, on the (structural) matrix of contemporaneous coefficients. These priors on the structural parameters should be explicitly acknowledged and defended trough extraneous information from economic literature and theory.[7] However, it is generally challenging to find such an additional information beyond restrictions already imposed in the existing literature, especially in medium- and large-size models. On the other hand, any marginal priors on $\boldsymbol{\theta}$ is implicitly informative on the structural responses in a manner which does not necessarily coincide with the prior beliefs of the researcher about the structural responses.

3 Illustrative Example

This section analytically illustrates the shrinkage in the identified set implied by restrictions on the FEV decomposition in static bivariate and trivariate models. Remarkably, the reduction also affects structural objects which are not involved in the restrictions.

[7] For example, in a 2-variable labour and supply demand model they place a prior directly on the (structural) elasticities relying on micro and macro meta-analysis.

3.1 Bivariate Setting

The structural framework is the following:

$$\boldsymbol{A}\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \text{ t=1,...,T}, \tag{3.1}$$

where (y_{1t}, y_{2t}) are two endogenous variables, respectively. $(\epsilon_{1t}, \epsilon_{2t})$ denotes an i.i.d. normally distributed vector of structural shocks with variance-covariance the identity matrix. $\boldsymbol{\theta} = \boldsymbol{A}$ collects the structural parameters and the contemporaneous impulse responses are elements of $\boldsymbol{A}^{-1}$. The reduced-form model is indexed by $\boldsymbol{\Sigma}$, the variance-covariance matrix of the endogenous variables, which satisfies $\boldsymbol{\Sigma} = \boldsymbol{A}^{-1}(\boldsymbol{A}^{-1})'$. Let $\boldsymbol{\Sigma}_{tr} = \begin{pmatrix} \sigma_{11} & 0 \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$ denote its lower triangular Cholesky decomposition, where $\sigma_{11} \geq 0$ and $\sigma_{22} \geq 0$. Thus, $\boldsymbol{\phi} = (\sigma_{11}, \sigma_{21}, \sigma_{22}) \in \boldsymbol{\Phi} = \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ collects the reduced-form parameters. Following Uhlig (2005), $\boldsymbol{A}$ can be parametrized via the Cholesky matrix $\boldsymbol{\Sigma}_{tr}$ and a rotation matrix $\boldsymbol{Q} = \begin{pmatrix} cos\rho & -sin\rho \\ sin\rho & cos\rho \end{pmatrix}$ with spherical coordinate $\rho \in [0, 2\pi]$. The structural matrix of impact responses can be written as

$$\boldsymbol{IR}^0 = \boldsymbol{A}^{-1} = \boldsymbol{\Sigma}_{tr}\boldsymbol{Q} = \begin{pmatrix} \sigma_{11}\cos\rho & -\sigma_{11}\sin\rho \\ \sigma_{21}\cos\rho + \sigma_{22}\sin\rho & -\sigma_{21}\sin\rho + \sigma_{22}\cos\rho \end{pmatrix}.$$

Without loss of generality, let the structural object of interest α be the response of output to a unit positive demand shock, $\alpha \equiv \sigma_{11}\cos\rho$.

3.1.1 Traditional Sign Restrictions

Two standard sign restrictions (SR) are imposed on IRFs:

- *SR1*

  On impact, positive shock ϵ_2 does not increase variable y_1: $\sigma_{11}\sin\rho \geq 0$. Under this assumption, the conditional covariance induced by ϵ_2 is negative.

- *SR2*

  Positive shock ϵ_1 does not reduce variable y_1 on impact: $-\sigma_{22}\sin\rho - \sigma_{21}\cos\rho \leq 0$.

Note that standard sign restrictions impose linear inequalities on ρ. Appendix A proves that the identified set for α is

$$IS_\alpha(\boldsymbol{\phi}) \equiv \begin{cases} \left[\sigma_{11}\cos\left(\arctan\left(\frac{\sigma_{22}}{\sigma_{21}}\right)\right), \sigma_{11}\right], & \text{for } \sigma_{21} > 0, \\ \left[0, \sigma_{11}\cos\left(\arctan\left(-\frac{\sigma_{21}}{\sigma_{22}}\right)\right)\right], & \text{for } \sigma_{21} \leq 0. \end{cases} \tag{3.2}$$

3.1.2 Restrictions on the FEV Decomposition

- *FEVR*

  Assume that shock ϵ_2 explains y_2 more than y_1 on impact, i.e., the contribution of shock ϵ_2 to the total error variance of y_2 is higher than its contribution to that one of y_1; this restriction ranks the contribution of the shock ϵ_2 in driving the variables. Following the notation introduced in Section ???, this restriction can be written as $\frac{FEV_{y_2,\epsilon_2}}{FEV_{y_2}} \geq \frac{FEV_{y_1,\epsilon_2}}{FEV_{y_1}}$. Appendix A proves this imposes the following quadratic constraints on ρ: $\frac{(-\sigma_{21}sin\rho+\sigma_{22}cos\rho)^2}{\sigma_{21}^2+\sigma_{22}^2} \geq \frac{(-\sigma_{11}sin\rho)^2}{\sigma_{11}^2}$.

SR1, *SR2* and *FEVR* deliver the following identified set for α:

$$IS_\alpha(\phi) \equiv \begin{cases} \left[\sigma_{11}\cos\left(\arctan\left(\frac{\sigma_{22}}{\sigma_{21}+\sqrt{\sigma_{21}^2+\sigma_{22}^2}}\right)\right), \sigma_{11}\right], & \text{for } \sigma_{21} > 0, \\ \left[\sigma_{11}\cos\left(\arctan\left(\frac{\sigma_{22}}{\sigma_{21}+\sqrt{\sigma_{21}^2+\sigma_{22}^2}}\right)\right), \sigma_{11}\cos\left(\arctan\left(-\frac{\sigma_{21}}{\sigma_{22}}\right)\right)\right], & \text{for } \sigma_{21} \leq 0. \end{cases} \tag{3.3}$$

For any σ_{21}, the restriction on the FEV shrinks the identified set of α with respect to the set induced by *SR1* and *SR2*; specifically, the lower bound increases, while the upper one remains unchanged. The degree of additional shrinkage depends on the reduced-form: the higher the error variance of variable y_1, namely $\sigma_{21}^2 + \sigma_{22}^2$, the stronger the shrinkage. Remarkably, the *FEVR* is never redundant and always adds information.

- *FEVR2*

  Note that *FEVR* is restricting one and one shock only. Assume that the researcher has now information on both shocks. For example, ϵ_1 explains y_2 more than ϵ_2: $(\sigma_{21}cos\rho + \sigma_{22}sin\rho)^2 \geq (-\sigma_{21}sin\rho + \sigma_{22}cos\rho)^2$.

Under *SR1*, *SR2* and *FEVR2* the identified set for α is

$$IS_\alpha(\phi) \equiv \begin{cases} \left[\sigma_{11}\cos\left(\arctan\left(\frac{\sigma_{22}}{\sigma_{21}}\right)\right), \sigma_{11}\cos\left(\arctan\left(\frac{\sigma_{22}-\sigma_{21}}{\sigma_{22}+\sigma_{21}}\right)\right)\right], & \text{for } \sigma_{21} > 0, \sigma_{22} \geq \sigma_{21}, \\ \left[\sigma_{11}\cos\left(\arctan\left(\frac{\sigma_{22}}{\sigma_{21}}\right)\right), \sigma_{11}\right], & \text{for } \sigma_{21} > 0, \sigma_{22} < \sigma_{21}, \\ \left[0, \sigma_{11}\cos\left(\arctan\left(\frac{\sigma_{22}-\sigma_{21}}{\sigma_{22}+\sigma_{21}}\right)\right)\right], & \text{for } \sigma_{21} \leq 0, \sigma_{22} \geq |\sigma_{21}|, \\ \emptyset, & \text{for } \sigma_{21} \leq 0, \sigma_{22} < |\sigma_{21}|. \end{cases} \tag{3.4}$$

For $\sigma_{21} > 0$, *FEVR2* shrinks the identified set of α with respect to the set induced by *SR1* and *SR2* if $\sigma_{22} \geq \sigma_{21}$, otherwise the restriction is redundant; specifically, the upper bound is now lower. For $\sigma_{21} \leq 0$, *FEVR2* restricts $IS_\alpha(\boldsymbol{\phi})$ if $\sigma_{22} \geq |\sigma_{21}|$; otherwise, it is at odds with *SR1* and delivers an empty identified set.

3.1.3 Restrictions on the FEV decomposition without Sign Restrictions

So far I showed that restrictions on the FEV adds information to the sign restrictions and shrink the identified set. However, credible sign restrictions are not always available to the researcher, at least for some variables in the model; in such a scenario, he/she may want to impose restrictions on the FEV decomposition only. For instance, most economists would agree that technological shocks are a more credible driver than monetary shocks to explain the hours worked, especially in the long-run; however, there is a huge controversy about the sign of the response of hours worked to a technological shock (Galí and Rabanal, 2004; McGrattan, 2004).

The following example analytically shows that restrictions on the FEV shrink the identified set even if not combined with sign restrictions; specifically, they are generally more informative than traditional sign restrictions.

Example 3.1 *Consider the bivariate setting; suppose that the researcher imposes a sign restriction on the first variable, but he/she has not credible information to impose sign restrictions on y_2; however, he/she knows that ϵ_2 explains y_1 more than y_2. The set of assumptions is $\boldsymbol{IR}^0_{12} \leq 0$ and $\frac{FEV_{y_2,\epsilon_2}}{FEV_{y_2}} \leq \frac{FEV_{y_1,\epsilon_2}}{FEV_{y_1}}$. The correspondent identified set is*

$$IS_\alpha(\boldsymbol{\phi}) \equiv \begin{cases} \left[\sigma_{11}\cos\left(\arctan\left(\frac{\sigma_{21}}{\sigma_{22}}\right)\right), \sigma_{11}\cos\left(\arctan\left(\frac{\sigma_{22}}{\sigma_{21}+\sqrt{\sigma_{21}^2+\sigma_{22}^2}}\right)\right)\right], & \textit{for } \sigma_{21} > 0, \\ \left[0, \sigma_{11}\cos\left(\arctan\left(\frac{\sigma_{22}}{\sigma_{21}+\sqrt{\sigma_{21}^2+\sigma_{22}^2}}\right)\right)\right], & \textit{for } \sigma_{21} \leq 0. \end{cases} \tag{3.5}$$

For any $\boldsymbol{\phi}$, the Appendix proves that $IS_\alpha(\boldsymbol{\phi})$ in equation (3.5) is smaller than the set induced by replacing the restriction on the FEV with a sign restrictions on y_2, i.e., the identified set in equation (3.2).

3.2 Trivariate Setting

Bivariate illustration shows that the impulse response of interest belongs to a strictly smaller identified set with restrictions on the FEV decomposition compared with sign restrictions under conditions on the reduced-form conditional covariance. Higher dimensional cases are more complicated. However, the trivariate case is useful to characterize the shrinkage in the identified set of the variable which is not involved in the restrictions on the FEV. In parallel with the bivariate setting, the degree of additional shrinkage depends on the reduced-form.

The structural framework is the following:

$$\boldsymbol{A}\begin{pmatrix} y_{1t} \\ y_{2t} \\ y_{3t} \end{pmatrix} = \begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \\ \epsilon_{3t} \end{pmatrix} \tag{3.6}$$

The reduced-form model is indexed by $\boldsymbol{\Sigma}$, the variance-covariance matrix of the endogenous variables, which satisfies $\boldsymbol{\Sigma} = \boldsymbol{A}^{-1}(\boldsymbol{A}^{-1})'$. Let $\boldsymbol{\Sigma}_{tr} = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$ denote its lower triangular Cholesky decomposition, where $\sigma_{11} \geq 0$, $\sigma_{22} \geq 0$ and $\sigma_{33} \geq 0$. $\boldsymbol{\phi} = (\sigma_{11}, \sigma_{21}, \sigma_{22}, \sigma_{31}, \sigma_{32}, \sigma_{33})$ collects the reduced-form parameters. Let the structural object of interest α be the response of y_1 to a unit positive shock ϵ_1, $\alpha \equiv \sigma_{11}\cos\rho$, where $\rho \in [0, 2\pi]$.

For simplicity, I consider a subspace of the set of all solutions in the trivariate context. This simplifies the analytical characterization of the identified set for α and the comparison between sign restrictions and restrictions on the FEV; for details about such a subspace and its specific features, see Appendix A. However, the results from the bivariate context still persist.

Three standard sign restrictions (*SR*) are imposed.

- *SR1*

  On impact, positive shock ϵ_2 does not increase variable y_1: $\sigma_{11}\sin\rho \geq 0$.

- *SR2*

  Positive shock ϵ_1 does not reduce variable y_1 on impact: $-\sigma_{22}\sin\rho - \sigma_{21}\cos\rho \leq 0$.

- *SR3*

  Positive shock ϵ_2 does not decrease variable y_3 on impact: $-\sigma_{31}\sin\rho + \sigma_{32}\cos\rho \geq 0$.

For $\sigma_{21} > 0$, $\sigma_{31} < 0$ and $\sigma_{32} < 0$, *SR1*, *SR2* and *SR3* imply that

$$IS_{\alpha}(\boldsymbol{\phi}) \equiv \begin{cases} \left[\sigma_{11}\cos\left(\arctan\left(\frac{\sigma_{22}}{\sigma_{21}}\right)\right), \sigma_{11}\cos\left(\arctan\left(\frac{\sigma_{32}}{\sigma_{31}}\right)\right)\right], & \text{for } \sigma_{21}\sigma_{32} \geq \sigma_{22}\sigma_{31}, \\ \emptyset, & \text{for } \sigma_{21}\sigma_{32} < \sigma_{22}\sigma_{31}. \end{cases} \tag{3.7}$$

Thus, limitation of the space for $\boldsymbol{\phi}$ allows to focus on a single case, i.e., $\sigma_{21}\sigma_{32} \geq \sigma_{22}\sigma_{31}$.

- *FEVR*

  Assume that shock ϵ_2 explains y_2 more than y_3; this restricts contribution of the shock ϵ_2 in driving y_2 and y_3. However, it does not involve y_1, namely the object of interest.

As with the bivariate case, *FEVR* imposes quadratic constraints on the columns of $\boldsymbol{Q}$: $\frac{(-\sigma_{21} sin\rho+\sigma_{22} cos\rho)^2}{\sigma_{21}^2+\sigma_{22}^2} \geq \frac{(-\sigma_{31} sin\rho+\sigma_{32} cos\rho)^2}{\sigma_{31}^2+\sigma_{32}^2+\sigma_{33}^2}$.

As long as $\sigma_{21}\sigma_{32} \geq \sigma_{22}\sigma_{31}$, *SR1*, *SR2*, *SR3* and *FEVR* characterize α as follows:

$$IS_\alpha(\boldsymbol{\phi}) \equiv \left[\sigma_{11} \cos\left(\arctan\left(\frac{\sigma_{32}\sqrt{\sigma_{21}^2+\sigma_{22}^2} - \sigma_{22}\sqrt{\sigma_{31}^2+\sigma_{32}^2+\sigma_{33}^2}}{\sigma_{31}\sqrt{\sigma_{21}^2+\sigma_{22}^2} - \sigma_{21}\sqrt{\sigma_{31}^2+\sigma_{32}^2+\sigma_{33}^2}}\right)\right), \sigma_{11}\cos\left(\arctan\left(\frac{\sigma_{32}}{\sigma_{31}}\right)\right)\right]. \tag{3.8}$$

Although *FEVR* does not involve the variable of interest y_1, it shrinks $IS_\alpha(\boldsymbol{\phi})$; in the appropriate subspace and for $\sigma_{21}\sigma_{32} \geq \sigma_{22}\sigma_{31}$, the lower bound of identified set (3.8) gets closer to the upper bound, which remains unchanged.

3.3 Relationship with Heterogeneity Restrictions

Amir-Ahmadi and Drautzburg (2017) employ heterogeneity restrictions, namely a ranking of impulse responses, on micro aggregates to derive informative inference for macro variables. For instance, consider elasticities of different industries to a defense spending shock. Manufacturing industry A might be more exposed to these shocks relative to sector B if the military is a key client of the former industry but not of the latter industry. Heterogeneity assumptions then restrict industry A to respond more than industry B to a defense spending shock.

However, restrictions on the FEV differ from heterogeneity assumptions. First, in the baseline specification the latter nest sign restrictions. With regard to the previous example, the heterogeneity assumptions also imply that shipment of all industries rise after a defense spending shock, but more when the government is an important client of industry. Second, heterogeneity restrictions rank straight the IRFs, while this paper restricts the contribution of shocks in explaining the (forecast error) variance of target variables. For the reasons mentioned above, the application of the two typologies of restrictions is dramatically different. Heterogeneity assumptions restrict micro variables and, as such, rely on micro data; on the other hand, restrictions on the FEV are derived from a large variety of theoretical models and affect macro variables only.

The following example shows that heterogeneity assumptions and restrictions on the FEV are generally different and deliver distinct identified sets.

Example 3.2 *Consider the bivariate example in Section 3.1 and recall the matrix of contemporaneous impulse response:*

$$\boldsymbol{IR}^0 = \begin{pmatrix} \sigma_{11}\cos\rho & -\sigma_{11}\sin\rho \\ \sigma_{21}\cos\rho + \sigma_{22}\sin\rho & -\sigma_{21}\sin\rho + \sigma_{22}\cos\rho \end{pmatrix}.$$

Suppose to have the assumptions SR1, SR2 of Section 3.1.1 and the following heterogeneity restriction:

$$-\sigma_{21}\sin\rho + \sigma_{22}\cos\rho \geq \sigma_{11}\sin\rho, \tag{3.9}$$

which assumes that the impulse response of y_2 to shock ϵ_2 is higher than that of y_1. By excluding the degenerative case in which the heterogeneity restriction delivers an empty identified set, Appendix A proves that the correspondent identified set for α is

$$IS_\alpha(\boldsymbol{\phi}) \equiv \begin{cases} \left[\sigma_{11}\cos\left(\arctan\left(\frac{\sigma_{22}}{\sigma_{11}+\sigma_{21}}\right)\right), \sigma_{11}\right], & \textit{for } \sigma_{21} \geq 0, \\ \left[\sigma_{11}\cos\left(\arctan\left(\frac{\sigma_{22}}{\sigma_{11}+\sigma_{21}}\right)\right), \sigma_{11}\cos\left(\arctan\left(-\frac{\sigma_{21}}{\sigma_{22}}\right)\right)\right], & \textit{for } \sigma_{21} < 0, \sigma_{11}+\sigma_{21} \geq 0, \\ \left[0, \sigma_{11}\cos\left(\arctan\left(-\frac{\sigma_{21}}{\sigma_{22}}\right)\right)\right], & \textit{for } \sigma_{21} < 0, \sigma_{11}+\sigma_{21} < 0. \end{cases} \tag{3.10}$$

Compare $IS_\alpha(\boldsymbol{\phi})$ in (3.10) with that in (3.3), where assumption that shock ϵ_2 explains y_2 more than y_1 (restriction FEVR in Section 3.1.2) replaces the heterogeneity restriction:

$$IS_\alpha(\boldsymbol{\phi}) \equiv \begin{cases} \left[\sigma_{11}\cos\left(\arctan\left(\frac{\sigma_{22}}{\sigma_{21}+\sqrt{\sigma_{21}^2+\sigma_{22}^2}}\right)\right), \sigma_{11}\right], & \textit{for } \sigma_{21} > 0, \\ \left[\sigma_{11}\cos\left(\arctan\left(\frac{\sigma_{22}}{\sigma_{21}+\sqrt{\sigma_{21}^2+\sigma_{22}^2}}\right)\right), \sigma_{11}\cos\left(\arctan\left(-\frac{\sigma_{21}}{\sigma_{22}}\right)\right)\right], & \textit{for } \sigma_{21} \leq 0. \end{cases} \tag{3.11}$$

For $\sigma_{21} \geq 0$ and $\sigma_{21} < 0, \sigma_{11}+\sigma_{21} \geq 0$, the two identified sets are equivalent if and only if $\sigma_{11} = \sqrt{\sigma_{21}^2+\sigma_{22}^2}$, namely the variance of the two endogenous variables is the same. For $\sigma_{21} < 0, \sigma_{11}+\sigma_{21} < 0$, the identified set induced by the restrictions on the FEV is always tighter.

4 Non-Emptiness and Convexity of the Identified Set

SECTION IN PROGRESS

5 Empirical Application: Monetary Policy Shocks

5.1 A Monte-Carlo Experiment

In order to derive robust implications for the responses to monetary policy shocks and FEV decomposition, I run the following Monte Carlo simulation. I employ Smets and Wouters (2007) as data-generating process (DGP); this model features real rigidities, such as adjustment costs for investment and variable capacity utilization, and nominal rigidities, namely sticky prices and wages. It consists of seven endogenous variables: $\Delta y_t, \Delta c_t, \Delta I_t, \Delta w_t, l_t, \Delta \pi_t, i_t$, denoting output growth rate, consumption growth rate, investment growth rate, real wage growth rate, hours worked, inflation rate, interest rate, respectively.

I assume that all structural parameters of the DGP are uniformly and independently distributed over sufficiently wide ranges. Table [TO BE ADDED] summarizes the ranges of the uniform distributions for the parameters of the model including real and nominal frictions. Since we are interested in implications in terms of the FEV decomposition across a broad range of parameterizations of the model, with and without nominal, these ranges cover reasonable values for the parameters, encompassing a large variety of theoretical models.

I then draw 10000 structural pareeters vectors. For each of them, I consider the responses and the correspondent FEV decomposition to a 1 standard deviation positive (contractionary) monetary policy shock,[8] and compute the 2.5% and 97.5% percentiles of their distributions. Table 1 reports the signs of impulse responses and the FEV decomposition at horizon $h = 0, 1$. Specifically, $+(-)$ indicate that a certain variable has the 90% probability to response positively (negatively) on impact; the bounds of the FEV decomposition represent the 5% and 95% percentiles.

5.1.1 Analysis without Estimation Uncertainty

First, I consider analysis without estimation uncertainty, i.e., population analysis. Suppose that there is an infinite amount of data on observables; it implies that $\boldsymbol{\phi}$, i.e., the reduced-form VAR, is estimated without error and is fixed at values implied by the data-generating process. As a result, the only unknown object is the matrix $\boldsymbol{A}_0$ in equation (3.1). In order to recover $\boldsymbol{A}_0$, the researcher uses the true covariance matrix $\boldsymbol{\Sigma}$ and set-identifying restrictions. The setting of this Monte-Carlo experiment isolates the identification uncertainty and excludes sample uncertainty by construction.

- *Model 1 (Sign Restrictions)*

  Model 1 identifies interest rate shock through sign restrictions on impact. Specifically, it employs robust sign restrictions in Table 1. Contractionary interest rate shock reduces inflation, consumption, investment, real wages, hours worked and increases interest rates: $IR^0_{\Delta ci} \leq 0$, $IR^0_{\Delta li} \leq 0$, $IR^0_{\Delta wi} \leq 0$,$IR^0_{Ii} \leq 0$, $IR^0_{\Delta \pi i} \leq 0$, $IR^0_{ii} \geq 0$. $IR_{\Delta yi}$. The object of interest, i.e, the output response, is left unrestricted.

[8] I focus on this shock only because the model has implications that would allow us to disentangle other shocks considered in the literature from monetary policy shocks.

- *Model 2 (Sign and Zero Restrictions)*

  Since literature recommends to combine sign restrictions with exclusion assumptions, Model 2 relies on restrictions in Model 1 and assumes long-run neutrality of monetary policy: $IR_{yi}^{\infty} = 0$.

- *Model 3 (FEV Restrictions)*

  Model 3 relies on the robust restrictions on the FEV decomposition in Table 1. In particular, on top of sign restrictions in Model 1, at horizons $h = 0, 1$ monetary policy shock explains the fluctuations in interest rates more than movements of consumption, investment, real wages, hours worked and inflation rate: $FEV_{ki}^{h} \leq FEV_{ii}^{h}$ at $h = 0, 1$, where $k = \{\Delta c, \Delta I, \Delta w, l, \Delta \pi\}$. Note that output response is left unrestricted by restrictions on the FEV decomposition.

For each model, Figure 1 reports the true output response to (contractionary) monetary policy shock and the 90% range of theory-consistent impulse responses indicating the identification uncertainty. This range is defined by the maximum and minimum response at each horizon; as long as there is no estimation of reduced-form VAR, such a range captures the identification uncertainty implied by each set of identifying restrictions, namely the true identified set. Sign restrictions alone or mixed with parametric assumptions are unlikely to provide informative results and recover the theoretical response (Model 1 and Model 2); however, once they are combined with FEV restrictions, Model 3 shrinks the identified set and is fully able to identify the sign of output response.

5.1.2 Estimation Uncertainty

SECTION IN PROGRESS

5.2 Seven-Variable SVAR

Given the promising results of the Monte-Carlo experiment, this section estimates the three models above with real data. Specifically, I use the dataset constructed by Stock and Watson (2008). This includes 149 quarterly variables from 1959Q1 to 2008Q4; several of them are monthly and transformed into quarterly by taking averages. In order to get annualized log levels, I take logs and multiply by 4 most of variables, except federal funds rate and bond rate. I discussed at length the controversy about placing a flat prior on rotation matrix $\boldsymbol{Q}$. In order to address concerns of the recent literature, I also implement the approach in Giacomini and Kitagawa (2015) as robustness check.

For Model 1-3, Figure 2 displays the output responses. FEV restrictions dramatically shrink the identified set of output response and lead to informative inference, while alternative restrictions support neutrality of monetary policy and are largely uninformative. Remarkably, the inference implied by FEV restrictions is robust to Giacomini and Kitagawa (2015) algorithm. Note that the posterior median under FEV restrictions is consistently negative, as opposed to posterior medians induced by Model 1-2.

6 Empirical Application: Technology Shock

SECTION IN PROGRESS

7 Conclusion

Set-identified SVARs, which relax exclusion restrictions and rely on weaker assumptions such as sign restrictions, are increasingly common. However, a known drawback is that the inference is rarely informative. This paper shows that robust restrictions on the Forecast Error Variance (FEV) decomposition may dramatically shrink the inference. Specifically, these restrictions are consistent with the implications of a variety of different DSGE models, with both real and nominal frictions, and with sufficiently wide ranges for their parameters. First, in a bivariate and trivariate setting, this paper analytically proves restrictions on the FEV decomposition are more informative than traditional sign restrictions. Second, sufficient conditions are provided to guarantee that the identified set is non-empty and convex. Finally, two applications are provided: using models of monetary policy and technology shocks, restrictions on the FEV decomposition tend to be highly informative, greatly shrink and even change the inference of models originally identified via traditional sign restrictions. Remarkably, shrinkage in inference is robust to the recent concerns over the unintended consequences of rotation matrix prior (Baumeister and Hamilton, 2015).

8 Tables and Figures

Table 1

	Δy_t	Δc_t	ΔI_t	Δw_t	l_t	$\Delta \pi_t$	i_t
IRFs	-	-	-	-	-	-	+
FEV, $h = 0$	$[0.05, 0.15]$	$[0.03, 0.13]$	$[0.02, 0.12]$	$[0.00, 0.02]$	$[0.02, 0.09]$	$[0.00, 0.03]$	$[0.37, 0.66]$
FEV, $h = 1$	$[0.07, 0.17]$	$[0.06, 0.13]$	$[0.03, 0.12]$	$[0.01, 0.03]$	$[0.04, 0.11]$	$[0.03, 0.10]$	$[0.30, 0.60]$

Sign of impact responses and FEV decomposition at horizon $h = 0, 1$ to positive monetary policy shock from Smets and Wouters (2007). $+(-)$ indicate that a certain variable has the 90% probability to response positively (negatively) on impact. The bounds of the FEV decomposition represent the 5% and 95% percentiles.

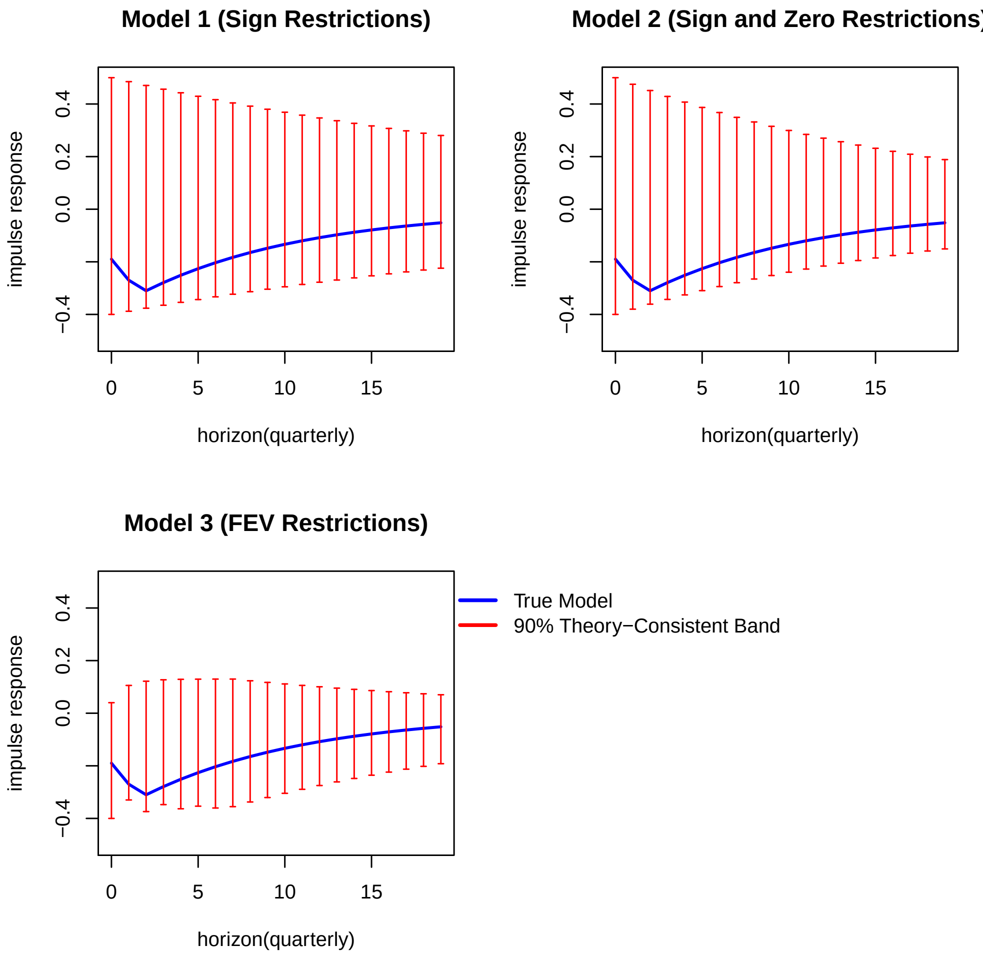

Figure 1: Population Analysis, Monte-Carlo Simulation

Notes: Figure 1 reports the theoretical DSGE impulse responses (blue) to contractionary monetary policy shock and the 90% range of theory-consistent responses (red vertical bars). See Section 5.1 for details. The shock size is set to one standard deviation.

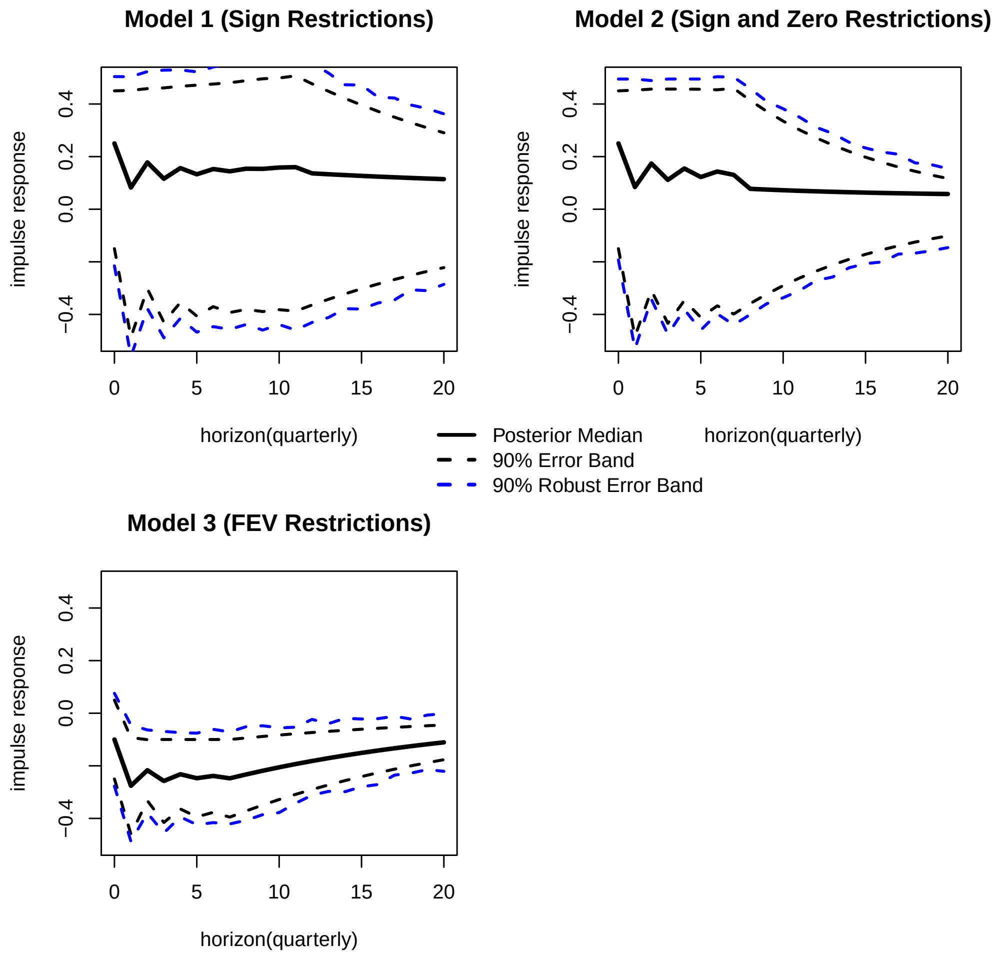

Figure 2: Output Impulse Responses to Contractionary Monetary Policy Shock, 7-variable SVAR

Notes: In each figure, the solid lines plot the posterior median and the dashed black lines show the correspondent 90% Bayesian credibility region. The dashed blue lines display the 90% robust Bayesian error band under multiple prior approach on $\boldsymbol{Q}$ a la Giacomini and Kitagawa (2015). Shock size is set to one standard deviation.

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Appendices

A Omitted Proofs

A.1 Bivariate Setting

Derivation of identified set (3.2).

Following Uhlig (2005), $\boldsymbol{A}$ can be parametrized via the Cholesky matrix $\boldsymbol{\Sigma}_{tr}$ and a rotation matrix $\boldsymbol{Q} = \begin{pmatrix} cos\rho & -sin\rho \\ sin\rho & cos\rho \end{pmatrix}$ with spherical coordinate $\rho \in [0, 2\pi]$. The structural matrix of impact responses can be written as

$$\boldsymbol{IR}^0 = \boldsymbol{A}^{-1} = \boldsymbol{\Sigma}_{tr}\boldsymbol{Q} = \begin{pmatrix} \sigma_{11}\cos\rho & -\sigma_{11}\sin\rho \\ \sigma_{21}\cos\rho + \sigma_{22}\sin\rho & -\sigma_{21}\sin\rho + \sigma_{22}\cos\rho \end{pmatrix}$$

and the parameter of interest is $\alpha \equiv \sigma_{11}\cos\rho$. Following Christiano, Eichenbaum, and Evans (1999), I impose the sign normalization restrictions by constraining the diagonal elements of $\boldsymbol{A}$ to being nonnegative,

$$\sigma_{22}\cos\rho - \sigma_{21}\sin\rho \geq 0 \tag{A.1}$$

and

$$\sigma_{11}\cos\rho \geq 0. \tag{A.2}$$

The identifying sign restrictions *SR1* and *SR2* in Section 3.1 are expressed as

$$\sigma_{11}\sin\rho \geq 0, \tag{A.3}$$

$$-\sigma_{22}\sin\rho - \sigma_{21}\cos\rho \geq 0. \tag{A.4}$$

Given $\boldsymbol{\phi}$, the identified set for $\alpha = \sigma_{11}\cos\rho$ is given by its range as ρ varies over the range characterized by the restrictions (A.1) - (A.4).

Assume $\sigma_{21} > 0$. Constraints (A.2) and (A.3) induce $\rho \in [0, \frac{\pi}{2}]$; constraints (A.1) and (A.4) imply $\rho \in [\arctan(-\sigma_{21}/\sigma_{22}), \arctan(\sigma_{22}/\sigma_{21})]$. Intersecting the two intervals leads to $[0, \arctan(\sigma_{22}/\sigma_{21})]$ as the identified set for ρ. Thus, for $\sigma_{21} > 0$ the identified set for α follows. A similar argument applies for $\sigma_{21} \leq 0$. ■

Derivation of identified set (3.3).

FEVR assumes that shock ϵ_2 explains y_2 more than y_1 on impact, i.e., the contribution of shock ϵ_2 to the total error variance of y_2 is higher than its contribution to that one of y_1. Following the notation introduced in Section ???, this restriction can be written as

$$\frac{FEV_{y_2,\epsilon_2}}{FEV_{y_2}} \geq \frac{FEV_{y_1,\epsilon_2}}{FEV_{y_1}}. \tag{A.5}$$

Given specification of $\boldsymbol{IR}^0$, note that

$$\begin{aligned} FEV_{y_2,\epsilon_2} &= (-\sigma_{21}sin\rho + \sigma_{22}cos\rho)^2, \\ FEV_{y_2} &= (-\sigma_{21}sin\rho + \sigma_{22}cos\rho)^2 + (\sigma_{21}cos\rho + \sigma_{22}sin\rho)^2 = \sigma_{21}^2 + \sigma_{22}^2, \\ FEV_{y_1,\epsilon_2} &= \sigma_{11}sin^2\rho, \\ FEV_{y_1} &= \sigma_{11}^2 sin^2\rho + \sigma_{11}^2 cos^2\rho = \sigma_{11}^2. \end{aligned}$$

Thus, restriction (A.5) can be written as

$$\frac{(-\sigma_{21} sin\rho + \sigma_{22} cos\rho)^2}{\sigma_{21}^2 + \sigma_{22}^2} \geq \frac{(-\sigma_{11} sin\rho)^2}{\sigma_{11}^2} \tag{A.6}$$

and imposes quadratic constraints on the two columns of $\boldsymbol{Q}$. Under the constraints (A.1) - (A.4) and (A.6), the argument used in the previous proof leads to the identified set for α in equation (3.3). $IS_\alpha(\boldsymbol{\phi})$ in equation (3.3) is restricted relative to the identified set in equation (3.2) as its lower bound gets closer to the upper bound. ■

Derivation of identified set (3.4).

In Section 3.1, *FEVR2* assumes that ϵ_1 explains y_2 more than ϵ_2:

$$(\sigma_{21} cos\rho + \sigma_{22} sin\rho)^2 \geq (-\sigma_{21} sin\rho + \sigma_{22} cos\rho)^2. \tag{A.7}$$

Following the same argument as the previous proof, constraints A.1-A.4 and A.7 deliver the identified set in equation (3.4). In order to evaluate the shrinkage induced by *FEVR2*, compare $IS_\alpha(\boldsymbol{\phi})$ in equation (3.3) and (3.4): for $\sigma_{21} > 0$, *FEVR2* shrinks the identified set of α with respect to the set induced by *SR1* and *SR2* if $\sigma_{22} \geq \sigma_{21}$, otherwise the restriction is redundant; specifically, the upper bound is now lower. For $\sigma_{21} \leq 0$, *FEVR2* restricts $IS_\alpha(\boldsymbol{\phi})$ if $\sigma_{22} \geq |\sigma_{21}|$; otherwise, it is at odds with *SR1* and delivers an empty identified set. ■

Example 3.1.

Restrictions are the following:

$$\sigma_{11} sin\rho \geq 0, \tag{A.8}$$

$$\frac{(-\sigma_{21} sin\rho + \sigma_{22} cos\rho)^2}{\sigma_{21}^2 + \sigma_{22}^2} \leq \frac{(-\sigma_{11} sin\rho)^2}{\sigma_{11}^2}, \tag{A.9}$$

and the usual sign normalizations on the main diagonal of $\boldsymbol{A}$.

The same argument as the previous proof delivers the identified set in equation (3.5):

$$IS_\alpha(\boldsymbol{\phi}) \equiv \begin{cases} \left[\sigma_{11} \cos\left(\arctan\left(\frac{\sigma_{21}}{\sigma_{22}}\right)\right), \sigma_{11} \cos\left(\arctan\left(\frac{\sigma_{22}}{\sigma_{21}+\sqrt{\sigma_{21}^2+\sigma_{22}^2}}\right)\right)\right], & \text{for } \sigma_{21} > 0, \\ \left[0, \sigma_{11} \cos\left(\arctan\left(\frac{\sigma_{22}}{\sigma_{21}+\sqrt{\sigma_{21}^2+\sigma_{22}^2}}\right)\right)\right], & \text{for } \sigma_{21} \leq 0. \end{cases} \tag{A.10}$$

Replacing restriction (A.9) with sign restriction on y_1, namely $-\sigma_{21} cos\rho - \sigma_{22} sin\rho \leq 0$, leads to the identified set in equation (3.2):

$$IS_\alpha(\boldsymbol{\phi}) \equiv \begin{cases} \left[\sigma_{11} \cos\left(\arctan\left(\frac{\sigma_{22}}{\sigma_{21}}\right)\right), \sigma_{11}\right], & \text{for } \sigma_{21} > 0, \\ \left[0, \sigma_{11} \cos\left(\arctan\left(-\frac{\sigma_{21}}{\sigma_{22}}\right)\right)\right], & \text{for } \sigma_{21} \leq 0. \end{cases} \tag{A.11}$$

Identified set in (A.10) is smaller than that in (A.11): for both $\sigma_{21} > 0$ and $\sigma_{21} \leq 0$, the upper bound in (A.10) is lower. ■

Example 3.2.

In order to derive the identified set in (3.2), the restrictions are the following:

$$\sigma_{11} sin\rho \geq 0, \tag{A.12}$$

$$-\sigma_{22} \sin\rho - \sigma_{21} \cos\rho \geq 0, \tag{A.13}$$

$$-\sigma_{21} \sin\rho + \sigma_{22} \cos\rho \geq \sigma_{11} \sin\rho, \tag{A.14}$$

where the last inequality ranks the impulse responses of variable y_1 and y_2 to shock ϵ_2. The usual sign normalizations on the main diagonal of $\boldsymbol{A}$ also apply. The same argument as the previous proofs delivers the identified set in (3.2); note that for $\sigma_{21} < 0, \sigma_{11} + \sigma_{21} \geq 0$, $IS_\alpha(\boldsymbol{\phi})$ is non-empty if and only if $\sigma_{22}^2 + \sigma_{21}^2 \geq -\sigma_{11}\sigma_{21}$. ■

A.2 Trivariate Setting

I consider a subspace of the set of all solutions in the trivariate context; this simplifies the analytical characterization of the identified set for α and the comparison between sign restrictions and restrictions on the FEV. However, the results from the bivariate context still persist.

Derivation of identified set (3.7).

In the trivariate setting, $\boldsymbol{Q}$ can be written as the product of three Givens matrices $\boldsymbol{Q}_{12}$, $\boldsymbol{Q}_{13}$ and $\boldsymbol{Q}_{23}$, each rotating a different pair of columns of the matrix to be transformed:

$$\boldsymbol{Q} = \begin{pmatrix} cos\rho_{12} & -sin\rho_{12} & 0 \\ sin\rho_{12} & cos\rho_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cos\rho_{13} & 0 & -sin\rho_{13} \\ 0 & 1 & 0 \\ sin\rho_{13} & 0 & cos\rho_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\rho_{23} & -sin\rho_{23} \\ 0 & sin\rho_{23} & cos\rho_{23} \end{pmatrix}$$

For simplicity, the main text limits the analysis to the case where $\rho_{13} = \rho_{23} = 0$, namely $\boldsymbol{Q}_{13} = \boldsymbol{Q}_{23} = \boldsymbol{I}_3$, $\boldsymbol{Q} = \boldsymbol{Q}_{12}$ and $\rho = \rho_{12}$. Thus, there are the following sign normalizations:

$$\sigma_{22} \cos\rho - \sigma_{21} \sin\rho \geq 0, \tag{A.15}$$

and

$$\sigma_{11} \cos\rho \geq 0. \tag{A.16}$$

The identifying sign restrictions *SR1*, *SR2* and *SR3* in Section 3.2 are

$$\sigma_{11} \sin\rho \geq 0 \tag{A.17}$$

$$-\sigma_{22} \sin\rho - \sigma_{21} \cos\rho \geq 0, \tag{A.18}$$

$$-\sigma_{31} \sin\rho + \sigma_{32} \cos\rho \geq 0. \tag{A.19}$$

Limiting the subspace of reduced-parameters such that $\sigma_{21} > 0$, $\sigma_{31} < 0$ and $\sigma_{32} < 0$, constraints (A.15) - (A.19) lead to the identified set for α in equation (3.7). ∎

Derivation of identified set (3.8).

This derivation still relies on the parameter subspace used to characterize identified set (3.7). In Section 3.2, *FEVR* assumes that shock ϵ_2 explains y_2 more than y_3: $\frac{FEV_{y_2,\epsilon_2}}{FEV_{y_2}} \geq \frac{FEV_{y_3,\epsilon_2}}{FEV_{y_3}}$. This implies that

$$\frac{(-\sigma_{21}sin\rho + \sigma_{22}cos\rho)^2}{\sigma_{21}^2 + \sigma_{22}^2} \geq \frac{(-\sigma_{31}sin\rho + \sigma_{32}cos\rho)^2}{\sigma_{31}^2 + \sigma_{32}^2 + \sigma_{33}^2} \tag{A.20}$$

Following the same argument as the previous proof, constraints A.15-A.19 and A.20 deliver the identified set in equation (3.8). Comparing the lower bound in equation (3.8) and (3.7), it is easy to see that *FEVR* sharpens the identified set of α. ∎