INFORMATION FLOW DEPENDENCE IN FINANCIAL MARKETS

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Abstract

In response to empirical evidence we propose a continuous-time model for multivariate asset returns providing a two-layered dependence structure. The price process is subject to multivariate information arrivals driving the market activity modeled by non-decreasing pure-jump Lévy processes. The jump dependence is determined by a Lévy copula allowing for a generic multivariate information flow and flexible dependence beyond the conditional aspects of the return distribution. Assuming that conditioning on the information flow asset returns are jointly normal, their dependence is modeled by a Brownian motion allowing for correlation. Furthermore, we provide an estimation framework based on maximum simulated likelihood. We apply novel multivariate models to equity data and obtain estimates which meet an economic intuition with respect to the two-layered dependence structure.

Key Words: Lévy processes, Lévy copulas, dependence modeling, weak multivariate subordination, variance gamma, simulated likelihood

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1 Introduction

Several studies provide evidence that financial returns – and in particular return volatilities – are primarily related to information arrivals measured by trading activity, e.g. Ross (1989), Gallant, Rossi and Tauchen (1992), Andersen (1996), and Ané and Geman (2000). When dealing with multivariate return data, a model should incorporate dependencies not only conditional on the accumulated information but also in the multivariate information flow process itself, as shown by Andersen, Bollerslev, Diebold and Ebens (2001) and references therein. This work proposes a continuous-time model for multivariate asset returns providing a two-layered dependence structure. We model the dependence of the information flow process using Lévy copulas, introduced by Kallsen and Tankov (2006). We overcome strong limitations of traditional multivariate approaches leading to more natural dynamics. We provide returns which are, conditional on the accumulated information, jointly normal and correlated, as we apply weak subordination of Buchmann, Madan and Lu (2017).

Frequently used continuous-time models in finance assume a log-return process

$$R(t) = \mu t + B(T(t)), \quad t \geq 0,$$

where $\mu \in \mathbb{R}$, B is a Brownian motion with drift and T is a non-decreasing stochastic process.[1] In terms of this representation, T serves as information flow process driving the return volatility and B is the directing process. Assuming independence of B and T, the return distribution is conditionally normal but obtains distortionary effects driven by the local information intensity, also considered as market activity. Barndorff-Nielsen and Shephard (2001) consider $T(t) = \int_0^t v(t)\,\mathrm{d}t$ and model the instantaneous variance of the log-return process by a non-Gaussian Ornstein-Uhlenbeck process $\mathrm{d}v(t) = -\lambda v(t)\,\mathrm{d}t + \mathrm{d}\tau(\lambda t)$, where τ is a non-decreasing Lévy process, called subordinator. The variance process is driven by the jumps of τ reflecting information arrivals, where $\lambda > 0$ controls for their persistence. Considering T as subordinator itself yields popular examples in the class of exponential Lévy market models constructed via subordination. The variance gamma

[1] In particular, Monroe (1978) proofed that every local semimartingale R can be represented as time changed Brownian motion, i.e. there exists a filtered probability space with a Brownian motion B and a stochastic time change T such that $R \overset{\mathcal{D}}{=} (B(T(t)))_{t\geq 0}$.

(VG) process (Madan and Seneta, 1990; Madan, Carr and Chang, 1998) and the normal inverse Gaussian (NIG) process (Barndorff-Nielsen, 1994), originated from gamma or inverse Gaussian processes as subordinator, respectively, are well known for their information flow interpretation.

In response to Andersen et al. (2001) a multivariate model should incorporate dependencies both in the directing process and the information flow process. A simplistic approach to measure these dependencies beyond log-return correlation is to investigate local standard deviations. Figure 1 shows the standardized 3-month log-volatility of major stock indices, clearly indicating moderate to strong dependencies in the return volatility and, thus, in the marginal market activity. A straightforward approach to obtain correlated subordinators is based on superpositioning, much in the spirit of Semeraro (2008) and Michaelsen and Szimayer (2017). However, strong limitations are apparent for modeling dependencies of subordinators with different marginal distributions or quite different marginal variance rates, since the multivariate Lévy measure does not have full support as is obtained from linear combinations of independent Lévy processes.

In this work we model the dependence structure of information flow processes using Lévy copulas, introduced by Kallsen and Tankov (2006) to model the jump-dependence of Lévy processes separately from the univariate margins. In particular, we use a pure-jump subordinator to model the joint structure of information arrivals driving the return process in financial models. The Lévy measure of the subordinator has in general full jump support leading to more natural dynamics and potentially stronger dependencies (see Figure 2). In contrast to traditional approaches, there are no limitations arising from the choice of the marginal distributions providing flexible dependence beyond plain correlations of asset returns. Since the application is technically challenging, we limit ourselves for now to the case of an exponential Lévy market model. Besides these aspects of the information flow process, dependencies in the directing process are indispensable when modeling financial return data. However, traditional multivariate subordination of Barndorff-Nielsen, Pedersen and Sato (2001) requires the subordinated process to have independent components. Using a Brownian motion as directing process implies marginal returns which are conditionally normal but independent. To overcome this shortcoming we apply weak subordination of Buchmann et al. (2017). We introduce dependencies also in

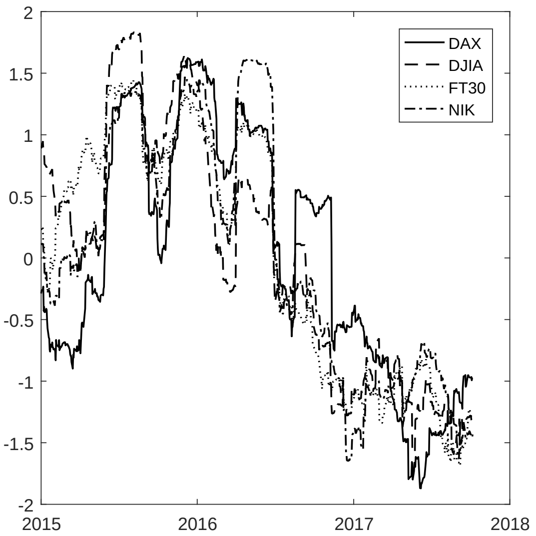

Figure 1: *Standardized 3-month log-volatility based on daily percentage log-returns of major stock indices.*

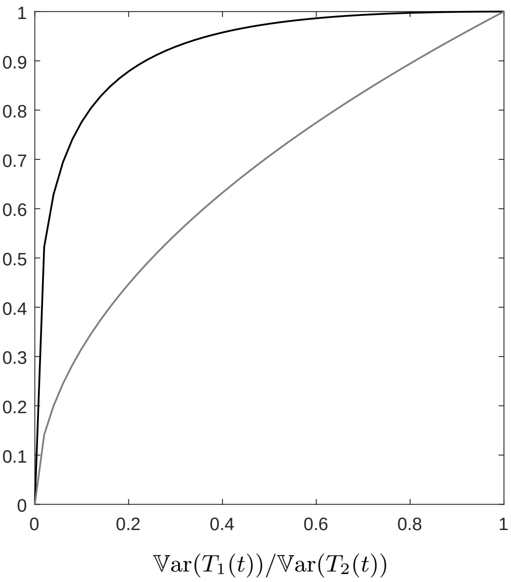

Figure 2: *Maximum correlation for a bivariate Gamma process* $\boldsymbol{T}$ *based on linear combinations (gray plot) or Lévy copulas (black plot).*

the directing process and provide returns which are conditionally normal and correlated. The choice of the subordinator yields a wide range of marginal distributions, e.g. VG or NIG, which are flexibly modeled together in the stochastic time as well as the space dimension. We state the moments and comoments and analyze the multivariate conditional distribution. Furthermore, we present an approach for simulation and likelihood-based estimation. As an example we study novel multivariate variance gamma models, estimate these models based on quarterly financial data and interpret the results particularly with regard to the dependence of the marginal market activities.

The paper is organized as follows. Section 2 contains the preliminaries related to Lévy processes, weak subordination and Lévy copulas. Section 3 introduces the model framework and states properties. Simulation and estimation procedures are given by Section 4. The empirical study is contained in Section 5. Section 6 concludes the paper.

2 Prelimiaries

We recall the necessary theory of Lévy processes and positive Lévy copulas. For a detailed analysis we refer to the books of Sato (1999) and Cont and Tankov (2004). Furthermore, Lévy copulas are investigated by Kallsen and Tankov (2006).

2.1 *Lévy Processes and Multivariate Subordination*

For any set S let $S_* := S \setminus \{\mathbf{0}\}$. For the row vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ and the square matrix $\mathrm{M} \in \mathbb{R}^{d\times d}$ let $\boldsymbol{x} \diamond \boldsymbol{y} := (x_1y_1, \ldots, x_dy_d)$, $(\boldsymbol{x} \diamond \mathbf{M})_{kl} := (x_k \wedge x_l)\mathrm{M}_{kl}$, $1 \leq k, l \leq d$, and set $\langle \boldsymbol{x}, \boldsymbol{y}\rangle_{\mathbf{M}} := \langle \boldsymbol{x}\mathbf{M}, \boldsymbol{y}\rangle$ and $\|\boldsymbol{x}\|^2_{\mathbf{M}} := \langle \boldsymbol{x}, \boldsymbol{x}\rangle_{\mathbf{M}}$. Let $\mathbb{D} := \{\boldsymbol{x} \in \mathbb{R} \mid \|\boldsymbol{x}\| \leq 1\}$.

Let $\boldsymbol{X}$ be a Lévy process on $\mathbb{R}^d$. The distribution of $\boldsymbol{X}(t)$ is infinitely divisible for any $t \geq 0$, and conversely for any infinitely divisible distribution F on $\mathbb{R}^d$ there exists a Lévy process $\boldsymbol{X}$ with $\boldsymbol{X}(1) \sim F$. The characteristic function of $\boldsymbol{X}$ fulfills the Lévy-Khintchine representation $\varphi_{\boldsymbol{X}(t)}(\boldsymbol{z}) := \mathbb{E}[\exp(i\langle \boldsymbol{z}, \boldsymbol{X}(t)\rangle)] = \exp(t\Psi_{\boldsymbol{X}}(\boldsymbol{z}))$ with

$$\Psi_{\boldsymbol{X}}(\boldsymbol{z}) = -\frac{1}{2}\|\boldsymbol{z}\|^2_{\boldsymbol{\Sigma_X}} + i\langle \boldsymbol{\gamma_X}, \boldsymbol{z}\rangle + \int_{\mathbb{R}^d_*} (e^{i\langle \boldsymbol{z},\boldsymbol{x}\rangle} - 1 - i\langle \boldsymbol{z}, \boldsymbol{x}\rangle \mathbb{1}_{\mathbb{D}}(\boldsymbol{x}))\, \Pi_{\boldsymbol{X}}(\mathrm{d}\boldsymbol{x}), \quad (2.1)$$

for $\boldsymbol{z} \in \mathbb{R}^d$ where $\boldsymbol{\Sigma_X}$ is a symmetric nonnegative-definite $d \times d$ matrix, $\boldsymbol{\gamma_X} \in \mathbb{R}^d$, and $\Pi_{\boldsymbol{X}}$ is a Borel measure on $\mathbb{R}^d_*$ satisfying $\int_{\mathbb{R}^d_*} 1 \wedge \|\boldsymbol{x}\|^2\, \Pi_{\boldsymbol{X}}(\mathrm{d}\boldsymbol{x}) < \infty$. The triplet $(\boldsymbol{\gamma_X}, \boldsymbol{\Sigma_X}, \Pi_{\boldsymbol{X}})$ is called *Lévy triplet* of $\boldsymbol{X}$, and $\boldsymbol{\Sigma_X}$, $\Pi_{\boldsymbol{X}}$ and $\Psi_{\boldsymbol{X}}$ are called the *(Gaussian) covariance matrix*, *Lévy measure* and *characteristic exponent* of $\boldsymbol{X}$, respectively. The distribution of $\boldsymbol{X}$ is uniquely determined by $\boldsymbol{\gamma_X}$, $\boldsymbol{\Sigma_X}$ and $\Pi_{\boldsymbol{X}}$. Conversely, if $\boldsymbol{\gamma_X}$, $\boldsymbol{\Sigma_X}$ and $\Pi_{\boldsymbol{X}}$ are of the forms in (2.1), there exists a Lévy process $\boldsymbol{X}$ whose characteristic exponent is given by (2.1). Therefore, we write $\boldsymbol{X} \sim L^d(\boldsymbol{\gamma_X}, \boldsymbol{\Sigma_X}, \Pi_{\boldsymbol{X}})$ if $\boldsymbol{X}$ is a Lévy process on $\mathbb{R}^d$ with Lévy triplet $(\boldsymbol{\gamma_X}, \boldsymbol{\Sigma_X}, \Pi_{\boldsymbol{X}})$. The process $\boldsymbol{B} \sim BM^d(\boldsymbol{m}, \boldsymbol{\Sigma}) := L^d(\boldsymbol{m}, \boldsymbol{\Sigma}, 0)$ is called *d-dimensional Brownian Motion* with drift $\boldsymbol{m}$ and covariance matrix $\boldsymbol{\Sigma}$. In particular, $\boldsymbol{B}(t)$ has normal (or Gaussian) distribution denoted $\boldsymbol{B}(t) \sim \mathcal{N}\{\mathrm{d}\boldsymbol{x} \mid t\boldsymbol{m}, t\boldsymbol{\Sigma}\}$ for $t \geq 0$.

A Lévy process $\boldsymbol{X} \sim L^d(\boldsymbol{\gamma_X}, \boldsymbol{\Sigma_X}, \Pi_{\boldsymbol{X}})$ is of *finite variation* if and only if $\boldsymbol{\Sigma_X} = \mathbf{0}$ and $\int_{\mathbb{R}^d_*} 1 \wedge \|\boldsymbol{x}\|\, \Pi_{\boldsymbol{X}}(\mathrm{d}\boldsymbol{x}) < \infty$. In this case, $\boldsymbol{X}$ can be expressed as the sum of its jumps and a linear term,

$$\boldsymbol{T}(t) = \boldsymbol{d_X} t + \sum_{s\in[0,t]} \Delta \boldsymbol{T}(s),$$

where $\boldsymbol{d_X} = \boldsymbol{\gamma} - \int_{0<\|\boldsymbol{x}\|\leq 1} \boldsymbol{x}\, \Pi_{\boldsymbol{X}}(\mathrm{d}\boldsymbol{x})$ is called *drift* of $\boldsymbol{X}$. We write $\boldsymbol{X} \sim FV^d(\boldsymbol{d_X}, \Pi_{\boldsymbol{X}})$ if $\boldsymbol{X}$ is of finite variation with drift $\boldsymbol{d_X}$.

Definition 2.1 (Strong Subordination). Let $\boldsymbol{T} \sim S^d(\boldsymbol{d_T}, \Pi_{\boldsymbol{T}}) = FV^d(\boldsymbol{d_T}, \Pi_{\boldsymbol{T}})$ a *subordinator*, that is, a Lévy process with almost surely non-decreasing sample paths in every component. Let $\boldsymbol{X} \sim L^d(\boldsymbol{\gamma_X}, \boldsymbol{\Sigma_X}, \Pi_{\boldsymbol{X}})$, independent of $\boldsymbol{T}$. One introduces a new process $\boldsymbol{Y}$ on $\mathbb{R}^d$, by setting

$$\boldsymbol{Y}(t) = (\boldsymbol{X} \circ \boldsymbol{T})(t) := \big(X_1(T_1(t)), \ldots, X_d(T_d(t))\big) \qquad t \geq 0.$$

Then $\boldsymbol{Y} = \boldsymbol{X} \circ \boldsymbol{T}$ is called $\boldsymbol{T}$ *subordinating* $\boldsymbol{X}$ *(in the strong sense).*

In literature, there are two important cases known where $\boldsymbol{Y}$ is again a Lévy process:

(1) Univariate subordination, nested in multivariate theory by taking indistinguishable $T_1, \ldots T_d$, always creates Lévy processes. The Lévy triplet, characteristic exponent and distribution of $\boldsymbol{Y}$ is known, see Sato (1999) for a detailed analysis.

(2) If $\boldsymbol{X}$ has independent components $X_1, \ldots, X_d$ then $\boldsymbol{Y} = \boldsymbol{X} \circ \boldsymbol{T}$ is again a Lévy process. We refer to Barndorff-Nielsen et al. (2001) for an extensive theory.

Buchmann et al. (2017) introduce a more general way of multivariate subordination, called weak subordination. Weak subordination extends the traditional theory of e.g. Barndorff-Nielsen et al. (2001) consistently and is especially able to deal with dependent components of the subordinated process. We restrict our perspective to the less complex case of zero-drift subordinators, i.e. we assume $\boldsymbol{d_T} = \boldsymbol{0}$ is fulfilled. Let $\boldsymbol{X}(\boldsymbol{t}) := (X_1(t_1), \ldots, X_d(t_d))$ be the random vector resulting from the Lévy process $\boldsymbol{X} \sim L^d(\boldsymbol{\gamma_X}, \boldsymbol{\Sigma_X}, \Pi_{\boldsymbol{X}})$ evaluated at multivariate time $\boldsymbol{t} \in [0, \infty)^d$, see Barndorff-Nielsen et al. (2001) for details on multiparameter Lévy processes.

Definition 2.2 (Weak Subordination). Let $\boldsymbol{T} \sim S^d(\boldsymbol{0}, \Pi_{\boldsymbol{T}})$ be a subordinator and $\boldsymbol{X} \sim L^d(\boldsymbol{\gamma_X}, \boldsymbol{\Sigma_X}, \Pi_{\boldsymbol{X}})$ a Lévy process on $\mathbb{R}^d$. A process $\boldsymbol{Y}$ is called $\boldsymbol{T}$ *subordinating* $\boldsymbol{X}$ *in the weak sense*, meaning that $\boldsymbol{Y} \overset{\mathcal{D}}{=} \boldsymbol{X} \odot \boldsymbol{T}$, whenever $\boldsymbol{Y} \sim L^d(\boldsymbol{\gamma_Y}, \boldsymbol{\Sigma_Y}, \Pi_{\boldsymbol{Y}})$ is a Lévy process with the characteristics determined by the following formulae:

$$\begin{aligned} \boldsymbol{\gamma_Y} &= \int_{[0,\infty)^d_*} \mathbb{E}[\boldsymbol{X}(\boldsymbol{t}) \mathbb{1}_{\mathbb{D}}(\boldsymbol{X}(\boldsymbol{t}))] \, \Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t}), \qquad (2.2) \\ \boldsymbol{\Sigma_Y} &= \boldsymbol{0}, \\ \Pi_{\boldsymbol{Y}}(\mathrm{d}\boldsymbol{x}) &= \int_{[0,\infty)^d_*} \mathbb{P}\{\boldsymbol{X}(\boldsymbol{t}) \in \mathrm{d}\boldsymbol{x}\} \, \Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t}). \end{aligned}$$

Buchmann et al. (2017) show that these quantities are valid Lévy characteristics. The characteristic exponent of $\boldsymbol{Y} \overset{\mathcal{D}}{=} \boldsymbol{X} \odot \boldsymbol{T}$ as defined above is given by

$$\Psi_{\boldsymbol{Y}}(\boldsymbol{z}) = \int_{[0,\infty)^d_*} \left(\mathbb{E}\left[e^{i\langle \boldsymbol{z}, X(\boldsymbol{t})\rangle}\right] - 1\right) \Pi_T(\mathrm{d}\boldsymbol{t}), \qquad \boldsymbol{z} \in \mathbb{R}^d. \tag{2.3}$$

Furthermore, the following consistency results are valid. The properties (i)–(ii) even hold in a semi-pathwise sense (see Buchmann et al. (2017) for proofs and details).

PROPOSITION 2.3. *Let $\boldsymbol{T} \sim S^d(\boldsymbol{0}, \Pi_{\boldsymbol{T}})$ and $\boldsymbol{X} \sim L^d(\boldsymbol{\gamma_X}, \boldsymbol{\Sigma_X}, \Pi_{\boldsymbol{X}})$.*

(i) If $\boldsymbol{X}$ and $\boldsymbol{T}$ are independent and either $T_1 \equiv \ldots \equiv T_d$, or $\boldsymbol{X}$ has independent components, then $\boldsymbol{X} \odot \boldsymbol{T} \overset{\mathcal{D}}{=} \boldsymbol{X} \circ \boldsymbol{T}$.

(ii) It holds that $(\boldsymbol{X} \odot \boldsymbol{T})_k \overset{\mathcal{D}}{=} X_k \odot T_k \overset{\mathcal{D}}{=} X_k \circ T_k$.

(iii) If $\boldsymbol{T}$ has independent components, then so are the components of $\boldsymbol{X} \odot \boldsymbol{T}$.

(iv) If $\boldsymbol{X}$ and $\boldsymbol{T}$ are independent and $\boldsymbol{T}$ has monotone components $T_1 \leq \ldots \leq T_d$, then $\boldsymbol{X} \odot \boldsymbol{T}(t) \overset{\mathcal{D}}{=} \boldsymbol{X} \circ \boldsymbol{T}(t)$ for all fixed $t \geq 0$.

(v) Let $\widetilde{\boldsymbol{T}} \sim S^d(\boldsymbol{0}, \Pi_{\widetilde{\boldsymbol{T}}})$ be independent from $\boldsymbol{T}$. Then $\boldsymbol{X} \odot (\boldsymbol{T} + \widetilde{\boldsymbol{T}}) \overset{\mathcal{D}}{=} \boldsymbol{X} \odot \boldsymbol{T} + \boldsymbol{X} \odot \widetilde{\boldsymbol{T}}$.

(vi) If $\boldsymbol{d_T} = \boldsymbol{0}$ and $\int_{[0,1]^d_*} \|\boldsymbol{t}\|^{1/2} \Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t})$, then $\boldsymbol{X} \odot \boldsymbol{T}$ is of finite variation.

2.2 *Positive Lévy Copulas*

Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. For $\boldsymbol{a}, \boldsymbol{b} \in \overline{\mathbb{R}}^d$ we write $\boldsymbol{a} \leq \boldsymbol{b}$ if $a_k \leq b_k$, $k = 1, .., d$, and $(\boldsymbol{a}, \boldsymbol{b}] = (a_1, b_1] \times \ldots \times (a_d, b_d]$. Analogously, we define $(\boldsymbol{a}, \boldsymbol{b})$, $[\boldsymbol{a}, \boldsymbol{b})$, $[\boldsymbol{a}, \boldsymbol{b}]$.

DEFINITION 2.4. Let $C\colon S \to \overline{\mathbb{R}}$ for $S \subseteq \overline{\mathbb{R}}^d$. For $\boldsymbol{a}, \boldsymbol{b} \in S$, $\boldsymbol{a} \leq \boldsymbol{b}$, the *$C$-volume* of $(\boldsymbol{a}, \boldsymbol{b}] \subseteq S$ is defined by

$$V_C((\boldsymbol{a}, \boldsymbol{b}]) := \sum_{\boldsymbol{u} \in \{a_1, b_1\} \times \ldots \times \{a_d, b_d\}} (-1)^{N_{\boldsymbol{a}}(\boldsymbol{u})} C(\boldsymbol{u}),$$

where $N_{\boldsymbol{a}}(\boldsymbol{u}) := \#\{k \mid u_k = a_k\}$. If $V_C((\boldsymbol{a}, \boldsymbol{b}]) \geq 0$ for all $(\boldsymbol{a}, \boldsymbol{b}] \subseteq S$, C is called d-increasing.

DEFINITION 2.5 (Positive Lévy Copula). A d-increasing function $C\colon [0,\infty]^d \to [0,\infty]$ is called *positive Lévy copula* if

(i) $C(\boldsymbol{u}) = 0$ if $u_k = 0$ for at least one $k \in \{1, \ldots, d\}$,

(ii) $C(\boldsymbol{u}) = u_k$ if $u_j = \infty$ for all $j \in \{1, \ldots, d\} \setminus \{k\}$,

(iii) $C(\boldsymbol{u}) \neq \infty$ for $\boldsymbol{u} \neq (\infty, \ldots, \infty)$.

If $\boldsymbol{T} \sim S^d(\boldsymbol{d_T}, \Pi_{\boldsymbol{T}})$, the *tail integral* of $\boldsymbol{T}$ (or $\Pi_{\boldsymbol{T}}$) is the function $U_{\boldsymbol{T}}\colon [0,\infty]^d \to \mathbb{R}$ defined by

$$U_{\boldsymbol{T}}(\boldsymbol{t}) := \begin{cases} \infty, & \boldsymbol{t} = \boldsymbol{0} \\ \Pi_{\boldsymbol{T}}\left([\boldsymbol{t}, \boldsymbol{\infty})\right), & \boldsymbol{t} \in [0,\infty)^d_* \\ 0, & \text{if } t_k = \infty \text{ for at least one } k \in \{1,\ldots,d\}. \end{cases}$$

The technical requirement $U_{\boldsymbol{T}}(\boldsymbol{0}) := \infty$, also valid for $\Pi_{\boldsymbol{T}}$ finite, does not change anything, since a Lévy measure is completely determined on $[0,\infty)^d_*$. Furthermore, $U_{T_k}(t_k) = U_{\boldsymbol{T}}(0,\ldots,0,t_k,0,\ldots,0)$, $t_k \geq 0$, $k = 1,\ldots,d$.

PROPOSITION 2.6 (Kallsen and Tankov, 2006). *Let $\boldsymbol{T} \sim S^d(\boldsymbol{d_T}, \Pi_{\boldsymbol{T}})$. Then there exists a positive Lévy copula C such that the tail integrals of $\boldsymbol{T}$ satisfy*

$$U_{\boldsymbol{T}}(\boldsymbol{t}) = C\left(U_{T_1}(t_1),\ldots,U_{T_d}(t_d)\right) \qquad \textit{for all } \boldsymbol{t} \in \mathbb{R}^d_*. \tag{2.4}$$

If the marginal tail integrals $U_{T_1},\ldots,U_{T_d}$ are continuous then C is unique, otherwise it is unique on $\times_{k=1}^d \operatorname{Ran}(U_{T_k})$. Conversely, if C is a positive Lévy copula on $[0,\infty]^d$ and $T_k \sim S^1(d_{T_k}, \Pi_{T_k})$, $k = 1,\ldots,d$, then $C\left(U_{T_1}(t_1),\ldots,U_{T_d}(t_d)\right)$ is the tail integral of a d-dimensional subordinator $\boldsymbol{T} \sim S^d(\boldsymbol{d_T}, \Pi_{\boldsymbol{T}})$.

3 Subordination using Positive Lévy Copulas

We model the log-return process $\boldsymbol{R}$ by a pure-jump Lévy process $\boldsymbol{Y} \sim L^d(\boldsymbol{\gamma_Y}, \boldsymbol{0}, \Pi_{\boldsymbol{Y}})$, possibly shifted by a location parameter $\boldsymbol{\mu} \in \mathbb{R}^d$,

$$\boldsymbol{R}(t) = \boldsymbol{\mu} t + \boldsymbol{Y}(t)$$

We assume $\boldsymbol{Y} \overset{\mathcal{D}}{=} \boldsymbol{X} \odot \boldsymbol{T}$ with directing process $\boldsymbol{X} \sim L^d(\boldsymbol{\gamma_X}, \boldsymbol{\Sigma_X}, \Pi_{\boldsymbol{X}})$ and information flow process $\boldsymbol{T} \sim S^d(\boldsymbol{0}, \Pi_{\boldsymbol{T}})$, independent from $\boldsymbol{X}$. The jumps by which $\boldsymbol{T}$ increases reflect information arrivals driving the log-return process. In particular, these jumps occur multivariately which controls for the amount of information relevant for the individual components. The jump-dependence of $\boldsymbol{T}$ is modeled with a Lévy copula, thus, the tail measure of $\boldsymbol{T}$ is of the form (2.4). This allows for a separate treatment of the dependence structure from arbitrary marginal distributions of the information flow process and, furthermore, for full support of the Lévy measure $\Pi_{\boldsymbol{T}}$. The directing process $\boldsymbol{X}$ controls for the jump size distribution of $\boldsymbol{Y}$, i.e.

$$\Pi_{\boldsymbol{Y}}(\mathrm{d}\boldsymbol{x}) = \int_{[0,\infty)^d_*} \mathbb{P}\{\boldsymbol{X}(\boldsymbol{t}) \in \mathrm{d}\boldsymbol{x}\}\, \Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t}), \qquad \boldsymbol{x} \in \mathbb{R}^d_*.$$

In economic terms, the directing process determines how the information arrivals are processed. Thus, we can exploit the additional dependence provided by $\boldsymbol{X}$. The process $\boldsymbol{Y}$ is a pure-jump Lévy process with jumps inherited from the subordinator.[2] By Proposition 2.3, we observe marginal consistency $Y_k \overset{\mathcal{D}}{=} X_k \circ T_k$, thus, we model univariate Lévy processes originated by subordination together both in the time and space dimension. In general, the dependence structure will be affected by the Lévy copula C and the dependence structure of the directing process $\boldsymbol{X}$, which is Gaussian in popular applications. The framework provides returns which are conditionally normal and correlated.

Proposition 3.1. *Let $\boldsymbol{Y} \overset{\mathcal{D}}{=} \boldsymbol{B} \odot \boldsymbol{T}$, where $\boldsymbol{B} \sim BM^d(\boldsymbol{m}, \boldsymbol{\Sigma})$ and $\boldsymbol{T} \sim S^d(\boldsymbol{0}, \Pi_{\boldsymbol{T}})$ are independent. Then, for $t > 0$,*

$$\mathbb{P}\big\{\boldsymbol{Y}(t) \in \mathrm{d}\boldsymbol{x} \,\big|\, \boldsymbol{T}\big\} = \mathcal{N}\Big\{\mathrm{d}\boldsymbol{x} \,\Big|\, \sum_{s\in[0,t]} \Delta\boldsymbol{T}(s) \diamond \boldsymbol{m}, \sum_{s\in[0,t]} \Delta\boldsymbol{T}(s) \diamond \boldsymbol{\Sigma}\Big\}. \tag{3.1}$$

Proof. Let $\mathcal{S}$ be the set of all permutations of $\{1,\dots,d\}$. There exists a partition $[0,\infty)^d_* = \bigcup_{\sigma\in\mathcal{S}} A_\sigma$ such that every $\boldsymbol{t} \in A_\sigma$ fulfills $t_{\sigma(1)} \le \ldots \le t_{\sigma(d)}$, $\sigma \in \mathcal{S}$. Now

$$\boldsymbol{T}(t) = \sum_{s\in[0,t]} \Delta\boldsymbol{T}(s) = \sum_{\sigma\in\mathcal{S}} \sum_{s\in[0,t]} \mathbb{1}_{A_\sigma}(\Delta\boldsymbol{T}(s))\Delta\boldsymbol{T}(s) =: \sum_{\sigma\in\mathcal{S}} \boldsymbol{T}^\sigma(t),$$

and $\boldsymbol{T}^\sigma$, $\sigma \in \mathcal{S}$, are independent with $T^\sigma_{\sigma(1)} \le \ldots \le T^\sigma_{\sigma(d)}$. Thus, with Proposition 2.3,

$$\boldsymbol{B} \odot \boldsymbol{T}(t) \overset{\mathcal{D}}{=} \boldsymbol{B} \odot \Big(\sum_{\sigma\in\mathcal{S}} \boldsymbol{T}^\sigma\Big)(t) \overset{\mathcal{D}}{=} \sum_{\sigma\in\mathcal{S}} \boldsymbol{B} \circ \boldsymbol{T}^\sigma(t).$$

In particular, $\boldsymbol{B} \circ \boldsymbol{T}^\sigma$, $\sigma \in \mathcal{S}$, are independent and

$$\mathbb{P}\{\boldsymbol{B} \circ \boldsymbol{T}^\sigma(t) \in \mathrm{d}\boldsymbol{x} \mid \boldsymbol{T}\} = \mathcal{N}\big\{\mathrm{d}\boldsymbol{x} \,\big|\, \boldsymbol{T}^\sigma(t) \diamond \boldsymbol{m}, \boldsymbol{T}^\sigma(t) \diamond \boldsymbol{\Sigma}\big\}$$

[2] Such a pathwise representation exists on a possibly augmented probability space, see Buchmann et al. (2017).

$$= \mathcal{N}\Big\{\mathrm{d}\boldsymbol{x} \,\Big|\, \sum_{s\in[0,t]} \Delta\boldsymbol{T}^{\sigma}(s) \diamond \boldsymbol{m}, \sum_{s\in[0,t]} \Delta\boldsymbol{T}^{\sigma}(s) \diamond \boldsymbol{\Sigma}\Big\},$$

Finally, note that the sum of independent Gaussian random vectors is again Gaussian with the sum of the mean vectors and covariance matrices. □

Note that (3.1) shows that the conditional covariance is subject to the amount of relevant cross information represented by the minimum jump process, i.e.

$$\mathbb{C}\mathrm{ov}(Y_k(t), Y_l(t) \mid \boldsymbol{T}) = \Sigma_{kl} \sum_{s\in[0,t]} \Delta T_k(s) \wedge \Delta T_l(s) \tag{3.2}$$

for $1 \le k, l \le d$. To recap, strong multivariate subordination of Barndorff-Nielsen et al. (2001) requires $\Sigma_{kl} = 0$ for $k \neq l$, which implies conditional independence. The subordinator $\boldsymbol{T}$ introduces non-Gaussian properties such as skewness and excess kurtosis. Thus, its jump-dependence controls for higher order comoments such as coskeness and cokurtosis and, in particular, the dependence of the conditional variance processes, i.e.

$$\mathbb{C}\mathrm{orr}\big(\mathbb{V}\mathrm{ar}(Y_1(t) \mid \boldsymbol{T}), \mathbb{V}\mathrm{ar}(Y_2(t) \mid \boldsymbol{T})\big) = \mathbb{C}\mathrm{orr}(T_1(t), T_2(t)). \tag{3.3}$$

With (3.2) and the law of total expectation, $\boldsymbol{Y} \stackrel{\mathcal{D}}{=} \boldsymbol{B} \odot \boldsymbol{T}$ has covariance

$$\mathbb{C}\mathrm{ov}(Y_k(t), Y_l(t)) = t\,\Sigma_{kl} \int_{[0,\infty)^d_*} t_k \wedge t_l \,\Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t}) + t\, m_k m_l \int_{[0,\infty)^d_*} t_k t_l \,\Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t})$$

for $1 \le k, l \le d$. The expected value of the minimum jump processes and the joint cumulants of $\boldsymbol{T}$ can be represented in terms of the tail integral.

LEMMA 3.2. *Let $\boldsymbol{T} \sim S^2(\boldsymbol{d_T}, \Pi_{\boldsymbol{T}})$ with tail integral $U_{\boldsymbol{T}}$. Then*

$$\int_{[0,\infty)^2_*} t_1 \wedge t_2 \,\Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t}) = \int_{(0,\infty)} U_{\boldsymbol{T}}(u,u)\,\mathrm{d}u,$$
$$\int_{[0,\infty)^2_*} t_1^{m_1} t_2^{m_2} \,\Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t}) = \int_{[0,\infty)^2_*} U_{\boldsymbol{T}}\big(u_1^{1/m_1}, u_2^{1/m_2}\big)\,\mathrm{d}\boldsymbol{u},$$

for $m_1, m_2 > 0$, provided the corresponding integrals are finite.

Proof. Note that $t_1 \wedge t_2 = \int_{(0,\infty)} \mathbb{1}_{[u,\infty)}(t_1)\mathbb{1}_{[u,\infty)}(t_2)\mathrm{d}u$ and $t_1 t_2 = \int_{[0,\infty)^2_*} \mathbb{1}_{[\boldsymbol{u},\boldsymbol{\infty})}(\boldsymbol{t})\,\mathrm{d}\boldsymbol{u}$. The result follows from Fubini's Theorem. □

Analogously, higher order comoments of $\boldsymbol{T}$ are obtained.

PROPOSITION 3.3. *Let* $\boldsymbol{Y} \overset{\mathcal{D}}{=} \boldsymbol{X} \odot \boldsymbol{T}$ *and* $\overline{\boldsymbol{Y}} := \boldsymbol{Y}(t) - \mathbb{E}[\boldsymbol{Y}(t)]$ *for a* $t \geq 0$, *where* $\boldsymbol{T} \sim S^d(\boldsymbol{0}, \Pi_{\boldsymbol{T}})$ *and* $\boldsymbol{X} \sim L^d(\boldsymbol{\gamma_X}, \Sigma_{\boldsymbol{X}}, \Pi_{\boldsymbol{X}})$ *are independent. Then*

$$
\begin{aligned}
\mathbb{E}[\overline{Y}_i\overline{Y}_j] &= t\int_{[0,\infty)^d_*} \mathbb{E}[X_i(s_i)X_j(s_j)]\,\Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{s}),\\
\mathbb{E}[\overline{Y}_i\overline{Y}_j\overline{Y}_k] &= t\int_{[0,\infty)^d_*} \mathbb{E}[X_i(s_i)X_j(s_j)X_k(s_k)]\,\Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{s}),\\
\mathbb{E}[\overline{Y}_i\overline{Y}_j\overline{Y}_k\overline{Y}_k] &= t\int_{[0,\infty)^d_*} \mathbb{E}[X_i(s_i)X_j(s_j)X_k(s_k)X_l(s_l)]\,\Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{s}) + \mathbb{E}[\overline{Y}_i\overline{Y}_j]\mathbb{E}[\overline{Y}_k\overline{Y}_l]\\
&\qquad + \mathbb{E}[\overline{Y}_i\overline{Y}_k]\mathbb{E}[\overline{Y}_j\overline{Y}_l] + \mathbb{E}[\overline{Y}_i\overline{Y}_l]\mathbb{E}[\overline{Y}_j\overline{Y}_k],
\end{aligned}
$$

for $i, j, k, l \in \{1, \ldots, d\}$, *provided the corresponding absolute moments are finite.*

Proof. The result follows by applying Fubini's theorem to the the partial derivatives of the characteristic function based on (2.3). For the $\Pi_{\boldsymbol{T}}$-integrability of the comoments of $\boldsymbol{X}$ we refer to Buchmann et al. (2017). □

To describe the jump dependence of the subordinator $\boldsymbol{T}$ we consider known classes of positive Lévy copulas, e.g. mentioned in Cont and Tankov (2004).

(i) The *independence Lévy copula* is defined by

$$
C^{\perp}(\boldsymbol{u}) = \sum_{k=1}^{d} u_k \prod_{\substack{j=1\\ j\neq k}}^{d} \mathbb{1}_{\{\infty\}}(u_j).
$$

(ii) The *complete dependence Lévy copula* is defined by

$$
C^{\|}(\boldsymbol{u}) = \min(u_1, \ldots, u_d).
$$

(iii) Let $\phi\colon [0,\infty] \to [0,\infty]$ be strictly decreasing with $\phi(0) = \infty$ and $\phi(\infty) = 0$, having derivatives of order up to d on $(0,\infty)$, satisfying $(-1)^k \frac{\partial^k \phi(u)}{\partial u^k} \geq 0$, for all $k = 0, \ldots, d$. Then

$$
C^{\phi}(\boldsymbol{u}) = \phi\Big(\sum_{k=1}^{d} \phi^{-1}(u_i)\Big)
$$

is called an *Archimedean Lévy copula* and ϕ is called *generator* of C^{ϕ}.

If both the Lévy copula and the marginal tail integrals U_{T_k}, $k = 1, \dots, d$ of the subordinator are sufficiently smooth, the Lévy density can be computed by differentiation,

$$\Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t}) = \left.\frac{\partial^d C(\boldsymbol{u})}{\partial u_1 \cdots \partial u_d}\right|_{\boldsymbol{u}=\left(U_{T_1}(t_1),\dots,U_{T_d}(t_d)\right)} \prod_{k=1}^{d} \Pi_{T_k}(\mathrm{d}t_k). \tag{3.4}$$

3.1 *Archimedean Subordinators*

Archimedean Lévy copulas will be of particular interest, since this class covers nontrivial examples of multivariate subordinators with valuable properties such as full jump support. Furthermore, Archimedean Lévy copulas fulfill (3.4), such that these Lévy measures are absolutely continuous if the marginal Lévy measures are. Let $T_k \sim S^1(0, \Pi_{T_k})$ be univariate subordinators with tail integrals $U_{T_k}(t_k)$, $k = 1, \dots, d$. Based on (3.4),

$$\Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t}) = \phi^{(d)}\left(\sum_{k=1}^{d} \phi^{-1}(U_{T_k}(t_k))\right) \prod_{k=1}^{d} \left(\phi^{-1}\right)^{(1)} (U_{T_k}(t_k))\Pi_{T_k}(\mathrm{d}t_k),$$

where $\boldsymbol{t} \in [0,\infty)^d_*$ and the superscript (k) denotes the k-th derivative, defines a d-dimensional *Archimedean subordinator* $\boldsymbol{T} \sim S^d(\boldsymbol{0}, \Pi_{\boldsymbol{T}})$. In particular, Archimedean Lévy copulas depend on the generator $\phi\colon [0,\infty] \to [0,\infty]$.

(i) For $\phi(u) = u^{-1/\theta}$, $\theta > 0$, the Lévy copula,

$$C^{\phi}(\boldsymbol{u}) = \left(\sum_{k=1}^{d} u_k^{-\theta}\right)^{-1/\theta},$$

is called *Clayton Lévy copula.*

(ii) For $\phi(u) = \exp\left(u^{-1/\theta}\right) - 1$, $\theta > 0$ the Lévy copula,

$$C^{\phi}(\boldsymbol{u}) = \exp\left(\left[\sum_{k=1}^{d} \left[\log(1+u_k)\right]^{-\theta}\right]^{-1/\theta}\right) - 1,$$

is called *Gumbel Lévy copula.*

(iii) For $\phi(u) = -\frac{1}{\theta}\log(1 - \exp(-u))$, $\theta > 0$ the Lévy copula,

$$C^{\phi}(\boldsymbol{u}) = -\frac{1}{\theta}\log\left(1 - \prod_{k=1}^{d}\left(1 - e^{-\theta u_k}\right)\right),$$

is called *Frank Lévy copula.*

(iv) For $\phi(u) = -\frac{1}{\theta}\log(1-\exp(-u^{1/\delta}))$, $\theta > 0, \delta \geq 1$, the Lévy copula ,

$$C^{\phi}(\boldsymbol{u}) = -\frac{1}{\theta}\log\left(1-\exp\left(\Big[\sum_{k=1}^{d}\big[\log\left(1-e^{-\theta u_k}\right)\big]^{\delta}\Big]^{1/\delta}\right)\right),$$

is called *Transformed Frank Lévy copula.*

(v) For $\phi(u) = \theta/(\exp(u)-1)$, $\theta > 0$ the Lévy copula,

$$C^{\phi}(\boldsymbol{u}) = \frac{\theta\prod_{k=1}^{d} u_k}{\prod_{k=1}^{d}(\theta+u_k) - \prod_{k=1}^{d} u_k},$$

is called *Ali-Mikhail-Haq Lévy copula.*

(vi) For $\phi(u) = (1-\exp(u))^{-1/\theta} - 1$, $\theta > 0$ the Lévy copula,

$$C^{\phi}(\boldsymbol{u}) = \left(1-\prod_{k=1}^{d}\left(1-(1+u_k)^{-\theta}\right)\right)^{-1/\theta} - 1,$$

is called *Joe Lévy copula.*

The names are based on well known distributional Archimedean copulas, see e.g. Nelsen (2007) for an overview. Note that for $\theta \to \infty$ ($\theta \to 0$) the Clayton, Gumbel, Frank and Joe Lévy copula converge to complete dependence (independence) Lévy copulas.

Example 3.4. The Lévy measure of a *Clayton subordinator* is defined by

$$\Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t}) = \left(\sum_{k=1}^{d} U_{T_k}(t_k)^{-\theta}\right)^{-1/\theta-d}\prod_{k=1}^{d}\frac{1+\theta(k-1)}{U_{T_k}(t_k)^{\theta+1}}\Pi_{T_k}(\mathrm{d}t_k), \tag{3.5}$$

where $\boldsymbol{t} \in [0,\infty)^d_*$. Using gamma processes $T_k \sim \Gamma(1/\alpha_k, 1/\alpha_k)$ in the margins defines a *Clayton-Gamma subordinator*. Note that the marginal Lévy measures are $\Pi_{T_k}(\mathrm{d}t_k) = \mathbb{1}_{(0,\infty)}(t)(\alpha_k t_k)^{-1}e^{-t_k/\alpha_k}\,\mathrm{d}t_k$ and the marginal tail integrals are given from $U_{T_k}(t_k) = E_1(t_k/\alpha_k)/\alpha_k$, where $E_1(x) := \int_x^{\infty} e^{-x}/x\,\mathrm{d}x$ is the exponential integral. Using a Clayton-Gamma subordinator $\boldsymbol{T}$, we define the *Weak Variance Clayton-Gamma process* $\boldsymbol{Y} \overset{\mathcal{D}}{=} \boldsymbol{B}\odot\boldsymbol{T}$, where $\boldsymbol{B} \sim BM^d(\boldsymbol{m},\boldsymbol{\Sigma})$ is independent from $\boldsymbol{T}$. The margins $Y_k \overset{\mathcal{D}}{=} B_k \circ T_k$, $k = 1,\ldots,d$, admit a representation as VG processes with parameters α_k, m_k and Σ_{kk}. The remaining entries of the covariance matrix $\boldsymbol{\Sigma}$ solely affect the dependence structure together with the parameter $\theta > 0$ of the Lévy copula.

3.2 *Completely Dependent Subordinators*

The Clayton Lévy copula converges to the complete dependence Lévy copula $C^{\|}$ for $\theta \to \infty$. This enables us to unravel the case of complete dependence, as already investigated by Cont and Tankov (2004). Consider Equation (3.5) and note that

$$\lim_{\theta\to\infty}\left(\sum_{k=1}^d u_k^{-\theta}\right)^{-1/\theta}\prod_{k=1}^d \frac{u_k^{-1}\big(1+\theta(k-1)\big)}{\sum_{j=1}^d\big[u_k/u_j\big]^\theta} = \begin{cases}\infty, & u_1=\ldots=u_d,\\ 0, & \text{else.}\end{cases}$$

As a result, the Lévy measure $\Pi_{\boldsymbol{T}}^{\|}$ of a completely dependent subordinator has support $\{\boldsymbol{t}\in[0,\infty)_*^d \mid U_{T_1}(t_1)=\ldots=U_{T_d}(t_d)\}$. If the marginal tail integrals $U_1,\ldots,U_d$ are continuous and strictly decreasing, $\Pi_{\boldsymbol{T}}^{\|}$ is singular and the support is a curve in $[0,\infty)_*^d$. For a $\Pi_{\boldsymbol{T}}^{\|}$-integrable function f, one has

$$\int_{[0,\infty)_*^d} f(\boldsymbol{t})\,\Pi_{\boldsymbol{T}}^{\|}(\mathrm{d}\boldsymbol{t}) = \int_{(0,\infty)} f\big(U_{T_1}^{-1}(u),\ldots,U_{T_d}^{-1}(u)\big)\,\mathrm{d}u,$$

where $U_{T_k}^{-1}$ is the inverse marginal tail integral for $u\geq 0$, $k=1,\ldots,d$.

Completely dependent subordinators have a predefined structure of joint jumps. As a result, the maximum amount of dependence is incorporated while the margins are still completely flexible. In terms of the economic interpretation, a completely dependent subordinator means that the assets are subject to a comonotonic information flow. Thus, the relevance of an information arrival increases jointly for all components of the log-return process, which is possibly the case for the components of a major stock index. However, if we use a completely dependent subordinator $\boldsymbol{T}$ to (weakly) subordinate a Lévy process $\boldsymbol{X}$ with support $\mathbb{R}^d$, the Lévy measure of $\boldsymbol{X}\odot\boldsymbol{T}$ has still full support $\mathbb{R}_*^d$.

Example 3.5. Let $\boldsymbol{B}\sim BM^d(\boldsymbol{m},\boldsymbol{\Sigma})$ and $\boldsymbol{T}\sim S^d(\boldsymbol{0},\Pi_{\boldsymbol{T}}^{\|})$ be independent with $T_k\sim\Gamma(1/\alpha_k,1/\alpha_k)$ for $k=1,\ldots,d$. The *Variance Dependent-Gamma process* $\boldsymbol{Y}\overset{\mathcal{D}}{=}\boldsymbol{B}\odot\boldsymbol{T}$ has characteristic exponent

$$\Psi_{\boldsymbol{Y}}(\boldsymbol{z}) = \int_{(0,\infty)} \exp\left(i\langle\boldsymbol{z},\boldsymbol{t}\diamond\boldsymbol{m}\rangle - \frac{1}{2}\|\boldsymbol{z}\|_{\boldsymbol{t}\diamond\boldsymbol{\Sigma}}\right)\Bigg|_{\boldsymbol{t}=\left(U_{T_1}^{-1}(u),\ldots,U_{T_d}^{-1}(u)\right)} - 1\,\mathrm{d}u,$$

where $U_{T_k}^{-1}(u)=\alpha_k E_1^{-1}(\alpha_k u)$ for $k=1,\ldots,d$. The margins $Y_k\overset{\mathcal{D}}{=}B_k\circ T_k$, $k=1,\ldots,d$, admit a representation as VG processes with parameters α_k, m_k and Σ_{kk}. The remaining entries of the covariance matrix $\boldsymbol{\Sigma}$ solely affect the dependence structure.

4 Simulation and Estimation

In several models originated by weak or pathwise subordination, e.g. the characteristic functions is known in closed form. Since we apply positive Lévy copulas to create the Lévy measure $\Pi_{\boldsymbol{T}}$, explicit formulae are rarely available. We state an algorithm for simulating these processes based on the series representation of the subordinator. To estimate the dependence structure, conditioning on the (simulated) subordinator encourages a simulated likelihood estimation procedure.

4.1 *Truncation of Small Jumps and Simulation*

In general, small jumps of Lévy processes have to be truncated to simulate their paths. Conditional on such path approximations of the subordinator $\boldsymbol{T}$, we obtain approximated paths of the weakly subordinated Lévy process $\boldsymbol{X} \odot \boldsymbol{T}$.

PROPOSITION 4.1. *Let* $\boldsymbol{Y} \overset{\mathcal{D}}{=} \boldsymbol{X} \odot \boldsymbol{T}$, *where* $\boldsymbol{X} \sim L^d(\boldsymbol{\gamma}_{\boldsymbol{X}}, \boldsymbol{\Sigma}_{\boldsymbol{X}}, \Pi_{\boldsymbol{X}})$ *and* $\boldsymbol{T} \sim S^d(\boldsymbol{0}, \Pi_{\boldsymbol{T}})$ *are independent. Define* $\boldsymbol{T}^{\boldsymbol{\varepsilon}} \sim S(\boldsymbol{0}, \Pi_{\boldsymbol{T}}^{\boldsymbol{\varepsilon}})$ *for all* $\boldsymbol{\varepsilon} \in (0,\infty)^d$, *where* $\Pi_{\boldsymbol{T}}^{\boldsymbol{\varepsilon}}(\mathrm{d}\boldsymbol{t}) := \mathbb{1}_{\mathbb{R}^d \setminus [\boldsymbol{0},\boldsymbol{\varepsilon})}(\boldsymbol{t})\Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t})$. *Then* $\boldsymbol{Y}^{\boldsymbol{\varepsilon}} \overset{\mathcal{D}}{=} \boldsymbol{X} \odot \boldsymbol{T}^{\boldsymbol{\varepsilon}}$ *fulfills*

$$\lim_{\|\boldsymbol{\varepsilon}\| \to 0} \boldsymbol{Y}^{\boldsymbol{\varepsilon}} \overset{\mathcal{D}}{=} \boldsymbol{Y}.$$

Let $\{\boldsymbol{\mathcal{X}}(t,\boldsymbol{t}) \in \mathbb{R}^d \mid (t,\boldsymbol{t}) \in [0,\infty) \times [0,\infty)^d_*\}$ *be a family of independent random vectors, satisfying* $\boldsymbol{\mathcal{X}}(t,\boldsymbol{t}) \overset{\mathcal{D}}{=} \boldsymbol{X}(\boldsymbol{t})$ *for all* $(t,\boldsymbol{t}) \in [0,\infty) \times [0,\infty)^d_*$. *If* $\boldsymbol{T}(t) = \sum_{s\in[0,t]} \Delta\boldsymbol{T}(s)$, *then* $\boldsymbol{T}^{\boldsymbol{\varepsilon}}(t) \overset{\mathcal{D}}{=} \sum_{s\in[0,t]} \mathbb{1}_{\mathbb{R}^d \setminus [\boldsymbol{0},\boldsymbol{\varepsilon})}(\Delta\boldsymbol{T}(s))\Delta\boldsymbol{T}(s)$ *and*

$$\boldsymbol{Y}^{\boldsymbol{\varepsilon}}(t) \overset{\mathcal{D}}{=} \sum_{s\in[0,t]} \mathbb{1}_{\mathbb{R}^d \setminus [\boldsymbol{0},\boldsymbol{\varepsilon})}(\Delta\boldsymbol{T}(s))\boldsymbol{\mathcal{X}}(s, \Delta\boldsymbol{T}(s)) \tag{4.1}$$

for all $\boldsymbol{\varepsilon} \in (0,\infty)^d$.

Proof. Let $\boldsymbol{\varepsilon}_n \in (0,\infty)^d$ with $\boldsymbol{\varepsilon}_n \geq \boldsymbol{\varepsilon}_{n+1}$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty}\|\boldsymbol{\varepsilon}_n\| = 0$. With (2.3) one has

$$\lim_{n\to\infty} \Psi_{\boldsymbol{Y}^{\boldsymbol{\varepsilon}_n}}(\boldsymbol{z}) = \lim_{n\to\infty} \int_{[0,\infty)^d_*} \left(\mathbb{E}\left[e^{i\langle \boldsymbol{z}, \boldsymbol{X}(\boldsymbol{t})\rangle}\right] - 1\right) \mathbb{1}_{\mathbb{R}^d \setminus [\boldsymbol{0},\boldsymbol{\varepsilon})}(\boldsymbol{t})\, \Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t})$$

$$= \int_{[0,\infty)^d_*} \Big(\mathbb{E}\Big[e^{i\langle \boldsymbol{z}, \boldsymbol{X}(\boldsymbol{t})\rangle}\Big] - 1\Big)\, \Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t}) = \Psi_{\boldsymbol{Y}}(\boldsymbol{z}), \quad \boldsymbol{z} \in \mathbb{R}^d,$$

due to dominated convergence, where the $\Pi_{\boldsymbol{T}}$-integrability of the majorant $|\mathbb{E}\big[e^{i\langle \boldsymbol{z}, \boldsymbol{X}(\boldsymbol{t})\rangle}\big]-1|$ follows from Lemma 5.1 in Buchmann et al. (2017). Thus, $\lim_{\|\boldsymbol{\varepsilon}\|\to 0} \boldsymbol{Y}^{\boldsymbol{\varepsilon}} \stackrel{\mathcal{D}}{=} \boldsymbol{Y}$.

Let $\boldsymbol{\varepsilon} \in (0,\infty)^d$ and $\mathbb{Y}^{\boldsymbol{\varepsilon}} = \sum_{t>0} \delta_{(t, \boldsymbol{\mathcal{X}}(t, \Delta \boldsymbol{T}^{\boldsymbol{\varepsilon}}(t)))}$ be a Poisson random measure on $[0,\infty)\times \mathbb{R}^d$. Let $\Pi^{\boldsymbol{\varepsilon}}_{\boldsymbol{Y}}(\mathrm{d}\boldsymbol{x}) := \int_{[0,\infty)^d_*} \mathbb{P}\{\boldsymbol{X}(\boldsymbol{t}) \in \mathrm{d}\boldsymbol{x}\}\Pi^{\boldsymbol{\varepsilon}}_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t})$, then $\mathbb{Y}^{\boldsymbol{\varepsilon}}$ has intensity measure $\mathrm{d}t \otimes \Pi^{\boldsymbol{\varepsilon}}_{\boldsymbol{Y}}$. As a result, $\mathbb{Y}^{\boldsymbol{\varepsilon}}$ is the point measure of jumps of a Lévy process

$$\boldsymbol{Z}^{\boldsymbol{\varepsilon}}(t) = \int_{[0,t]\times\mathbb{R}^d_*} \boldsymbol{y}\, \mathbb{Y}^{\boldsymbol{\varepsilon}}(\mathrm{d}s, \mathrm{d}\boldsymbol{y}) = \sum_{s\in[0,t]} \boldsymbol{\mathcal{X}}(s, \Delta \boldsymbol{T}^{\boldsymbol{\varepsilon}}(s)) \stackrel{\mathcal{D}}{=} \sum_{s\in[0,t]} \mathbb{1}_{[\boldsymbol{\varepsilon},\boldsymbol{\infty})}(\Delta \boldsymbol{T}(s)) \boldsymbol{\mathcal{X}}(s, \Delta \boldsymbol{T}(s)).$$

Using the Lévy-Itô decomposition of the process $\boldsymbol{Y}^{\boldsymbol{\varepsilon}}$, with characteristics as defined in Definition 2.2, one obtains $\boldsymbol{Z}^{\boldsymbol{\varepsilon}} \stackrel{\mathcal{D}}{=} \boldsymbol{Y}^{\boldsymbol{\varepsilon}}$. □

The error of the approximation scheme (4.1) depends on the truncation of the subordinator. The following proposition states a uniform error bound for (weakly) subordinated Brownian motions.

PROPOSITION 4.2. *In the situation of Theorem 4.1, let* $\boldsymbol{X} \sim BM^d(\boldsymbol{m}, \boldsymbol{\Sigma})$. *Then*

$$\mathbb{E}\big[\sup_{s\in[0,t]} \|\boldsymbol{Y}(s) - \boldsymbol{Y}^{\boldsymbol{\varepsilon}}(s)\|\big] \le 2\left(tC_{\boldsymbol{\varepsilon}} + \sqrt{tC_{\boldsymbol{\varepsilon}}}\right), \tag{4.2}$$

where $C_{\boldsymbol{\varepsilon}} = \left(\frac{1}{2}\|\boldsymbol{m}\| + \mathrm{trace}(\boldsymbol{\Sigma})\right) \int_{[\boldsymbol{0},\boldsymbol{\varepsilon})_*} \|\boldsymbol{t}\|\Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t}) + 2\|\boldsymbol{m}\|^2 \int_{[\boldsymbol{0},\boldsymbol{\varepsilon})_*} \|\boldsymbol{t}\|^2\Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t})$.

Proof. Let $\widetilde{\boldsymbol{T}}^{\boldsymbol{\varepsilon}} \sim S(\boldsymbol{0}, \widetilde{\Pi}^{\boldsymbol{\varepsilon}}_{\boldsymbol{T}})$ for $\boldsymbol{\varepsilon} \in (0,\infty)^d$, where $\widetilde{\Pi}^{\boldsymbol{\varepsilon}}_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t}) := \mathbb{1}_{[\boldsymbol{0},\boldsymbol{\varepsilon})}(\boldsymbol{t}))\Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t})$. Then $\boldsymbol{R}^{\boldsymbol{\varepsilon}} \stackrel{\mathcal{D}}{=} \boldsymbol{X} \odot \widetilde{\boldsymbol{T}}^{\boldsymbol{\varepsilon}} \sim L^d(\boldsymbol{\gamma}^{\boldsymbol{\varepsilon}}_{\boldsymbol{R}}, \boldsymbol{0}, \Pi^{\boldsymbol{\varepsilon}}_{\boldsymbol{R}})$ fulfills $\boldsymbol{R}^{\boldsymbol{\varepsilon}} \stackrel{\mathcal{D}}{=} \boldsymbol{Y} - \boldsymbol{Y}^{\boldsymbol{\varepsilon}}$. Using the Lévy-Itô decomposition, $\boldsymbol{R}^{\boldsymbol{\varepsilon}}(t) \stackrel{\mathcal{D}}{=} \boldsymbol{\gamma}^{\boldsymbol{\varepsilon}}_{\boldsymbol{R}} t + R^{\boldsymbol{\varepsilon}}_{FV}(t) + R^{\boldsymbol{\varepsilon}}_{M}(t)$ for all $t \ge 0$, where $R^{\boldsymbol{\varepsilon}}_{FV} \sim FV^d(\boldsymbol{0}, \boldsymbol{0}, \Pi^{\boldsymbol{\varepsilon}}_{\boldsymbol{R}}|_{\mathbb{D}^c})$ and $\boldsymbol{R}^{\boldsymbol{\varepsilon}}_{M} \sim L^d(\boldsymbol{0}, \boldsymbol{0}, \Pi^{\boldsymbol{\varepsilon}}_{\boldsymbol{R}}|_{\mathbb{D}})$ is a martingale. By Doob's martingale inequality,

$$\begin{aligned}
\mathbb{E}\big[\sup_{s\in[0,t]} \|\boldsymbol{R}^{\boldsymbol{\varepsilon}}(s)\|\big] &\le t\|\boldsymbol{\gamma}^{\boldsymbol{\varepsilon}}_{\boldsymbol{R}}\| + \mathbb{E}\big[\sup_{s\in[0,t]} \|\sum_{\tau\in[0,s]} \Delta \boldsymbol{R}^{\boldsymbol{\varepsilon}}_{FV}(\tau)\|\big] + \mathbb{E}\big[\sup_{s\in[0,t]} \|\boldsymbol{R}^{\boldsymbol{\varepsilon}}_{M}(s)\|^2\big]^{1/2} \\
&\le t\|\boldsymbol{\gamma}^{\boldsymbol{\varepsilon}}_{\boldsymbol{R}}\| + \mathbb{E}\big[\sum_{s\in[0,t]} \|\Delta \boldsymbol{R}^{\boldsymbol{\varepsilon}}_{FV}(s)\|\big] + 2\mathbb{E}\big[\|\boldsymbol{R}^{\boldsymbol{\varepsilon}}_{M}(t)\|^2\big]^{1/2} \\
&\le t\|\boldsymbol{\gamma}^{\boldsymbol{\varepsilon}}_{\boldsymbol{R}}\| + t\int_{[\boldsymbol{0},\boldsymbol{\varepsilon})_*} \mathbb{E}\big[\|\boldsymbol{X}(\boldsymbol{t})\|\mathbb{1}_{\mathbb{D}^c}(\boldsymbol{X}(\boldsymbol{t}))\big]\, \Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t})
\end{aligned}$$

$$+ 2 \left(t \int_{[\mathbf{0},\boldsymbol{\varepsilon})_*} \mathbb{E}\big[\|\boldsymbol{X}(\boldsymbol{t})\|^2 \mathbb{1}_{\mathbb{D}}(\boldsymbol{X}(\boldsymbol{t}))\big] \, \Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t}) \right)^{1/2}.$$

Obviously,

$$\begin{aligned} \mathbb{E}\big[\|\boldsymbol{X}(\boldsymbol{t})\|^2 \mathbb{1}_{\mathbb{D}}(\boldsymbol{X}(\boldsymbol{t}))\big] &\leq \mathbb{E}\big[\|\boldsymbol{X}(\boldsymbol{t})\|^2\big], \\ \mathbb{E}\big[\|\boldsymbol{X}(\boldsymbol{t})\| \mathbb{1}_{\mathbb{D}^c}(\boldsymbol{X}(\boldsymbol{t}))\big] &\leq \mathbb{E}\big[\|\boldsymbol{X}(\boldsymbol{t})\|^2\big]. \end{aligned}$$

By (2.2), observe that $\|\boldsymbol{\gamma}^{\boldsymbol{\varepsilon}}_{\boldsymbol{R}}\| \leq \int_{[\mathbf{0},\boldsymbol{\varepsilon})^d_*} \|\mathbb{E}[\boldsymbol{X}(\boldsymbol{t})\mathbb{1}_{\mathbb{D}}(\boldsymbol{X}(\boldsymbol{t}))]\| \, \Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t})$. Furthermore

$$\begin{aligned} \|\mathbb{E}[\boldsymbol{X}(\boldsymbol{t})\mathbb{1}_{\mathbb{D}}(\boldsymbol{X}(\boldsymbol{t}))]\| &\leq \|\mathbb{E}[\boldsymbol{X}(\boldsymbol{t})]\| + \|\mathbb{E}[\boldsymbol{X}(\boldsymbol{t})\mathbb{1}_{\mathbb{D}^c}(\boldsymbol{X}(\boldsymbol{t}))]\| \\ &\leq \|\mathbb{E}[\boldsymbol{X}(\boldsymbol{t})]\| + \mathbb{E}\big[\|\boldsymbol{X}(\boldsymbol{t})\|^2\big] \end{aligned}$$

For all $\boldsymbol{t} \in [0,\infty)^d$ one has $\boldsymbol{X}(\boldsymbol{t})$ is a Gaussian random vector with mean $\boldsymbol{t} \diamond \boldsymbol{m}$ and covariance matrix $\boldsymbol{t} \diamond \boldsymbol{\Sigma}$. Since $\|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|$, one has

$$\begin{aligned} \|\mathbb{E}[\boldsymbol{X}(\boldsymbol{t})]\| &\leq \|\boldsymbol{m}\|\|\boldsymbol{t}\|, \\ \mathbb{E}\big[\|\boldsymbol{X}(\boldsymbol{t})\|^2\big] &\leq \operatorname{trace}(\boldsymbol{\Sigma})\|\boldsymbol{t}\| + 2\|\boldsymbol{m}\|^2\|\boldsymbol{t}\|^2, \end{aligned}$$

which completes the proof. □

Remark 4.3. For general Lévy processes $\boldsymbol{X}$ similar error bounds can be obtained if $\boldsymbol{X}$ has a finite second moment. Asymptotically, the quality of the approximation depends on the quantity $\int_{[\mathbf{0},\boldsymbol{\varepsilon})_*} \|\boldsymbol{t}\|\Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t})$. For the sake of convenience, note that $\|\boldsymbol{t}\| \leq \sum_{k=1}^d t_k$.

(i) If $\boldsymbol{T}$ has gamma margins $\int_{[\mathbf{0},\boldsymbol{\varepsilon})_*} \|\boldsymbol{t}\|\Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t}) = \mathcal{O}(\|\boldsymbol{\varepsilon}\|)$, for $\|\boldsymbol{\varepsilon}\| \to 0$.
(ii) If $\boldsymbol{T}$ has inverse Gaussian margins $\int_{[\mathbf{0},\boldsymbol{\varepsilon})_*} \|\boldsymbol{t}\|\Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t}) = \mathcal{O}(\|\boldsymbol{\varepsilon}\|^{1/2})$, for $\|\boldsymbol{\varepsilon}\| \to 0$.
(iii) If $\boldsymbol{T}$ has α-stable margins $\int_{[\mathbf{0},\boldsymbol{\varepsilon})_*} \|\boldsymbol{t}\|\Pi_{\boldsymbol{T}}(\mathrm{d}\boldsymbol{t}) = \mathcal{O}(\|\boldsymbol{\varepsilon}\|^{1-\alpha})$, for $\|\boldsymbol{\varepsilon}\| \to 0$, $\alpha \in (0,1)$.

Tankov (2006, 2016) states a method for simulating finite as well as infinite variation Levy processes with dependence given by a general Lévy copula. The algorithm is based on series representation, much in the spirit of Rosiński (2001). The procedure for our setting is most simple, since subordinators are of finite variation and just require positive Lévy copulas to introduce dependencies. After sampling a path of the (truncated) subordinator $\boldsymbol{T}^{\boldsymbol{\varepsilon}}$, one draws independent random vectors distributed as $\boldsymbol{X}(\Delta T^{\boldsymbol{\varepsilon}}(s))$ for each jump time

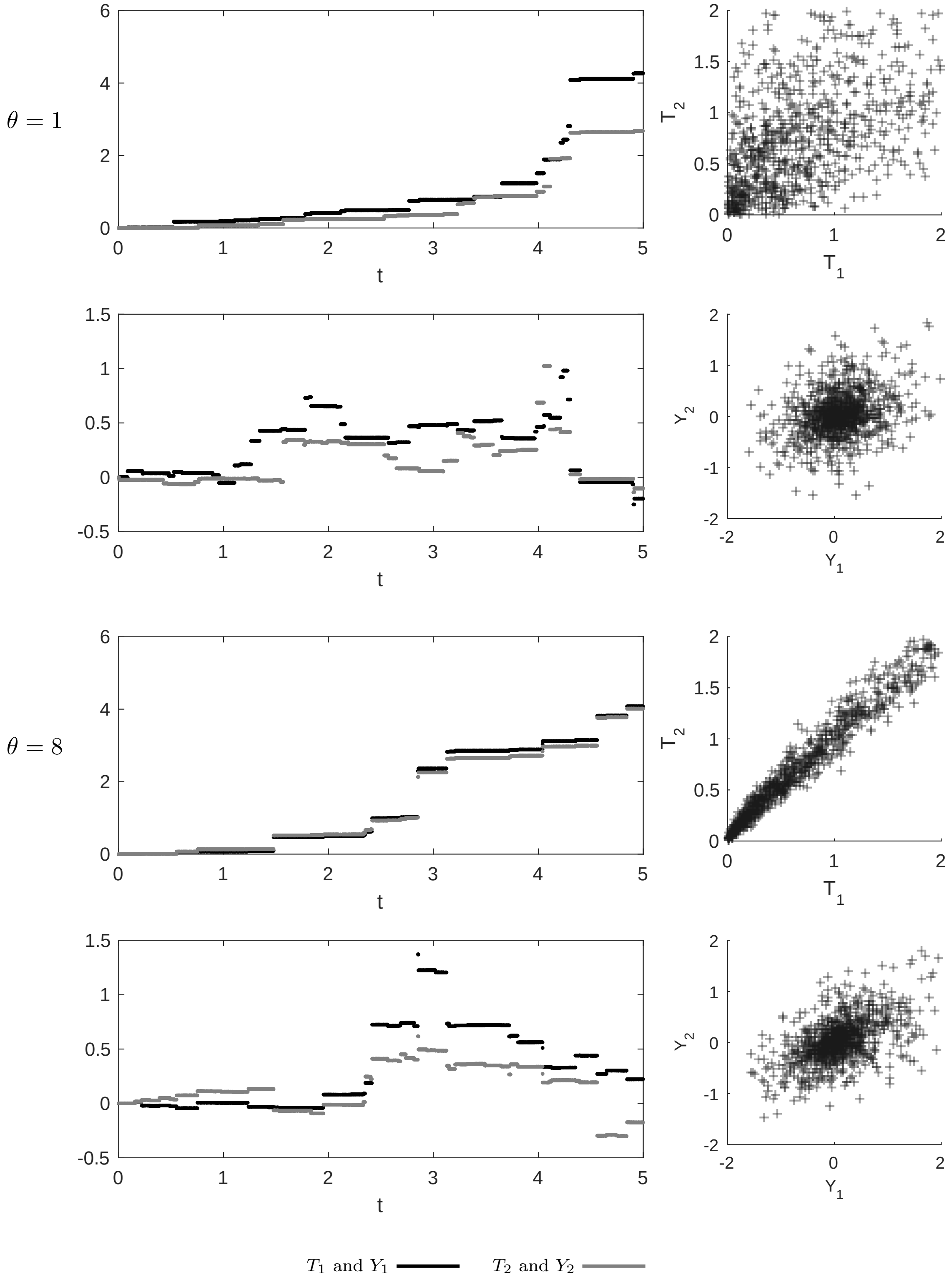

Figure 3: *Simulated paths of bivariate Clayton-Gamma processes T and respective paths of of the bivariate Weak Variance Clayton-Gamma processes $Y \stackrel{\mathcal{D}}{=} B \odot T$. Furthermore, 1000 realisations at t=1 are plotted as a scatter plot. The simulation of the subordinator is based on Tankov (2016), with $\tau = 1000$. The remaining parameters are $\alpha_1 = 1$, $\alpha_2 = 0.8$, $m_1 = 0.10$, $m_2 = 0.05$, $\Sigma_{11} = 0.20$, $\Sigma_{22} = 0.15$, $\Sigma_{12} = 0.10$.*

s to sample the jumps of $\boldsymbol{X} \odot \boldsymbol{T}^{\varepsilon}$. If $\boldsymbol{X} \sim BM^d(\boldsymbol{m}, \boldsymbol{\Sigma})$, then $\boldsymbol{X}(\Delta T^{\varepsilon}(s))$ are Gaussian random vectors with mean $\Delta\boldsymbol{T}^{\varepsilon}(s) \diamond \boldsymbol{m}$ and covariance matrix $\Delta\boldsymbol{T}^{\varepsilon}(s) \diamond \boldsymbol{\Sigma}$.

Figure 3 illustrates simulated paths of bivariate Clayton-Gamma processes as well as the corresponding paths of of the bivariate Weak Variance Clayton-Gamma processes. The algorithm of Tankov (2006, 2016) truncates the subordinator at $\boldsymbol{\varepsilon} = (U_{T_k}^{-1}(\tau))_{k=1,\ldots,d}$, where τ is a positive integer corresponding to the expected number of jumps. As a result, the error (4.2) for the path approximations in Figure 3 is smaller than 10^{-160}.[3]

Remark 4.4. The practicability of the algorithm of Tankov (2006, 2016) depends on the invertibility of the conditional distribution function

$$F_{x_1}(x_2, \ldots, x_n) = \frac{\partial}{\partial x_1} V_C\big((0, x_1] \times (0, x_2] \times \ldots \times (0, x_d]\big).$$

For $d = 2$, we observe closed and semi-closed solutions for some Lévy copulas.

(i) If C is a Clayton Lévy copula,

$$F_{x_1}^{-1}(y) = x_1\left(y^{-\theta/(\theta+1)} - 1\right)^{-1/\theta}.$$

(ii) If C is a Gumbel Lévy copula,

$$F_{x_1}^{-1}(y) = \exp\left((\theta+1)\left[W\left(\frac{A_{x_1}(y)}{\theta+1}\right)^{-\theta} - \left(\frac{\log(1+x_1)}{\theta+1}\right)^{-\theta}\right]^{-1/\theta}\right) - 1,$$

where $A_{x_1}(y) = (y + x_1 y)^{1/(\theta+1)} \log(1+x_1)$ and W is the Lambert-W function, which can be efficiently computed using the algorithm of Fritsch, Shafer and Crowley (1973).

(iii) If C is a Frank Lévy copula,

$$F_{x_1}^{-1}(y) = \frac{1}{\theta}\log\left(1 - \exp(\theta x_1)\frac{y}{y-1}\right).$$

(iv) If C is an Ali-Mikhail-Haq Lévy copula,

$$F_{x_1}^{-1}(y) = \frac{\theta - 2y(\theta + x_1) - \sqrt{\theta^2 + 4yx_1(\theta + x_1)}}{2(y-1)}.$$

[3] Of course, there is an additional error arising from the numerical inversion of the tail integral, if no analytic form is available. Regarding Gamma subordinators, note that $E_1^{-1}(x) \approx \exp(-x - \gamma_E)$ is very accurate for $x \geq 40$, where γ_E is the Euler-Mascheroni constant.

4.2 *Simulated Likelihood Estimation*

In exponential Lévy market models, the d-dimensional market price process $\boldsymbol{S}$ with components $S_k(t) = S_k(0)\exp(R_k(t))$, $t \geq 0$, $S_k(0) > 0$, $k = 1, \dots, d$, is driven by the d-dimensional log-price process $\boldsymbol{R}(t) := \boldsymbol{\mu} t + \boldsymbol{Y}(t)$, where $\boldsymbol{\mu} \in \mathbb{R}^d$ is an additional location parameter. While Proposition 3.3 provides at least semi-closed formulae for moments and comoments of $\boldsymbol{Y} \stackrel{\mathcal{D}}{=} \boldsymbol{X} \odot \boldsymbol{T}$, the probability density or likelihood function of $\boldsymbol{Y}$ is not obtainable. However, dealing with the conditional distribution of $\boldsymbol{Y}|\boldsymbol{T}$ might be feasible, especially if $\boldsymbol{X}$ is a Brownian motion. Note that

$$\mathbb{P}\{\boldsymbol{Y}(t) \in \mathrm{d}\boldsymbol{x}\} = \mathbb{E}_{\boldsymbol{T}}\big[\mathbb{P}\{\boldsymbol{Y}(t) \in \mathrm{d}\boldsymbol{x} \mid \boldsymbol{T}\}\big].$$

Under the approach of simulated likelihood, the expected value is approximated by Monte Carlo simulation. Since the paper of Berry, Levinsohn and Pakes (1995), simulated likelihood methods are frequently applied in a growing number of studies. For a technical overview we refer to Gourieroux and Monfort (1996). As a true simulation of the subordinator $\boldsymbol{T}$ is in general not possible, we apply its approximation $\boldsymbol{T}^{\boldsymbol{\varepsilon}}$.

Let $\boldsymbol{Y} \stackrel{\mathcal{D}}{=} \boldsymbol{B} \odot \boldsymbol{T}$, where $\boldsymbol{B} \sim BM^d(\boldsymbol{m}, \boldsymbol{\Sigma})$ and $\boldsymbol{T} \sim S^d(\boldsymbol{0}, \Pi_{\boldsymbol{T}})$ are independent. Note that $\boldsymbol{B}(\boldsymbol{t})$ is a Gaussian random vector with probability density $f_{\mathcal{N}}(\boldsymbol{x} \mid \boldsymbol{t} \diamond \boldsymbol{m}, \boldsymbol{t} \diamond \boldsymbol{\Sigma}) := \mathcal{N}\{\mathrm{d}\boldsymbol{x} \mid \boldsymbol{t} \diamond \boldsymbol{m}, \boldsymbol{t} \diamond \boldsymbol{\Sigma}\}(\boldsymbol{x})/\mathrm{d}\boldsymbol{x}$. Furthermore $f_{\mathcal{N}}(\boldsymbol{x} \mid \boldsymbol{t} \diamond \boldsymbol{m}, \boldsymbol{t} \diamond \boldsymbol{\Sigma})$ is bounded in $\boldsymbol{t} \in [0,\infty)^d_*$ for $\boldsymbol{x} \in \mathbb{R}^d_*$. Let $\boldsymbol{T}^{\boldsymbol{\varepsilon}}(t) \stackrel{\mathcal{D}}{=} \sum_{s \in [0,t]} \mathbb{1}_{\mathbb{R}^d \setminus [\boldsymbol{0}, \boldsymbol{\varepsilon})}(\Delta \boldsymbol{T}(s)) \Delta \boldsymbol{T}(s)$, $\boldsymbol{\varepsilon} \in (0,\infty)^d$, be a truncated version of $\boldsymbol{T}$, also independent from $\boldsymbol{B}$. Due to Proposition 3.1 and weak convergence,

$$\begin{aligned}
\frac{\mathbb{P}\{\boldsymbol{Y}(t) \in \mathrm{d}\boldsymbol{x}\}}{\mathrm{d}\boldsymbol{x}}(\boldsymbol{x}) &= \mathbb{E}_{\boldsymbol{T}}\left[\frac{\mathbb{P}\{\boldsymbol{Y}(t) \in \mathrm{d}\boldsymbol{x} \mid \boldsymbol{T}\}}{\mathrm{d}\boldsymbol{x}}(\boldsymbol{x})\right] \\
&= \mathbb{E}_{\boldsymbol{T}}\Big[f_{\mathcal{N}}\Big(\boldsymbol{x} \,\Big|\, \sum_{s \in [0,t]} \Delta \boldsymbol{T}(s) \diamond \boldsymbol{m}, \sum_{s \in [0,t]} \Delta \boldsymbol{T}(s) \diamond \boldsymbol{\Sigma}\Big)\Big] \\
&\approx \mathbb{E}_{\boldsymbol{T}^{\boldsymbol{\varepsilon}}}\Big[f_{\mathcal{N}}\Big(\boldsymbol{x} \,\Big|\, \sum_{s \in [0,t]} \Delta \boldsymbol{T}^{\boldsymbol{\varepsilon}}(s) \diamond \boldsymbol{m}, \sum_{s \in [0,t]} \Delta \boldsymbol{T}^{\boldsymbol{\varepsilon}}(s) \diamond \boldsymbol{\Sigma}\Big)\Big] \\
&\approx \frac{1}{M} \sum_{i=1}^{M} f_{\mathcal{N}}\Big(\boldsymbol{x} \,\Big|\, \sum_{s \in [0,t]} \Delta \boldsymbol{T}^{\boldsymbol{\varepsilon},i}(s) \diamond \boldsymbol{m}, \sum_{s \in [0,t]} \Delta \boldsymbol{T}^{\boldsymbol{\varepsilon},i}(s) \diamond \boldsymbol{\Sigma}\Big), \qquad (4.3)
\end{aligned}$$

for $\boldsymbol{x} \in \mathbb{R}^d_*$, where $\boldsymbol{T}^{\boldsymbol{\varepsilon},i}$ are Monte Carlo realizations of $\boldsymbol{T}^{\boldsymbol{\varepsilon}}$ for $i = 1, \dots, M$. As a result, $\boldsymbol{Y}$ obtains a probability density function which can be almost everywhere approximated by

Monte Carlo simulation of the subordinator. This allows for simulated likelihood estimation of the parameters characterizing the distribution of $\boldsymbol{Y}$.

The models of Section 3 are affected by several parameters. Obviously, there is the location parameter $\boldsymbol{\mu}$, and the drift $\boldsymbol{m}$ and covariance matrix $\boldsymbol{\Sigma}$ of the Brownian motion. Furthermore, the parameters characterizing the Lévy measure of the subordinator $\Pi_{\boldsymbol{T}}$ are affecting the distribution of $\boldsymbol{Y}$. In the framework of Lévy copulas, these are given from the (parametric) marginal Lévy measures and the positive Lévy copula. Since these parameters are also affecting the simulation required for the approximation (4.3), we are facing a simulation error. This makes a gradient based maximization of the simulated log-likelihood difficult. However, the marginal consistency $Y_k \overset{\mathcal{D}}{=} B_k \circ T_k$ makes a decoupled estimation procedure tractable.

In the first step, we estimate the parameters affecting the marginal distributions. Moment conditions imply possible starting values. Thus, we obtain estimates for $\boldsymbol{\mu}$, $\boldsymbol{m}$, the diagonal entries of $\boldsymbol{\Sigma}$, and the parameters of the marginal Lévy measures of the subordinator. In the second step, we estimate the parameters solely affecting the dependence structure, i.e. the remaining entries of $\boldsymbol{\Sigma}$ and the parameters of the positive Lévy copula. To do so, we consider a reasonable grid on the parameter space of the Lévy copula parameters. For each point, we simulate M paths of the subordinator and maximize the simulated log-likelihood with respect to the remaining entries of $\boldsymbol{\Sigma}$. Finally, we consider the locally estimated scatterplot smoothing (LOESS) curve fitted to the simulated log-likelihood entries to deal with the simulation error. The maximum of this smooth determines the estimates for the Lévy copula parameters. Accordingly, we obtain estimates for the covariances of the Brownian motion.

5 Modeling Financial Data

For modeling log-returns, we focus on the case of VG distributed margins. Thus,

$$\boldsymbol{R}(t) = \boldsymbol{\mu} t + \boldsymbol{Y}(t), \qquad t \geq 0,$$

with $\boldsymbol{Y} \overset{\mathcal{D}}{=} \boldsymbol{B} \odot \boldsymbol{T}$, where $\boldsymbol{B} \sim BM^d(\boldsymbol{m}, \boldsymbol{\Sigma})$ and $\boldsymbol{T} \sim S^d(\boldsymbol{0}, \Pi_{\boldsymbol{T}})$ are independent with $T_k(t) \sim \Gamma(t/\alpha_k, 1/\alpha_k)$, $k = 1, \ldots, d$. Each marginal process is specified by the location parameter μ_k, drift m_k and variance Σ_{kk} of the Brownian motion, and variance rate α_k

of the gamma subordinator. While Σ_{kk} also controls for the variance of the marginal VG distribution, the parameters m_k and α_k are primarily addressing the skewness and excess kurtosis, respectively. In terms of the common economic interpretation, the variance rate α_k of the gamma subordinator represents the market activity for the modelled asset. Thus, the choice of the Lévy copula is introducing dependencies between the marginal market activities. Note that these dependencies are different to correlations between log-returns, which are primarily controlled by the Brownian correlation.

5.1 *Moderate Market Dependencies*

To model moderate market dependencies, we first apply a Clayton-Gamma subordinator. We fit the model to pairs of major stock indices and interpret the parameter estimates in an economic sense. Furthermore, we compare model alternatives originated by different choices of the Lévy copula.

Clayton-Gamma subordination. We consider a Weak Variance Clayton-Gamma process of Example 3.4 as driving Lévy process for the log-return process. We select a sample based on quarterly log-returns of the German DAX, the US-American Dow Jones Industrial Average (DJIA), the British Financial Times Ordinary Index (FT30) and the Japanese Nikkei 225 (NIK). We choose an observation period from the third quarter of 1950 to the third quarter of 2017 (269 observations).[4] Using the simulated likelihood approach of Section 4.2, we estimate bivariate Weak Variance Clayton-Gamma models based on the six pairs $(\mathrm{DAX}, \mathrm{DJIA})$, $(\mathrm{DAX}, \mathrm{FT30})$, $(\mathrm{DAX}, \mathrm{NIK})$, $(\mathrm{DJIA}, \mathrm{FT30})$, $(\mathrm{DJIA}, \mathrm{NIK})$ and $(\mathrm{FT30}, \mathrm{NIK})$. Thus, we first compute the maximum likelihood estimates for the four marginal VG models. To estimate the Clayton parameter $\theta \in (0, \infty)$ for each pair, we consider the transformation $\widetilde{\theta} = 1 - 1/(\theta + 1) \in (0, 1)$ on 999 equally spaced points on [0.001, 0.999]. For each point, we simulate $M = 100{,}000$ paths of the subordinator and maximize the simulated likelihood with respect to the Brownian correlation $\rho = \Sigma_{12}/\sqrt{\Sigma_{11}\Sigma_{22}}$. Finally we consider the LOESS curve fitted to the simulated log-likelihoods and the estimate of $\widetilde{\theta}$ is the maximum of the smooth. The estimate of ρ is given by evaluating the LOESS curve fitted to the corresponding Brownian correlations.

[4] The data is retrieved from Thomson Reuters. The data of the DAX from 1950–1954 and 1955–1987 is based on Deutsches Aktieninstitut (2011) and Stehle, Maier and Huber (1996), respectively.

Table 1: *Summary of maximum likelihood parameter estimates and moments of the Weak Variance Clayton-Gamma models. The data is given by quarterly log-returns (30/09/1950 – 30/09/2017, 269 observations) of major stock indices.*

Panel A: Marginal VG parameter estimates (standard errors in parenthesis)

	DAX	DJIA	FT30	NIK
α_k	0.6798	0.4775	0.7178	0.3576
	(0.1816)	(0.1162)	(0.1557)	(0.1096)
m_k	-0.0260	-0.0452	-0.0364	-0.0682
	(0.0085)	(0.0067)	(0.0075)	(0.0105)
Σ_{kk}	0.0114	0.0046	0.0085	0.0098
	(0.0014)	(0.0005)	(0.0010)	(0.0011)
μ_k	-0.0507	0.0626	0.0480	0.0885
	(0.0055)	(0.0056)	(0.0056)	(0.0089)

Panel B: Marginal model moments (empirical moments in parenthesis)

	DAX	DJIA	FT30	NIK
Mean	0.0248	0.0174	0.0117	0.0203
	(0.0248)	(0.0174)	(0.0116)	(0.0203)
Var.	0.0119	0.0056	0.0094	0.0115
	(0.0124)	(0.0056)	(0.0098)	(0.0113)
Skew.	-0.4793	-0.8178	-0.7793	-0.6502
	(-0.7885)	(-0.8613)	(-0.1315)	(-0.4843)
Kurt.	5.1935	4.8914	5.5651	4.3614
	(5.8743)	(4.9091)	(6.9500)	(3.8101)

Panel C: Dependence parameter estimates (standard errors in parenthesis)

	$\widetilde{\theta}$	ρ	log-L.
(DAX, DJIA)	0.5755	0.9908	612.05
	(0.0482)	(0.1075)	
(DAX, FT30)	0.8546	0.5956	541.73
	(0.0847)	(0.0758)	
(DAX, NIK)	0.6044	0.5598	478.41
	(0.1656)	(0.2252)	
(DJIA, FT30)	0.6677	0.9897	680.40
	(0.0406)	(0.0714)	
(DJIA, NIK)	0.4230	0.8773	580.23
	(0.0646)	(0.2828)	
(FT30, NIK)	0.4453	0.8905	512.46
	(0.0589)	(0.2351)	

Panel D: Model comoments (empirical comoments in parenthesis)

	ρ_{T_1,T_2}	ρ_{R_1,R_2}	$\rho_{R_1^2,R_2^2}$
(DAX, DJIA)	0.8142	0.5853	0.5948
		(0.6001)	(0.5535)
(DAX, FT30)	0.9867	0.5529	0.5521
		(0.5719)	(0.4011)
(DAX, NIK)	0.8324	0.3638	0.4209
		(0.3944)	(0.2873)
(DJIA, FT30)	0.8948	0.7027	0.6987
		(0.6977)	(0.5963)
(DJIA, NIK)	0.5980	0.3790	0.4348
		(0.4011)	(0.2195)
(FT30, NIK)	0.6264	0.3860	0.4310
		(0.4040)	(0.1682)

Table 1 shows the results of the estimation. Accordingly, Figure 5 shows the LOESS curves of the simulated log-likelihoods and the corresponding Brownian correlations, and Figure 6 illustrates the fit of the model (see below). Panel A of Table 1 shows the parameter estimates for the marginal VG models. Panel B shows the marginal fit in terms of the first four standardized moments. Most importantly, Panel C shows the pairwise estimates of the transformed Clayton parameter $\widetilde{\theta}$ and the corresponding estimates of the Brownian correlation ρ, respectively. Figure 4 shows 90 % confidence ellipses for these estimates. Panel D of Table 1 compares the model and empirical comoments. The marginal fit in terms of the moments is satisfactory, as usual for the VG model and index data. Furthermore, the bivariate Weak Variance Clayton-Gamma models are able to reproduce the empricial correlation and to roughly capture the proportions of squared return correlations.

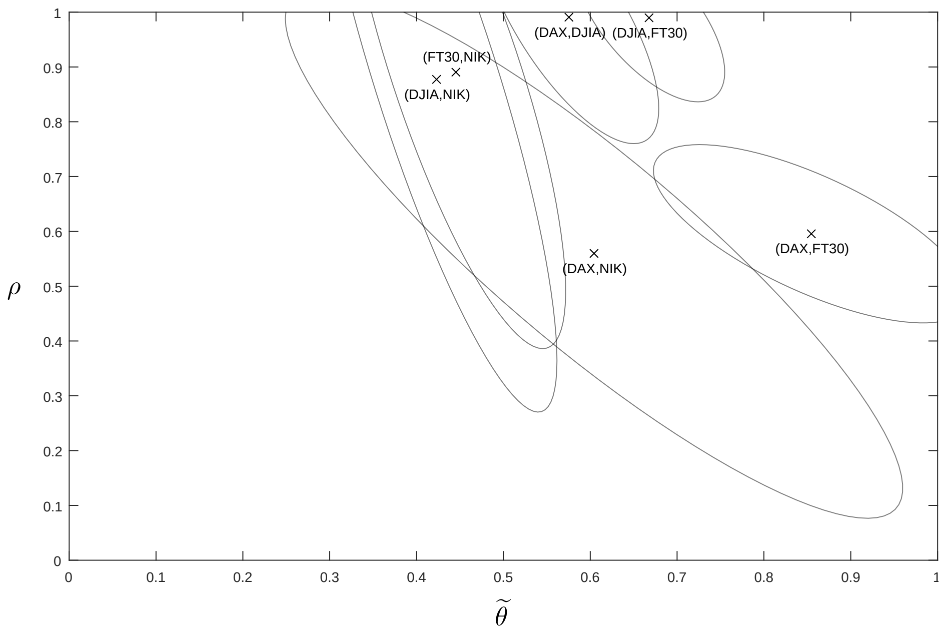

Figure 4: *Asymtotic 90 % confidence ellipses for the parameter estimates according to Table 1.*

For the pairs (DAX, NIK), (DJIA, NIK) and (FT30, NIK) the estimates for the transformed Clayton parameter are 0.60, 0.42 and 0.45, respectively. The underlying Clayton-Gamma processes have correlations 0.83, 0.60, and 0.63, respectively, indicating moderately strong dependencies in the market activity between the West and Japan. In other words, these markets are subject to a moderately comonotonic information flow within the scope of the model, which also seems reasonable from a macroeconomic perspective. The slightly stronger dependence in the market activity for (DAX, NIK) may be caused by the fact that the German and Japanese markets both heavily rely on the automotive industry. However, note that these estimates also obtain relatively large standard errors. For the transatlantic pairs (DAX, DJIA) and (DJIA, FT30) the respective estimates for the Clayton parameter 0.58 and 0.67 indicate semi-strong dependencies in the market activity. The underlying Clayton-Gamma processes have correlations 0.81 and 0.89, respectively. In addition, the directing process is almost perfectly correlated in both cases. Within the scope of the model, these markets are subject to a semi-strongly related information flow, but the proportion of cross information is processed in the same way, i.e. either 'good' or 'bad' for both markets. From a macroeconomic perspective, this may correspond to the leading economic power of the US-markets. For the European pair (DAX, FT30) the estimate for the Clayton parameter is 0.85. The underlying Clayton-Gamma process has correlation 0.99, indicating (almost) perfect dependence in the market activity. Within the scope of the model, these markets are subject to a comonotonic information flow, which may be caused by the geographical proximity. Note that the confidence ellipses in Figure 4 also emphasize these three clusters of market dependencies, i.e. the West and Japan, the transatlantic pairs and the European pair.

Model Comparison. The optimal Lévy copula to describe the jump dependence of the subordinator is hard to come by, but selecting a model from different candidate models

Table 2: *Joint parameter estimates (standard errors in parenthesis), log-likelihood and Bayesian information criterion (BIC) of models originated by different subordinators for the data corresponding to Table 1.*

Subordinator	$\widetilde{\theta}$	ρ	log-L.	BIC	$\widetilde{\theta}$	ρ	log-L.	BIC
		(DAX, DJIA)				(DAX, FT30)		
Indep.-Gamma			557.91	-1071.06			489.85	-934.93
Clayton-Gamma	0.5755	0.9908	612.05	-1168.15	0.8546	0.5956	541.73	-1027.51
	(0.0482)	(0.1075)			(0.0847)	(0.0758)		
Gumbel-Gamma	0.6757	0.9941	612.20	-1168.45	0.9025	0.5960	541.74	-1027.54
	(0.0389)	(0.0795)			(0.0793)	(0.0873)		
Frank-Gamma	0.9990	0.5930	610.17	-1164.00	0.9990	0.5260	541.60	-1027.26
	(0.0338)	(0.0562)			(0.0350)	(0.0602)		
Dep.-Gamma		0.5940	610.15	-1169.95		0.5265	541.59	-1032.82
		(0.0067)				(0.0040)		
		(DAX, NIK)				(DJIA, FT30)		
Indep.-Gamma			455.87	-866.98			593.28	-1141.82
Clayton-Gamma	0.6044	0.5598	478.41	-900.88	0.6677	0.9897	680.40	-1304.86
	(0.1656)	(0.2252)			(0.0406)	(0.0714)		
Gumbel-Gamma	0.5378	0.9563	478.90	-901.85	0.7628	0.9821	680.72	-1305.50
	(0.0878)	(0.3755)			(0.0598)	(0.1210)		
Frank-Gamma	0.9990	0.3748	478.09	-900.24	0.9990	0.7220	679.32	-1302.00
	(0.0492)	(0.0773)			(0.0304)	(0.0470)		
Dep.-Gamma		0.3750	478.08	-905.81		0.7232	679.29	-1308.22
		(0.0804)				(0.0146)		
		(DJIA, NIK)				(FT30, NIK)		
Indep.-Gamma			559.41	-1074.07			491.35	-937.94
Clayton-Gamma	0.4230	0.8773	580.23	-1104.52	0.4453	0.8905	512.46	-968.97
	(0.0646)	(0.2828)			(0.0589)	(0.2351)		
Gumbel-Gamma	0.5174	0.9101	580.67	-1105.40	0.5460	0.8877	512.62	-969.30
	(0.0691)	(0.2014)			(0.0822)	(0.2268)		
Frank-Gamma	0.9990	0.2750	577.94	-1099.94	0.9990	0.3740	510.22	-964.49
	(0.0505)	(0.0697)			(0.0689)	(0.0757)		
Dep.-Gamma		0.2754	577.93	-1105.52		0.3742	510.23	-970.11
		(0.0717)				(0.1002)		

is possible. Given the data and marginal parameter estimates from Panel A of Table 1, we use a Clayton, Gumbel or Frank Lévy copula to describe the jump dependence of the multivariate gamma process. Moreover, we include the limit-cases of the complete dependence and independence Lévy copula to our comparison.

Table 2 shows the results of the estimation. All considered models have 2×4 parameters to specify the margins, and 1 parameter to specify the correlation of the Brownian motion, which is redundant for the Independent-Gamma subordinator. The Clayton, Gumbel and Frank Lévy copula have 1 additional parameter to model the jump dependence of the subordinator. The results show that the Independent-Gamma subororordinator leads to a poor fit in comparison, as no dependence modeling is possible. For all datasets, the Gumbel-Gamma subordinator generates the highest likelihood. However, the Clayton-Gamma subordinator leads to almost the same results in all cases also having comparable estimates for $\widetilde{\theta}$ and ρ. This likewise underlines the results of the previous section. Overall, models based on Clayton and the Gumbel Lévy Copulas seem to be highly similar. However, remember from Remark 4.4 that the Clayton Lévy copula is more accessible for numerical application. The Frank Lévy Copula seems not suitable to model the jump dependence of the subordinator. In all cases, the estimate of $\widetilde{\theta}$ is 0.9990 which indicates convergence against the complete dependence subcase. Since the Dependent-Gamma subordinator has one parameter less, the BIC of this model is smallest for all pairs of data. However, note that the number of observations is rather low and that our main issue is to flexibly model the dependence of the market activity, which is not possible for the rigid complete dependence Lévy copula.

5.2 *Strong Market Dependencies*

Andersen et al. (2001) provides evidence that instantaneous volatilities and correlations within the Dow Jones Industrial Average systematically move together and are, thus, driven by a latent factor structure. Within our model framework, this feature corresponds to strong dependencies in the marginal market activities, see (3.3). Thus, we apply a Dependent-Gamma subordinator to introduce the Variance Dependent-Gamma process as given by Example 3.5. Remember that a completely dependent subordinator economically reflects a comonotonic information flow, which is reasonable within a major stock index,

Table 3: *Summary of maximum likelihood parameter estimates of the Weak Variance Dependent-Gamma model. The data is given by quarterly log-returns (01/01/1946 – 30/09/2017, 287 observations) the 10 Fama and French (1997) industry portfolios.*

	CD	EN	HT	HC	MN	CN	OT	SH	TC	UT
					Panel A: Marginal parameter estimates (standard errors in parenthesis)					
α_k	0.5139	0.2318	0.7720	0.4881	0.5919	0.5593	0.3674	0.7251	0.3249	0.3798
	(0.1617)	(0.0545)	(0.2067)	(0.1505)	(0.1416)	(0.1521)	(0.0838)	(0.1722)	(0.1211)	(0.1006)
m_k	-0.0424	-0.0895	-0.0246	-0.0434	-0.0444	-0.0257	-0.0764	-0.0171	-0.0350	-0.0446
	(0.0148)	(0.0109)	(0.0109)	(0.0097)	(0.0073)	(0.0075)	(0.0109)	(0.0085)	(0.0072)	(0.0058)
Σ_{kk}	0.0119	0.0061	0.0122	0.0071	0.0067	0.0057	0.0073	0.0081	0.0057	0.0042
	(0.0014)	(0.0007)	(0.0015)	(0.0008)	(0.0008)	(0.0471)	(0.0009)	(0.0010)	(0.0006)	(0.0005)
μ_k	0.0662	0.1176	0.0513	0.0738	0.0710	0.0537	0.1006	0.0439	0.0580	0.0693
	(0.0134)	(0.0094)	(0.0092)	(0.0090)	(0.0055)	(0.0271)	(0.0100)	(0.0068)	(0.0057)	(0.0048)
					Panel B: Brownian correlation ρ_{kj}					
CD	1.0000									
EN	0.4865	1.0000								
HT	0.6981	0.5210	1.0000							
HC	0.3902	0.3757	0.6580	1.0000						
MN	0.8099	0.7063	0.8495	0.7003	1.0000					
CN	0.6198	0.4373	0.6446	0.7567	0.7825	1.0000				
OT	0.8075	0.5687	0.8142	0.6758	0.9500	0.8714	1.0000			
SH	0.7787	0.4223	0.6764	0.6876	0.8133	0.8992	0.9168	1.0000		
TC	0.5599	0.3662	0.6494	0.5201	0.6366	0.7255	0.6430	0.7280	1.0000	
UT	0.4060	0.4992	0.4296	0.4609	0.5215	0.6878	0.5581	0.5596	0.6193	1.0000
					Panel C: Model correlation					
CD	1.0000									
EN	0.4451	1.0000								
HT	0.6359	0.3873	1.0000							
HC	0.4391	0.4042	0.5940	1.0000						
MN	0.7945	0.6012	0.7856	0.7052	1.0000					
CN	0.6310	0.3955	0.6048	0.7471	0.7837	1.0000				
OT	0.7465	0.6091	0.6466	0.6748	0.8494	0.7716	1.0000			
SH	0.7117	0.3239	0.6744	0.6217	0.7602	0.8369	0.7198	1.0000		
TC	0.5224	0.4034	0.5161	0.5049	0.5697	0.6388	0.6494	0.5800	1.0000	
UT	0.4337	0.5332	0.3846	0.5027	0.5370	0.6427	0.6336	0.4782	0.6267	1.0000
					Panel D: Empirical correlation					
CD	1.0000									
EN	0.5054	1.0000								
HT	0.6938	0.5019	1.0000							
HC	0.5178	0.4357	0.6606	1.0000						
MN	0.8367	0.6503	0.8005	0.7476	1.0000					
CN	0.6892	0.4546	0.6381	0.7937	0.8286	1.0000				
OT	0.8122	0.6006	0.7549	0.7159	0.9023	0.8338	1.0000			
SH	0.7717	0.4279	0.7243	0.7121	0.8233	0.8714	0.8312	1.0000		
TC	0.5910	0.4596	0.6671	0.5600	0.6483	0.6462	0.6955	0.6569	1.0000	
UT	0.4843	0.5492	0.4811	0.5323	0.5888	0.6695	0.6482	0.5543	0.6244	1.0000

10 Fama and French (1997) industry portfolios: Consumer Durables (CD), Energy (EN), High-Tech (HT), Health Care (HC), Manufacturing (MN), Consumer Non-Durables (CN), Others (OT), Shops (SH), Telecommunication (TL), Utilities (UT)

since the relevance of information arrivals increases jointly for all index constituents. By way of example, we fit a 10-dimensional VG model to the 10 Fama and French (1997) industry portfolios based on quarterly log-returns from the first quarter of 1946 to the third quarter of 2017 (287 observations).[5] Note that application to even higher dimensions is feasible, even with different univariate marginal distributions. As previously implemented, we first compute the maximum likelihood estimates for the univariate marginal VG models. The multivariate distribution of the subordinator is completely specified by the marginal parameters, since we assume complete dependence. Based on the simulated likelihood approach of Section 4.2, we then simulate $M = 100,000$ paths of the subordinator and maximize the simulated likelihood with respect to the Brownian correlation matrix.

The parameters of the log-price process are all well identifiable. The marginal VG distributions are separately specified by 4 parameters. Panel A of Table 3 shows the corresponding estimates. The joint distribution of the log-price processes and, in particular, the pairwise correlation is primarily affected by the Brownian correlation matrix. The estimates are given by Panel B. The absence of Lévy-copula parameter allows to extend univariate VG models very easily to a multivariate framework with a wide-ranging correlation structure. Panel C and D of Table 3 show that the empirical correlation is mostly covered by the correlation of the Variance Dependent-Gamma process, although slightly underestimated by the maximum likelihood estimates. The χ^2-statistic of a likelihood-ratio test against a multivariate Black-Scholes model is 85.10. Considering the 20 degrees of freedom this results in a p-value less than 6×10^{-10}.

6 Conclusion

Our continuous time model framework bases on a multivariate exponential Lévy market model with a log-return process driven by a Lévy process $\boldsymbol{Y}$ originated from weak subordination. In particular, $\boldsymbol{Y} \stackrel{\mathcal{D}}{=} \boldsymbol{X} \odot \boldsymbol{T}$ is based on a directing process $\boldsymbol{X}$ and an information flow process $\boldsymbol{T}$, independent from $\boldsymbol{X}$. We consider Lévy copulas to describe the jump dependence of the information flow process $\boldsymbol{T}$ driving the financial return process. The Lévy measure of the subordinator has full jump support leading to more natural dynamics and potentially stronger dependencies. As a result, the price process is subject to possibly

[5] The data is retrieved from the website of Kenneth French.

non-linear dependencies in the marginal market activities, different to plain correlations of returns. In particular, the multivariate information flow process introduces correlations in the marginal conditional variance processes, e.g. empirically investigated by Andersen et al. (2001). Furthermore, we allow for dependencies also in the directing process providing returns which are conditionally normal and correlated. In fact, we model univariate Lévy processes originated by subordination, e.g. VG or NIG processes, jointly both in the stochastic time and space dimension. We provide novel examples of multivariate subordinators based on the theory of Lévy copulas, especially within the Archimedean class.

We present an approach for simulation of the log-return process and state an error bound as well as convergence rates in case the directing process $\boldsymbol{X}$ is Gaussian. In this case also an estimation procedure based on the simulated likelihood is given. As an example, we study novel multivariate VG models and estimate these models based on quarterly financial data. In particular, we consider the Clayton Lévy copula to accommodate moderate market dependencies for six pairs of major regional stock indices. We find that the parameters are well identifiable and obtain estimates which meet a reasonable interpretation from a macroeconomic perspective. The results indicate three clusters of market dependencies, i.e. moderate dependencies between the West and Japan, semi-strong dependencies between Europe and the US, and strong dependencies within Europe. To model strong market dependencies within the US, we apply a completely dependent information flow process. The marginal distributions are still flexible but the jumps follow a comontonic structure, which means that the relevance of information arrivals increases jointly for all modeled assets. The strong dependence structure of the 10 Fama and French (1997) industry portfolios is supported by the model, while marginal models can be chosen arbitrarily, which overcomes the shortcomings of traditional model alternatives, e.g. Semeraro (2008) and Luciano, Marena and Semeraro (2016).

References

ANDERSEN, T. G. (1996), 'Return volatility and trading volume: An information flow interpretation of stochastic volatility', The Journal of Finance 51(1), 169–204.

ANDERSEN, T. G., BOLLERSLEV, T., DIEBOLD, F. X., AND EBENS, H. (2001), 'The distribution of realized stock return volatility', Journal of Financial Economics 61(1), 43–76.

Ané, T. and Geman, H. (2000), 'Order flow, transaction clock, and normality of asset returns', The Journal of Finance 55(5), 2259–2284.

Barndorff-Nielsen, O. E. (1994), 'Normal inverse gaussian processes and the modelling of stock returns', Research Report 300, Department of Mathematical Sciences, Aarhus University.

Barndorff-Nielsen, O. E., Pedersen, J., and Sato, K.-i. (2001), 'Multivariate subordination, self-decomposability and stability', Advances in Applied Probability 160–187.

Barndorff-Nielsen, O. E. and Shephard, N. (2001), 'Non-gaussian ornstein–uhlenbeck-based models and some of their uses in financial economics', Journal of the Royal Statistical Society: Series B (Statistical Methodology) 63(2), 167–241.

Berry, S., Levinsohn, J., and Pakes, A. (1995), 'Automobile prices in market equilibrium', Econometrica 63(4), 841–890.

Buchmann, B., Madan, D. B., and Lu, K. (2017), 'Weak subordination of multivariate Lévy processes and variance generalised gamma convolutions', To appear in Bernoulli Available at https://arxiv.org/abs/1609.04481.

Cont, R. and Tankov, P. (2004), Financial modelling with jump processes, vol. 2, CRC press.

Deutsches Aktieninstitut (2011), DAI factbook, Dt. Aktieninst.

Fama, E. F. and French, K. R. (1997), 'Industry costs of equity', Journal of financial economics 43(2), 153–193.

Fritsch, F. N., Shafer, R. E., and Crowley, W. P. (1973), 'Solution of the transcendental equation $we^w = x$', Communications of the ACM 16(2), 123–124.

Gallant, A. R., Rossi, P. E., and Tauchen, G. (1992), 'Stock prices and volume', The Review of Financial Studies 5(2), 199–242.

Gourieroux, C. and Monfort, A. (1996), Simulation-based econometric methods, Oxford University Press.

Kallsen, J. and Tankov, P. (2006), 'Characterization of dependence of multidimensional Lévy processes using Lévy copulas', Journal of Multivariate Analysis 97(7), 1551–1572.

Luciano, E., Marena, M., and Semeraro, P. (2016), 'Dependence calibration and portfolio fit with factor-based subordinators', Quantitative Finance 1–16.

Madan, D. B., Carr, P. P., and Chang, E. C. (1998), 'The variance gamma process and option pricing', European Finance Review 2(1), 79–105.

Madan, D. B. and Seneta, E. (1990), 'The variance gamma (VG) model for share market returns', The Journal of Business 63(4), 511–524.

Michaelsen, M. and Szimayer, A. (2017), 'Marginal consistent dependence modeling using weak subordination for Brownian motions', Available at https://ssrn.com/abstract=3079478.

Monroe, I. (1978), 'Processes that can be embedded in Brownian motion', The Annals of Probability 42–56.

Nelsen, R. B. (2007), An introduction to copulas, Springer Science & Business Media.

Rosiński, J. (2001), 'Series representations of Lévy processes from the perspective of point processes', in 'Lévy processes', 401–415, Springer.

Ross, S. A. (1989), 'Information and volatility: The no-arbitrage martingale approach to timing and resolution irrelevancy', The Journal of Finance 44(1), 1–17.

Sato, K.-i. (1999), Lévy processes and infinitely divisible distributions, Cambridge university press.

Semeraro, P. (2008), 'A multivariate variance gamma model for financial applications', International Journal of Theoretical and Applied Finance 11(1), 1–18.

Stehle, R., Maier, J., and Huber, R. (1996), Rückberechnung des DAX für die Jahre 1955 bis 1987, Humboldt-Universität zu Berlin, Wirtschaftswissenschaftliche Fakultät.

Tankov, P. (2006), 'Simulation and option pricing in Lévy copula models', in M. Avellaneda and R. Cont, eds., 'Mathematical Modelling of Financial Derivatives', Springer.

Tankov, P. (2016), 'Lévy copulas: review of recent results', in M. Podolskij, R. Stelzer, S. Thorbjørnsen, and A. E. Veraart, eds., 'The Fascination of Probability, Statistics and their Applications', 127–151, Springer.

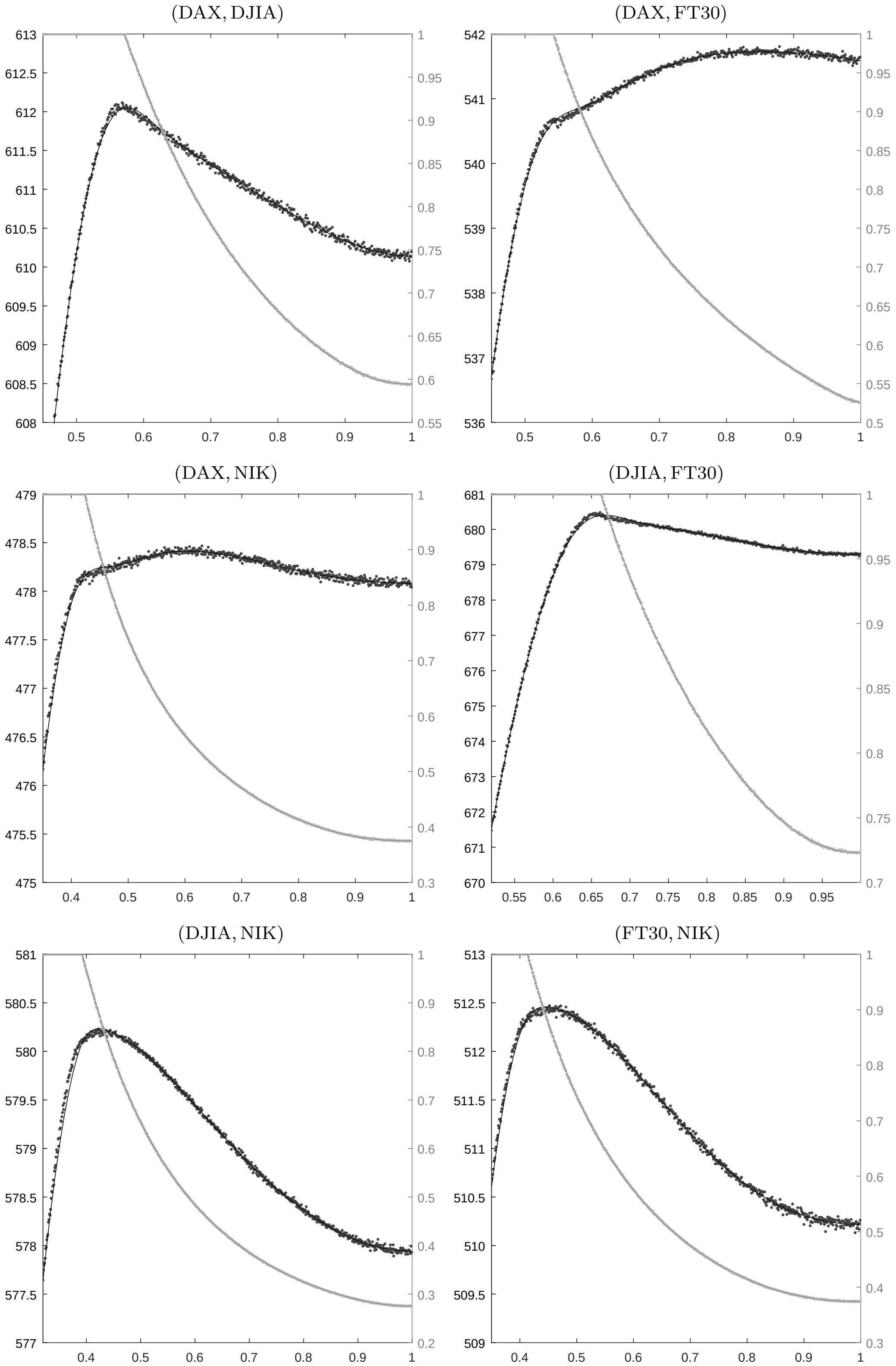

Figure 5: *The x-axis refers to the transformed Clayton parameter* $\widetilde{\theta}$*. The black plot shows the LOESS curve fitted to the simulated log-likelihoods (left y-axis) maximized with respect to the Brownian correlation parameter. The gray plot shows the LOESS curve fitted to the corresponding maximum likelihood estimates of the Brownian correlation parameter* ρ *(right y-axis).*

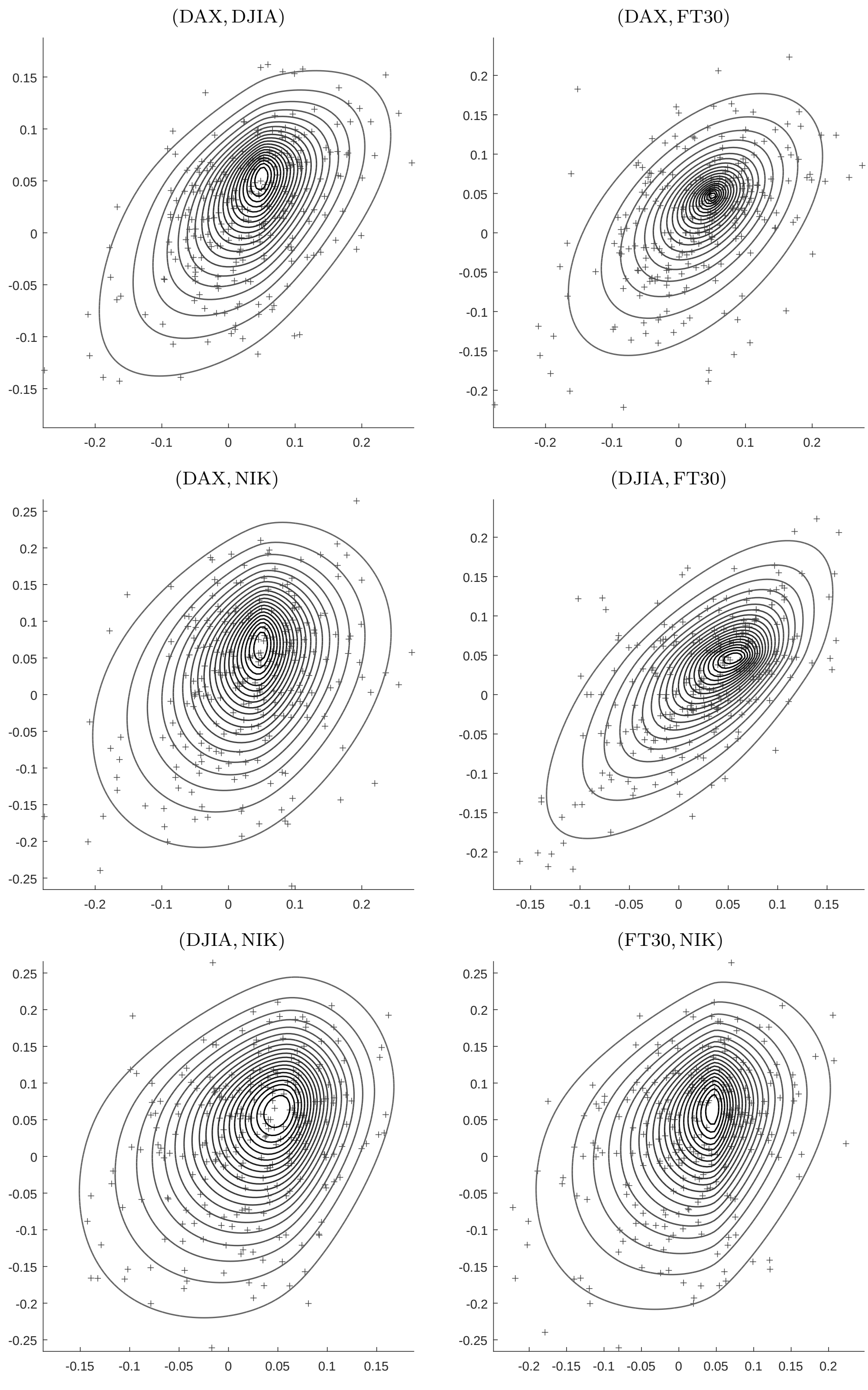

Figure 6: *Contour plots of the estimated Weak Variance Clayton-Gamma models according to Table 1.*