Confidence Sets for the Date of a Structural Change at the End of a Sample

Eiji Kurozumi[1]

Department of Economics, Hitotsubashi University

September 1, 2017

This version: January 31, 2018

Abstract

This paper proposes constructing a confidence set for the date of a structural change at the end of a sample in a linear regression model. While the break fraction, the ratio of the number of observations before the break to the sample size, is typically assumed to take a value in the $(0,1)$ open interval, we consider the case where a permissible break date is included in a fixed number of observations at the end of the sample and thus the break fraction approaches one as the sample size goes to infinity. We propose inverting the test for the break date to construct a confidence set, while critical values are obtained by using the subsampling method. By using Monte Carlo simulations, we show that the confidence set proposed in this paper can control the coverage rate in finite samples well, while the average length of the confidence set is comparable to existing methods based on asymptotic theory with a fixed break fraction in the $(0,1)$ interval.

JEL classification: C12, C15, C22

Keywords: structural change, coverage rate, subsampling method

[1]This research was supported by JSPS KAKENHI Grant Number 16K03594. Address correspondence to Eiji Kurozumi, Department of Economics, Hitotsubashi University, 2-1 Naka, Kunitachi, Tokyo 186-8601, Japan; e-mail: kurozumi@stat.hit-u.ac.jp

1. Introduction

This paper proposes constructing a confidence set for the date of a structural change at the end of a sample in a linear regression model. Structural change is a crucial issue in time series analysis when dealing with relatively long samples and many tests for structural change have been proposed in the literature. In empirical analyses, once evidence of structural change is found, the next task is to construct a confidence set for the break date. One frequently used method for constructing such a confidence set is based on the limiting distribution of the break point estimator. For example, Bai (1994) derived this in the case of a one-time shift in the mean of linear processes, while more general regression models were considered by Bai (1997). The latter model was further extended to the case of multiple structural changes by Bai and Perron (1998) and Qu and Perron (2007). In these cases, because the pivotal limiting distribution is obtained by normalizing the break point estimator using unknown parameters, the confidence interval is constructed by using consistent estimators of those parameters. In the case of the level-shift model, Hušková and Kirich (2008) showed that the block bootstrap is available and proposed constructing the confidence interval by using the empirical distribution of the resampled time series. On the contrary, because confidence intervals based on the above method tend to be too liberal for small or moderate structural change, according to the results of Monte Carlo simulations, Elliott and Müller (2007), Eo and Morley (2015), Kurozumi and Yamamoto (2015), and Yamamoto (2016) among others, proposed constructing a confidence set by inverting tests for the location of the break date, showing that their methods control the coverage rate well.

One common and important assumption for constructing a confidence set made in most of these studies is that a sufficient number of observations exists in both the pre-change and the post-change samples, suggesting that the break fraction, the ratio of the number of observations in the pre-change sample to that in the whole sample, is a fixed value in the $(0, 1)$ open interval. However, in empirical analyses, we may be interested in the case where structural change occurs only in a small number of observations at the end of the sample. In other words, because the number of observations before the break is much greater than that thereafter, it is natural to assume that the break fraction approaches one under asymptotic theory. For example, Dufour, Ghysels, and Hall (1994) considered the case where the size of the first sample goes to infinity, while that of the second sample is fixed; they then proposed tests for structural change in the second sample. In a similar setting, Andrews (2003) proposed tests for a one-time change in the second sample assuming stationarity and ergodicity in the error term, in which the critical values are obtained by using the subsampling method. This latter method was further extended to cointegration models by Andrews and Kim (2006), while Kim (2010) proposed an improvement in the end-of-sample tests. Further, Astill, Harvey, Leybourne, and Taylor (2017) tested for the presence of end-of-sample bubbles by using the subsampling method. However, to the best of our knowledge, constructing a confidence set for a break date at the end of a sample has not thus far been investigated.

To bridge this gap in the literature, in this paper, we propose constructing a confidence set for a one-time shift in a linear regression model at the end of a sample. We consider inverting a test for the location of the break date with the critical values obtained by using the subsampling method as in Andrews (2003). We show that the critical values based on the empirical distribution of the subsampled time series are asymptotically valid. The finite sample performance of our method is investigated by using Monte Carlo simulations, which show that our method controls the coverage rate relatively well, while the average length of the confidence set falls as the magnitude of the break rises.

The remainder of the paper is organized as follows. The model and assumptions are introduced and the test for the location of the break date is explained in Section 2. The finite sample properties of the confidence set proposed in the paper are examined in Section 3. We apply our method to the investigation of the Japanese inflation rate in 2013 in Section 4. Concluding remarks are provided in Section 5.

2. Construction of a Confidence Set

Let us consider the following linear regression model:

$$y_t = x_t'\beta + x_t'\delta 1\left(t > T + k_0\right) + u_t \qquad \text{for} \qquad t = 1, \cdots, T, \cdots, T+m, \tag{1}$$

where x_t is a p-dimensional regressor, $\delta \neq 0$, $1(\cdot)$ is an indicator function that takes one if the argument is true and zero otherwise, $T + k_0$ is an unknown break point with $0 \leq k_0 \leq m-1$, and $\{u_t\}$ is a zero-mean stationary and ergodic sequence. We suppose that m is fixed. In this model, the coefficient associated with x_t is β for $t = 1, \cdots, T + k_0$; in other words, there is no structural change before $T + k_0$, while a one-time shift occurs at $t = T + k_0$. We call $t = 1, \cdots, T$ the stable period, while the unstable period corresponds to $t = T+1, \cdots, T+m$. Our goal is to construct a confidence set for $T + k_0$ (or k_0) at a confidence level of $1 - \alpha$. Note that the break fraction, $(T + k_0)/(T + m)$, goes to one as $T \to \infty$ in this setting. Thus, existing asymptotic methods may be unsuitable because they assume that the break fraction is positive but below one.

Since the asymptotic distribution of the break point estimator cannot be derived for fixed m, we consider constructing a confidence set by inverting the test for the location of the break point. That is, we test for

$$H_0: \ k_0 = k_1 \qquad \text{vs.} \qquad H_1: \ k_0 \neq k_1 \tag{2}$$

at a significance level of α and include k_1 in the confidence set if the null hypothesis is accepted. By performing this procedure for $k_1 = 0, \cdots, m-1$, we can construct the confidence set at a confidence level of $1 - \alpha$.

To construct the test statistic, we first consider the simple alternative hypothesis given by $k_0 = k_2$, where $k_2 \neq k_1$ and $|k_2 - k_1| \geq p$. Let

$$V_j(k_1, k_2) = \left(\sum_{t \in \mathcal{A}(k_1,k_2)} x_{j+t} \hat{u}_{j+t} \right)' \left(\sum_{t \in \mathcal{A}(k_1,k_2)} x_{j+t} x'_{j+t} \right)^{-1} \left(\sum_{t \in \mathcal{A}(k_1,k_2)} x_{j+t} \hat{u}_{j+t} \right) \tag{3}$$

where $\mathcal{A}(k_1, k_2) = \{t : \ k_2 + 1 \leq t \leq k_1\}$ for $k_2 < k_1$ and $\mathcal{A}(k_1, k_2) = \{t : \ k_1 + 1 \leq t \leq k_2\}$ for $k_1 < k_2$,

$$\hat{u}_{j+t} = \begin{cases} y_{j+t} - x'_{j+t} \hat{\beta}_{T+k_1} & : \ t = 0, \cdots, k_1 \\ y_{j+t} - x'_{j+t} \tilde{\beta}_{j,k_1} & : \ t = k_1 + 1, \cdots, m \end{cases}$$

with $\hat{\beta}_{T+k_1} = \left(\sum_{t=1}^{T+k_1} x_t x'_t \right)^{-1} \sum_{t=1}^{T+k_1} x_t y_t$ and $\tilde{\beta}_{j,k_1} = \left(\sum_{t=k_1+1}^{m} x_{j+t} x'_{j+t} \right)^{-1} \sum_{t=k_1+1}^{m} x_{j+t} y_{j+t}$.

We cannot obtain $\tilde{\beta}_{j,k_1}$ when $m - k_1 < p$ and we define $\hat{u}_{j+t} = 0$ for $t = k_1 + 1, \cdots, m$ in this case. Then, as shown by Elliott and Müller (2007) and Kurozumi and Yamamoto (2015), the test statistic $V_T(k_1, k_2)$ maximizes the weighted average of power over δ for the simple hypothesis under the assumption of the i.i.d. normality of $\{u_t\}$. Since the possible alternatives are the integer values of $k_2 \in [0, m-1]$ ($|k_2 - k_1| \geq p$), the test statistic we consider is given by

$$S_T(k_1) = \max_{\substack{0 \leq k_2 \leq m-1 \\ |k_2 - k_1| \geq p}} V_T(k_1, k_2). \tag{4}$$

The critical values for the test statistic $S_T(k_1)$ are obtained by using the subsampling method suggested by Andrews (2003). Let

$$S_j^0(k_1) = \max_{\substack{0 \leq k_2 \leq m-1 \\ |k_2 - k_1| \geq p}} V_j^0(k_1, k_2) \qquad \text{for } j = 1, \cdots, T-m \text{ and } T,$$

where $V_j^0(k_1, k_2)$ is defined as (3) with $\hat{u}_{j+t}$ replaced with

$$\hat{u}_{j+t}^0 = \begin{cases} u_{j+t} & : \quad t = 0, \cdots, k_1 \\ u_{j+t} - x'_{j+t} \left(\sum_{t=k_1+1}^{m} x_{j+t} x'_{j+t} \right)^{-1} \sum_{t=k_1+1}^{m} x_{j+t} u_{j+t} & : \quad t = k_1 + 1, \cdots, m \end{cases}.$$

Since $\hat{\beta}_{T+k_1}$ converges to β in probability under both the null and the alternative, the null limiting distribution of $S_T(k_1)$ is given by $S_T^0(k_1) \stackrel{d}{=} S_1^0(k_1)$ because of the stationarity and ergodicity of $\{(x_t, u_t)\}$, where $\stackrel{d}{=}$ denotes equality in distribution. Since $S_j^0(k_1)$ for $j = 1, \cdots, T-m$ have the same distributions under both the null and the alternative, the distribution function (df) of $S_1^0(k_1)$ can be consistently estimated by the empirical df of

$$S_j(k_1) = \max_{\substack{0 \leq k_2 \leq m-1 \\ |k_2 - k_1| \geq p}} V_j(k_1, k_2) \qquad \text{for} \qquad j = 1, \cdots, T-m.$$

Thus, the $1-\alpha$ quantile $\hat{q}_{S,1-\alpha}$ of $S_j(k_1)$ for $j = 1, \cdots, T-m$ is asymptotically valid as the critical value for $S_T(k_1)$ at a significance level of α, where

$$\hat{q}_{S,1-\alpha} = \inf \left\{ x \in \mathbb{R} : \ \hat{F}_S(x) \geq 1 - \alpha \right\} \quad \text{with} \quad \hat{F}_S(x) = \frac{1}{T-m} \sum_{j=1}^{T-m} 1\left(S_j(k_1) \leq x\right).$$

To state our result more precisely, we make the following assumption.

Assumption 1 *(a)* $\{(x_t, u_t)\}$ *is a stationary and ergodic sequence with* $E[x_1 u_1] = 0$, $E[u_1^2] < \infty$, $E[\|x_1\|^{2+\gamma_1}] < \infty$ *for some* $\gamma_1 > 0$, $Var[x_1]$ *is positive definite, and* $E[|x_{i,1} u_1|^{2+\gamma_2}] < \infty$

for some $\gamma_2 > 0$*, where* $x_{i,1}$ *is the* i*th element of* x_1 $(i = 1, \cdots, p)$*.*

(b) $T^{-1/2} \sum_{t=1}^{T+m} x_t u_t = O_p(1)$*.*

(c) The df of $S_1^0(k_1)$ *is continuous and increasing at its* $1-\alpha$ *quantile.*

We need the stationary and ergodic assumption to consistently estimate the critical values of the test statistic by using the subsampling method. Note that the assumption of $\{u_t\}$ is general such that it is allowed to be serially correlated.

Let $S_\infty(k_1)$ be a random variable with the same distribution as $S_T(k_1)$ with $\hat{\beta}_{T+k_1}$ replaced with β. Note that $S_\infty(k_1) \stackrel{d}{=} S_T^0(k_1) \stackrel{d}{=} S_1^0(k_1)$ under the null hypothesis, whereas $S_\infty(k_1)$ has a different distribution because of the misallocation of the break point ($k_1 \neq k_0$). Let $F_S^0(k_1)$ be the df of $S_1^0(k_1)$ and $q_{S,1-\alpha}^0$ be the $1-\alpha$ quantile of $S_1^0(k_1)$.

Theorem 1 *Suppose that Assumption 1 holds. Then, as* $T \to \infty$ *while* m *is fixed,*

(a) $S_T(k_1) \stackrel{d}{\longrightarrow} S_\infty(k_1)$ *under both* H_0 *and* H_1*,*

(b) $\hat{F}_S(x) \stackrel{p}{\longrightarrow} F_S^0(x)$ *for all* x *in a neighborhood of* $q_{S,1-\alpha}^0$ *under both* H_0 *and* H_1*,*

(c) $\hat{q}_{S,1-\alpha} \stackrel{p}{\longrightarrow} q_{S,1-\alpha}^0$ *under both* H_0 *and* H_1*,*

(d) $P\left(S_T(k_1) > \hat{q}_{S,1-\alpha}\right) \to \alpha$ *under* H_0*,*

where $\stackrel{p}{\longrightarrow}$ *and* $\stackrel{d}{\longrightarrow}$ *signify convergence in probability and convergence in distribution, respectively.*

Theorem 1 implies that the critical values obtained by using the subsampling method are asymptotically valid. We can also see that the p-values obtained by

$$\frac{1}{T-m} \sum_{j=1}^{T-m} 1\left(S_T(k_1) \leq S_j(k_1)\right)$$

is asymptotically valid as far as the corresponding quantiles are continuous points.

To summarize our procedure to construct the confidence set for the break date, we construct the test statistic $S_T(k_1)$ and use the $1-\alpha$ quantile $S_j(k_1)$ for $j = 1, \cdots, T-m$ as the critical value. If the null hypothesis of $k_0 = k_1$ is not rejected, then k_1 is included in the confidence set. We conduct this procedure for $k_1 = 0, \cdots, m-1$ and finally obtain the confidence set at an asymptotic confidence level of $1-\alpha$.

3. Finite Sample Performance

3.1. Homoskedastic case

In this section, we investigate the finite sample properties of the confidence set constructed in the previous section. The data-generating process we consider is given by

$$y_t = x_t'\beta + x_t'\delta 1\left(t > T + k_0\right) + u_t, \qquad u_t = \rho u_{t-1} + \varepsilon_t, \tag{5}$$

for $t = 1, \cdots, T, \cdots, T+m$. We consider two cases for the regressor: $x_t = 1$ with $\beta = 0$ and $\delta = d$ ($p = 1$) and $x_t = [1, x_{2t}]'$ with $\beta = [0, 1]'$ and $\delta = [d, d]'$ ($p = 2$), where x_{2t} has the same AR structure as u_t; $x_t = \rho x_{t-1} + \varepsilon_{xt}$ with $\{\varepsilon_{xt}\}$ independent of $\{\varepsilon_t\}$. Sample size T in the stable period is 100 and 200, whereas sample size m in the unstable period, in which a one-time shift occurs, is 5 and 10 when $T = 100$, while it is 10 and 20 when $T = 200$. In other words, we consider the case where m/T is at most 0.1. k_0 is set to 0, 2, and 4 for $m = 5$, to 2, 5, and 8 for $m = 10$, and to 4, 10, and 16 for $m = 20$. The AR(1) coefficient ρ is 0, 0.4, and 0.8 and the innovations $\{\varepsilon_t\}$ and $\{\varepsilon_{xt}\}$ are i.i.d. with mean zero and variance one. We consider four distributions for $\{\varepsilon_t\}$ and $\{\varepsilon_{xt}\}$: $N(0,1)$, χ_2^2 (rescaled and recentered), t_3 (rescaled), and $U(-\sqrt{12}/2, \sqrt{12}/2)$. Although we conduct simulations with various sets of parameters, we tabulate only selected cases to save space.

For comparison purposes, we also construct the confidence sets by using three other methods. One is the confidence set based on the asymptotic distribution of the break point estimator by Bai (1994, 1997). Another confidence set applicable to the case of $x_t = 1$ is based on the circular block bootstrap proposed by Hušková and Kirich (2008). For the latter method, we set block sizes of 2, 4, and 10. We construct the confidence sets for k_0 within the $[0, m-1]$ closed interval and thus truncate the confidence sets obtained by these methods at $k = 0$ and $m - 1$, when the left end of the confidence set is less than zero and the right end is greater than $m - 1$. The third method is the modified Elliott and Müller (2007) test proposed by Yamamoto (2016). To implement this test for the case where the possible break date k_1 is close to the end point ($k_1 > m - p$), we set $\hat{u}_t = 0$ for $t = T + k_1 + 1, \cdots, T + m$ when we cannot estimate the coefficient after $T + k_1$ (the sample period after the break is too short). The number of replications is 5000, and all computations are conducted by using the GAUSS matrix language.

Table 1 reports the coverage rates and average lengths of the confidence sets relative to m at a confidence level of 0.9 when $x_t = 1$ and the innovations are i.i.d.N(0,1) with $T = 100$ and $m = 5$. When the break occurs at the end of the stable period ($k_0 = 0$), the coverage rate based on asymptotic theory is too conservative and close to one in most cases (see the column headed "Asy."), while the average length of the confidence interval becomes shorter as the magnitude of the break increases, although the confidence interval covers most of the unstable period (the average length is close to one) when the errors are strongly serially correlated. A similar tendency is observed when $k_0 = 2$ and 4. The coverage rates and average lengths of the confidence intervals based on the bootstrap method are reported in the columns headed "B(2)," "B(4)," and "B(10)." Here, the numbers in parentheses correspond to the block size. The table shows no significant difference caused by the choice of block size. When $k_0 = 0$, the coverage rates tend to be too liberal for small breaks but too conservative for large breaks, while the average lengths tend to be shorter than those of the other methods, although this tendency is partially due to the liberal coverage rates. As the break date approaches the end of the sample, the coverage rates become too liberal, particularly when $\rho = 0.8$. These results suggest that it is difficult to control the coverage rates of the confidence intervals by using the bootstrap method when the break occurs at the end of the sample.

On the contrary, the modified Elliott and Müller method (see the column headed "MEM") can control the coverage rate when k_0 is not too close to the end of the sample and the errors are not strongly serially correlated, although it tends to be too conservative when $\rho = 0.8$. We also note that the average length of the confidence set by MEM is very long and thus this method is less informative in this setting. By contrast, the coverage rate of the confidence set based on the subsampling method proposed in this paper (see the column headed "SS") is relatively close to 0.9, although it becomes slightly liberal in the case of strong serial correlation. Although it is difficult to compare the average length of SS with that of the other methods because the coverage rates are different, it is shorter than or comparable to for $k_0 = 0$ and 2, while it is longer than the others in the case of $k_0 = 4$.

Table 2 reports the case when $p = 1$ and $m = 10$, while Tables 3 and 4 correspond to the case of $p = 2$, for which we did not use the bootstrap method. We can see that relative performance is preserved for these cases. A similar tendency is observed when $T = 200$. The effect of the distribution of the innovations is relatively minor and we omit the details.

Note that the choice of m depends on the researcher, and m might be taken to be relatively long, as explained in Astill, Harvey, Leybourne, and Taylor (2017). For example, we conducted simulations with $m = 5$ when $T = 100$ as in Table 1 but a researcher may choose $m = 10$ in a similar situation to cover the break date at the end of the sample. In this case, the performances of the tests can be deduced by comparing Tables 1 and 2 or 3 and 4. Roughly speaking, when a larger m is taken, there is no significant effect on the coverage rate based on the subsampling method, while the length of the confidence set tends to be longer.

3.2. Heteroskedastic case

In this section, we consider extending test statistic (4) to accommodate unconditional heteroskedasticity. Similar to Astill, Harvey, Leybourne, and Taylor (2017), we modify test statistic (4) as

$$S_T^h(k_1) = \max_{0 \le k_2 \le m-1} V_T^h(k_1, k_2),$$

$$\text{where} \quad V_j^h(k_1, k_2) = \frac{\left(\sum_{t \in \mathcal{A}(k_1,k_2)} x_{j+t}\hat{u}_{j+t}\right)' \left(\sum_{t \in \mathcal{A}(k_1,k_2)} x_{j+t}x_{j+t}'\right)^{-1} \left(\sum_{t \in \mathcal{A}(k_1,k_2)} x_{j+t}\hat{u}_{j+t}\right)}{\frac{1}{m}\sum_{t=1}^{m} \hat{u}_{j+t}^2}$$

for $j = 1, \cdots, T - m$ and T. The data-generating process is the same as (5) with $\rho = 0$ and $\varepsilon_{xt} \sim i.i.d.N(0, 1)$ for $t = 1, \cdots, T + m$, while $\varepsilon_t \sim i.i.d.N(0, 1)$ for $t = 1, \cdots, T_\sigma$ and $i.i.d.N(0, \sigma^2)$ for $t = T_\sigma + 1, \cdots, T + m$. We consider two cases for the variance change point: $T_\sigma = T/2$ as DGP(h1) and $T_\sigma = T + m - 5$ as DGP(h2). That is, in DGP(h1), the variance of the error term changes in the middle of the stable period, while it changes five periods before the end of the sample. We set $\sigma^2 = 1/10$, $1/5$, 1, 5, and 10, while the magnitude of the break is fixed at $d = 2$. The sample size T is 100 with $m = 10$, $k_0 = 5$ and 200 with $m = 20$, $k_0 = 10$, respectively.

Table 5 reports the coverage rates and relative average lengths (we do not report them based on the bootstrap method). In DGP(h1), the coverage rate of the confidence set based on the original subsampling method as well as the other two existing asymptotic methods tend to be too close to 1 when σ^2 is small, while it is too small and far below 0.9. On the contrary, the subsampling method based on modified statistic $S_T^h(k_1)$ (see the column headed "SS(h)") can control the coverage rate well. Although the change in the variance in the unstable period does affect the coverage rates of all the methods as in the panels of GDP(h2), the coverage rate based on $S_T^h(k_1)$ is less affected than those based on the other methods.

4. Empirical Application

We next implement the proposed method to construct a confidence set for the date of the mean shift, using Japanese economic data. For the Japanese economy to address its long-term continued deflation, the Bank of Japan introduced a 2% inflation target in January 2013. Although this target has not yet been achieved, we investigate whether the inflation rate began to increase immediately after the introduction of this new policy, say within one year.[2]

We investigate the annual growth rate of the consumer price index (all items, excluding fresh food) from April 1998 to avoid the effect of the rise in the consumption tax rate in 1997. We first test for structural changes in the sample period ranging from April 1998 to December 2012 (just before the introduction of the inflation target), because structural changes are not allowed in the stable period ($t = 1, \cdots, T$). We implement the UD max and WDmax tests for the null hypothesis of no break against the alternative that there are at most three breaks, as proposed by Bai and Perron (1998), and find no evidence of breaks at the 10% significance level, as reported in Panel (a) of Table 6. Because we implement these tests by assuming that the possible break fractions are within the $[0.1, 0.9]$ interval (the trimming parameter is set to 0.1), we also test for parameter stability at the end of the sample from July 2011

[2]The consumption tax rate was increased by three percentage points (from 5% to 8%) in 2014. Although we may be able to modify our method to construct the confidence set by taking into account the jump in the inflation rate due to the rise in the consumption tax rate, we do not investigate 2014 data to avoid a complicated modification.

to December 2012 (the number of observations in this period is 18, the ratio of which to the whole observations is around 0.1) with the stable period from April 1998 to June 2011, using the test proposed by Andrews (2003). The null of end-of-sample stability is accepted, as reported in Panel (b) in Table 6.

We next test for a possible structural change from January 2013 to December 2013 ($m = 12$) with the stable sample period running from April 1998 to December 2012 ($T = 177$) using the same test as in Panel (c). Since the p-value of the test is almost 10%, we find weak evidence of a level shift in the inflation rate. Hence, we estimate the break date by minimizing the sum of the squared residuals. The break point estimate is May 2013, implying that the new policy of the Bank of Japan became effective in five months after its introduction. However, the confidence sets for the break date obtained by our method as well as the existing ones are wide, as reported in Figure 1. This result may be interpreted such that although the new policy raised the inflation rate, its effect was weak and gradual and thus the confidence sets for the break date became too wide. Judging by the result based on the subsampling method, the 90% confidence set includes three months before and after the estimated break date.

5. Concluding Remarks

In this paper, we proposed constructing a confidence set for a break date by inverting the test for the location of the break date when the break occurs at the end of the sample. The critical values can be easily obtained by using the subsampling method, and these are shown to be asymptotically valid. Although the settings of Monte Carlo simulations are limited, the confidence set proposed in the present paper can control the coverage rate well in finite samples. Further, the average length of the confidence set is short when the break occurs early in the unstable period, but it may be long when the break point is close to the end of the sample. In this sense, our method is not uniformly best among the other methods considered in the paper. However, by taking into account the good coverage rate, our method is nonetheless useful in practical analyses when the research interest is the break date at the end of the sample.

Appendix

Proof of Theorem 1: Note that because $S_j(k_1)$ is continuous with respect to $\hat{\beta}_{T+k_1}$ but not differentiable, our test statistic does not satisfy Assumption 3 in Andrews (2003) and we need to modify the proof.

(a) From Assumptions 1(a) and (b), $\hat{\beta}_{T+k_1} - \beta = O_p(T^{-1/2})$ as $T \to \infty$ for a given k_1 under both H_0 and H_1 and thus we have $P(S_T(k_1) \le x) \to P(S_\infty(k_1) \le x)$.

(b) Let $\hat{F}_S^0(x)$ be the empirical df of $S_j^0(k_1)$. Then, we have

$$\left|\hat{F}_S(x) - F_S^0(x)\right| \le \left|\hat{F}_S(x) - \hat{F}_S^0(x)\right| + \left|\hat{F}_S^0(x) - F_S^0(x)\right|. \tag{6}$$

The second term on the right-hand side of (6) converges to zero in probability under both H_0 and H_1 by the ergodic theorem, because $S_j^0(k_1)$ for $j = 1, \cdots, T-m$ are functions of only $(x_1, u_1), \cdots, (x_T, u_T)$.

For the first term of (6), note that $V_j(k_1, k_2)$ for $j = 1, \cdots, T-m$ can be expressed as (3) with

$$\hat{u}_{j,t} = \begin{cases} u_{j+t} - x'_{j+t}(\hat{\beta}_{T+k_1} - \beta) & : \ t = 0, \cdots, k_1 \\ u_{j+t} - x'_{j+t}\left(\sum_{t=k_1+1}^{m} x_{j+t}x'_{j+t}\right)^{-1} \sum_{t=k_1+1}^{m} x_{j+t}u_{j+t} & : \ t = k_1+1, \cdots, m \end{cases}$$

under both H_0 and H_1. Then, we can see that $V_j(k_1, k_2) = V_j^0(k_1, k_2)$ for $k_1 < k_2$, while for $k_2 < k_1$,

$$\begin{aligned} V_j(k_1, k_2) &= \left[\sum_{t=k_2+1}^{k_1} x_{j+t}u_{j+t} - \sum_{t=k_2+1}^{k_1} x_{j+t}x'_{j+t}(\hat{\beta}_{T+k_1} - \beta)\right]' \left(\sum_{t=k_2+1}^{k_1} x_{j+t}x'_{j+t}\right)^{-1} \\ &\qquad \left[\sum_{t=k_2+1}^{k_1} x_{j+t}u_{j+t} - \sum_{t=k_2+1}^{k_1} x_{j+t}x'_{j+t}(\hat{\beta}_{T+k_1} - \beta)\right] \\ &= V_j^0(k_1, k_2) - 2(\hat{\beta}_{T+k_1} - \beta)' \sum_{t=k_2+1}^{k_1} x_{j+t}u_{j+t} \\ &\quad + (\hat{\beta}_{T+k_1} - \beta)' \sum_{t=k_2+1}^{k_1} x_{j+t}x'_{j+t}(\hat{\beta}_{T+k_1} - \beta). \end{aligned} \tag{7}$$

Regarding the second term, for any $\varepsilon > 0$,

$$\begin{aligned} P\left(\max_{1\le j\le T-m}\left|\frac{1}{\sqrt{T}}\sum_{t=k_2+1}^{k_1} x_{i,j+t}u_{j+t}\right| \ge \varepsilon\right) &\le P\left(\frac{m}{\sqrt{T}}\max_{1\le j\le T}|x_{i,j}u_j| \ge \varepsilon\right) \\ &\le TP\left(\frac{m}{\sqrt{T}}|x_1u_1| \ge \varepsilon\right) \\ &\le T^{1-(2+\gamma)/2}\frac{m^{2+\gamma}}{\varepsilon^{2+\gamma}}E\left[|x_{i,1}u_1|^{2+\gamma}\right], \qquad (8) \end{aligned}$$

where the first and second inequalities hold because $0 \le k_1, k_2 \le m$ and the stationarity of $\{(x_t, u_t)\}$, respectively, while the third inequality is obtained from the generalized Markov inequality. Because $E\left[|x_{i,1}u_1|^{2+\gamma}\right] < \infty$ according to Assumption 1(a) and $1-(2+\gamma)/2 < 0$, the right-hand side of (8) converges to zero as $T \to \infty$. Similarly, we can show that the maximum of the third term on the right-hand side of (7) over $j = 1, \cdots, T-m$ converges to zero in probability. Thus, from (7), we can see that

$$V_j(k_1, k_2) = V_j^0(k_1, k_2) + o_p(1),$$

where the $o_p(1)$ term is uniformly over $j = 1, 2, \cdots, T-m$ and thus $S_j(k_1) = S_j^0(k_1) + o_p(1)$ uniformly over j. This implies that there exists a sequence of positive constants $\{h_T : T \ge 1\}$ such that $h_T \to 0$ and $P(K_T) \to 1$, where $K_T = \{|S_j(k_1) - S_j^0(k_1)| \le h_T, \forall j = 1, \cdots, T-m\}$. In this case, for any $\epsilon > 0$,

$$\begin{aligned} P\left(\left|\hat{F}_S(x) - \hat{F}_S^0(x)\right| \ge \epsilon\right) &\le P\left(\left\{\left|\hat{F}_S(x) - \hat{F}_S^0(x)\right| \ge \epsilon\right\} \cup K_T\right) + P\left(K_T^c\right) \\ &\le P\left(\frac{1}{T-m}\sum_{j=1}^{T-m} 1\left(x - h_T \le S_j^0(k_1) \le x + h_T\right) \ge \epsilon\right) + o(1) \\ &\le \frac{1}{\epsilon}E\left[1\left(x - h_T \le S_1^0(k_1) \le x + h_T\right)\right] + o(1), \qquad (9) \end{aligned}$$

where the second inequality holds by noting that on K_T, $1(S_j^0(k_1) \le x) = 1(S_j(k_1) \le x) = 1$ when $S_j^0(k_1) < x - h_T$ and $1(S_j^0(k_1) \le x) = 1(S_j(k_1) \le x) = 0$ when $S_j^0(k_1) > x + h_T$, while the last inequality holds because of the Markov inequality and stationarity. The right-hand side of (9) goes to zero as $T \to \infty$ by the dominated convergence theorem, because $S_1^0(k_1) \ne x$ a.s. from Assumption 1(c). As a result, we can see that part (b) is established.

From Assumption 1(c), part (b) implies part (c), which implies part (d).■

References

[1] Andrews, D. W. K. (2003). End-of-Sample Instability Tests. *Econometrica* 71, 1661-1694.

[2] Andrews, D. W. K. and J-Y. Kim (2006). Tests for Cointegration Breakdown over a Short Time Period. *Journal of Business and Economic Statistics* 24, 379-394.

[3] Astill, S., D. I. Harvey, S. J. Leybourne and A. M. R. Taylor (2017). Tests for an End-of-sample bubble in Financial Time Series. *Econometric Reviews* 36, 651-666.

[4] Bai, J. (1994). Least Squares Estimation of a Shift in Linear Processes. *Journal of Time Series Analysis* 15, 453-472.

[5] Bai, J. (1997). Estimation of a Change Point in Multiple Regression Models. *Review of Economics and Statistics* 79, 551-563.

[6] Bai, J. and P. Perron (1998). Estimating and Testing Linear Models with Multiple Structural Changes. *Econometrica* 66, 47-78.

[7] Dufour, J. M., E. Ghysels and A. Hall (1994). Generalized Predictive Tests and Structural Change Analysis in Econometrics. *International Economic Review* 35, 199-229.

[8] Eo, Y. and J. Morley (2015). Likelihood-ratio-based Confidence Sets for the Timing of Structural Breaks. *Quantitative Economics* 6, 463-497.

[9] Elliott, G. and U. K. Müller (2007). Confidence Sets for the Date of a Single Break in Linear Time Series Regressions. *Journal of Econometrics* 141, 1196-1218.

[10] Hušková, M. and C. Kirich (2008). Bootstrapping Confidence Intervals for the Change-point of Time Series. *Journal of Time Series Analysis* 29, 947-972.

[11] Kim, D. (2010). Improved and Extended End-of-Sample Instability Tests Using a Feasible Quasi-Generalized Least Squares Procedure. *Econometric Theory* 26, 994-1031.

[12] Kurozumi, E. and Y. Yamamoto (2015). Confidence Sets for the Break Date Based on Optimal Tests. *Econometrics Journal* 18, 412-435.

[13] Qu, Z. and P. Perron (2007). Estimating and Testing Structural Changes in Multivariate Regressions. *Econometrica* 75, 459-502.

[14] Yamamoto, Y. (2016). A Modified Confidence Set for the Structural Break Date in Linear Regression Models. *Econometric Reviews*, DOI:10.1080/00927872.2016.1178892.

Table 1: Coverage rates and lengths of the confidence sets ($p = 1$, $T = 100$, $m = 5$, $N(0, 1)$)

ρ	d	Asy.	B(2)	B(4)	B(10)	MEM	SS	Asy.	B(2)	B(4)	B(10)	MEM	SS
		Coverage rates						Average lengths					
$k_0 = 0$													
	1.0	0.922	0.811	0.811	0.811	0.885	0.875	0.944	0.388	0.388	0.388	0.890	0.646
	1.5	0.950	0.894	0.894	0.896	0.887	0.875	0.860	0.328	0.328	0.328	0.870	0.448
0.0	2.0	0.969	0.938	0.938	0.939	0.887	0.875	0.712	0.277	0.277	0.277	0.839	0.308
	2.5	0.981	0.953	0.954	0.955	0.889	0.875	0.569	0.240	0.240	0.240	0.792	0.236
	3.0	0.991	0.958	0.959	0.958	0.890	0.875	0.469	0.215	0.215	0.215	0.738	0.202
	1.0	0.974	0.782	0.783	0.769	0.924	0.879	0.982	0.398	0.398	0.394	0.940	0.717
	1.5	0.975	0.872	0.872	0.859	0.924	0.879	0.957	0.340	0.340	0.336	0.930	0.558
0.4	2.0	0.979	0.923	0.922	0.909	0.924	0.879	0.900	0.291	0.289	0.285	0.916	0.403
	2.5	0.986	0.941	0.937	0.932	0.925	0.879	0.809	0.250	0.249	0.247	0.896	0.290
	3.0	0.989	0.955	0.953	0.950	0.925	0.879	0.704	0.224	0.222	0.222	0.872	0.226
	1.0	1.000	0.810	0.810	0.769	0.985	0.890	0.998	0.383	0.382	0.368	0.980	0.807
	1.5	0.999	0.853	0.852	0.811	0.985	0.890	0.996	0.358	0.357	0.344	0.977	0.740
0.8	2.0	0.999	0.884	0.881	0.849	0.985	0.890	0.992	0.323	0.322	0.311	0.975	0.656
	2.5	1.000	0.913	0.911	0.886	0.985	0.890	0.986	0.293	0.292	0.283	0.971	0.563
	3.0	0.999	0.945	0.941	0.921	0.985	0.890	0.975	0.265	0.263	0.257	0.966	0.466
$k_0 = 2$													
	1.0	0.977	0.702	0.703	0.702	0.887	0.877	0.951	0.607	0.607	0.607	0.867	0.765
	1.5	0.971	0.799	0.799	0.797	0.887	0.877	0.933	0.664	0.661	0.662	0.819	0.635
0.0	2.0	0.978	0.858	0.856	0.856	0.888	0.877	0.893	0.657	0.656	0.655	0.758	0.492
	2.5	0.986	0.896	0.897	0.894	0.890	0.877	0.809	0.594	0.596	0.597	0.694	0.370
	3.0	0.991	0.925	0.925	0.926	0.891	0.877	0.709	0.495	0.496	0.497	0.635	0.283
	1.0	0.999	0.639	0.639	0.631	0.947	0.874	0.987	0.576	0.575	0.564	0.931	0.803
	1.5	0.997	0.732	0.732	0.716	0.948	0.874	0.979	0.646	0.646	0.626	0.911	0.711
0.4	2.0	0.997	0.817	0.816	0.796	0.948	0.874	0.970	0.685	0.683	0.651	0.879	0.588
	2.5	0.996	0.871	0.870	0.854	0.949	0.874	0.952	0.663	0.656	0.621	0.840	0.459
	3.0	0.996	0.909	0.907	0.897	0.949	0.874	0.919	0.587	0.572	0.547	0.793	0.350
	1.0	1.000	0.465	0.465	0.435	0.984	0.863	0.999	0.475	0.474	0.445	0.982	0.835
	1.5	1.000	0.551	0.549	0.520	0.985	0.863	0.998	0.546	0.544	0.498	0.980	0.795
0.8	2.0	1.000	0.643	0.641	0.612	0.985	0.863	0.998	0.620	0.614	0.546	0.977	0.738
	2.5	1.000	0.724	0.719	0.689	0.985	0.863	0.997	0.675	0.664	0.572	0.973	0.666
	3.0	1.000	0.799	0.792	0.764	0.986	0.863	0.996	0.707	0.689	0.573	0.967	0.580
$k_0 = 4$													
	1.0	0.998	0.718	0.717	0.717	0.977	0.884	0.916	0.430	0.430	0.429	0.854	0.869
	1.5	0.997	0.789	0.789	0.788	0.977	0.884	0.848	0.382	0.382	0.382	0.791	0.850
0.0	2.0	0.993	0.857	0.857	0.856	0.978	0.884	0.760	0.337	0.337	0.337	0.704	0.817
	2.5	0.992	0.918	0.917	0.917	0.978	0.884	0.662	0.300	0.300	0.299	0.607	0.765
	3.0	0.990	0.947	0.948	0.948	0.978	0.884	0.574	0.269	0.269	0.269	0.500	0.694
	1.0	0.996	0.597	0.597	0.597	0.972	0.876	0.977	0.428	0.428	0.425	0.931	0.870
	1.5	0.996	0.668	0.667	0.668	0.972	0.876	0.949	0.384	0.384	0.380	0.908	0.859
0.4	2.0	0.994	0.750	0.749	0.750	0.973	0.876	0.904	0.339	0.339	0.334	0.870	0.843
	2.5	0.992	0.818	0.818	0.818	0.973	0.876	0.838	0.300	0.300	0.295	0.816	0.813
	3.0	0.991	0.881	0.880	0.880	0.974	0.876	0.757	0.268	0.268	0.263	0.745	0.767
	1.0	0.997	0.388	0.387	0.387	0.981	0.859	0.999	0.388	0.388	0.372	0.983	0.863
	1.5	0.997	0.439	0.438	0.437	0.981	0.859	0.998	0.363	0.363	0.346	0.981	0.859
0.8	2.0	0.997	0.501	0.500	0.499	0.982	0.859	0.996	0.336	0.336	0.319	0.979	0.852
	2.5	0.996	0.563	0.562	0.561	0.982	0.859	0.993	0.304	0.303	0.289	0.976	0.842
	3.0	0.996	0.634	0.632	0.630	0.983	0.859	0.986	0.276	0.276	0.263	0.971	0.828

Table 2: Coverage rates and lengths of the confidence sets ($p = 1$, $T = 100$, $m = 10$, $N(0,1)$)

ρ	d	Asy.	B(2)	B(4)	B(10)	MEM	SS	Asy.	B(2)	B(4)	B(10)	MEM	SS
		Coverage rates						Average lengths					
$k_0 = 2$													
	1.0	0.904	0.720	0.719	0.718	0.884	0.880	0.828	0.467	0.465	0.463	0.839	0.575
	1.5	0.946	0.824	0.826	0.825	0.888	0.880	0.661	0.435	0.434	0.432	0.766	0.372
0.0	2.0	0.968	0.884	0.885	0.883	0.889	0.880	0.521	0.377	0.376	0.377	0.670	0.245
	2.5	0.980	0.926	0.927	0.925	0.892	0.880	0.427	0.309	0.310	0.312	0.561	0.172
	3.0	0.985	0.939	0.938	0.937	0.894	0.880	0.346	0.250	0.249	0.250	0.464	0.130
	1.0	0.934	0.600	0.605	0.597	0.900	0.879	0.920	0.448	0.453	0.452	0.891	0.700
	1.5	0.945	0.716	0.719	0.713	0.901	0.879	0.843	0.442	0.449	0.450	0.856	0.529
0.4	2.0	0.957	0.805	0.807	0.805	0.903	0.879	0.733	0.418	0.425	0.426	0.812	0.375
	2.5	0.971	0.859	0.859	0.856	0.905	0.879	0.621	0.373	0.378	0.382	0.757	0.258
	3.0	0.981	0.893	0.894	0.891	0.908	0.879	0.532	0.310	0.312	0.315	0.691	0.184
	1.0	0.997	0.453	0.455	0.433	0.966	0.854	0.985	0.386	0.389	0.382	0.958	0.811
	1.5	0.996	0.539	0.541	0.518	0.966	0.854	0.975	0.403	0.408	0.398	0.951	0.755
0.8	2.0	0.995	0.627	0.631	0.603	0.967	0.854	0.961	0.413	0.420	0.407	0.942	0.678
	2.5	0.996	0.706	0.711	0.684	0.968	0.854	0.939	0.416	0.423	0.407	0.930	0.588
	3.0	0.997	0.780	0.779	0.760	0.968	0.854	0.912	0.412	0.417	0.397	0.915	0.490
$k_0 = 5$													
	1.0	0.927	0.688	0.690	0.688	0.885	0.885	0.843	0.583	0.578	0.574	0.805	0.709
	1.5	0.942	0.816	0.815	0.813	0.887	0.885	0.735	0.573	0.567	0.563	0.704	0.535
0.0	2.0	0.950	0.885	0.886	0.884	0.888	0.885	0.583	0.476	0.474	0.475	0.598	0.365
	2.5	0.964	0.924	0.921	0.922	0.888	0.885	0.452	0.361	0.363	0.368	0.507	0.236
	3.0	0.980	0.935	0.935	0.936	0.890	0.885	0.361	0.266	0.268	0.274	0.437	0.159
	1.0	0.977	0.578	0.576	0.588	0.919	0.884	0.928	0.520	0.527	0.528	0.879	0.782
	1.5	0.973	0.691	0.693	0.701	0.921	0.884	0.883	0.545	0.564	0.564	0.827	0.669
0.4	2.0	0.966	0.782	0.787	0.788	0.922	0.884	0.807	0.514	0.546	0.547	0.760	0.530
	2.5	0.968	0.854	0.860	0.857	0.924	0.884	0.707	0.439	0.477	0.488	0.685	0.390
	3.0	0.971	0.886	0.895	0.890	0.925	0.884	0.598	0.341	0.371	0.393	0.615	0.272
	1.0	0.996	0.375	0.376	0.377	0.970	0.864	0.988	0.402	0.409	0.396	0.959	0.832
	1.5	0.995	0.459	0.461	0.458	0.970	0.864	0.983	0.439	0.452	0.433	0.950	0.799
0.8	2.0	0.991	0.558	0.561	0.558	0.972	0.864	0.975	0.474	0.497	0.466	0.938	0.750
	2.5	0.989	0.660	0.666	0.657	0.973	0.864	0.964	0.494	0.531	0.490	0.923	0.685
	3.0	0.986	0.742	0.749	0.735	0.975	0.864	0.947	0.493	0.540	0.490	0.904	0.607
$k_0 = 8$													
	1.0	0.991	0.626	0.623	0.618	0.896	0.883	0.774	0.401	0.400	0.399	0.806	0.848
	1.5	0.984	0.731	0.728	0.725	0.897	0.883	0.654	0.370	0.369	0.367	0.701	0.798
0.0	2.0	0.980	0.814	0.812	0.812	0.898	0.883	0.531	0.334	0.333	0.332	0.568	0.720
	2.5	0.975	0.883	0.882	0.880	0.900	0.883	0.432	0.293	0.292	0.291	0.435	0.608
	3.0	0.978	0.927	0.926	0.926	0.902	0.883	0.365	0.250	0.250	0.249	0.325	0.471
	1.0	0.986	0.563	0.564	0.564	0.953	0.868	0.906	0.405	0.405	0.407	0.883	0.862
	1.5	0.983	0.656	0.657	0.656	0.954	0.868	0.833	0.375	0.376	0.378	0.836	0.838
0.4	2.0	0.978	0.749	0.750	0.751	0.956	0.868	0.737	0.333	0.334	0.337	0.768	0.798
	2.5	0.975	0.825	0.827	0.829	0.957	0.868	0.634	0.294	0.296	0.297	0.681	0.737
	3.0	0.971	0.883	0.887	0.888	0.958	0.868	0.537	0.256	0.260	0.260	0.581	0.654
	1.0	0.981	0.397	0.401	0.400	0.963	0.852	0.988	0.346	0.349	0.343	0.961	0.855
	1.5	0.981	0.445	0.450	0.449	0.963	0.852	0.981	0.329	0.332	0.325	0.956	0.851
0.8	2.0	0.981	0.511	0.515	0.515	0.964	0.852	0.969	0.309	0.313	0.306	0.948	0.844
	2.5	0.979	0.575	0.581	0.580	0.965	0.852	0.950	0.283	0.287	0.280	0.937	0.833
	3.0	0.977	0.647	0.653	0.652	0.966	0.852	0.923	0.260	0.265	0.258	0.923	0.819

Table 3: Coverage rates and lengths of the confidence sets ($p = 2$, $T = 100$, $m = 5$, $N(0, 1)$)

ρ	d	Asy.	MEM	SS	Asy.	MEM	SS
		Coverage rates			Average lengths		
$k_0 = 0$							
	1.0	0.896	0.917	0.875	0.829	0.928	0.601
	1.5	0.932	0.920	0.875	0.660	0.885	0.474
0.0	2.0	0.947	0.921	0.875	0.520	0.834	0.414
	2.5	0.962	0.921	0.875	0.452	0.783	0.391
	3.0	0.974	0.922	0.875	0.432	0.731	0.377
	1.0	0.923	0.948	0.875	0.887	0.943	0.654
	1.5	0.948	0.948	0.875	0.771	0.909	0.526
0.4	2.0	0.959	0.949	0.875	0.635	0.869	0.454
	2.5	0.964	0.949	0.875	0.530	0.830	0.417
	3.0	0.969	0.949	0.875	0.468	0.793	0.395
	1.0	0.971	0.989	0.878	0.951	0.969	0.730
	1.5	0.978	0.989	0.878	0.907	0.946	0.635
0.8	2.0	0.980	0.989	0.878	0.838	0.919	0.560
	2.5	0.985	0.989	0.878	0.751	0.894	0.507
	3.0	0.987	0.989	0.878	0.669	0.869	0.468
$k_0 = 2$							
	1.0	0.989	0.949	0.876	0.872	0.912	0.729
	1.5	0.981	0.949	0.876	0.798	0.868	0.603
0.0	2.0	0.962	0.950	0.876	0.697	0.816	0.507
	2.5	0.956	0.950	0.876	0.631	0.772	0.440
	3.0	0.958	0.950	0.876	0.609	0.729	0.389
	1.0	0.991	0.970	0.873	0.910	0.939	0.740
	1.5	0.981	0.971	0.873	0.866	0.908	0.626
0.4	2.0	0.967	0.972	0.873	0.801	0.872	0.528
	2.5	0.958	0.972	0.873	0.725	0.837	0.461
	3.0	0.953	0.973	0.873	0.661	0.802	0.412
	1.0	0.989	0.990	0.856	0.951	0.970	0.736
	1.5	0.983	0.991	0.856	0.936	0.951	0.636
0.8	2.0	0.976	0.991	0.856	0.910	0.929	0.555
	2.5	0.968	0.992	0.856	0.876	0.907	0.493
	3.0	0.959	0.993	0.856	0.833	0.885	0.443
$k_0 = 4$							
	1.0	0.885	0.991	0.882	0.857	0.916	0.852
	1.5	0.871	0.991	0.882	0.813	0.878	0.816
0.0	2.0	0.861	0.991	0.882	0.771	0.834	0.773
	2.5	0.861	0.991	0.882	0.740	0.792	0.727
	3.0	0.871	0.991	0.882	0.721	0.752	0.686
	1.0	0.893	0.988	0.872	0.878	0.946	0.847
	1.5	0.875	0.988	0.872	0.835	0.919	0.807
0.4	2.0	0.867	0.989	0.872	0.796	0.886	0.762
	2.5	0.865	0.989	0.872	0.764	0.854	0.718
	3.0	0.864	0.990	0.872	0.740	0.821	0.679
	1.0	0.917	0.983	0.853	0.909	0.970	0.814
	1.5	0.894	0.984	0.853	0.868	0.952	0.766
0.8	2.0	0.879	0.985	0.853	0.828	0.934	0.720
	2.5	0.874	0.986	0.853	0.798	0.914	0.679
	3.0	0.870	0.988	0.853	0.772	0.895	0.644

Table 4: Coverage rates and lengths of the confidence sets ($p = 2$, $T = 100$, $m = 10$, $N(0, 1)$)

ρ	d	Asy.	MEM	SS	Asy.	MEM	SS
		Coverage rates			Average lengths		
$k_0 = 2$							
	1.0	0.898	0.906	0.873	0.640	0.840	0.489
	1.5	0.923	0.909	0.873	0.469	0.736	0.328
0.0	2.0	0.908	0.911	0.873	0.355	0.632	0.248
	2.5	0.919	0.914	0.873	0.305	0.548	0.206
	3.0	0.939	0.915	0.873	0.298	0.481	0.181
	1.0	0.910	0.918	0.874	0.744	0.869	0.577
	1.5	0.930	0.923	0.874	0.580	0.795	0.401
0.4	2.0	0.924	0.927	0.874	0.454	0.719	0.300
	2.5	0.916	0.929	0.874	0.370	0.652	0.243
	3.0	0.918	0.930	0.874	0.324	0.595	0.206
	1.0	0.970	0.959	0.851	0.883	0.913	0.699
	1.5	0.972	0.961	0.851	0.789	0.873	0.562
0.8	2.0	0.966	0.962	0.851	0.679	0.831	0.446
	2.5	0.961	0.964	0.851	0.577	0.789	0.358
	3.0	0.952	0.965	0.851	0.497	0.749	0.295
$k_0 = 5$							
	1.0	0.883	0.917	0.883	0.678	0.813	0.652
	1.5	0.882	0.921	0.883	0.509	0.711	0.483
0.0	2.0	0.889	0.923	0.883	0.380	0.617	0.355
	2.5	0.906	0.925	0.883	0.320	0.540	0.274
	3.0	0.927	0.926	0.883	0.304	0.480	0.225
	1.0	0.917	0.943	0.884	0.778	0.865	0.712
	1.5	0.904	0.945	0.884	0.640	0.794	0.568
0.4	2.0	0.897	0.947	0.884	0.501	0.721	0.444
	2.5	0.898	0.949	0.884	0.401	0.653	0.352
	3.0	0.907	0.950	0.884	0.344	0.594	0.289
	1.0	0.947	0.962	0.859	0.903	0.915	0.772
	1.5	0.939	0.966	0.859	0.843	0.882	0.679
0.8	2.0	0.925	0.971	0.859	0.754	0.844	0.580
	2.5	0.914	0.974	0.859	0.655	0.807	0.491
	3.0	0.911	0.978	0.859	0.567	0.770	0.420
$k_0 = 8$							
	1.0	0.900	0.989	0.871	0.652	0.820	0.820
	1.5	0.890	0.990	0.871	0.525	0.720	0.746
0.0	2.0	0.889	0.991	0.871	0.435	0.624	0.650
	2.5	0.904	0.991	0.871	0.380	0.545	0.559
	3.0	0.917	0.992	0.871	0.351	0.486	0.480
	1.0	0.891	0.984	0.867	0.751	0.872	0.835
	1.5	0.871	0.985	0.867	0.637	0.806	0.773
0.4	2.0	0.871	0.987	0.867	0.536	0.738	0.695
	2.5	0.875	0.989	0.867	0.463	0.673	0.617
	3.0	0.885	0.991	0.867	0.411	0.615	0.547
	1.0	0.899	0.955	0.852	0.878	0.914	0.832
	1.5	0.882	0.959	0.852	0.808	0.881	0.792
0.8	2.0	0.869	0.963	0.852	0.730	0.846	0.740
	2.5	0.866	0.967	0.852	0.655	0.809	0.685
	3.0	0.864	0.972	0.852	0.594	0.774	0.632

Table 5: Coverage rates and lengths of the confidence sets (heteroskedasticity case)

T	σ^2	Asy.	MEM	SS	SS(h)	Asy.	MEM	SS	SS(h)
			Coverage rates				Average lengths		
DGP(h1), $p = 1$									
	0.1	1.000	0.967	1.000	0.881	0.389	0.497	0.252	0.093
	0.2	0.999	0.960	1.000	0.885	0.418	0.511	0.269	0.142
100	1	0.950	0.888	0.885	0.888	0.583	0.598	0.365	0.539
	5	0.870	0.775	0.768	0.898	0.754	0.700	0.597	0.806
	10	0.854	0.747	0.766	0.899	0.782	0.723	0.674	0.853
	0.1	1.000	0.966	1.000	0.894	0.193	0.391	0.126	0.045
	0.2	1.000	0.961	1.000	0.895	0.212	0.402	0.131	0.051
200	1	0.949	0.900	0.890	0.894	0.301	0.473	0.188	0.265
	5	0.834	0.787	0.775	0.900	0.573	0.633	0.488	0.700
	10	0.788	0.763	0.773	0.904	0.645	0.683	0.612	0.807
DGP(h1), $p = 2$									
	0.1	0.984	0.980	1.000	0.884	0.304	0.530	0.357	0.330
	0.2	0.969	0.977	1.000	0.886	0.308	0.545	0.354	0.379
100	1	0.889	0.923	0.883	0.889	0.380	0.617	0.355	0.579
	5	0.808	0.823	0.758	0.897	0.581	0.717	0.546	0.747
	10	0.798	0.799	0.756	0.902	0.643	0.745	0.635	0.780
	0.1	0.984	0.978	1.000	0.889	0.150	0.383	0.170	0.096
	0.2	0.970	0.974	1.000	0.895	0.151	0.394	0.171	0.117
200	1	0.872	0.913	0.890	0.896	0.187	0.453	0.179	0.294
	5	0.755	0.788	0.781	0.905	0.372	0.594	0.417	0.659
	10	0.722	0.755	0.780	0.909	0.472	0.653	0.558	0.765
DGP(h2), $p = 1$									
	0.1	0.976	0.976	0.907	0.738	0.575	0.644	0.340	0.322
	0.2	0.977	0.973	0.907	0.771	0.581	0.641	0.344	0.357
100	1	0.950	0.888	0.885	0.888	0.583	0.598	0.365	0.539
	5	0.757	0.380	0.495	0.887	0.497	0.379	0.283	0.734
	10	0.643	0.170	0.270	0.853	0.444	0.265	0.185	0.769
	0.1	0.959	0.955	0.899	0.810	0.289	0.514	0.182	0.185
	0.2	0.958	0.951	0.899	0.824	0.290	0.511	0.183	0.193
200	1	0.949	0.900	0.890	0.894	0.301	0.473	0.188	0.265
	5	0.847	0.595	0.758	0.970	0.328	0.320	0.192	0.527
	10	0.671	0.371	0.561	0.976	0.305	0.226	0.168	0.666
DGP(h2), $p = 2$									
	0.1	0.919	0.990	0.899	0.695	0.369	0.655	0.368	0.469
	0.2	0.917	0.988	0.899	0.743	0.371	0.651	0.368	0.490
100	1	0.889	0.923	0.883	0.889	0.380	0.617	0.355	0.579
	5	0.762	0.493	0.515	0.913	0.371	0.444	0.234	0.670
	10	0.678	0.262	0.296	0.869	0.354	0.341	0.147	0.681
	0.1	0.879	0.966	0.895	0.798	0.183	0.489	0.179	0.226
	0.2	0.881	0.961	0.895	0.812	0.184	0.486	0.179	0.234
200	1	0.872	0.913	0.890	0.896	0.187	0.453	0.179	0.294
	5	0.807	0.581	0.750	0.976	0.197	0.305	0.160	0.490
	10	0.685	0.339	0.509	0.981	0.197	0.214	0.121	0.601

Table 6: Empirical results

(a) Test for multiple structural breaks (before 2012.12)	
UDmax= 1.499	WDmax= 1.827
(b) Test for the end of sample stability (2011.07 to 2012.12)	
Andrews' test= 0.059	(p-value= 0.887)
(c) Test for the end of sample stability (2013.01 to 2013.12)	
Andrews' test= 2.315	(p-value= 0.108)
(d) Estimated break date	
$\hat{k}_0 = 2013.05$	

Note: $***$, $**$ and $*$ denote 1%, 5% and 10% significance, respectively. The WDmax test statistic depends on the significance level and the result in the table is when it is 0.10. The same test does not reject the null hypothesis with 5% and 1% levels, respectively.

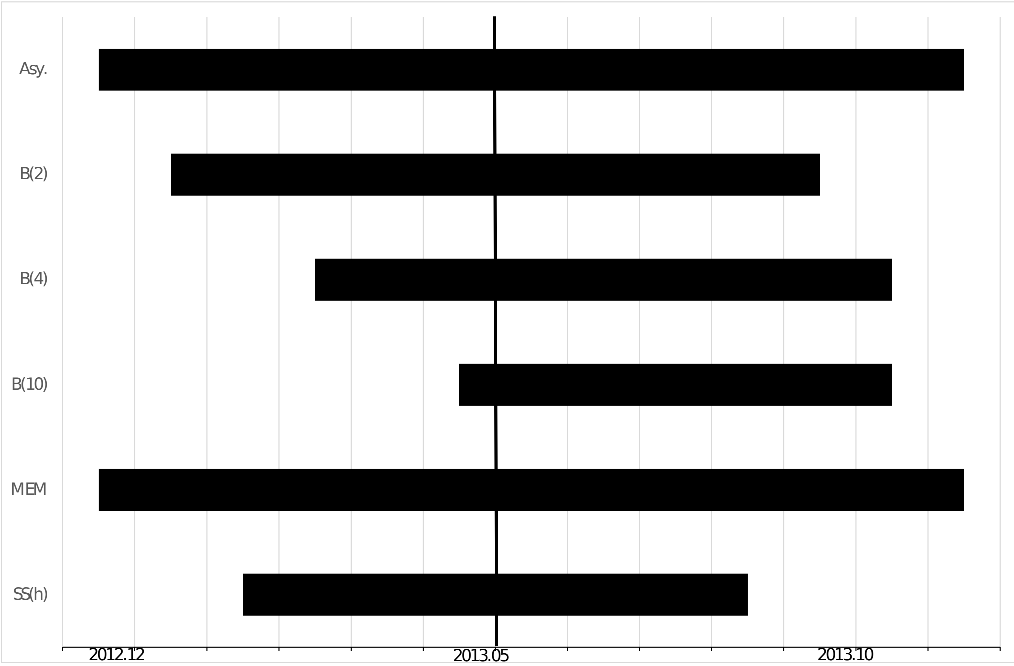

Figure 1: Confidence sets for the break date for the Japanese inflation rate