

Optimal Monetary Policy in a Regime-Switching DSGE Model with Time-Varying Concern for Model Uncertainty *

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Abstract

This paper analyzes how the optimal behavior of a central bank changes if the central bank has a concern for robustness regarding model uncertainty when there is a possibility of a regime switch in the economy in which the transmission mechanism of monetary policy weakens. The aim is to stress the expectational effects arising from the regime-switching structure. The framework allows identifying the contribution of time-varying doubts about model misspecification on top of the risk of a future weakening of the policy transmission. The result implies a more active policy stance to reduce the possibility to experience a deterioration of monetary transmission mechanism even in normal times.

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1 Introduction and Background

Awareness of the uncertainties regarding the structure of the economy might lead to shifts in the expectation formation of economic agents, and call for fundamental changes in the way policy is conducted. Policy makers seek for policies that are robust across different (uncertain) states of the world, and optimality of the decision rules necessitates minimizing the impact of any type of uncertainty. How best to react uncertainty, depending on different forms, is widely studied by academics to provide guidance for policymakers.

The literature on uncertainty has pioneered by Brainard (1967) which advises a more cautious behavior for the policymaker when faced with ambiguity about parameters of the model. However, there may be fundamental uncertainties regarding the true data generating process. The monetary authority facing uncertainty about her model, or the exogenous disturbances might find it optimal to respond to fluctuations in the economy in a stronger manner. Therefore, introducing uncertainty into an otherwise standard New Keynesian model can improve the propagation of the shocks, the fit of the model with the data, and relieve the DSGE models from the criticism of several, unfounded shocks. Further providing a sound economic intuition, allowing model uncertainty can improve the reliability of the policy suggestions derived from the model.

When the primary monetary policy tool is constrained by the zero bound, lowering it further to produce further stimulus has not been an option. Especially when a negative shock to aggregate demand hits the economy, the recession might become persistent if the central bank cannot do any policy.

Several models predict the possibility of a long-lasting liquidity trap when monetary authorities rely exclusively on the short-term nominal interest rate as their policy instrument (which is constrained from below). Eggertsson and Woodford (2003) is a widely-cited example of how an exogenous fundamental shock drives the economy into a liquidity trap. Other stand of the literature relies on the property of self-fulfilling expectations as drivers of the trap. When economic agents expect the situation to worsen as long as the recession

persists, the existence of the lower bound has the potential to produce a self-fulfilling crisis due to pessimistic expectations.¹

In contrast to the predictions of standard models, recent crisis experience has proven that central banks can further stimulate the economy even if the short-term nominal rate has reached zero. To begin with, the policy rate can go below zero. Euro area, Denmark, Sweden, Switzerland and Japan are examples of countries which decline the interest rate modestly below zero starting from 2012 as illustrated in Figure 1. The idea is to encourage cash holdings rather than keeping assets paying negative rates.

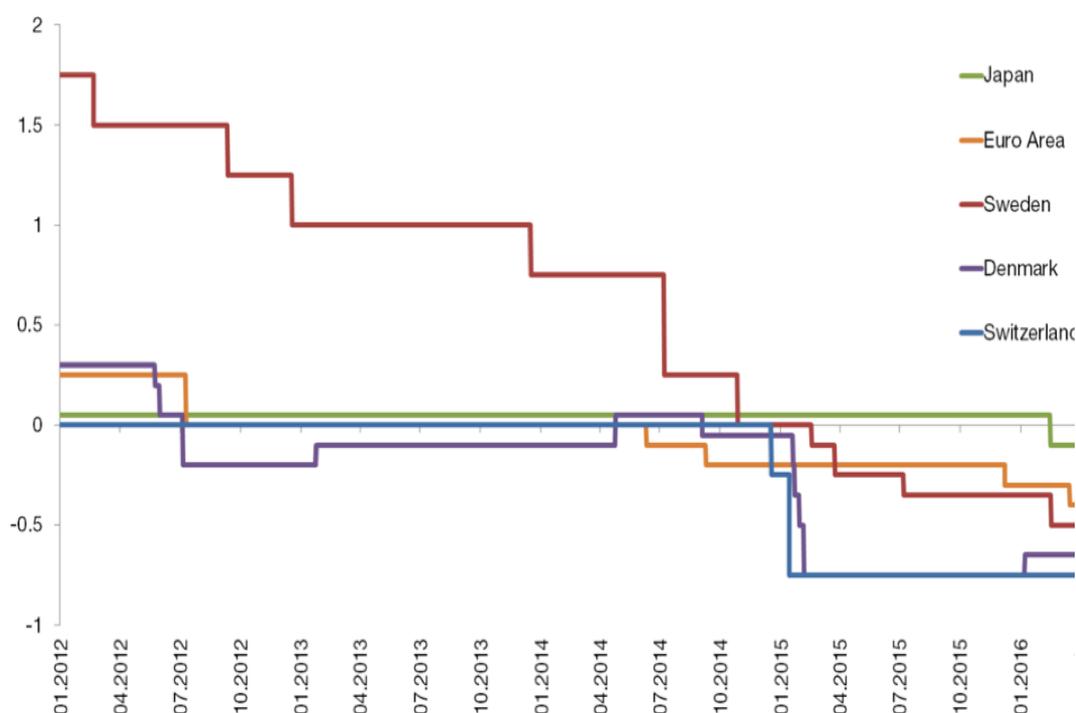


Figure 1: Negative Policy Rates

Moreover, central banks have developed unconventional policy tools such as forward-guidance, i.e. promises for future policy actions and asset purchase programs, i.e. quantitative easing as a substitute for the short rate to influence the economy by affecting long-term interest rates.²

Though, monetary policy can theoretically stimulate aggregate spending, and hence

¹Benhabib et al. (2001*b,a*, 2002) shows the existence of multiple equilibria consistent with rational expectations due to the zero lower bound. Mertens and Ravn (2014) studies how a sudden loss in confidence can produce multiple equilibria in a New Keynesian model.

²Central banks have continued to use these unconventional tools when the zero lower bound is no longer binding to complement normal time policy.

inflation, through unconventional channels; the ability of the central bank to effect the private sector expectations is the key in the effectiveness of these policies. For instance, forming inflationary expectations by promising to keep the policy rate low -by increasing the inflation target even after the zero bound is no longer binding can reduce the duration of a liquidity trap (Eggertsson and Woodford, 2003, 2004).

Theoretically, these unconventional tools can work as a substitute to the short rate; however, empirical evidence casts a suspicion on the implications of the models. Several studies suggest that these are not as powerful as the normal time monetary policy in stimulating the economy. Although, monetary authority has access to unconventional tools such as forward-guidance and asset purchase programs to effect real economy, these tools are proven not to work well in practice. Filardo and Hofmann (2014) empirically show that the ability of forward guidance policies in affecting market expectations has declined over time. Contributions by Giannoni et al. (2016) and McKay et al. (2016) are theoretical (and empirical) demonstrations in this direction.

Only two decades ago, the concept of a *liquidity trap* has been evaluated as an academic curiosity. Large declines in aggregate output have been seen as extreme tail events. Models used to estimate the frequency of this negative tail risk are misleading as they mostly rely on data from U.S. postwar period. Williams (2014) document that long-lasting and deep recessions are not as rare as the models predict. On the other hand, a binding zero lower bound has become a relevant problem in practice only when the interest rates are low to begin with. Central banks' low inflation targets together with declining long run real interest rates are to blame as they account for low nominal rates. Policymakers' ability to reverse economic downturns has been restricted by a binding zero lower bound in most advanced countries with a target level of inflation close to two percent. Studies on how to prevent the crisis stress the potential role for higher inflation targets as they can provide more room for nominal rate reductions in the first place.³

³Billi (2011), Coibion et al. (2012), and Ball (2014) dissect the impact of increasing the inflation target in reducing the costs of zero lower bound periods.

Whether weakening of monetary policy transmission is due to different data generating process or increasing pessimism is both academically and practically relevant. This paper aims to study optimal monetary policy design in a changing economic structure. To model a possible weakening of monetary policy subject to an effective lower bound on nominal interest rate, I allow elasticity of substitution to vary across different states of the economy.

In an environment where the central bank is not able to use its conventional instrument, she has to rely on alternative channels. Although, theoretically, the monetary authority has other unconventional tools such as forward guidance and asset purchase programs to effect real economy, these tools are proven not to work well in practice as discussed above. Hence, alternating states of the economy where monetary authority is forced to switch to unconventional tools is similar to the economic environment in this study especially when unconventional tools do not work as the usual one.

Further, by allowing the nominal interest rate to fall below zero, I aim to abstract from model complexity of switching to unconventional tools. The idea is to use the policy rate as a shadow rate in the place of unconventional tools to summarize the overall stance of monetary policy -yet in a less effective manner. Wu and Zhang (2017) present an elegant description of modeling the shadow rate in a New Keynesian model from microfoundations. As discussed in this study, the idea is to use the short rate as the shadow rate during normal times. They also find out that the negative shadow rate summarizes unconventional monetary policy when the zero lower bound binds.

I examine the link between time-varying fear of model uncertainty and changing effectiveness of monetary policy. To capture the idea that monetary policy responds endogenously to possible different states of the economy, I utilize the regime switching approach. Earlier contributions to the literature relies on Markov switching processes for different states so they treat regime changes as exogenous. A novel contribution by Davig and Leeper (2008) allows for systematic responses of the decision makers to possible changes

in economic structure. I utilize the approach developed by Barthélemy and Marx (2017) which also allows for endogenous or state-dependent transition probabilities in the spirit of Davig and Leeper (2008).

I also utilize piecewise linear perturbation technique which is extensively-used in the literature on occasionally binding constraints. I present the results of this methodology as a benchmark ignoring the expectations channel of the possibility of a future regime change. By providing a comparison with the regime switching structure, I focus on the expectation formation effects of the possibility of alternating regimes.

The aim of this study is to analyze the optimal commitment behavior of a central bank facing uncertainty regarding the structural equations governing the economy and becomes pessimistic under an effective lower bound on nominal interest rate; and how the policy recommendation changes if the central bank has a time-varying concern for robustness regarding model uncertainty when the transmission mechanism of the monetary policy weakens.

When there is a possibility that monetary policy transmission mechanism will not work as usual, the optimal response of the monetary authority is to respond aggressively to a negative demand shock in order to reduce the probability of switching to the second regime. This result is due to the expectations channel inherent in regime switching structure. Even in normal circumstances, optimal policy becomes more active when a deterioration of monetary transmission mechanism is highly possible.

The paper is organized as follows. In section 2, the model is constructed and details of the optimal monetary policy design is discussed. Section 3 discusses the solution strategies and presents the results. Section 4 concludes.

2 Model

This section presents the micro-foundations of the general equilibrium model. The model is a small-scale New Keynesian model with price stickiness. I first explain the characteristics of the model. Then, I will show the linearized versions of the set of conditions characterizing the general equilibrium.

The representative household derives utility from consumption and leisure by maximizing the following separable utility function:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\eta}}{1+\eta} \right\} \quad (1)$$

where $0 < \beta < 1$ is the discount factor, $\sigma \geq 0$ is the inverse of the intertemporal elasticity of substitution, $\eta \geq 0$ is the inverse of Frisch labor supply elasticity. C_t and L_t are consumption and hours worked respectively. At the beginning of each period t , representative household has B_{t-1} nominal bonds, with a nominal gross interest rate of R_{t-1} ; and buys new bonds B_t . Also, household earns $w_t L_t$ labor income, and Φ_t firm's profits. Each household maximize (1) subject to the sequence of intertemporal budget constraints:

$$P_t C_t + B_t \leq B_{t-1} R_{t-1} + P_t [w_t L_t] + \Phi_t \quad (2)$$

The household's optimality conditions are as follows:

$$\begin{aligned} w_t &= L_t^\eta C_t^\sigma \\ \left(\frac{C_t^{-\sigma} / P_t}{E_t C_{t+1}^{-\sigma} / P_{t+1}} \right) &= \beta R_t \\ C_t^{-\sigma} &= \beta E_t \left[C_{t+1}^{-\sigma} R_t \frac{P_t}{P_{t+1}} \right] \end{aligned} \quad (3)$$

For the production side, I follow the typical intermediate-final good producer setup. Final

good Y_t is produced using the following intermediate goods aggregator:

$$Y_t = \left[\int_0^1 Y_t(i)^{\frac{\xi-1}{\xi}} \right]^{\frac{\xi}{\xi-1}}$$

Intermediate good firms are monopolistically competitive. The corresponding demand for an intermediate good i is:

$$Y_t(i) = Y_t \left(\frac{P_t(i)}{P_t} \right)^{-\xi}$$

where ξ denotes for the constant price elasticity of demand, $P_t(i)$ is the price of the intermediate good i , and P_t is the price of the final good. The aggregate production function is

$$F(L_t) = L_t^{1-\alpha}$$

The first order conditions with respect to the cost minimization problem of firm's yields:

$$w_t = z_t F'(L_t) = (1-\alpha) z_t L_t^{-\alpha} = (1-\alpha) z_t \frac{Y_t}{L_t} \quad (4)$$

where z_t is the marginal cost of production. Prices are sticky à la Calvo (1983). Each period a fraction $(1-\theta)$ of firms gets a signal to reoptimize their prices. Firms maximize the sum of discounted profits:

$$E_t \sum_{j=t}^{\infty} \theta^j \left(\frac{1}{\prod_{i=0}^j R_i} \right)^j \left[\left(\frac{P_t(i)}{P_t} \right)^{-\xi} Y_t \left(\frac{P_t(i)}{P_t} - z_t \right) \right] \quad (5)$$

where z denotes for the marginal cost. The profit maximization conditions produce a log-linearized New Keynesian Phillips Curve (NKPC) of the form:

$$\pi_t = \beta E_t \pi_{t+1} + \left\{ \frac{(1-\theta)(1-\beta\theta)}{\theta} \right\} z_t \quad (6)$$

Log-linearizing the optimality conditions leads to the following New Keynesian Phillips equation with a supply shock.⁴

$$\pi_t = \beta E_t \pi_{t+1} + \kappa y_t + e_t \quad (7)$$

where π_t is the inflation rate, y_t is the output gap, and e_t is the cost-push shock which follows an $AR(1)$ process as follows:

$$e_t = \rho_e e_{t-1} + \epsilon_t, \quad \epsilon_t \sim N(0, 1), \quad |\rho_e| < 1 \quad (8)$$

The output gap is described by the following IS equation:

$$y_t = E_t y_{t+1} - \Omega(r_t - E_t \pi_{t+1}) + u_t \quad (9)$$

where u_t denotes for the demand shock (or preference shock) which follows a stationary $AR(1)$ process:

$$u_t = \rho_u u_{t-1} + v_t, \quad v_t \sim N(0, 1), \quad |\rho_u| < 1 \quad (10)$$

2.1 Optimal Monetary Policy

A welfare criterion for the monetary authority can be derived by taking a second-order approximation to the representative household's utility function;⁵ however, these are proven to be very model dependent and not robust. Instead, many central banks are assigned to pursue a simple objective that involves only a small set of economic variables. A simple mandate offers more transparent monetary policy and easier communication with the public. Moreover, a simple objective is found to be more robust to structured model and

⁴Derivations of the model can be found in the appendix.

⁵Several papers derive a utility-based welfare criterion following Rotemberg and Woodford (1998). See Edge (2003) for the derivation of a welfare criterion for a model with capital accumulation; Leith et al. (2012) for a model with external habit formation

parameter uncertainty compared to a complicated state-contingent policy.⁶

I assume an ad hoc objective for the central bank so the policy maker chooses $\{y_t, \pi_t, r_t\}$ which minimizes the following quadratic loss function:

$$\max_{\{r_t\}_{t=0}^{\infty}} E_t \sum_{t=t_0}^{\infty} \beta^t \{[\pi_t^2 + \lambda_y y_t^2 + \lambda_r (\Delta r_t)^2]\} \quad (11)$$

where π , y and r denotes for the inflation rate, the output gap and the nominal interest rate respectively. $\lambda_y, \lambda_r > 0$ are weights of the stabilization of the output gap, and the change in the nominal interest rate.

This welfare criterion implies triple mandate for the policymaker; namely deviation of the inflation rate from its optimal rate, deviation of output from its natural level and interest rate smoothing.⁷ The rationale for assigning a positive weight for the nominal interest rate in the objective of the central bank can be derived from a model with transaction frictions which justifies the presence of real money balances together with consumption and leisure in utility as shown by Woodford (2011). An ad hoc reason for including the interest rate is to prevent large fluctuations in the short-term nominal rate that might increase the term premium and the long-term interest rate. Hence, by implying a degree of commitment to future rates, interest rate smoothing enables the policymaker to keep aggregate demand and inflation under control by guiding long-term bond rates. Furthermore, a policymaker incorporating the zero lower bound on the policy rate reduces the likelihood of hitting the bound by decreasing the financial market volatility when the economy is subject to large shocks.

The policymaker designs optimal monetary policy by minimizing the expected loss using all information available up to the current period (which does not include the knowledge about the true model) and taking the structural equations governing the economy as

⁶Svensson (2010), and recently Debortoli et al. (2016) elaborate more on this point.

⁷A recent contribution by Davig and Gürkaynak (2015) advocates a case for the single mandate for monetary authority in an environment where multiple objectives can create disincentive motives for other policymakers. Here, I abstract from these kind of strategic interaction among policy authorities.

constraints. I assume credible commitment on the side of the policymaker. By committing to a policy rule, policymaker determines optimality conditions to hold in any future period; hence she takes advantage of being able to affect the formation of private sector expectations.

2.2 Different States of the Economy

I will use the term *risk* as suggested by Hansen and Sargent (2008) for the type of unknowns that the economic agents can quantify. I will allow the risk to be represented by the possibility of the economy shifting between two different models. Accordingly, this study combines both the uncertainty caused by model specification doubts of the monetary authority and the risk of weakening transmission of monetary policy on the real economy.

In the analysis, I distinguish two different regimes ($s_t = N, W$): i) normal regime (N) in which monetary policy transmission mechanism works as usual, and ii) a weak regime (W) in which monetary policy transmission weakens. Here, I describe the regime switching as a change in the sensitivity of the economy to policy rate, and IS equation varies depending on the two regimes as follows:

$$\Omega_{s_t} = \begin{cases} \Omega_N & \text{if } s_t = N \\ \Omega_W & \text{if } s_t = W \end{cases} \quad (12)$$

where the real interest-rate elasticity of output (intertemporal substitution effects in consumption) is higher in the normal regime, i.e. $\Omega_N > \Omega_W$.

This possible change in the structure of the economy is reflected on the monetary policy as follows. In normal regime, decision makers trust their model and do not worry about model uncertainty, yet in the second regime, decision makers guard against possible model misspecification. Hence, this structure provides a basis for time-varying uncertainty aversion. This type of unstructured uncertainty is modeled à la Hansen and Sargent

(2008).

In the next section, I derive optimal commitment policy for normal regime and for the weak regime.

2.2.1 Normal Time Specification for Monetary Policy

Following Woodford (2011), I form the Lagrangian of the policymaker for the benchmark model as

$$\begin{aligned} \min_{\{\pi_t, y_t, i_t\}_{t=0}^{\infty}} E_0^{RE} \sum_{t=0}^{\infty} \beta^t \{ & [(\frac{1}{2})\pi_t^2 + (\frac{1}{2})\lambda_y y_t^2 + (\frac{1}{2})\lambda_r (r_t - r_{t-1})^2] \\ & + s_{1,t}(\pi_t - \beta\pi_{t+1} - \Omega y_t - e_t) \\ & + s_{3,t}(y_t - y_{t+1} + \frac{1}{\sigma}(r_t - \pi_{t+1}) - u_t) \} \end{aligned} \quad (13)$$

The first-order conditions with respect to π_t , y_t and r_t are as follows:

$$\begin{aligned} \pi_t + s_{1t} - s_{1t-1} - \frac{1}{\beta\sigma} s_{3t-1} &= 0 \\ \lambda_y y_t - \Omega s_{1t} + s_{3t} - \frac{1}{\beta} s_{3t-1} &= 0 \\ \lambda_r (r_t - r_{t-1}) - \beta \lambda_r (r_{t+1} - r_t) + \frac{1}{\sigma} s_{3t} &= 0 \end{aligned} \quad (14)$$

at each date $t \geq 0$, with the initial conditions⁸

$$s_{1,-1} = s_{3,-1} = 0 \quad (15)$$

implying that the monetary authority has no previous commitment at the initial period.

Here, E^{RE} is the expectations operator derived from the model under rational expecta-

⁸According to Woodford (2011), the Lagrange multipliers can be specified in a way that make the policy optimal in a timeless perspective. Time-zero optimal solution is subject to time-inconsistency problem since policymakers gain by reoptimizing in later periods. By ignoring the first order conditions for the initial period Woodford's approach is time-invariant. I leave the comparison of both types of commitment policies for further research.

tions.

The optimal policy under rational expectations for the benchmark model is the commitment solution $\{y_t, \pi_t, r_t, s_{1t}, s_{3t}\}$ to the system of first-order conditions (14) initial conditions (15) together with the IS and Phillips equations discussed in the previous section.

2.2.2 Worst-Case Specification for Monetary Policy

When modelling uncertainty, I assume that the policymaker cannot assign probabilities to alternative models of the economy. In this case, policymaker has a reference model of the economy; however, she has doubts about her model and thinks that her model can only approximate the structure of the economy. To accommodate potential model misspecification, she considers a set of alternative models that are difficult to distinguish from her reference model. Decision-making under robust control can be interpreted as a dynamic game between the policymaker and the nature. In this game, the nature, or the hypothetical evil agent forms the worst-case strategy that would deteriorate the performance of the policymaker as bad as possible, and the policy maker designs the best decision rule given the decision of the nature. Hence, a robust policymaker seeks for a min-max solution to the policy problem. This type of uncertainty is additive and reflected as additional disturbances, which can feed back on state variables, in the additive shock processes. As opposed to structured model uncertainty,⁹ robust control approach relies on designing optimal policy that would work on the worst possible outcome irrespective of how likely this outcome might be. Hence, in the case of designing robust policy under unstructured uncertainty, policymaker has to set a prior judgment on the worst-case

⁹When the policymaker can put a structure on the model uncertainty, she can assign prior probabilities to possible different views of the economy. To design a robust policy across competing models, a policymaker first computes economic losses implied by optimal policy prescribed by each model. And she minimizes average expected loss across models, in which the policymaker assigns larger weights for most likely model and smaller weights for least likely model. Hence, she utilizes Bayesian model averaging to insure against undesirable outcomes (as in Levin et al. (2003)). A robust policy under structured model uncertainty is set to work well across alternative models. See Kara (2002); Giannoni (2002) for examples of a comparison of structured and unstructured model uncertainty.

outcome depending on how much she is concerned about model misspecification. Hansen and Sargent (2008) suggest interpreting possible deviations from policymaker's reference models as a collection of the likelihood ratios whose relative entropies with respect to the reference model are bounded by the policymaker's desired degree of robustness.

On modeling uncertainty, this paper explicitly relies on the robust control literature as suggested by Hansen and Sargent (2008) in general equilibrium models.¹⁰

To allow for model uncertainty, I introduce a second type of disturbances in the shock processes.

$$\begin{aligned} e_t &= \rho_e e_{t-1} + \epsilon_t, & \epsilon_t &\sim N(0, 1), & |\rho_e| &< 1 \\ u_t &= \rho_u u_{t-1} + v_t, & v_t &\sim N(0, 1), & |\rho_u| &< 1 \end{aligned} \tag{16}$$

which leads:

$$\begin{aligned} e_t &= \rho_e e_{t-1} + [\epsilon_t + v_t^e] \\ u_t &= \rho_u u_{t-1} + [v_t + v_t^u] \end{aligned} \tag{17}$$

When (17) is the true shock process, the error terms in (16) are distributed as $N(v_t^e, 1)$ and $N(v_t^u, 1)$ rather than as $N(0, 1)$. Therefore, the misspecification in (16) is captured by allowing the conditional mean of the shock vector (17) that actually generates the data to feed back arbitrarily on the history of the state variables.¹¹

Moreover, the model with (16) is assumed to be a good approximation under the following restriction of the distortion.¹²

¹⁰Hansen and Sargent (2001), Gilboa and Schmeidler (1989), Epstein and Schneider (2003), Klibanoff et al. (2005), Maccheroni et al. (2006) and Strzalecki (2011) are examples of earlier decision-theoretical studies.

¹¹In the New Keynesian models studied in this paper, the state variables are the disturbance terms. Thus, the presence of v_t^e and v_t^u indicates the model misspecification.

¹²The distortion can also be interpreted as a measure of the discrepancy between the distorted and approximating models.

$$E_t \sum_{\tau=0}^{\infty} \{[v_{t+\tau}^e]^2 + [v_{t+\tau}^u]^2\} \leq \eta_0, \quad \eta_0 > 0 \quad (18)$$

Under model uncertainty, the robust policymaker considers the model presented in the previous section as the reference model, which represents the most likely description of the economic structure. However, the policymaker knows that this model could be subject to a wide range of distortions with (17) satisfying (18). Such a policymaker reformulates its optimization problem to obtain a policy rule that performs well even if the underlying structure of the economy deviates from the reference model. Following Hansen and Sargent (2008), to reformulate the problem under robust control, I introduce a fictitious evil agent who shares the same reference model that the policymaker considers and tries to maximize the same objective function. Hence the policymaker's problem can be represented by a two-person zero-sum game or a Stackelberg problem. While the evil agent chooses a model from the available set of alternative models, the central bank designs its policy optimally to perform well in this worst-case scenario. Note that the most likely outcome of the model, the approximating model,¹³ is when the policymaker sets policy -assuming that the private agents share the same worst-case model and form expectations accordingly-, yet there is no such misspecification in reality. To put differently, an approximating model will be the model where the structure of the economy evolves without any misspecification in the disturbances; however, the monetary authority finds it optimal to consider the worst possible scenario when conducting policy.

The game between the policymaker and the evil agents is represented by the following extremization problem that is the joint maximization and the minimization problem subject to the distorted model where the shock process is given by (21) and the entropy

¹³The details of the solution algorithm for the approximating model is described in Appendix C.

constraint (24).

$$\min_{\{v_{t+1}^e, v_{t+1}^u\}_{t=0}^{\infty}} \max_{\{\pi_t, y_t, r_t\}_{t=0}^{\infty}} E_t^{RC} \sum_{t=l_0}^{\infty} \beta^t \{[\pi_t^2 + \lambda_y y_t^2 + \lambda_r (r_t - r_{t-1})^2]\} \quad (19)$$

subject to

$$\pi_t = \beta E_t^{RC} \pi_{t+1} + \Omega y_t + e_t \quad (20)$$

$$e_t = \rho_e e_{t-1} + [\epsilon_t + v_t^e] \quad (21)$$

$$y_t = E_t^{RC} y_{t+1} - \frac{1}{\sigma} (r_t - E_t^{RC} \pi_{t+1}) + u_t \quad (22)$$

$$u_t = \rho_u u_{t-1} + [v_t + v_t^u] \quad (23)$$

$$E_t^{RC} \sum_{\tau=0}^{\infty} \{[v_{t+\tau}^e]^2 + [v_{t+\tau}^u]^2\} \leq \eta_0 \quad (24)$$

where E^{RC} denotes for the expectations operator of the distorted model. To make the model compatible with the rational expectations framework discussed in the previous section, I assume that the policymaker and the private agents have the same information set, and they form expectations about future variables in a homogeneous manner. Hence, the private agents share the policymaker's approximating model and the degree of concern for robustness as in Giordani and Söderlind (2004). Note that expectations under model misspecification will differ from those under rational expectations since the former includes the behavior of the hypothetical evil agent as well.

The monetary authority chooses the paths for $\{\pi_t, y_t, r_t\}$ that minimizes her loss function. Under model misspecification, the evil agent is supposed to choose v_{t+1}^e and v_{t+1}^u that maximizes the loss function under the budget constraint (24).

Following Walsh (2004), I write the Lagrangian as the multiplier version of the Stack-

elberg problem defined in Hansen and Sargent (2008) as follows:

$$\begin{aligned}
\min_{\{v_{t+1}^e, v_{t+1}^u\}_{t=0}^{\infty}} \max_{\{\pi_t, y_t, r_t\}_{t=0}^{\infty}} E_t^{RC} \sum_{i=0}^{\infty} \beta^i \{ & [(\frac{1}{2})\pi_{t+i}^2 + (\frac{1}{2})\lambda_y y_{t+i}^2 + (\frac{1}{2})\lambda_r (r_{t+i} - r_{t+i-1})^2 \\
& - (\frac{1}{2})\beta\Theta([v_{t+i+1}^e]^2 + [v_{t+i+1}^u]^2)] \\
& + s_{1t+i}(\pi_{t+i} - \beta\pi_{t+i+1} - \Omega y_{t+i} - e_{t+i}) \\
& + s_{2t+i}(\rho_e e_{t+i-1} + [\epsilon_{t+i} + v_{t+i}^e] - e_{t+i}) \\
& + s_{3t+i}(y_{t+i} - y_{t+i+1} + \frac{1}{\sigma}(r_{t+i} - \pi_{t+i+1}) - u_{t+i}) \\
& + s_{4t+i}(\rho_u u_{t+i-1} + [v_{t+i} + v_{t+i}^u] - u_{t+i}) \} \quad (25)
\end{aligned}$$

where $0 < \Theta < \infty$ represents the monetary authority's preference for the degree of robustness. Θ can be interpreted as a Lagrange multiplier on the constraint (24) measuring the size of the set of models surrounding the reference model. Hansen and Sargent (2008) show that Θ is positively related to η_0^{-1} . As Θ rises (or as η_0 gets smaller in (24)) the policymaker becomes less concerned about model uncertainty, and $\Theta = \infty$ (equivalently $\eta_0 = 0$) corresponds to the rational expectations solution with $v_t = 0$ for all t .¹⁴

I obtain the following first order conditions with respect to the choice variables of the policymaker and the evil agents, π_t , y_t , r_t , e_t , u_t , v_{t+1}^e , and v_{t+1}^u for the optimal commitment policy:

¹⁴For now, I take the connection between Θ and η_0 as given but in Section D I describe how to calibrate a reasonable value for the robustness parameter Θ using detection probabilities.

$$\begin{aligned}
\pi_t + s_{1t} - s_{1t-1} - \frac{1}{\beta\sigma} s_{3t-1} &= 0 \\
\lambda_y y_t - \Omega s_{1t} + s_{3t} - \frac{1}{\beta} s_{3t-1} &= 0 \\
\lambda_r (r_t - r_{t-1}) - \beta \lambda_r (r_{t+1} - r_t) + \frac{1}{\sigma} s_{3t} &= 0 \\
-s_{1t} + \rho_e s_{2t} - \frac{1}{\beta} s_{2t-1} &= 0 \\
-s_{3t} + \rho_u s_{4t} - \frac{1}{\beta} s_{4t-1} &= 0 \\
-\beta \Theta v_{t+1}^e + s_{2t} &= 0 \\
-\beta \Theta v_{t+1}^u + s_{4t} &= 0
\end{aligned} \tag{26}$$

The associated equilibrium conditions for the robust policy maker are the first-order conditions¹⁵ summarized in Equation (26) together with the following structural equations:

$$\begin{aligned}
\pi_t &= \beta E_t^{RC} \pi_{t+1} + \Omega y_t + e_t \\
e_t &= \rho_e e_{t-1} + [\epsilon_t + v_t^e] \\
y_t &= E_t^{RC} y_{t+1} - \frac{1}{\sigma} (r_t - E_t^{RC} \pi_{t+1}) + u_t \\
u_t &= \rho_u u_{t-1} + [v_t + v_t^u]
\end{aligned} \tag{27}$$

¹⁵One has to take into consideration the first order condition for the initial period by letting $s_{1,t-1}, s_{2,t-1} = 0$ for $t = 0$. Since there are no previous commitment in the initial period, the monetary authority finds it optimal to consider expectations as fixed for $t = 0$.

3 Deterministic Switching versus Regime Switching

3.1 Solution Methods

To capture the impact of time-varying nature of the key parameters, I first consider two different cases, namely *deterministic switch*, and *regime switch*. Change of regimes is a deterministic switch, in the terminology of Goldfeld and Quandt (1973), when it depends on an observable variable with a threshold. In other words, the switching mechanism is determined and triggered by this threshold. The latter is the case when the transition of the system between two states of the economy is controlled by a transition matrix.

To obtain model solutions, I use two different methodologies: i) a piecewise linear solution method, and ii) a regime switching approach.

Although this paper is mainly interested in accounting for expectational effects due to the possibility of a future regime change, I exploit the discreet configuration of piecewise linear solution. The simplicity of the method provides a natural benchmark by excluding any form of forward-looking behavior on an alternative regime. When compared with the resulting dynamics from regime switching, piecewise linear solution brings out the expectations channel inherent in regime switching structure.

Jung et al. (2005) is the first to propose a piecewise linear solution method to apply occasionally binding constraints. A widely-cited work by Eggertsson and Woodford (2003) generalizes their method for a stochastic environment where the economy alternates between two states conditional on exogenously determined values of the natural interest rate. Guerrieri and Iacoviello (2015) extended the solution method, and proposed a Occbin toolkit that implements the codes in Dynare. I use this toolkit to illustrate how the dynamics of the model changes when a threshold interest rate is introduced in the model.

Piecewise linear solution considers model with occasionally binding constraints as a model with multiple regimes. The method requires linearizing the model under each regime around the non-stochastic steady state of the first regime. When the constraint does not bind, the linearized model without the constraint determines the state of the economy. In addition, when the constraint binds (in response to a shock), a future time period in which the constraint does not bind is chosen and the model is solved backwards to obtain the sequence of policy functions consistent with the constraint binding in the current period.

This methodology is criticized by not accounting for the risk of the constraint binding in the future in response to unanticipated shocks (Binning and Maih, 2016). Here, agents are unaware of the possibility that the constraint might bind until it actually binds. However, the drawback of this methodology becomes a virtue and serves well for the purposes of this study.

Here, the regimes are determined by a threshold interest rate as follows:

$$s_t = \begin{cases} N \text{ (Normal Regime)} & \text{if } r_t > r^* \\ W \text{ (Weak Regime)} & \text{if } r_t \leq r^* \end{cases} \quad (28)$$

To take into account the possibility of future switches, I adopt the endogenous regime switching structure proposed by Barthélemy and Marx (2017) in the spirit of Davig and Leeper (2008).

The probability to remain in regime $j \in \{N, W\}$ is

$$Pr(s_t = j | s_{t-1} = j, z_{t-1}, u_t) = p_{jj} + \lambda_{jj} g(z_{t-1}, u_t) \quad (29)$$

where p_{jj} is the steady state level of the probability of remaining in regime j , λ_{jj} is the sensitivity of transition probabilities to a function ($g(\cdot)$) of the vector of endogenous variables, z , and multi-dimensional stochastic process, u_t . Hence, the transition probability

is allowed to depend on the information available at time $t - 1$, i.e. past endogenous variables, and/or current shocks for $\lambda_{jj} \neq 0$. Accordingly, the probability of regime shift is

$$\Pr(s_t = i | s_{t-1} = j, z_{t-1}, u_t) = p_{ji} \equiv 1 - (p_{jj} + \lambda_{jj}g(z_{t-1}, u_t)) \quad (30)$$

I also analyze constant transition probabilities, the case where $\lambda_{jj} = 0$ as a special case.

Then the state evolves exogenously according to the following transition matrix:

$$P = \begin{pmatrix} p_N & 1 - p_W \\ 1 - p_N & p_W \end{pmatrix} \quad (31)$$

where $p_j = \Pr(s_t = i | s_{t-1} = j)$, $j \in \{N, W\}$.

When $\lambda_{jj} \neq 0$, the transition probabilities are endogenous. Then, the probability to remain in a particular regime is assumed to follow:

$$\Pr(s_t = N | s_{t-1} = N, r_{t-1}) = p_N - (p_{jj} + \lambda_{jj}g(r_{t-1} - r^N)) \quad (32)$$

$$\Pr(s_t = W | s_{t-1} = W) = p_W \quad (33)$$

where r^N represents the interest rate threshold for entering the second regime.

3.2 Calibration

Table 1 summarizes the calibration. Under this calibration, this section provides a comparison of the results obtained under different solution methods.

Table 1: Calibration

Parameter	Description	Values
β	Discount Factor	0.99
Ω_N	Elasticity of Intertemporal Substitution	0.5
Ω_W	Elasticity of Intertemporal Substitution	0.2
κ	Slope of the Phillips Relation	0.75
λ_y	Weight of the output gap in loss function	0.25
λ_r	Weight of the interest rate in loss function	0.1
Θ_W	Robustness Parameter in Weak Regime	22

3.3 Piecewise Linear Solution

By using the piecewise linear solution, impulse response functions are represented as a combination of the two regimes: the first regime in which the constraint is never binding, and the second regime in which the constraint is binding all the time.

Figure 2 demonstrates the dynamics of the economy in response to two unforeseen demand shocks. All results are reported in percentage values. In period 5, the economy is hit by a positive demand shock. Both the output gap and inflation rate increase above their steady-state values. Monetary authority responds by increasing its instrument gradually acknowledging the impact of her actions on expectations on future values of the variables under commitment and the interest rate inertia in her objective function. For positive rates of nominal interest rate, the economy stays in the first regime in which monetary policy transmission works efficiently. Hence, the dynamics implied by the piecewise linear and the linear models overlap. In the sequent 20 years, the impact of the shock dissipates and the economy returns back to the zero inflation steady state.

In period 25, the economy experiences an unexpected negative demand shock which is symmetric (around the steady-state) to the first shock in magnitude. Since a negative shock to demand necessitates decreasing the policy rate below its steady state, linear and piecewise linear solutions differs substantially. Piecewise linear solution shows how the economy switches to the second regime in which transmission mechanism of monetary policy weakens and the central bank becomes pessimistic about her model. Both the change in the structure of the economy and the uncertainty aversion of the central bank

reduce the policy rate well below the one in the linear model. Further, output gap and inflation rate fluctuates more under the piecewise linear model.

Piecewise linear solution shows that output falls by almost 4% when agents perceive the constraint to bind forever. This creates an approximately 5% annual inflation. Since the monetary policy transmission weakens for low values of the policy rate, central bank has to react by decreasing the policy rate by more than 60 basis points.

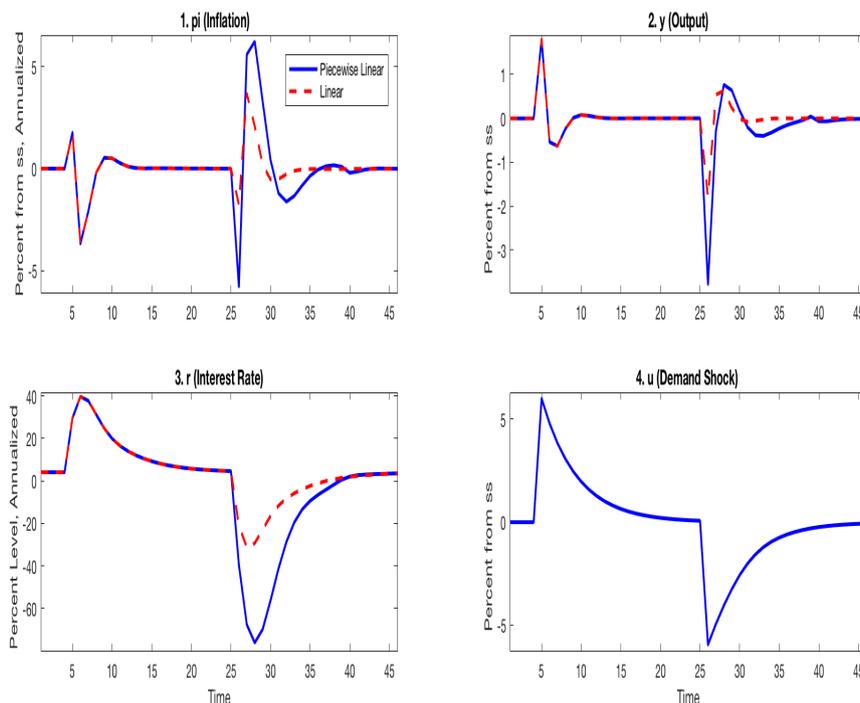


Figure 2: Impulse Responses for Piecewise Linear Solution-Demand Shock

3.4 Regime Switching

Under regime switching, the impulse responses are generated conditional on a specific regime. Agents form their decisions by internalizing the possibility that the economy can switch to the other regime. However, ex post, the economy stays in the current regime over the impulse response period. Hence, the dynamics of the model is affected by other regime through the expectations formation effects.

I distinguish two cases depending whether the transition probabilities are exogenous or endogenous.

I consider $p_N = 0.90$, $p_W = 0.80$ to represent conditional probability of remaining in regimes N and W for the exogenous case. These probabilities also serve as the steady state values of the probabilities in the endogenous switching model.

3.4.1 Exogenous Transition Probabilities

The economy is assumed to start in the normal regime where monetary policy transmission works effectively. Figure 3 plots the impulse response functions in response to a negative demand shock conditional on staying in the same regime. That is to say, economic agents forms their expectations by taking into account the possibility of switching to the weak regime with a constant exogenous probability. The differences from the deterministic

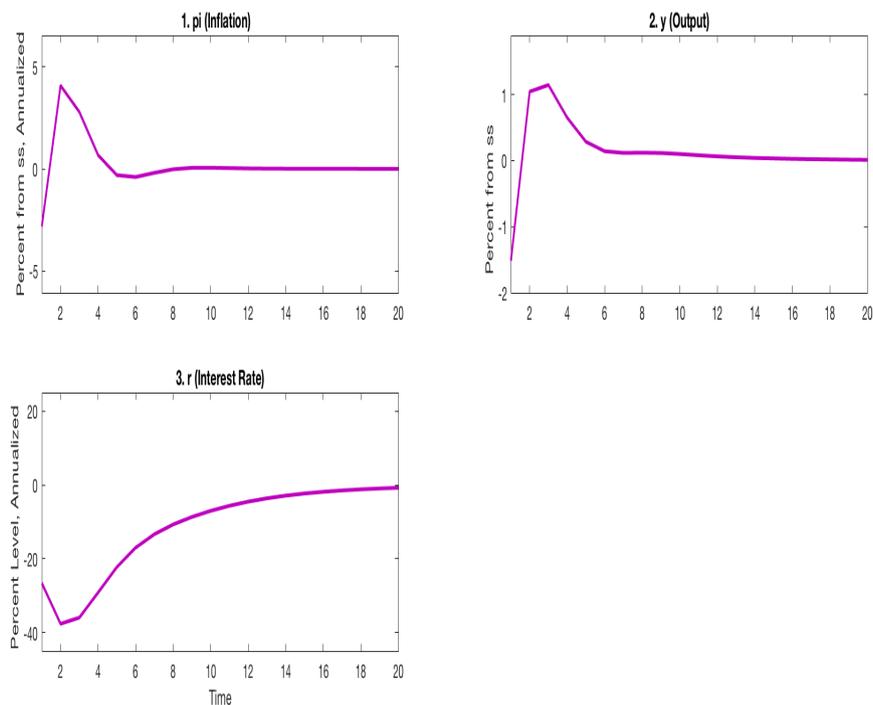


Figure 3: Impulse Responses with exogenous transition probabilities

switch is substantial. Since the monetary authority now internalizes the possibility of a future weakening of the policy transmission, her response to the shock becomes 2 times stronger. Central bank lowers the policy instrument by 40 basis points (in comparison to the 20 basis point decline in the linear model) to avoid the inconvenience even before it happens.

3.4.2 Endogenous Transition Probabilities

As opposed to the previous case, Figure 4 shows the impulse responses for the inflation, output gap, and the nominal interest rate when agents consider endogenous transition probabilities in the formation of expectations. I allow the transition to depend on the interest rate decisions of the policymaker. To be specific, the probability of staying in the first regime increases with the nominal interest rate. The higher the interest rate the lower the probability to switch to the weak regime in which the monetary policy does not work as usual. Compared to the piecewise linear solution, central banks reaction is more

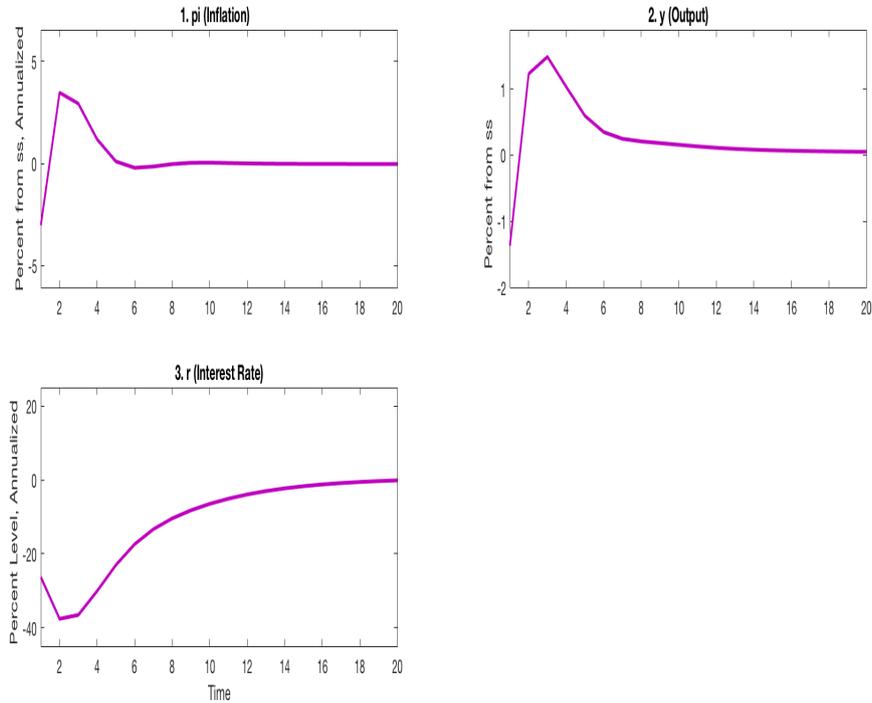


Figure 4: Impulse Responses with endogenous transition probabilities

aggressive to prevent the economy from switching to the weak regime. Resulting impulse responses are similar to the ones for the case where regime switch is triggered by constant transition probabilities. As pointed out by Barthélemy and Marx (2017), this result is due to steady state being constant over regimes.

4 Concluding Remarks and Future Work

This study combines both risk and uncertainty by allowing the possibility of the economy shifting between these two different models, as well as time-varying model specification doubts of the monetary authority.

In this paper, I first allow for regime-dependent elasticity of substitution as a short-cut to model a possible weakening of monetary policy subject to an effective lower bound on nominal interest rate. In an environment where the central bank is not able to use its conventional instrument, she has to rely on alternative channels. Although, theoretically, the monetary authority has other unconventional tools such as forward-guidance and asset purchase programs to effect the real economy, these tools are proven not to work uniformly well in practice. Hence, alternating states of the economy where monetary authority is forced to switch to unconventional tools is similar to the economic environment in this study especially when unconventional tools do not work as the standard policy tool.

To stress the role of expectations of regime switching structure, I utilize the deterministic switch as a natural framework excluding this channel. I use the resulting dynamics of the model economy as a benchmark to assess the implications of forward-looking nature inherent in regime switches.

The results of this paper first demonstrate that weakening of the monetary policy transmission mechanism calls for stronger policy actions in response to shocks. When agents acknowledge this possible future change in the economic structure, optimal policy becomes more active even in normal times.

When there is a possibility that monetary policy transmission mechanism will not work as usual, the optimal response of the monetary authority is to respond aggressively to a negative demand shock in order to reduce the probability to switch to the second regime. This result is because of the expectations channel inherent in regime switching structure. The model where the regime switching probabilities endogenously depend on the policy instrument confirms the results.

This paper shows the importance of model uncertainty and risks about future effectiveness of the monetary policy in a standard New Keynesian model. The findings has important implications for more detailed descriptions of the real world including several frictions.

A natural follow-up work will be studying a model allowing for positive steady-state inflation. In an environment where a positive inflation target for the monetary authority is justified, the risk of future obstacles to policymaking might involve in aggressive policy stance even in normal times. Policymaker might guard against future unfavorable output costs by increasing the inflation target.

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A Model Derivation

A.1 A Simple New Keynesian Model with Sticky Prices

Household's problem:

$$\begin{aligned} \mathcal{L} = E_t \sum_{t=0}^{\infty} \beta^t \left\{ \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\eta}}{1+\eta} \right. \\ \left. + \lambda_t (-B_t - P_t C_t + B_{t-1} R_{t-1} + P_t w_t L_t + T_t + \Phi_t) \right\} \end{aligned} \quad (34)$$

with the first-order conditions:

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \Rightarrow \frac{C_t^{-\sigma}}{P_t} = \lambda_t \quad (35)$$

$$\frac{\partial \mathcal{L}}{\partial L_t} = 0 \Rightarrow \frac{L_t^\eta}{w_t P_t} = \lambda_t \quad (36)$$

$$\frac{\partial \mathcal{L}}{\partial B_t} = 0 \Rightarrow \frac{\lambda_t}{E_t \lambda_{t+1}} = \beta R_t \quad (37)$$

Equation (35) together with equation (36) yields:

$$w_t = L_t^\eta C_t^\sigma \quad (38)$$

(38) in log-linearized form (using $w_t = z_t F_L = z_t(1-\alpha)L_t^{-\alpha} = z_t(1-\alpha)\frac{Y_t}{L_t}$) is:

$$\begin{aligned} \eta \hat{L}_t + \sigma \hat{C}_t &= \hat{z}_t + \hat{Y}_t - L_t \\ \hat{z}_t &= \sigma \hat{C}_t + (1+\eta) \hat{L}_t - \hat{Y}_t \end{aligned} \quad (39)$$

Equation (35) together with equation (37) gives:

$$\begin{aligned} \frac{\lambda_t}{E_t \lambda_{t+1}} &= \frac{C_t^{-\sigma}/P_t}{E_t C_{t+1}^{-\sigma}/P_{t+1}} = \beta R_t \\ \Rightarrow C_t^{-\sigma} &= \beta E_t [C_{t+1}^{-\sigma} R_t \frac{P_t}{P_{t+1}}] \end{aligned} \quad (40)$$

Equation (40) is written in log-linearized form as:

$$-\sigma \hat{C}_t = -\sigma E_t \hat{C}_{t+1} + \hat{R}_t - E_t \hat{\Pi}_{t+1} \quad (41)$$

Goods market clearing condition $Y_t = C_t$ in log-linearized form is:

$$\hat{Y}_t = \hat{C}_t \quad (42)$$

Using equation (42) in (41) gives the IS relation as follows:

$$\sigma \hat{Y}_t = \sigma E_t \hat{Y}_{t+1} - \hat{R}_t + E_t \hat{\Pi}_{t+1} \quad (43)$$

Inserting equation (42) in (39) yields:

$$\hat{z}_t = \sigma \hat{Y}_t + (1 + \eta) \frac{\hat{Y}_t}{(1 - \alpha)} - \hat{Y}_t = \left(\frac{1 + \eta}{1 - \alpha} - (1 - \sigma) \right) \hat{Y}_t \quad (44)$$

By substituting equations (42) and (44) in the New Keynesian Phillips curve, one obtains:

$$\hat{\Pi}_t = \beta E_t \hat{\Pi}_{t+1} + \frac{(1 - \theta)(1 - \beta\theta)}{\theta} \left(\frac{1 + \eta}{1 - \alpha} - (1 - \sigma) \right) \hat{Y}_t \quad (45)$$

B State-space representation of the Model

A convenient representation of a New Keynesian model is given by the following set of difference equations:

$$z_{t+1} = Az_t + Br_t + \varepsilon_{t+1} \quad (46)$$

where z_t is the state vector, defining each period the state space. A is a matrix that represent the structural coefficients of the economy. B is a vector capturing the impact of the choice of the policymaker. ε_{t+1} is a vector of shocks that drive the system. Since the variance covariance matrix is diagonal, individual shocks can be treated as structural shocks.

Structural equations of the model can be represented in the following state-space form in terms of predetermined and forward-looking variables as follows:

$$\begin{bmatrix} z_{1,t+1} \\ E_t z_{2,t+1} \end{bmatrix} = A \begin{bmatrix} z_{1,t} \\ z_{2,t} \end{bmatrix} + Br_t + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix}$$

where

$$z_t = \begin{bmatrix} z_{1,t} \\ z_{2,t} \end{bmatrix}$$

Optimization problem of the central bank is as follows:

$$J_0 = E_0 \sum_{t=0}^{\infty} \beta^t \begin{bmatrix} z'_t & r'_t \end{bmatrix} \begin{bmatrix} Q & U \\ U' & R \end{bmatrix} \begin{bmatrix} z_t \\ r_t \end{bmatrix}$$

For the specification of the model explained in text and derived from microfoundations in the previous section, we have:

$$z_{1,t} = \begin{bmatrix} e_t \\ u_t \end{bmatrix}, z_{2,t} = \begin{bmatrix} \pi_t \\ y_t \end{bmatrix}, \varepsilon_t = \begin{bmatrix} \epsilon_{\pi_t} \\ \epsilon_{y_t} \end{bmatrix}$$

where $z_{1,t}$ and $z_{2,t}$ denote for the vectors of predetermined and forward-looking variables respectively.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & \frac{1}{\sigma} & 1 \end{bmatrix} \begin{bmatrix} e_{t+1} \\ u_{t+1} \\ E_t \pi_{t+1} \\ E_t y_{t+1} \end{bmatrix} = \begin{bmatrix} \rho_e & 0 & 0 & 0 \\ 0 & \rho_u & 0 & 0 \\ -1 & 0 & 1 & -\Omega \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_t \\ u_t \\ \pi_t \\ y_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\sigma} \end{bmatrix} r_t + \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \epsilon_{\pi,t+1} \\ \epsilon_{y,t+1} \end{bmatrix}$$

with the variances of the two shocks are $\sigma_1 = \sigma_2 = 1$

C Solution Algorithm for the Approximating Equilibrium

The approximating equilibrium represents the dynamics of the economy under robust decision-making by the policymaker and the private agents. To form the approximating equilibrium, one needs the equilibrium dynamics under the worst-case model. The worst-

case model can be represented by the following linear law of motion

$$\begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = A \begin{bmatrix} x_t \\ y_t \end{bmatrix} + B u_t + C(\varepsilon_{t+1} + v_{t+1}), \quad C = \begin{bmatrix} C_1 \\ 0 \end{bmatrix} \quad (47)$$

where x_t and y_t represent n_1 predetermined and n_2 the forward-looking variables respectively. u_t is the control variable of the policymaker. The predetermined variables has shocks ε_{t+1} ($n_1 \times 1$ vector with zero mean and identity covariance matrix). v_{t+1} is $n_1 \times 1$ vector of misspecification which is the control variables of the hypothetical evil agent. The misspecification is bounded as

$$E_0 \sum_{\tau=0}^{\infty} \beta^{\tau+1} v'_{t+1} v_{t+1}^u \leq \eta_0, \quad \eta_0 > 0 \quad (48)$$

The policymaker and the evil agent share the following linear quadratic problem ¹⁶

$$\min_{\{u_t\}} \max_{\{v_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t [z'_t Q z_t + u'_t R u_t + 2z'_t U u_t - 2\beta \Theta v'_{t+1} v_{t+1}] \quad (49)$$

subject to

$$z_{t+1} = A z_t + B u_t + C(\varepsilon_{t+1} + v_{t+1}), \quad (50)$$

where $z'_t = (x'_t \quad y'_t)$ is the vector of all endogenous variables ¹⁷

The problem can be written in the standard state-space form by defining $R^* = \begin{bmatrix} R & 0 \\ 0 & -\beta \Theta I \end{bmatrix}$, $u_t^* = \begin{bmatrix} u_t \\ v_{t+1} \end{bmatrix}$, $B^* = \begin{bmatrix} B & C \end{bmatrix}$, and $U^* = \begin{bmatrix} U & 0 \end{bmatrix}$ as follows:

$$\min_{\{u_t\}} \max_{\{v_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t [z'_t Q z_t + u_t'^* R^* u_t^* + 2z_t' U^* z_t] \quad (51)$$

subject to

$$z_{t+1} = A z_t + B^* u_t^* + C(\varepsilon_{t+1} + v_{t+1}), \quad (52)$$

¹⁶This is the representation of the multiplier version of the Stackelberg problem defined in the text.

¹⁷The control problem under rational expectations corresponds to the case where $\Theta = 0$ (so the maximization is irrelevant) and there is no misspecification, i.e. $v_{t+1} = 0$ in equation (50).

In the worst-case equilibrium, Giordani and Söderlind (2004) shows how to write the evolution of the state variables, the forward-looking variables and the decision rules as follows:

$$\begin{bmatrix} x_{t+1} \\ \lambda_{t+1} \end{bmatrix} = M \begin{bmatrix} x_t \\ \lambda_t \end{bmatrix} + C\varepsilon_{t+1}, \quad (53)$$

$$\begin{bmatrix} y_t \\ u_t^* \end{bmatrix} = N \begin{bmatrix} x_t \\ \lambda_t \end{bmatrix}, \quad (54)$$

$$\begin{bmatrix} u_t \\ v_{t+1} \end{bmatrix} = \begin{bmatrix} N_2 \\ N_3 \end{bmatrix} P^{-1} \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} -F_u \\ -F_v \end{bmatrix} P^{-1} \begin{bmatrix} x_t \\ \lambda_t \end{bmatrix} \quad (55)$$

where λ_t is $n_2 \times 1$ vector of shadow prices of the forward-looking variables with $\lambda_0 = 0$.

$$M = P(A - BF_u - CF_v)P^{-1}, \quad P = \begin{bmatrix} I & 0 \\ & N_1 \end{bmatrix} \text{ implying } \begin{bmatrix} x_t \\ y_t \end{bmatrix} = P \begin{bmatrix} x_t \\ \lambda_t \end{bmatrix}$$

The worst-case model in terms of the state variables is shown below:

$$x_{t+1} = M_{11}x_t + M_{12}\lambda_t + C_1\varepsilon_{t+1} \quad (56)$$

$$\lambda_{t+1} = M_{21}x_t + M_{22}\lambda_t \quad (57)$$

$$y_t = H_{21}x_t + H_{22}\lambda_t \quad (58)$$

$$u_t = F_{u11}x_t + F_{u12}\lambda_t \quad (59)$$

$$v_{t+1} = F_{v11}x_t + F_{v12}\lambda_t \quad (60)$$

where $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$, $\begin{bmatrix} F_{u11} & F_{u12} \end{bmatrix} \equiv -F_u P^{-1}$, and $\begin{bmatrix} F_{v11} & F_{v12} \end{bmatrix} \equiv -F_v P^{-1}$.

To obtain the approximating model, I first eliminate the worst-case distortions by equating F_{v11} and F_{v12} to zero in equation (60). Note that the state variables evolve according to the following equation in the absence of distortions:

$$x_{t+1} = A_{11}x_t + A_{21}y_t + B_1u_t + C_1\varepsilon_{t+1} \quad (61)$$

Inserting equations (59) and (60) into (60) yields

$$x_{t+1} = A_{11}x_t + A_{21}[H_{21}x_t + H_{22}\lambda_t] + B_1[F_{u11}x_t + F_{u12}\lambda_t] + C_1\varepsilon_{t+1} \quad (62)$$

The approximating equilibrium is summarized by equations (58)-(60) and (62). I calculate the matrices, and implement the procedure using Dynare.

D The Degree of Robustness

As in Hansen and Sargent (2008) and Giordani and Söderlind (2004), I use likelihood ratio test to calculate the detection error probabilities. A detection-error probability is the probability that an econometrician observing equilibrium outcomes would make a wrong deduction about whether two competing models generate the data. The idea is to connect the value of the robustness parameter Θ to the probability of making the incorrect choice of model between the reference model and the worst-case model. The probability of making this mistake is computed by simulations with

$$p(\Theta) = \frac{1}{2}[Pr(L^R > L^W|W) + Pr(L^W > L^R|R)] \quad (63)$$

where R and W denotes for the reference model and the worst-case model respectively. $(Pr(L^A > L^B|B))$ is the probability of the likelihood of model A i.e. (L^A) , being higher than the probability of the likelihood of model B, i.e. (L^B) , conditional on the hypothesis that model B is the true data generating process.

A detection error probability of 50% implies that two data generating processes are almost the same, and differentiating between the reference model and the worst-case model is not possible. With a detection error probability closer to zero, the econometrician is able to detect the true data generating process since the models are highly different. I choose the value of Θ to achieve 20% of detection error probability.

E Endogenous Regime Switch

To allow for allow for endogenous transition probabilities, I use the methodology of Barthélemy and Marx (2017) which relies on perturbation techniques. They use functional iteration method to obtain the solution of a fix-point problem, through successive approximations of the solution.