Permanent and Transitory Shocks to the U.S. Economy: Has Their Importance Been Constant over Time?

Jaeho Kim*[1] and Sora Chon†[2]

[1] *University of Oklahoma*

[2] *Korea Development Institute*

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Abstract: The main purpose of this paper is to scrutinize the time-varying relative importance of the permanent and the transitory shocks to the U.S. real GDP. After accounting for the cointegration relation between real GDP, consumption, and investment, we find that i) the real permanent shock played a more prominent role than the nominal transitory shock from the 1950s to the 1960s, but the two shocks have almost equally contributed to the stochastic movements of real GDP since the 1970s; (ii) the long-run growth of the U.S. economy has substantially slowed since the recession in 2001. The annualized growth rate of real GDP has declined to approximately 1.6 %, falling from a peak of nearly 3.3 % in the last decade. For the empirical analysis, we employ a multivariate unobserved component model that accommodates stochastic volatility. The multivativariate model is estimated by an efficient particle Gibbs sampler with particle rejuvenation that simultaneously draws latent state variables all at once.

Keywords: *Long-run Growth, Unobserved Component Model, Trend-cycle Decomposition, Particle Gibbs Sampling, Particle Rejuvenation*

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*Corresponds to 248 Cate Center Drive, Norman, OK 73019, U.S.A. Department of Economics, University of Oklahoma. E-mail: jaeho@ou.edu

†263, Namsejong-ro, Sejong-si 30149, Korea. Department of Macroeconomic Policy, KDI. E-mail: sora@kdi.re.kr

1 Introduction

Understanding the nature of the cycle and the trend of real GDP has broad implications for fiscal and monetary policy implementation. Hence, various econometric models, such as the Beveridge and Nelson (1981) decomposition and the unobserved component (UC) models introduced by Harvey (1985), have been used to investigate the relative importance of the real permanent shock to the trend and the nominal transitory shock to the cycle.

The central question in the long-standing literature is which shock is the dominant determinant of real GDP dynamics. For example, Perron and Wada (2009) and Luo and Startz (2014) raise the possibility that real GDP is decomposed differently into trend and cycle components, depending on how the long-run trend of real GDP is modeled in a UC model. Allowing for a structural break in the long-run trend, these authors conclude that the cyclical shock is the main driver of stochastic movements of U.S. real GDP. In contradistinction to this approach, Kim and Kim (2016) use reduced-form ARIMA models and a Bayesian estimation to produce evidence that the shock to the trend component explains most of the variations in real output, even when accounting for a break in the long-run trend. Moreover, Chan and Grant (forthcoming) compare different UC models with fixed break dates within a Bayesian framework and argue that the permanent trend shock is a more important factor in deriving U.S. real GDP.

In this paper, we raise a different empirical question. When did the trend or the cycle components play a prominent role in real output? Has their relative importance been changing over time? A notable shortcoming of the existing trend-cycle decomposition literature is the restrictive assumption that the relative importance of the two fundamental shocks remains static over time. However, as many studies including Fernald (2016) have pointed out, the productivity of U.S. economy improved remarkably from 1950s to 1960s, as explained by new scientific discoveries, high growth in education attainment, and the construction of highway systems during the period. Moreover, such high increases in economic efficiency led to unprecedented long economic expansion. In 1970s and 1980s, aggressive monetary and fiscal policies to fight high inflation and unemployment unexpectedly made the U.S. economy unstable. As emphasized by Kim and Nelson (1999) and Mcconnell and Perez-Quiros (2000), a substantial decline in the aggregate uncertainty of the U.S.

economy, referred to as "the Great Moderation", has emerged since the mid-1980s. Moreover, the recessions since the 1990s have not been accompanied by strong post-recession recoveries, which had been the main feature of the recessions prior to the 1990s (Kim (2008) and Huang et al. (2016)). The dramatic or gradual changes in productivity, economic policies, and economic environment during the post-World War period highlights the significance of the question we cast here. As such, our attempt to determine how the importance of the real permanent and the nominal transitory shocks evolves over time could be an important first step to complement the existing literature, which does not provide a proper answer to the question.

We undertake an empirical investigation on this issue using flexible unobserved components (UC) models. Specifically, we incorporate stochastic volatilities (SV) for the shocks to the trend and cycle components. The proposed UC model is designed to capture invisible changes in the relative contribution of the two shocks to U.S. real GDP. One of essential prerequisites for estimating the relative importance would be the precise identification of the latent trend and cycle components. For instance, a failure to account for or to accurately estimate a structural break in the long-run trend function could lead to substantial upward bias in a variance estimate of the permanent shock, as shown in Perron and Wada (2009). Therefore, motivated by neoclassical growth theory – which suggests that real GDP, real consumption, and real investment share a common stochastic trend – we employ a multivariate UC model of cointegration among those macro variables to obtain a reliable estimate of the latent long-run trend of real GDP. Following Morley (2007), all shocks (including permanent and transitory shocks) are allowed to be correlated to remove an arbitrary smoothness assumption with regard to the common trend.

The time-varying covariance matrix included in our model is a special case of the multivariate SV model suggested by Tsay (2005); his model allows the correlations among shocks to vary over time, while ours assumes them to be time-invariant. If all variables can be directly observed, as they are in a vector autoregressive (VAR) model, in practice, it is possible to estimate the time variations in the correlations among shocks. In contrast, the time-varying correlations are not clearly identified when the variables that are subject to correlated shocks are unobserved, as in our UC model. As such, we adopt a specific model setup in which the correlations are constant over time. Another reason is because, in our model, different priors with reasonable economic interpretations can be directly imposed on the correlation parameters. The alternative SV model by Tsay (2005) uses re-parameterization based on the Cholesky decomposition to preserve the positive definiteness of the time-varying covariance matrix. Such re-parameterization prevents imposing economically sensible priors on the correlation parameters.

For estimation of the proposed multivariate UC model, this paper introduces a new Bayesian algorithm that is constructed using the particle Gibbs (PG) method developed by Andrieu et al. (2010). Our PG algorithm accommodates particle rejuvenation, a recent innovation developed by Lindsten et al. (2015) and Bunch et al. (2015), which generates latent state variables at a substantially lower computational cost than the standard Andrieu et al. (2010) PG method. The PG algorithm with particle rejuvenation is robust to degenerate[1] transition and measurement equations, which is an important feature of our UC model, in contrast to the ancestor sampling approach by Lindsten et al. (2014). To the best of our knowledge, this paper is the first extension and application study of the particle rejuvenation method in the econometrics literature.

The proposed algorithm has meaningful advantages over alternative MCMC methods, which should rely on separate Gibbs or Metropolis-Hasting blocks to handle latent state variables. The Metropolis-Hastings algorithms developed by Pitt and Shephard (1999) and Aguilar and West (2000) can be applied to simulate the time-varying covariance matrix of our UC model. However, the methods are all based on a single-move sampling scheme and thus could be inefficient in practice in terms of MCMC convergence. Chib et al. (2005) propose a multi-move algorithm by using the mixture normal approximation approach proposed by Kim et al. (1998). However, it is not applicable to the case in which the shocks are contemporaneously correlated and the correlations are time-invariant. Gerlach et al. (2000) and Giordani and Kohn (2012) propose a single-move approach that draws a discrete regime indicator variable. Unfortunately, their sampler cannot be a suitable tool for estimating a model with absorbing states (e.g., structural breaks) in which the transition probability for a certain regime is exactly one. In contradistinction to the methods dis-

[1] In statistics, a distribution of a random variable is said to be degenerate when the random variable takes only a single value. Therefore, the degenerate density function is one at a single point and zero elsewhere.

cussed immediately above, neither the separate Gibbs sampling nor the Metroplis-Hastings method is necessary to perform posterior simulation for latent variables under our PG algorithm. The proposed PG algorithm is a multi-move sampler that enables one-step efficient posterior simulation for all discrete (regime indicator) and continuous (the trend/cycle components and the time-varying covariance matrix) state variables from their joint posterior distribution. Bayesian model comparisons become straightforward when applying the proposed PG algorithm together with the method developed by Chib (1995).

The main empirical findings obtained by applying the cointegrated UC model to U.S. real GDP, consumption and investment data from 1947:Q2 to 2016:Q2 are as follows. First, the stochastic trend component was more important than the cycle component from the 1950s to the 1960s. However, the two components have been almost equally important since the 1970s. Second, the annualized long-run growth rate of U.S. real GDP has declined to approximately 1.6 % since the recession in 2001, falling from a peak of nearly 3.3 %. The first empirical result confirms our initial conjecture that the relative importance of the permanent and the transitory shocks is not stable during the post-World War II period. Moreover, it emphasizes the non-negligible roles of the both shocks in U.S. economy after 1970s.

The remainder of this paper is organized as follows. In Section 2, we present the model specification and explain our proposed PG algorithm. Section 3 discusses the empirical results, and Section 4 concludes.

2 Bayesian Inference

2.1 Model Specification

In this section, we introduce a tri-variate unobserved components (UC) model, motivated by neoclassical growth theory, as in Kim and Piger (2002). The UC model is specified as follows:

Multivariate Unobserved Components Model

$$y_t = x_t + z_t, \tag{1}$$

$$c_t = \bar{c} + \lambda_c x_t + g_{c,t}, \tag{2}$$

$$i_t = \bar{i} + \lambda_i x_t + g_{i,t}, \tag{3}$$

where y_t is real GDP; c_t is real consumption for non-durables and services; i_t is real fixed investment. The three macro variables share a common latent stochastic trend that follows a random walk process with draft:

$$x_t = \mu_t + x_{t-1} + u_t. \tag{4}$$

Perron and Wada (2009), Luo and Startz (2014), and Chan and Grant (forthcoming) attempt to detect abrupt changes in the long-run growth rate under the above random walk specification. Following recent studies, we consider a structural break model for the trend growth rate μ_t:

$$\mu_t = \sum_{k}^{K} \mu_k \ I(s_t = k), \tag{5}$$

$$p(s_t = j|s_{t-1} = k) = \pi_{kj}, \ \sum_{j=1}^{K} \pi_{kj} = 1, \ j, k = 1, 2, ..., K.$$

where $I(s_t = k)$ is the indicator function that takes 1 if $s_t = k$ for $k = 1, 2, ..., K$; K is the total number of regimes; $p(s_t = j|s_{t-1} = k) = \pi_{k,j}$ is the transition probability from regime k to regime j. The hidden regime indicator variable denoted by s_t therefore governs the dynamics of the long-run growth rate of y_t. The transition probabilities are restricted to incorporate the breaks in the trend function as in Chib (1998).

The transitory cycle components of y_t, c_t, and i_t are assumed to follow stationary autoregressive processes of order 2 given by

$$z_t = \phi_1 z_{t-1} + \phi_2 z_{t-2} + e_t, \tag{6}$$

$$g_{c,t} = \gamma_{c,1} \ g_{c,t-1} + \gamma_{c,2} \ g_{c,t-2} + \eta_{c,t}, \tag{7}$$

$$g_{i,t} = \gamma_{i,1} \ g_{i,t-1} + \gamma_{i,2} \ g_{i,t-2} + \eta_{i,t}, \tag{8}$$

where all the roots of $\phi(L) = (1 - \phi_1 L - \phi_2 L^2) = 0$, $\gamma_c(L) = (1 - \gamma_{c,1} L - \gamma_{c,2} L^2) = 0$, and $\gamma_i(L) = (1 - \gamma_{i,1} L - \gamma_{i,2} L^2) = 0$ lie outside the complex unit circle.

Different from the existing literature, we allow for flexible time variations in the relative roles of the real permanent and the nominal transitory shocks under a non-zero correlation structure. The time-varying covariance matrix for the permanent and transitory shocks is specified by

$$\begin{bmatrix} u_t \\ e_t \\ \eta_{c,t} \\ \eta_{i,t} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2_{u,t} & \rho_{u,e}\sigma_{u,t}\sigma_{e,t} & \rho_{u,\eta_c}\sigma_{u,t}\sigma_{\eta_c,t} & \rho_{u,\eta_i}\sigma_{u,t}\sigma_{\eta_i,t} \\ \rho_{u,e}\sigma_{e,t}\sigma_{u,t} & \sigma^2_{e,t} & \rho_{e,\eta_c}\sigma_{e,t}\sigma_{\eta_c,t} & \rho_{e,\eta_i}\sigma_{e,t}\sigma_{\eta_i,t} \\ \rho_{u,\eta_c}\sigma_{\eta_c,t}\sigma_{u,t} & \rho_{e,\eta_c}\sigma_{\eta_c,t}\sigma_{e,t} & \sigma^2_{\eta_c,t} & \rho_{\eta_c,\eta_i}\sigma_{\eta_c,t}\sigma_{\eta_i,t} \\ \rho_{u,\eta_i}\sigma_{\eta_i,t}\sigma_{u,t} & \rho_{e,\eta_i}\sigma_{\eta_i,t}\sigma_{e,t} & \rho_{\eta_c,\eta_i}\sigma_{\eta_c,t}\sigma_{\eta_i,t} & \sigma^2_{\eta_i,t} \end{bmatrix} \right),$$

where heteroskedasticity is captured by the random walk stochastic volatility model:

$$\sigma^2_{w,t} = \sigma^2_{w,0}\ exp(h_{w,t}),$$

$$h_{w,t} = h_{w,t-1} + \epsilon_{w,t},\ \ \epsilon_{w,t} \sim N(0, \sigma^2_{h,w}).$$

For identification, the initial values of the volatility processes are assumed to be $h_{w,0} = 0$ for $w = u, e, \eta_c, \eta_i$. To capture the relative importance between the permanent and the transitory shocks to real GDP, we focus on estimating the following variance ratio:

$$\delta_t = \frac{\sigma^2_{u_t}}{\sigma^2_{e_t}}. \tag{9}$$

We refer to δ_t as the noise-to-signal ratio throughout this paper. Of course, the time-varying volatility setup is also important for decomposing the macro variables into the common trend and three individual cycles and for improving Bayesian inference on structural breaks in the long-run trend. The above UC model can be represented by a reduced-form error correction VARMA model with time-varying parameters and stochastic volatilities. Despite significant recent advances in Bayesian and classical statistics, direct estimation of the VARMA model remains a challenging task due to complex problems such as over-parameterization, parameter identification and a highly nonlinear likelihood function. We acknowledge that our model is somewhat restrictive in parameterization. However, its parsimonious parameterization enables Bayesian estimation and model comparisons with various UC models of U.S. real GDP that are widely used in the literature for trend-cycle decomposition.

The UC model is cast into the following state space representation to gain the joint smoothing distribution of the latent state variables:

$$Y_t = D + H\beta_t \tag{10}$$

$$\beta_t = \tilde{\mu}_t + F\beta_{t-1} + G\epsilon_t, \tag{11}$$

where $Y_t = [y_t \;\; c_t \;\; i_t]'$, $D = [0 \;\; \bar{c} \;\; \bar{i}]'$, $\beta_t = [x_t \;\; z_t \;\; z_{t-1} \;\; g_{c,t} \;\; g_{c,t-1} \;\; g_{i,t} \;\; g_{i,t-1}]'$, $\tilde{\mu}_t = [\mu_t \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]'$, $\epsilon_t = [u_t \;\; e_t \;\; \eta_{c,t} \;\; \eta_{i,t}]'$,

$$H = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \gamma_c & 0 & 0 & 1 & 0 & 0 & 0 \\ \gamma_i & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \; F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \phi_1 & \phi_2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_{c,1} & \gamma_{c,2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_{i,1} & \gamma_{i,2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here, we derive a valid Bayesian algorithm to estimate the UC model whose measurement and transition densities related to equations (10) and (11) are degenerate because of the reduced rank covariance matrix of $G\epsilon_t$ and a lack of measurement error. Our proposed Bayesian method can be easily applied to more general cases in which Y_t contains other macroeconomic variables and more latent components with non-linear specifications.

2.2 Target Posterior Distribution

By targeting at the following posterior density, we carry out a Bayesian inference on the model

$$p(\theta, \beta_{0:T}, s_{0:T}, h_{0:T}|Y_{1:T}) \propto \left[\prod_{t=1}^{T} g_\theta(Y_t|\beta_t, s_t, h_t) f_\theta(\beta_t, s_t, h_t|\beta_{t-1}, s_{t-1}, h_{t-1})\right] f_\theta(\beta_0, s_0, h_0)\pi(\theta), \tag{12}$$

where

$$f_\theta(\beta_t, s_t, h_t|\beta_{t-1}, s_{t-1}, h_{t-1}) = f_\theta(\beta_t|\beta_{t-1}, s_t, h_t) f_\theta(s_t|s_{t-1}) f_\theta(h_t|h_{t-1}),$$

where $g_\theta(.)$ and $f_\theta(.)$ are the probability densities associated with equations (1)-(8), given all the model parameters $\theta = \{\mu, \phi, \bar{c}, \bar{i}, \lambda_c, \lambda_i, \gamma_c, \gamma_i, \pi, \Sigma_0, \sigma^2_{h,u}, \sigma^2_{h,e}, \sigma^2_{h,\eta^c}, \sigma^2_{h,\eta^i}\}$; μ represents the vector of the long-run mean growth rates in K regimes; π represents the vector of the transition probabilities; $\phi, \gamma_c, \gamma_i, \pi$ are the sets of AR coefficients for the cyclical components of y_t, c_t, and i_t ; $\pi(\theta)$ represents the prior density of θ; Σ_0 is the covariance matrix at $t = 0$ that contains the volatilities of the initial time period and the time-invariant correlation terms; and T represents the sample size. Throughout this paper, we adhere to the notations for the latent variables as $\beta_{\kappa:\tau} = \{\beta_\kappa, \beta_{\kappa+1}, ..., \beta_\tau\}$, $s_{\kappa:\tau} = \{s_\kappa, s_{\kappa+1}, ..., s_\tau\}$, and $h_{\kappa:\tau} = \{h_\kappa, h_{\kappa+1}, ..., h_\tau\}$ for $\tau \geq \kappa$, where $h_t = \{h_{u,t}, h_{e,t}, h_{\eta_c,t}, h_{\eta_i,t}\}$.

2.3 Posterior Simulation for Latent State Variables

As the target density in equation (12) is not analytically tractable, we resort to a Markov Chain Monte Carlo (MCMC) algorithm. Particularly, in this section, we will concentrate on designing a valid posterior simulator for $\{\beta_{0:T}, s_{0:T}, h_{0:T}\}$ because posterior simulation for θ is relatively simple, conditional on $\{\beta_{0:T}, s_{0:T}, h_{0:T}\}$. Building on Lindsten et al. (2015) and Bunch et al. (2015), we employ particle rejuvenation to resolve estimation problems caused by the degenerate measurement and the transition equations of the proposed UC model in carrying out forward filtering and backward smoothing.

2.3.1 A Sequential Monte Carlo Method under a Degenerate Dynamic System

Suppose that we have the following approximate filtering density for the latent variables:

$$p_\theta(\beta_{t-1}, s_{t-1}, h_{t-1}|Y_{1:t-1}) \approx \sum_{n=1}^{N} \hat{\omega}^{(n)}_{t-1} \delta_{\{\beta^{(n)}_{t-1}, s^{(n)}_{t-1}, h^{(n)}_{t-1}\}}(\beta_{t-1}, s_{t-1}, h_{t-1}) \tag{13}$$

where $\{\beta^{(n)}_{t-1}, s^{(n)}_{t-1}, h^{(n)}_{t-1}\}^N_{n=1}$ and $\{\hat{\omega}^{(n)}_{t-1}\}^N_{n=1}$ denote N particles of the latent variables and their corresponding importance weights at time $t-1$; and $\delta_{\{\beta^{(n)}_{t-1}, s^{(n)}_{t-1}, h^{(n)}_{t-1}\}}(\beta_{t-1}, s_{t-1}, h_{t-1})$ is a Dirac measure for each particle in $\{\beta_{t-1}^{(n)}, s_{t-1}^{(n)}, h_{t-1}^{(n)}\}_{n=1}^{N}$. For forward filtering, an ancestor index denoted by $a_t^{(n)}$ is drawn using the following importance weights prior to sampling a new particle $\{\beta_t^{(n)}, s_t^{(n)}, h_t^{(n)}\}$:

$$Pr(a_t^{(n)} = r) = \hat{\omega}_{t-1}^{(r)},$$

where $r = 1, 2, ..., N$. Then, a properly constructed importance distribution is used to generate the new particle $\{\beta_t^{(n)}, s_t^{(n)}, h_t^{(n)}\}$ conditional on $\{\beta_{t-1}^{(a_t^{(n)})}, s_{t-1}^{(a_t^{(n)})}, h_{t-1}^{(a_t^{(n)})}\}$. Here, $\{\beta_{t-1}^{(a_t^{(n)})}, s_{t-1}^{(a_t^{(n)})}, h_{t-1}^{(a_t^{(n)})}\}$ represents the $a_t^{(n)}$-th particle in the particle swarm at time $t-1$.

It is important to recognize that in a case where measurement or transition densities are degenerate, sampling all latent variables from an importance distribution is not possible. For better understanding, see equations (10) and (11). From those equations, we know that certain elements of $\beta_t^{(n)}$, such as $z_{t-1}^{(n)}$, $g_{c,t-1}^{(n)}$, and $g_{i,t-1}^{(n)}$ are directly linked to some elements of $\beta_{t-1}^{(a_t^{(n)})}$, such as $z_{t-1}^{(n)} = z_{t-1}^{(a_t^{(n)})}$, $g_{c,t-1}^{(n)} = g_{c,t-1}^{(a_t^{(n)})}$ and $g_{i,t-1}^{(n)} = g_{i,t-1}^{(a_t^{(n)})}$. In addition, due to the degenerate measurement equation in equation (10), values that $z_t^{(n)}$, $g_{c,t}^{(n)}$, and $g_{i,t}^{(n)}$ can take are always restricted, conditional on $x_t^{(n)}$, y_t, c_t, and i_t. To cope with this problem, we sample x_t and deterministically recover the other cycle components, given x_t. We adopt this approach for both forward filtering and backward smoothing.

For constructing an importance distribution, we employ the transition densities for x_t, h_t, and s_t:

$$q(\beta_t, s_t, h_t|\beta_{t-1}, s_{t-1}, h_{t-1}) = f_\theta(x_t|x_{t-1}, s_t, h_t) f_\theta(s_t|s_{t-1}) f_\theta(h_t|h_{t-1}). \tag{14}$$

The densities of z_t, $g_{c,t}$, and $g_{i,t}$ are omitted in equation (14) because they are deterministic functions of x_t. The corresponding unnormalized importance weight is given by[2]:

$$\bar{\omega}_t^{(n)} = \frac{f_\theta(\beta_t^{(n)}|\beta_{t-1}^{(a_t^{(n)})}, s_t^{(n)}, h_t^{(n)})}{f_\theta(x_t^{(n)}|x_{t-1}^{(a_t^{(n)})}, s_t^{(n)}, h_t^{(n)})},$$

under the importance density in equation (14). After repeatedly sampling the ancestor index and the new particle and evaluating the unnormalized importance weight N times, we obtain

[2] $g_\theta(Y_t|\beta_t^{(n)}, s_t^{(n)}, h_t^{(n)}) = 1$ under the proposed sampling scheme. No observation is available at time $t = 0$. Therefore, a user-specific importance distribution or an unconditional distribution can be used to generate all the elements of β_0.

$\{a_t^{(n)}\}_{n=1}^N$, $\{\beta_t^{(n)}, s_t^{(n)}, h_t^{(n)}\}_{n=1}^N$, and $\{\bar{\omega}_t^{(n)}\}_{n=1}^N$. The importance weight for $\{\beta_t^{(n)}, s_t^{(n)}, h_t^{(n)}\}$ is computed through self-normalization as follows:

$$\hat{\omega}_t^{(n)} = \frac{\bar{\omega}_t^{(n)}}{\sum_{i=1}^N \bar{\omega}_t^{(i)}} \tag{15}$$

for $n = 1, 2, ..., N$[3]. Continuously performing the multi-sampling steps up to time T, we can collect $\Xi_{0:T} = \{\beta_0^{(n)}, \beta_1^{(n)}, ..., \beta_T^{(n)}\}_{n=1}^N$; $S_{0:T} = \{s_0^{(n)}, s_1^{(n)}, ..., s_T^{(n)}\}_{n=1}^N$; $H_{0:T} = \{h_0^{(n)}, h_1^{(n)}, ..., h_T^{(n)}\}_{n=1}^N$; $A_{1:T} = \{a_1^{(n)}, a_2^{(n)}, ..., a_T^{(n)}\}_{n=1}^N$. This simulation-based algorithm is referred to as a sequential Monte Carlo (SMC) method.

2.3.2 A Particle Gibbs Sampler with Particle Rejuvenation

Among $\{\Xi_{0:T}, S_{0:T}, H_{0:T}, A_{1:T}\}$ generated by the SMC method, we sample one particular particle path known as a reference particle trajectory. The reference particle trajectory is sampled by first drawing index $b_T \in \{1, 2, ..., N\}$ and then sequentially recovering $b_{t-1} = a_t^{(b_t)}$ for $t = T, ..., 1$. Here, the index b_t indicates the locations of the individual particles in the chosen reference particle trajectory as

$$\{\beta_0^{(b_0)}, \beta_1^{(b_1)}, ..., \beta_T^{(b_T)}, s_0^{(b_0)}, s_1^{(b_1)}, ..., s_T^{(b_T)}, h_0^{(b_0)}, h_1^{(b_1)}, ..., h_T^{(b_T)}\}.$$

For notational simplicity, let us label the reference particle trajectory with $\{\beta_{0:T}^{(b_{0:T})}, s_{0:T}^{(b_{0:T})}, h_{0:T}^{(b_{0:T})}\}$. The probability of $b_T = r$ for $r = 1, 2, ..., N$ is determined by the importance weight at time T:

$$Pr(b_T = r|\theta, X_{0:T}, S_{0:T}, A_{1:T}) = \hat{w}_T^{(r)}. \tag{16}$$

The proof of this step is provided in Kim (2015).

Then, we update the remaining particles and the ancestor indices denoted by $\{\Xi_{0:T}^{(-b_{0:T})}, S_{0:T}^{(-b_{0:T})}, H_{0:T}^{(-b_{0:T})}, A_{1:T}^{(-b_{0:T})}\}$ conditional on $\{\beta_{0:T}^{(b_{0:T})}, s_{0:T}^{(b_{0:T})}, h_{0:T}^{(b_{0:T})}, b_{0:T}\}$ that we obtain in the previous step. The update is simple. $N-1$ particles and corresponding ancestor indices are sampled by treating the accepted $\{\beta_t^{(b_t)}, s_t^{(b_t)}, h_t^{(b_t)}, b_{t-1}\}$ as the N-th particle and the N-th ancestor index for $t = 0, 1, ..., T$.

[3] If all the latent states are randomly drawn, the resulting importance weights $\{\hat{\omega}_t^{(n)}\}_{n=1}^N$ will become all zeros and, consequently, none of generated particles will have a positive probability mass.

This step is called the conditional sequential Monte Carlo (CSMC) method. A detailed exposition regarding its implementation is provided in the outline of the proposed PG algorithm.

The particle rejuvenation method is a procedure to update $\{\beta_{0:T}^{(b_{0:T})}, s_{0:T}^{(b_{0:T})}, h_{0:T}^{(b_{0:T})}, b_{0:T}\}$ by sequentially generating new values of certain selected states and their ancestor indices. For time period t, the method is composed of sampling $b_{t-1} \in \{1, 2, ..., N\}$ and generating new values of selected states in $\{\beta_t, s_t, h_t\}$ conditional on $\{\beta_{t-1}^{(b_{t-1})}, s_{t-1}^{(b_{t-1})}, h_{t-1}^{(b_{t-1})}\}$ and $\{\beta_{t+1}^{(b_{t+1})}, s_{t+1}^{(b_{t+1})}, h_{t+1}^{(b_{t+1})}\}$. After those two steps, the newly generated values are plugged back into $\{\beta_t^{(b_t)}, s_t^{(b_t)}, h_t^{(b_t)}\}$. The sampling procedure is carried out for the entire period. To construct a valid MCMC step, consider the following target density for particle rejuvenation, which is the backward kernel at time t:

$$\begin{aligned}\Phi(\beta_t, s_t, h_t, b_{t-1}) \propto\ & f_\theta(\beta_{t+1}^{(b_{t+1})}|\beta_t, s_{t+1}^{(b_{t+1})}, h_{t+1}^{(b_{t+1})}) f_\theta(s_{t+1}^{(b_{t+1})}|s_t) f_\theta(h_{t+1}^{(b_{t+1})}|h_t) \\ & \times g_\theta(Y_t|\beta_t, s_t, h_t) f_\theta(\beta_t|\beta_{t-1}^{(b_{t-1})}, s_t, h_t) f_\theta(s_t|s_{t-1}^{(b_{t-1})}) f_\theta(h_t|h_{t-1}^{(b_{t-1})}) \hat{\omega}_{t-1}^{(b_{t-1})},\end{aligned} \tag{17}$$

conditional on $\{\beta_{t-1}^{(b_{t-1})}, s_{t-1}^{(b_{t-1})}, h_{t-1}^{(b_{t-1})}\}$ and $\{\beta_{t+1}^{(b_{t+1})}, s_{t+1}^{(b_{t+1})}, h_{t+1}^{(b_{t+1})}\}$. See Kim (2015) for the derivation of equation (17). Because the normalizing constant of the backward kernel is unknown, we employ an importance sampling method to draw $\{\beta_t, s_t, h_t, b_{t-1}\}$. Consider the following importance density for time t:

$$W(\beta_t, s_t, h_t, b_{t-1}) = Q(\beta_t, s_t, h_t|\beta_{t-1}^{(b_{t-1})}, s_{t-1}^{(b_{t-1})}, h_{t-1}^{(b_{t-1})}, \beta_{t+1:T}^{(b_{t+1:T})}, s_{t+1:T}^{(b_{t+1:T})}, h_{t+1:T}^{(b_{t+1:T})}) \hat{\omega}_{t-1}^{(b_{t-1})}. \tag{18}$$

The above importance density suggests drawing $b_{t-1} \in \{1, 2, ..., N\}$ with $\hat{\omega}_{t-1}^{(b_{t-1})}$ and generating new values of the latent states from $Q(.)$. In particular, as in forward filtering, only x_t in β_t should be randomly sampled from $Q(.)$ to circumvent any problem associated with the degenerate transition and measurement equations. The importance density for $\{\beta_t, s_t, h_t\}$ is given by

$$\begin{aligned}& Q(\beta_t, s_t, h_t|\beta_{t-1}^{(b_{t-1})}, s_{t-1}^{(b_{t-1})}, h_{t-1}^{(b_{t-1})}, \beta_{t+1:T}^{(b_{t+1:T})}, s_{t+1:T}^{(b_{t+1:T})}, h_{t+1:T}^{(b_{t+1:T})}) \\ & \quad = f_\theta(x_t|x_{t-1}^{(b_{t-1})}, s_t, h_t) f_\theta(h_t|h_{t-1}^{(b_{t-1})}) \times \frac{1}{K}.\end{aligned} \tag{19}$$

where K is the total number of regimes. The first step of particle rejuvenation at time t, based on the importance densities in equations (18) and (19), is to draw $\{b_{t-1}^{(c)}\}_{c=1}^{C}$, using $\{\hat{\omega}_{t-1}^{(n)}\}_{n=1}^{N}$, where C is the total number of sampled b_{t-1}. Next, we deterministically assign fixed values for

$s_t \in \{1, 2, ..., K\}$; to do so, we first replicate $\{b_{t-1}^{(c)}\}_{c=1}^{C}$ K times to obtain:

$$\{b_{t-1}^{(l)}\}_{l=1}^{KC} = \{b_{t-1}^{(1)}, ..., b_{t-1}^{(C)}, b_{t-1}^{(1)}, ..., b_{t-1}^{(C)}, ..., b_{t-1}^{(1)}, ..., b_{t-1}^{(C)}\}.$$

Given $\{b_{t-1}^{(l)}\}_{l=1}^{KC}$, we set $s_t^{(l)}$ as $s_t^{(l)} = k$ for $l \in \{(k-1)C+1, (k-1)C+2, ..., kC\}$ for $k = 1, 2, ..., K$ and $l = 1, 2, ..., KC$. Because the proportion of $s_t^{(l)} = k$ is $\frac{1}{K}$ among all particles in $\{s_t^{(l)}\}_{l=1}^{KC}$, the importance density in equation (19) is also proportional to $\frac{1}{K}$. The proposed scheme controls the number of particles for each value of $s_t \in \{1, 2, ..., K\}$ and explores full support for s_t. The remaining states x_t and h_t are sampled from $f(x_t|x_{t-1}, s_t, h_t)$ and $f(h_t|h_{t-1})$. The normalized importance weights for $l = 1, 2, ..., KC$ is given by

$$\hat{\tau}_t^{(l)} = \frac{\bar{\tau}_t^{(l)}}{\sum_{j=1}^{KC} \bar{\tau}_t^{(j)}},$$

where,

$$\begin{aligned} \bar{\tau}_t^{(l)} &\propto \frac{\Phi(\beta_t^{(l)}, s_t^{(l)}, h_t^{(l)}, b_{t-1}^{(l)})}{W(\beta_t^{(l)}, s_t^{(l)}, h_t^{(l)}, b_{t-1}^{(l)})} \\ &= f_\theta(\beta_{t+1}^{(b_{t+1})}|\beta_t^{(l)}, s_{t+1}^{(b_{t+1})}, h_{t+1}^{(b_{t+1})}) f_\theta(s_{t+1}^{(b_{t+1})}|s_t^{(l)}) f_\theta(h_{t+1}^{(b_{t+1})}|h_t^{(l)}) \\ &\times \frac{f_\theta(\beta_t^{(l)}|\beta_{t-1}^{(b_{t-1}^{(l)})}, s_t^{(l)}, h_t^{(l)}) f_\theta(s_t^{(l)}|s_{t-1}^{(b_{t-1}^{(l)})})}{f_\theta(x_t^{(l)}|x_{t-1}^{(b_{t-1}^{(l)})}, s_t^{(l)}, h_t^{(l)}) \frac{1}{K}}. \end{aligned} \tag{20}$$

The importance sampling for particle rejuvenation is performed conditional on the previously accepted reference particle trajectory similar to the CSMC method. We denote the particle and its ancestor index at time t that are contained in the given reference particle path with $\{\beta_t^*, s_t^*, h_t^*, b_{t-1}^*\}$.

Algorithm 1: Conditional Importance Sampling (CIS) at time t

i) Draw ancestor index $b_{t-1}^{(c)} \in \{1, 2, ..., N\}$ with probability $\{\hat{\omega}_{t-1}^{(n)}\}_{n=1}^{N}$ for $c = 1, 2, ..., C-1$. Set $b_{t-1}^{(C)} = b_{t-1}^*$ to obtain $\{b_{t-1}^{(c)}\}_{c=1}^{C}$.

ii) Replicate $\{b_{t-1}^{(c)}\}_{c=1}^{C}$ K times to obtain $\{b_{t-1}^{(l)}\}_{l=1}^{KC}$.

iii) Set $s_t^{(l)} = k$ for $l \in \{(k-1)C+1, (k-1)C+2, ..., \text{kC}\}$ for $k = 1, 2, ..., K$.

iv) Draw $h_t^{(l)}$ and $x_t^{(l)}$ from $f_\theta(h_t|h_{t-1}^{(b_{t-1}^{(l)})})$ and $f_\theta(x_t|x_{t-1}^{(b_{t-1}^{(l)})}, s_t^{(l)}, h_t^{(l)})$ for $l = 1, 2, ..., KC$.

v) Obtain $\beta_t^{(l)} = \{x_t^{(l)}, z_t^{(l)}, z_{t-1}^{(l)}, g_{c,t}^{(l)}, g_{c,t-1}^{(l)}, g_{i,t}^{(l)}, g_{i,t-1}^{(l)}\}$ using the transition equations, $z_t = y_t - x_t$, $g_{c,t} = c_t - \bar{c} - \lambda_c x_t$, $g_{i,t} = i_t - \bar{i} - \lambda_i x_t$ given $x_t^{(i)}$ and $x_{t-1}^{(b_{t-1}^{(l)})}$ for $l = 1, 2, ..., KC$.

vi) Replace $\{\beta_t^{(m)}, s_t^{(m)}, h_t^{(m)}\}$ with $\{\beta_t^*, s_t^*, h_t^*\}$ where $m = s_t^{(b_t)}C$.

vii) Calculate $\bar{\tau}_t^{(l)}$ for $l = 1, 2, ..., KC$ and obtain the normalized weights $\hat{\tau}_t^{(l)} = \frac{\bar{\tau}_t^{(l)}}{\sum_{j=1}^{KC} \bar{\tau}_t^{(j)}}$.

viii) Draw $\{\beta_t, s_t, h_t, b_{t-1}\}$ from $\{\beta_t^{(l)}, s_t^{(l)}, h_t^{(l)}, b_{t-1}^{(l)}\}_{l=1}^{KC}$ with probability $\{\hat{\tau}_t^{(l)}\}_{l=1}^{KC}$.

Set $\{\beta_t^{(b_t)}, s_t^{(b_t)}, h_t^{(b_t)}, b_{t-1}\} = \{\beta_t, s_t, h_t, b_{t-1}\}$.

Using the derivation from Andrieu et al. (2010), it can be shown that the CIS scheme admits the target Markov kernel in equation (17) as a stationary distribution.

Our proposed PG algorithm is an extension of the PG algorithm developed by Lindsten et al. (2015) to estimate a degenerate non-linear switching state space model. The uniform ergodicity for the suggested algorithm can be demonstrated as in Lindsten et al. (2014). Combing with the CIS method, the below PG algorithm provides new MCMC samples of the latent states conditional on previously accepted $\{\beta_{0:T}^{(b_{0:T})}, s_{0:T}^{(b_{0:T})}, h_{0:T}^{(b_{0:T})}, b_{0:T}\}$:

Algorithm 2: Conditional SMC with Particle Rejuvenation

- For $t = 0$,

i-1) Draw $\{s_0^{(n)}\}_{n=1}^{N-1}$ from the $f_\theta(s_0)$ and set $\{h_0^{(n)} = 1\}_{n=1}^{N-1}$. Generate $\{x_0^{(n)}\}_{n=1}^{N-1}$ from $f_\theta(x_0|s_0^{(n)}, h_0^{(n)})$ and $\{z_0^{(n)}, g_0^{(n)}\}_{n=1}^{N-1}$ from $f_\theta(z_0, g_0|s_0^{(n)}, h_0^{(n)})$.

i-2) Update $\{\beta_0^{(b_0)}, s_0^{(b_0)}, h_0^{(b_0)}\}$ using *Algorithm 1* **(CIS algorithm)** and set $\{\beta_0^{(N)}, s_0^{(N)}, h_0^{(N)}\} = \{\beta_0^{(b_0)}, s_0^{(b_0)}, h_0^{(b_0)}\}$.

i-3) Set importance weights as $\{\hat{\omega}_0^{(n)} = \frac{1}{N}\}_{n=1}^{N}$.

- Iterate step ii-1), ii-2), ii-3), ii-4), and ii-5) for $t = 1, 2, ..., T$.

ii-1) Draw ancestor indices, $\{a_t^{(n)}\}_{n=1}^{N-1}$ with the normalized probabilities, $\{\hat{\omega}_{t-1}^{(n)}\}_{n=1}^{N}$.

ii-2) Draw $\{s_t^{(n)}, h_t^{(n)}, x_t^{(n)}\}_{n=1}^{N-1}$, from $f_\theta(s_t|s_{t-1}^{(a_t^{(n)})})$, $f_\theta(h_t|h_{t-1}^{(a_t^{(n)})})$, and $f_\theta(x_t|x_{t-1}^{(a_t^{(n)})}, s_t^{(n)}, h_t^{(n)})$.

ii-3) Obtain $\{\beta_t^{(n)}\}_{n=1}^{N-1} = \{[x_t^{(n)}, z_t^{(n)}, z_{t-1}^{(n)}, g_{c,t}^{(n)}, g_{c,t-1}^{(n)}, g_{i,t}^{(n)}, g_{i,t-1}^{(n)}]'\}_{n=1}^{N-1}$ using $z_t = y_t - x_t$, $g_{c,t} = c_t - \bar{c} - \lambda_c x_t$, $g_{i,t}^{(n)} = i_t - \bar{i} - \lambda_i x_t$ given $x_t^{(n)}$ and $x_{t-1}^{(a_t^{(n)})}$.

ii-4) Run *Algorithm 1* **(CIS algorithm)** to update $\{\beta_t^{(b_t)}, s_t^{(b_t)}, h_t^{(b_t)}, b_{t-1}\}$. Set $a_t^{(N)} = b_{t-1}$ and $\{\beta_t^{(N)}, s_t^{(N)}, h_t^{(N)}\} = \{\beta_t^{(b_t)}, s_t^{(b_t)}, h_t^{(b_t)}\}$.

ii-5) Calculate unnormalized weights $\bar{\omega}_t^{(n)}$and obtain normalized weights $\hat{\omega}_t^{(n)} = \frac{\bar{\omega}_t^{(n)}}{\sum_{j=1}^{N} \bar{\omega}_t^{(j)}}$ for $n = 1, 2, ..., N$.

- Perform step iii-1), and iii-2) for $t = T$.

iii-1) Draw b_T^* with $\{\hat{\omega}_T^{(n)}\}_{n=1}^N$ and construct $b_{0:T-1}^*$ using $b_{t-1}^* = a_t^{(b_t^*)}$ for $t = T, T-1, ..., 1$.

iii-2) Construct a new reference particle trajectory $\{\beta_{0:T}^{(b_{0:T}^*)}, s_{0:T}^{(b_{0:T}^*)}, h_{0:T}^{(b_{0:T}^*)}\}$.
Set $\{\beta_{0:T}^{(b_{0:T})}, s_{0:T}^{(b_{0:T})}, h_{0:T}^{(b_{0:T})}, b_{0:T}\} = \{\beta_{0:T}^{(b_{0:T}^*)}, s_{0:T}^{(b_{0:T}^*)}, h_{0:T}^{(b_{0:T}^*)}, b_{0:T}^*\}$.

Conditional on $\{\beta_{0:T}^{(b_{0:T})}, s_{0:T}^{(b_{0:T})}, h_{0:T}^{(b_{0:T})}\}$, it is much simpler to sample θ from proper conditional posterior distributions. Appendix A provides detailed exposition on how to directly simulate posterior samples of θ with conjugate priors[4].

3 Trend-Cycle Decomposition of U.S. Real GDP under Time-varying Importance of Permanent and Transitory Shocks

In this section, we focus on investigating how the importance of the permanent and the transitory shocks has changed over time based on UC models that incorporate a cointegration relationship among macro variables. Seasonally adjusted U.S. real GDP, real fixed investment, and consumption data for non-durables and services in logs from 1947:Q1 to 2016:Q2 are employed for our empirical analysis. The dataset is extracted from the Federal Reserve Bank of St. Louis data base.

3.1 Empirical Results

A main obstacle to obtaining a reliable estimate of the noise-to-signal ratio ($\delta_t = \frac{\sigma_{u,t}^2}{\sigma_{e,t}^2}$) and to performing credible trend-cycle decomposition lies in a lack of information on the low-frequency trend component that would contain structural breaks. In this regard, Bai et al. (1998) demonstrate that a confidence interval for a structural break date does not decrease even asymptotically as a sample size increases. Instead, it is shown that the confidence interval is inversely related with the dimension of time series data that share a common break point. Such a theoretical result implies

[4]We confirm via simulations that the Bayesian method developed herein performs well as a practical matter – even under the degenerate transition and measurement equations of our UC models – and provides reliable estimates of a break date and other latent variables.

that our multivariate UC model that considers the cointegration relation can improve inference on permanent regime changes in the trend growth rate of real GDP, which can eventually lead to more credible trend-cycle decomposition.

We estimate the common long-run trend using the multivariate UC model defined in equations (1)-(9). Let us refer to the multivariate model with a single break ($K = 1$) and two breaks ($K = 2$) as MUC-SB1-SV and MUC-SB2-SV, respectively. Another popular specification for the time-varying long-run growth rate is a random walk model intended to capture gradual changes in μ_t. The model is given as follows:

$$\mu_t = \mu_{t-1} + \nu_t,\ \nu_t \sim N(0, \sigma_\mu^2),$$

for $t = 0, 1, 2, ..., T$. We label the last UC model with MUC-RW-SV. At each MCMC literature, we employ *Algorithm 2* to generate the latent state variables of the UC models. The model parameters are simulated from their conditional posterior distributions as explained in *Appendix A*[5].

3.1.1 Model Comparison

For model comparison, we compute the marginal likelihood of each UC model using Chib's (1995) method. The marginal likelihoods are reported in the last row of Table 1. The marginal likelihoods of MUC-SB1-SV, MUC-SB2-SV, and MUC-RW-SV are -1004.68, -1008.04, and -1040.53. Among these models, MUC-SB1-SV is the most preferred model in the light of the computed marginal likelihoods. The marginal likelihood of MUC-RW-SV is too low to compare with two other structural break models, which implies that the long-run growth rate of U.S. real GDP is better described by infrequent and large structural breaks rather than frequent and small gradual

[5]We use informative priors for the variance of the shocks to the volatility processes to reflect our prior belief that changes in volatilities are smooth over time. Admittedly, completely diffuse priors for the stochastic volatilities do not produce sensible Bayesian estimates for all model parameters, including time-varying volatilities. Moreover, we impose a weakly informative prior to Σ_0, which represents our prior belief that the permanent and transitory shocks are negatively correlated. In a latter section, we introduce other UC models that assume that correlations between permanent and and transitory shocks are zeros. The nested UC models will be compared with the models without the correlation restriction presented here. For other model parameters, non-informative priors are used.

changes.

3.1.2 Trend-Cycle Decomposition

Even if the marginal likelihood strongly supports MUC-SB1-SV among the completing models, it would be still worth comparing with the results of the other models. Figure 1 shows that the trend-cycle decompositions based on the three models are quantitatively and qualitatively similar. For example, it is commonly observed that the cycle prior to the 1990s is relatively more persistent and larger in amplitude than that after the 1990s. In contrast to univariate UC models with correlated shocks, such as ones used in Chan and Grant (forthcoming), the estimated cycle also matches well with NBER's recession dates before the 1990s.

The disappearance of large cycle fluctuations after the 1990s is not irrelevant to the recent empirical findings presented by Kim (2008) and Huang et al. (2016). Kim (2008) claims that the recessions after the mid-1980s were not followed by rapid economic recoveries, resulting in permanent losses to U.S. real GDP. Huang et al. (2016) classify U.S. recessions together with subsequent recoveries as an "L-shaped" or "U-shaped" recession. The "L-shaped" recession is characterized by permanent output losses without any economic recovery, while the "U-shaped" recession is characterized by temporal output losses with rapid recoveries. Huang et al. (2016) also claim that permanent shock played a dominant role after the 1990s, providing evidence that all recessions are the "L-shaped" recessions during the time period. Figures 1 (a) and (b) visually illustrate this point. The UC models explain the two most recent recessions as permanent shifts in the long-run trend of real GDP. See Figure 1(b). As a result, only a small output gap appears around the economic downturns in Figure 1(a). Through simple visual inspection, we can immediately see that there was no noticeable sign of economic recovery after the two recessions.

3.1.3 Time-varying Noise-to-Signal Ratio

Figure 2 (a) shows the posterior mean estimates of $\delta_t = \frac{\sigma^2_{u,t}}{\sigma^2_{e,t}}$. MUC-SB1-SV indicates that neither the trend nor the cycle of U.S. real GDP is fully dominant over the entire post-War period. Instead, it shows that their relative weights have been changing (substantially) since 1947:Q1. In particular, the high noise-to-signal ratio until the 1960s implies that real GDP were largely derived by the permanent shock during the time period. Since then, the transitory shock and the permanent shock almost equally contributed to the stochastic variation in real GDP. The noise-to-signal ratios estimated by MUC-SB1-SV and MUC-SB2-SV largely coincide, while MUC-RW-SV produces a somewhat lower estimate.

Jones (2015) and Fernald (2016) elaborate on the rapid productivity growth during the 1950s and 1960s, which led to exceptionally high economic growth. The high growth in productivity is attributed to great scientific innovations, such as electricity, the telephone, and the internal combustion engine, among others. The prominent role of the permanent trend shock for almost two decades after World War II would be closely linked to those great inventions. Moreover, the role of the permanent shock in the U.S. economy has been gradually growing since the mid-1980s. The gradual increase in the relative importance of the permanent shock would be explained by the disappearance of the post-recession recoveries, as we mentioned in the previous section. However, overall, our empirical results indicate that both the permanent and the transitory shocks are important in explaining U.S. real output for the last 40 years.

3.1.4 Long-run Growth Rate

Figure 2(b) displays the posterior means of the annual long-run growth rate (μ_t) obtained from the three models. For the two structural break models, the posterior mean estimates can be interpreted as the expected long-run growth rate under uncertainty in the structural break dates. The common conclusion of the resulting estimates is that the long-run growth rate of the U.S. economy gradually declined before the 2000s but dramatically fell after the recession in 2001. The key implication regarding the trend growth is robust to different model specifications. Notably, UC-RW-SV produces an extremely volatile estimate for the long-run growth rate compared to the structural break models. The relatively low noise-to-signal ratio estimated by MUC-RW-SV in Figure 2 (a) is due to the excessive variations in μ_t that absorb the effect of the permanent shock u_t on real GDP. However, we should interpret the empirical results of MUC-RW-SV with caution because the model does not fit the data well in light of the low marginal likelihood value.

Exploiting more information on the common trend from real consumption and investment data substantially improves inferences on the trend growth rate of real GDP. The annualized long-run growth rate of 2016:Q2 is estimated to be 1.63% under MUC-SB1-SV and to be 1.55 % under MUC-SB2-SV. Those estimates are very close to the long-run forecasts of potential output growth derived from Gordon (2014) and Fernald (2016). Their forecasts are between 1.5%-1.6%. Fernald (2016) indicates that low productivity growth is a main reason for the recent slow growth of U.S. economy and is expected to last over the long run in that the high growth in education attainment that underlies the high productivity and the economic prosperity of the 20th century is no longer achievable in the U.S. economy. Although our Bayesian estimation does not provide a long-run prediction of real GDP, the consistent result would be an indication that our structural break UC models describe the dynamics of the trend component well. The posterior mean of the long-run growth rate from MUC-RW-SV is 2.41%. This figure may be too optimistic, considering other related studies such as Gordon (2014) and Fernald (2016).

3.1.5 Correlations between Permanent and Transitory Shocks

For U.S. real GDP, the posterior mean of the correlation between the trend shock and the cycle shock is estimated to be -0.69 by MUC-SB1-SV. Other UC models produce very similar estimates. A prevalent interpretation of this negative correlation is the time-to-build effect stressed by stylized RBC theories. When a real positive (negative) permanent shock shifts potential real GDP up (down), a negative (positive) cycle will result due to the sluggish reaction of output to the real shock. The role of nominal rigidity that triggers the initial negative impact of the shock on the cycle component is also emphasized in the trend-cycle decomposition literature.

The correlation between the shocks to the common trend and the cycle of real consumption is also strongly negative, which is consistent with Morley (2007). The mean estimate of the correlation parameter is -0.92 based on MUC-SB1-SV. This implies that consumption slowly – as opposed to immediately – adjusts to a permanent shock to the common trend component. This slow adjustment of real consumption suggests the importance of habit formation in consumer preferences and a precautionary savings behavior in deriving aggregate consumption fluctuations, as emphasized by

Morley (2007).

3.2 Alternative Model Specifications

3.2.1 Restricted Correlation Structure Between Permanent and Transitory Shocks

When extracting the trend and the cycle components of U.S. real GDP, one of the most common assumptions is that the permanent and transitory shocks are orthogonal. Harvey (1985), Clark (1987), and Perron and Wada (2009), among many others, use this assumption for trend-cycle decompositions. In this section, we carry out additional model comparisons to test for the popular zero correlation restriction in a multivariate setup. Let MUC0-SB1-SV and MUC0-SB2-SV denote the multivariate UC models with one and two breaks that assume $\rho_{u,e} = \rho_{u,\eta_c} = \rho_{u,\eta_i} = 0$. MUC0-RW-SV denotes the multivariate UC model in which the long-run growth rate follows a random walk process under the zero correlation restriction. The results for MUC-SB1-SV, MUC-SB2-SV, and MUC0-RW-SV are reported in Table 2.

Two interesting findings show up via Bayesian model comparisons. See the last row of Table 2. First, the number of structural breaks in μ_t, selected by the marginal likelihoods, is two in the case with the zero correlation restriction, while it was one in the case with the flexible correlation structure. More importantly, the zero correlation assumption that is frequently used in univariate analysis is overwhelmingly rejected in the multivariate setup, regardless of model specifications for the long-run growth rate. Different from previous studies that take univariate models, the gap between the two marginal likelihoods of MUC-SB1-SV and MUC0-SB1-SV is large enough to confer strong credibility on the unrestricted UC models rather than on the nested model with the zero correlation restriction. This conclusion does not change even when we consider the second break or the random walk process for the trend growth rate.

Berger et al. (2016) and Chan and Grant (forthcoming) suggest multivariate UC models of real GDP, inflation, and the unemployment rate for trend-cycle decomposition. Even if they use different sets of data from ours, we may be able to obtain some useful information by comparing our results with theirs. Berger et al. (2016) incorporate stochastic volatilities into the permanent and the transitory components of real GDP, as in our models. However, they do not consider the correlation between the shocks to the trend and cycle components, which is crucial for trend-cycle decomposition, as demonstrated in this section and by Chan and Grant (forthcoming). Our UC models here that assume zero correlation between the permanent and the transitory shocks produce results that generally correspond to the results presented by Berger et al. (2016)[6]. On the other hand, Chan and Grant (forthcoming) consider the correlation between the permanent and transitory shocks even if they do not consider time-varying volatilities. Chan and Grant (forthcoming) conclude that the zero correlation restriction is not supported by data. Here, we provide evidence to support Chan and Grant (forthcoming).

3.2.2 Is a Univariate UC model Enough to Capture the Relative Importance of Permanent and Transitory Shocks?

Our tentative answer is "No". We confirm that there exists much larger uncertainty in the estimation of the long-run growth rate when employing a univariate UC model than when employing the multivariate models. Such large uncertainty in estimating the trend function could cause an unexpected consequence that the role of the permanent shock is overestimated. In the UC models that we have estimated so far, a change in the trend component is attributed either to a change in μ_t or to a new arrival of the permanent shock u_t. This implies that imprecise estimation of μ_t due to a misspecified trend function or a lack of information can produce upward bias for the estimated variance of the permanent shock. Perron and Wada (2009) and Luo and Startz (2014) heavily cope with the issue related with the uncertainty transmission originated from a misspecified model and conclude that a fail to account for a break in the long-run trend function substantially inflates the variance of the permanent shock.

In this section, we investigate the issue of how the poor estimation of the long-run trend affects the trend-cycle decomposition of U.S. real GDP from the perspective of a lack of information. The univariate UC model we use is obtained by assuming $\sigma^2_{h,u} > 0, \sigma^2_{h,e} > 0$ and by excluding all the

[6]For MUC-SB1-SV, MUC-SB2-SV, and MUC-RW-SV, the time-varying noise-signal ratio δ_t is estimated to be close to zero under the zero correlation assumption. This implies that the role of the permanent shock is negligible for the entire sample period, which is consistent with Perron and Wada (2009), and Berger et al. (2016).

equations associated with c_t and i_t from equations (1)-(9). The time-varying covariance matrix for the univariate model is therefore given by

$$\begin{bmatrix} u_t \\ e_t \end{bmatrix} \sim i.i.d.N.(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2_{u,t} & \rho_{u,e}\sigma_{u,t}\sigma_{e,t} \\ \rho_{u,e}\sigma_{e,t}\sigma_{u,t} & \sigma^2_{e,t} \end{bmatrix}),$$

where $\sigma^2_{w,t} = \sigma^2_{w,0}\ exp(h_{w,t})$ for $w = u, e$. We do not impose any restriction on the correlation parameter $\rho_{u,e}$. Let UC-SB1-SV denote the univariate model[7].

Figure 3 (a) shows the posterior means of δ_t obtained from UC-SB1-SV and MUC-SB1-SV. For the entire sample period, the estimate of δ_t for UC-SB1-SV is much higher than the estimate from MUC-SB1-SV. We can explain the difference with high uncertainty in estimating μ_t under the univariate model. For example, in Table 3, the posterior standard deviation of μ_2 is 0.28 for UC-SB1-SV, while the estimate of the same parameter is only 0.13 for MUC-SB1-SV. The multivariate model, which contains more information on the low-frequency long-run trend, reduces the posterior standard deviation more than 50 %. As mentioned before, because it is difficult to precisely infer the trend component in a univariate setup, the univariate UC model could convey a misleading implication about the relative roles of the two shocks to real GDP. Figure 3 (b) also shows a noticeable difference between the extracted cycles from UC-SB1-SV and MUC-SB1-SV. The cycle from UC-SB1-SV is less persistent and smaller in amplitude than its multivariate counterpart. The results from the univariate UC model are broadly consistent with Chan and Grant (forthcoming) and Kim and Kim (2016).

Additionally, we allow for a second break in the long-run growth rate. ($K = 3$) Our modeling approach for μ_t differs from that used in Luo and Startz (2014) because we do not restrict the total number of breaks during the sample period. Simply speaking, the number of breaks could be estimated to be 0, 1, or 2 in our model, while the maximum and total numbers of estimated breaks are assumed to be same in the Luo and Startz (2014) models. We can re-confirm their conclusion that post-war real GDP data do not contain evidence of a second break. The probability of a second break is nearly zero for all periods, and posterior estimates are almost identical to those from the

[7]As in the multivariate models, we impose a weakly informative prior to Σ_0 in UC-SB1-SV, which represents our prior belief that the permanent and transitory shocks are negatively correlated.

single-break UC model. The additional results are not reported here for the sake of brevity.[8]

Lastly, following Harvey (1985), Clark (1987) and Perron and Wada (2009), who advocate using a UC model with orthogonal permanent and transitory shocks, we assume $\rho_{u,e} = 0$. The univariate UC model with the zero correlation assumption is labeled with UC0-SB1-SV. Consistent with the previous literature, the assumption on $\rho_{u,e}$ substantially affects Bayesian inferences in trend-cycle decomposition. See the right-side panel in Table 3. The sum of the AR coefficients of the cycle component is estimated 0.95 for UC0-SB1-SV, while it is 0.52 for UC-SB1-SV. Which univariate model better describes real GDP dynamics? As Iwata and Li (2015) articulate, choosing a better model between the two UC models that gives us very different implications is a difficult task on purely statistical grounds. The log marginal likelihoods of UC-SB1-SV and UC0-SB1-SV are -346.20 and -348.94, respectively[9]. Therefore, in contrast to the case of using the multivariate UC models, we could not gain decisive evidence on which univariate model better captures the natures of the trend and the cycle of U.S. real GDP[10].

4 Concluding Remarks

Our study has highlighted that the conventional assumption that the ratio of the variances of permanent and transitory shocks is constant over time does not hold for real GDP. Instead, the multivariate UC model proposed in this paper – which utilizes the cointegration relation between real GDP, consumption, and investment – provide new empirical evidence that the permanent shock played a more important role from the 1950s to the 1960s, but both shocks have been almost equally important in U.S. economy since the 1970s. Moreover, we find that long-run growth of

[8]As a robustness check, we also use a different inverse Wishart prior for UC-SB1-SV. The scale matrix for the prior is $3 \times diag(1.5^2, 1.5^2, 1.5^2)$, and the degree of freedom is 3. For other parameters, the same priors shown in Table 3 are used. The empirical results associated with the long-run trend and trend-cycle decomposition largely coincide.

[9]UC-SB1-SV and UC0-SB1-SV are strongly preferred to an univariate UC model with a constant covariance matrix. The log marginal likelihood of the UC model without stochastic volatility is -356.97. This result emphasizes the importance of embedding stochastic volatilities in a UC model.

[10]Based on the model selection criterion proposed by Kass and Raftery (1995) and Raftery (1995), Bayesian evidence in favor of one model against another is considered strong or very strong when the twice-log Bayes factor is larger than 6 or 10, respectively.

U.S. economy has significantly slowed down since 2001. The results for long-run economic growth is of great importance for policy makers because different perspectives on the long-run economic outlook of the U.S. economy could suggest contrasting implications for optimal monetary and fiscal policies.

A recent study by Grant and Chan (2017) introduces a univariate unobserved component model where the trend component is modeled as a second-order Markov process. Grant and Chan (2017) show that the new trend function nests the Hodrick-Prescott (1997) trend as a special case and generates a large and persistent cycle measure. Berger et al. (2016) suggest a multivariate UC model of real GDP, unemployment, and inflation based on the Phillips curve and Okun's law. Using the dynamic interactions among the cycle components of those variables, Berger et al. (2016) decompose real GDP into the cycle and the trend components. For future research, it would be interesting to embed the new type of the trend function and the cycle interactions in our multivariate UC model and evaluate its performance for trend-cycle decomposition.

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A Appendix: Posterior Simulation for Model Parameters

Kim and Chon (2016) show that, in a univariate UC model, the variance of the permanent shock could be severely biased if the autoregressive and the long-run mean growth rate parameters are sampled separately without considering the correlation structure of the permanent and the transitory shocks. Therefore, we sample the model parameters of our multivariate UC model simultaneously using their joint posterior distribution. Consider the following seemingly unrelated regression (SUR) model derived from the transition equations of the proposed UC model, conditional on $\{\beta_{0:T}, s_{0:T}, h_{0:T}\}$:

$$\begin{bmatrix} \Delta x_t^* \\ z_t^* \\ c_t^* \\ i_t^* \end{bmatrix} = \begin{bmatrix} I_{s_t=1}^* & I_{s_t=2}^* & ... & I_{s_t=K}^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & ... & 0 & z_{t-1}^* & z_{t-2}^* & 0 & 0 & 0 & 0 \\ 0 & 0 & ... & 0 & 0 & 0 & g_{c,t-1}^* & g_{c,t-2}^* & 0 & 0 \\ 0 & 0 & ... & 0 & 0 & 0 & 0 & 0 & g_{i,t-1}^* & g_{i,t-2}^* \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_K \\ \phi_1 \\ \phi_2 \\ \gamma_{c,1} \\ \gamma_{c,2} \\ \gamma_{i,1} \\ \gamma_{i,2} \end{bmatrix} + \begin{bmatrix} u_t^* \\ e_t^* \\ \eta_{c,t}^* \\ \eta_{i,t}^* \end{bmatrix}, \qquad (A.1)$$

$$(\tilde{Y}_t = \tilde{X}_t \Gamma + \tilde{\epsilon}_t),$$

$$\begin{bmatrix} u_t^* \\ e_t^* \\ \eta_{c,t}^* \\ \eta_{i,t}^* \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_{u,0}^2 & \rho_{u,e}\sigma_{u,0}\sigma_{e,0} & \rho_{u,\eta_c}\sigma_{u,0}\sigma_{\eta_c,0} & \rho_{u,\eta_i}\sigma_{u,0}\sigma_{\eta_i,0} \\ \rho_{u,e}\sigma_{e,0}\sigma_{u,0} & \sigma_{e,0}^2 & \rho_{e,\eta_c}\sigma_{e,0}\sigma_{\eta_c,0} & \rho_{e,\eta_i}\sigma_{e,0}\sigma_{\eta_i,0} \\ \rho_{u,\eta_c}\sigma_{\eta_c,0}\sigma_{u,0} & \rho_{e,\eta_c}\sigma_{\eta_c,0}\sigma_{e,0} & \sigma_{\eta_c,0}^2 & \rho_{\eta_c,\eta_i}\sigma_{\eta_c,0}\sigma_{\eta_i,0} \\ \rho_{u,\eta_i}\sigma_{\eta_i,0}\sigma_{u,0} & \rho_{e,\eta_i}\sigma_{\eta_i,0}\sigma_{e,0} & \rho_{\eta_c,\eta_i}\sigma_{\eta_c,0}\sigma_{\eta_i,0} & \sigma_{\eta_i,0}^2 \end{bmatrix} \right), \qquad (A.2)$$

$$(\tilde{\epsilon}_t \sim N(0, \Sigma_0)),$$

where $\Gamma = [\mu_1 \ \mu_2 \ ... \ \mu_K \ \phi_1 \ \phi_2 \ \gamma_{c,1} \ \gamma_{c,2} \ \gamma_{i,1} \ \gamma_{i,2}]'$; $\Delta x_t^* = \frac{(x_t - x_{t-1})}{exp(h_{u,t}/2)}$; $I_{s_t=k}^* = \frac{I(s_t=k)}{exp(h_{u,t}/2)}$ for $k = 1, 2, ..., K$; $z_{t-j}^* = \frac{z_{t-j}}{exp(h_{e,t}/2)}$, $g_{c,t-j}^* = \frac{g_{c,t-j}}{exp(h_{\eta_c,t}/2)}$, $g_{i,t-j}^* = \frac{g_{i,t-j}}{exp(h_{\eta_i,t}/2)}$ for $j = 0, 1, 2$. We apply the Bayesian Gibbs sampling introduced by Zellner and Ando (2010) to estimate the above SUR model. With a normal prior distribution for Γ and an inverse Wishart prior for Σ_0, we can obtain the conditional posterior distribution of Γ:

$$\Gamma | Y_{1:T}, \beta_{1:T}, s_{1:T}, \theta_{-\Gamma} \sim N(\bar{\Gamma}, \bar{\Omega}_\Gamma) I_\Gamma, \tag{A.3}$$

where $\bar{\Gamma} = \bar{\Omega}_\Gamma(\underline{\Omega}_\Gamma^{-1}\underline{\Gamma} + \tilde{X}'(\Sigma_0^{-1} \otimes I)\tilde{X}\hat{\Gamma})$, $\bar{\Omega}_\Gamma = (\underline{\Omega}_\Gamma^{-1} + \tilde{X}'(\Sigma_0^{-1} \otimes I)\tilde{X})^{-1}$, and $\hat{\Gamma} = (\tilde{X}'(\Sigma_0^{-1} \otimes I)\tilde{X})^{-1}\tilde{X}'(\Sigma_0^{-1} \otimes I)\tilde{Y}$. $\{\underline{\Gamma}, \underline{\Omega}_\Gamma\}$ are the prior mean and covariance matrices. The data matrices $\{\tilde{X}, \tilde{Y}\}$ are obtained by vertically stacking $\{\tilde{X}_t, \tilde{Y}_t\}$. The indication function I_Γ takes one if both the regime identification and stationarity conditions for $\{\mu, \phi, \gamma_c, \gamma_i\}$ hold and zero otherwise. We carry out the posterior simulation for Γ using the multivariate normal distribution given in equation (A.3) with rejection sampling. Conditional on Γ, the covariance matrix Σ_0 at $t = 0$ is simulated from a posterior inverse Wishart distribution with a conjugate prior $IW(\Lambda_0, \nu_0)$, where Λ_0 and ν_0 are prior scale and degree of freedom parameters.

Conditional on $\{\beta_{0:T}, s_{0:T}, h_{0:T}\}$, Γ, and Σ_0, it is relatively simple to draw other model parameters. We simulate the posterior sample of the transition probabilities for s_t from posterior beta distributions. The MCMC samples of $\{\sigma^2_{h,u}, \sigma^2_{h,e}, \sigma^2_{h,\eta_c}, \sigma^2_{h,\eta_i}\}$ are drawn from posterior inverse Gamma distributions using conventionally used conjugate priors. The sampling scheme for the cointegration parameters $\{\bar{c}, \lambda_c, \bar{i}, \lambda_i\}$ is based on the following Cholesky decomposition:

$$\begin{aligned}\begin{bmatrix}\eta_{c,t} \\ \eta_{i,t}\end{bmatrix} &= PA_tA_t^{-1}\begin{bmatrix}u_t \\ e_t \\ \eta_{c,t} \\ \eta_{i,t}\end{bmatrix} = PA_t\begin{bmatrix}\nu_{1,t} \\ \nu_{2,t} \\ \nu_{3,t} \\ \nu_{4,t}\end{bmatrix} \\ &= \begin{bmatrix}A_{t,3,1}\ \nu_{1,t} + A_{t,3,2}\ \nu_{2,t} \\ A_{t,4,1}\ \nu_{1,t} + A_{t,4,2}\ \nu_{2,t}\end{bmatrix} + \begin{bmatrix}A_{t,3,3}\ \nu_{3,t} \\ A_{t,4,3}\ \nu_{3,t} + A_{t,4,4}\ \nu_{4,t}\end{bmatrix}, \quad \begin{bmatrix}\nu_{3,t} \\ \nu_{4,t}\end{bmatrix} \sim N(0, I_2)\end{aligned} \tag{A.4}$$

where $P = \begin{bmatrix}0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix}$, A_t is the lower triangular matrix such that $A_tA_t' = \Sigma_t$, $A_{t,i,j}$ is the (i, j) element of A_t, and Σ_t is the time-varying covariance matrix of $[u_t\ e_t\ \eta_{c,t}\ \eta_{i,t}]'$. Given all other parameters, u_t $(= y_t - x_t)$, and e_t $(= z_t - \phi_1 z_{t-1} - \phi_2 z_{t-2})$, we can obtain A_t, and the normalized errors, $\{\nu_{1,t}, \nu_{2,t}\}$. Based on equations (2), (3), (7), and (8), we can derive the posterior mean and covariance matrix for $\{\bar{c}, \lambda_c, \bar{i}, \lambda_i\}$ with a conjugate prior based on the following VAR representation:

$$\begin{bmatrix} c_t^* \\ i_t^* \end{bmatrix} - \begin{bmatrix} A_{t,3,1}\ \nu_{1,t} + A_{t,3,2}\ \nu_{2,t} \\ A_{t,4,1}\ \nu_{1,t} + A_{t,4,2}\ \nu_{2,t} \end{bmatrix} = \begin{bmatrix} x_{c,0}^* & x_{c,1}^* & 0 & 0 \\ 0 & 0 & x_{i,0}^* & x_{i,1}^* \end{bmatrix} \begin{bmatrix} \bar{c} \\ \lambda_c \\ \bar{i} \\ \lambda_i \end{bmatrix} + \begin{bmatrix} A_{t,3,3}\ \nu_{3,t} \\ A_{t,4,3}\ \nu_{3,t} + A_{t,4,4}\ \nu_{4,t} \end{bmatrix}, \tag{A.5}$$

where $c_t^* = (c_t - \gamma_{c,1} c_{t-1} - \gamma_{c,2} c_{t-2})$, $i_t^* = (i_t - \gamma_{i,1} i_{t-1} - \gamma_{i,2} i_{t-2})$, $x_{c,0}^* = (1 - \gamma_{c,1} - \gamma_{c,2})$, $x_{i,0}^* = (1 - \gamma_{i,1} - \gamma_{i,2})$, $x_{c,1}^* = (x_t - \gamma_{c,1} x_{t-1} - \gamma_{c,2} x_{t-2})$, $x_{i,1}^* = (x_t - \gamma_{i,1} x_{t-1} - \gamma_{i,2} x_{t-2})$.

Table 1: Posterior Estimates - Multivariate UC models with Correlated Permanent and Transitory Shocks [Real GDP, Consumption, and Investment: 1947:Q1 ∼ 2016:Q2]

Model		MUC-SB1-SV				MUC-SB2-SV				MUC-RW-SV			
Parameter	Prior	Posterior				Posterior				Posterior			
		Mean	SD	Quantile		Mean	SD	Quantile		Mean	SD	Quantile	
($T = 278$)				5%	95%			5%	95%			5%	95%
		Permanent Component											
μ_1	$N(0.9, 0.5^2)$	0.83	0.08	0.70	0.96	0.89	0.10	0.74	1.08	-	-	-	-
μ_2	$N(0.6, 1^2)$	0.39	0.13	0.20	0.60	0.62	0.17	0.34	0.88	-	-	-	-
μ_3	$N(0.4, 1^2)$	-	-	-	-	0.33	0.16	0.04	0.57	-	-	-	-
μ_0	$N(0.9, 0.5^2)$	-	-	-	-	-	-	-	-	0.66	0.24	0.29	1.03
σ_μ^2	$IG(10, 0.01^2)$	-	-	-	-	-	-	-	-	0.01	0.00	0.00	0.01
$\bar{c}$	$N(-58.31, 10^2)$	-56.10	6.36	-66.47	-45.73	-58.00	6.55	-68.68	-47.21	-57.68	8.36	-71.25	-43.82
λ_c	$N(1.00, 1^2)$	1.01	0.01	0.99	1.02	1.01	0.01	1.00	1.02	1.01	0.01	0.99	1.02
$\bar{i}$	$N(-392.63, 10^2)$	-392.41	9.66	-408.28	-376.54	-392.35	9.72	-408.36	-376.43	-393.31	9.59	-409.08	-377.61
λ_i	$N(1.22, 1^2)$	1.22	0.01	1.20	1.24	1.22	0.01	1.20	1.24	1.22	0.01	1.20	1.24
		Cyclical Components											
$\phi_1 + \phi_2$	$N(0.5, 1^2)$	0.86	0.08	0.70	0.94	0.87	0.08	0.71	0.96	0.91	0.07	0.79	0.98
ϕ_2	$N(0, 1^2)$	0.03	0.09	-0.13	0.18	0.04	0.09	-0.11	0.19	-0.04	0.12	-0.22	0.16
$\gamma_{c,1} + \gamma_{c,2}$	$N(0.5, 1^2)$	0.92	0.02	0.88	0.95	0.92	0.03	0.88	0.96	0.94	0.03	0.89	0.98
$\gamma_{c,2}$	$N(0, 1^2)$	0.02	0.05	-0.05	0.09	0.02	0.05	-0.06	0.09	-0.00	0.07	-0.13	0.09
$\gamma_{i,1} + \gamma_{i,2}$	$N(0.5, 1^2)$	0.96	0.01	0.93	0.98	0.96	0.01	0.93	0.98	0.96	0.01	0.94	0.98
$\gamma_{i,2}$	$N(0, 1^2)$	-0.42	0.08	-0.56	-0.30	-0.41	0.07	-0.54	-0.30	-0.40	0.07	-0.52	-0.30
		Covariance Matrix											
$\sigma_{u,0}^2$	$IW(4, A)$	2.33	0.77	1.30	3.72	2.35	0.90	1.23	4.05	1.67	0.68	0.78	2.92
$\sigma_{e,0}^2$	$IW(4, A)$	1.06	0.53	0.46	2.10	1.12	0.58	0.47	2.23	1.26	0.65	0.50	2.49
$\sigma_{\eta_c,0}^2$	$IW(4, A)$	2.02	0.71	1.07	3.30	2.09	0.78	1.07	3.51	1.73	0.70	0.80	3.06
$\sigma_{\eta_i,0}^2$	$IW(4, A)$	5.83	2.53	2.97	9.74	5.94	2.52	2.98	10.62	7.14	3.07	3.44	13.04
$\rho_{u,e}$	$IW(4, A)$	-0.69	0.09	-0.82	-0.53	-0.70	0.09	-0.83	-0.52	-0.65	0.12	-0.82	-0.43
ρ_{u,η_c}	$IW(4, A)$	-0.92	0.02	-0.95	-0.87	-0.92	0.03	-0.96	-0.87	-0.90	0.04	-0.95	-0.82
ρ_{u,η_i}	$IW(4, A)$	-0.03	0.21	-0.38	0.30	-0.06	0.22	-0.42	0.29	-0.17	0.27	-0.65	0.26
ρ_{e,η_c}	$IW(4, A)$	0.70	0.09	0.53	0.83	0.71	0.10	0.52	0.84	0.68	0.12	0.47	0.83
ρ_{e,η_i}	$IW(4, A)$	0.39	0.20	0.01	0.68	0.42	0.20	0.04	0.70	0.53	0.20	0.15	0.80
ρ_{η_c,η_i}	$IW(4, A)$	0.07	0.23	-0.30	0.45	0.11	0.24	-0.28	0.49	0.22	0.28	-0.24	0.68
		Transition Probability											
π_{11}	$Beta(99, 1)$	0.99	0.00	0.98	0.99	0.99	0.01	0.97	0.99	-	-	-	-
π_{22}	$Beta(99, 1)$	-	-	-	-	0.98	0.01	0.95	0.99	-	-	-	-
Log Marginal Likelihood		-1004.687				-1008.046				-1040.535			

Total Iteration/Burn-in Iteration = 25,000/5,000; In *Algorithm 1* and *Algorithm 2*, the numbers of particles are set at $N = 400$ and $C = 200$, respectively; $A = 4 \times \begin{bmatrix} 1.5^2 & -0.5\times1.5^2 & -0.5\times1.5^2 \\ -0.5\times1.5^2 & 1.5^2 & 0 \\ -0.5\times1.5^2 & 0 & 1.5^2 \end{bmatrix}$; The posterior means of $\{\bar{c}, \lambda_c, \bar{i}, \lambda i\}$ are obtained by OLS estimation. $(y_t = \hat{\bar{c}} + \hat{\lambda}_c c_t + \hat{\omega}_{c,t},\ y_t = \hat{\bar{i}} + \hat{\lambda}_i i_t + \hat{\omega}_{i,t})$

Table 2: Posterior Estimates - Multivariate UC models with Independent Permanent and Transitory Shocks [Real GDP, Consumption, and Investment: 1947:Q1 $\sim$ 2016:Q2]

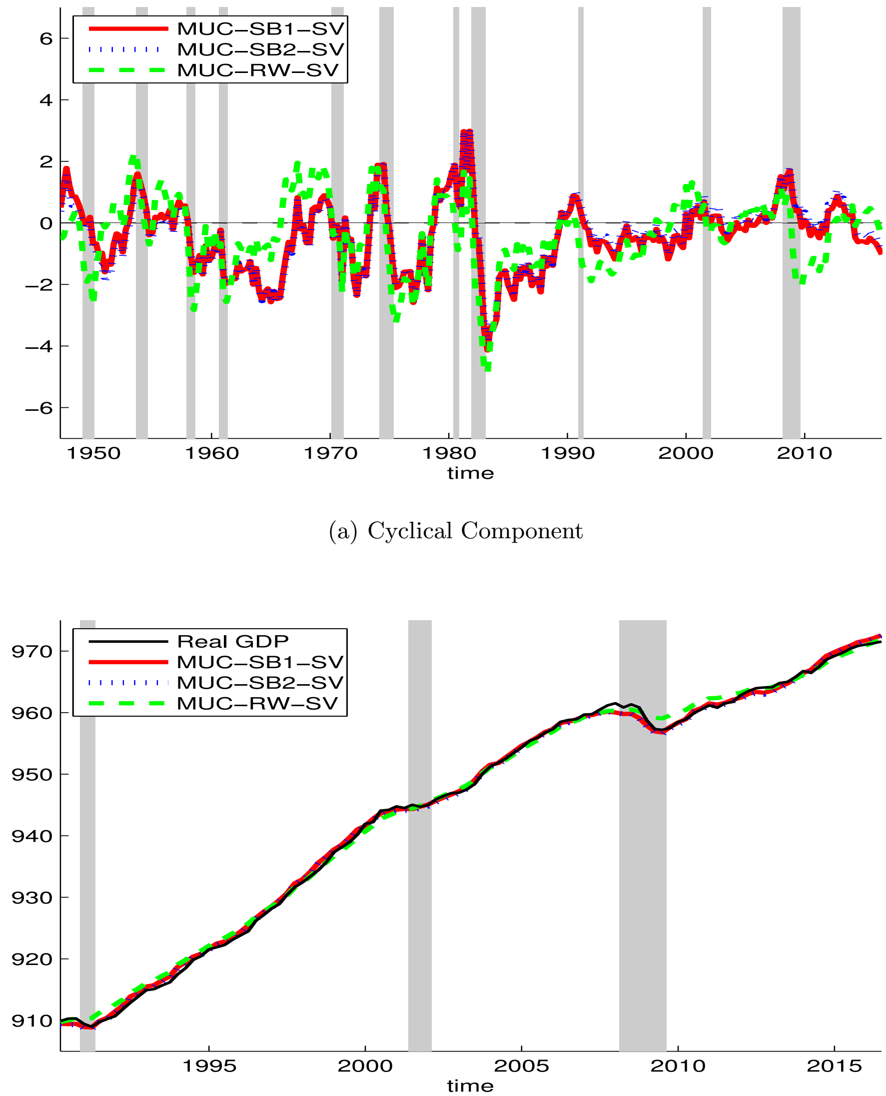
Model		MUC0-SB1-SV				MUC0-SB2-SV				MUC0-RW-SV			
Parameter	Prior	Posterior				Posterior				Posterior			
		Mean	SD	Quantile		Mean	SD	Quantile		Mean	SD	Quantile	
($T = 278$)				5%	95%			5%	95%			5%	95%
		Permanent Component											
μ_1	$N(0.9, 0.5^2)$	0.84	0.04	0.78	0.91	0.93	0.08	0.82	1.05	-	-	-	-
μ_2	$N(0.6, 1^2)$	0.36	0.09	0.24	0.53	0.72	0.10	0.50	0.83	-	-	-	-
μ_3	$N(0.4, 1^2)$	-	-	-	-	0.35	0.09	0.24	0.47	-	-	-	-
μ_0	$N(0.9, 0.5^2)$	-	-	-	-	-	-	-	-	0.58	0.24	0.20	0.98
σ^2_μ	$IG(10, 0.01^2)$	-	-	-	-	-	-	-	-	0.01	0.00	0.00	0.02
$\bar{c}$	$N(-58.31, 10^2)$	-52.17	9.71	-68.21	-36.41	-55.40	9.45	-70.55	-39.35	-57.84	8.15	-70.93	-44.12
λ_c	$N(1.00, 1^2)$	1.00	0.01	0.98	1.02	1.00	0.01	0.99	1.02	1.01	0.01	0.99	1.02
$\bar{i}$	$N(-392.63, 10^2)$	-393.70	9.67	-409.47	-377.73	-393.67	9.55	-409.53	-378.01	-393.83	9.57	-409.53	-378.01
λ_i	$N(1.22, 1^2)$	1.22	0.01	1.20	1.24	1.22	0.01	1.20	1.24	1.22	0.01	1.20	1.24
		Cyclical Components											
$\phi_1 + \phi_2$	$N(0.5, 1^2)$	0.97	0.02	0.94	0.99	0.96	0.02	0.93	0.99	0.94	0.02	0.90	0.98
ϕ_2	$N(0, 1^2)$	-0.15	0.06	-0.25	-0.06	-0.16	0.06	-0.26	-0.06	-0.14	0.06	-0.24	-0.04
$\gamma_{c,1} + \gamma_{c,2}$	$N(0.5, 1^2)$	0.98	0.01	0.96	0.99	0.97	0.02	0.94	0.99	0.89	0.08	0.74	0.98
$\gamma_{c,2}$	$N(0, 1^2)$	-0.31	0.11	-0.49	-0.11	-0.29	0.11	-0.47	-0.10	-0.04	0.13	-0.25	0.17
$\gamma_{i,1} + \gamma_{i,2}$	$N(0.5, 1^2)$	0.96	0.01	0.94	0.98	0.96	0.01	0.94	0.98	0.96	0.01	0.94	0.98
$\gamma_{i,2}$	$N(0, 1^2)$	-0.38	0.05	-0.46	-0.29	-0.38	0.05	-0.47	-0.29	-0.40	0.05	-0.48	-0.31
		Covariance Matrix											
$\sigma^2_{u,0}$	$IW(4, A)$	0.33	0.11	0.20	0.50	0.32	0.10	0.19	0.52	0.26	0.06	0.18	0.37
$\sigma^2_{e,0}$	$IW(4, A)$	0.97	0.27	0.62	1.48	0.91	0.25	0.59	1.37	0.72	0.17	0.49	1.05
$\sigma^2_{\eta_c,0}$	$IW(4, A)$	0.40	0.13	0.24	0.65	0.38	0.13	0.23	0.61	0.28	0.08	0.18	0.42
$\sigma^2_{\eta_i,0}$	$IW(4, A)$	8.99	2.73	5.46	14.05	8.21	2.37	5.08	12.53	6.69	1.73	4.29	9.93
ρ_{e,η_c}	$IW(4, A)$	0.32	0.08	0.18	0.44	0.31	0.08	0.17	0.44	0.22	0.09	0.06	0.36
ρ_{e,η_i}	$IW(4, A)$	0.64	0.04	0.57	0.71	0.64	0.04	0.57	0.71	0.60	0.05	0.52	0.68
ρ_{η_c,η_i}	$IW(4, A)$	0.26	0.09	0.12	0.40	0.27	0.08	0.12	0.40	0.19	0.10	0.02	0.35
		Transition Probability											
π_{11}	$Beta(99, 1)$	0.99	0.00	0.99	0.99	0.99	0.01	0.97	0.99	-	-	-	-
π_{22}	$Beta(99, 1)$	-	-	-	-	0.99	0.01	0.97	0.99	-	-	-	-
Log Marginal Likelihood		-1043.119				-1042.004				-1053.472			

Total Iteration/Burn-in Iteration = 25,000/5,000; In *Algorithm 1* and *Algorithm 2*, the numbers of particles are set at $N = 400$ and $C = 200$, respectively; for MUC-SB-SV, $A = 4\times \begin{bmatrix} 1.5^2 & -0.5\times 1.5^2 & -0.5\times 1.5^2 \\ -0.5\times 1.5^2 & 1.5^2 & 0 \\ -0.5\times 1.5^2 & 0 & 1.5^2 \end{bmatrix}$; for MUC0-SB-SV, the prior distribution of $\sigma^2_{u,0}$ is $IG(4, A_{1,1})$ and the prior distribution of $\{\sigma^2_{e,0}, \sigma^2_{\eta,0}, \rho_{e,\eta}\}$ is $IW(4, A_{2:3,2:3})$. The posterior means of $\{\bar{c}, \lambda_c, \bar{i}, \lambda i\}$ are obtained by OLS estimation. ($y_t = \hat{\bar{c}} + \hat{\lambda}_c c_t + \hat{\omega}_{c,t},\ y_t = \hat{\bar{i}} + \hat{\lambda}_i i_t + \hat{\omega}_{i,t}$)

Table 3: Posterior Estimates - Univariate UC model with a Single Structural Break in the Long-run Growth Rate and SV [Real GDP: 1947:Q1 $\sim$ 2016:Q2]

Model		UC-SB1-SV					UC0-SB1-SV				
Parameter	Prior	Posterior					Posterior				
		Mean	Median	SD	Quantile		Mean	Median	SD	Quantile	
($T = 278$)					5%	95%				5%	95%
μ_1	$N(0.9, 0.5^2)$	0.78	0.78	0.12	0.60	0.97	0.83	0.83	0.05	0.76	0.91
μ_2	$N(0.6, 1^2)$	0.30	0.35	0.28	-0.27	0.66	0.35	0.35	0.11	0.18	0.53
x_0	$N(756.76, 5^2)$	755.86	755.85	1.51	753.41	758.34	757.39	757.33	2.90	752.74	762.25
$\phi_1 + \phi_2$	$N(0.5, 1^2)$	0.52	0.53	0.11	0.34	0.71	0.95	0.95	0.03	0.90	0.99
ϕ_2	$N(0, 1^2)$	-0.11	-0.11	0.09	-0.25	0.04	-0.51	-0.51	0.08	-0.64	-0.37
$\sigma^2_{u,0}$	$IW(4, A)$	4.61	4.21	1.98	2.27	8.21	0.61	0.58	0.17	0.37	0.92
$\sigma^2_{e,0}$	$IW(4, A)$	1.88	1.67	0.93	0.89	3.58	0.73	0.71	0.19	0.46	1.08
ρ	$IW(4, A)$	-0.87	-0.87	0.05	-0.93	-0.78	-	-	-	-	-
π_{11}	$Beta(99, 1)$	0.99	1.00	0.01	0.98	0.99	0.99	0.99	0.00	0.99	1.00
Log Marginal Likelihood		-346.20					-348.94				

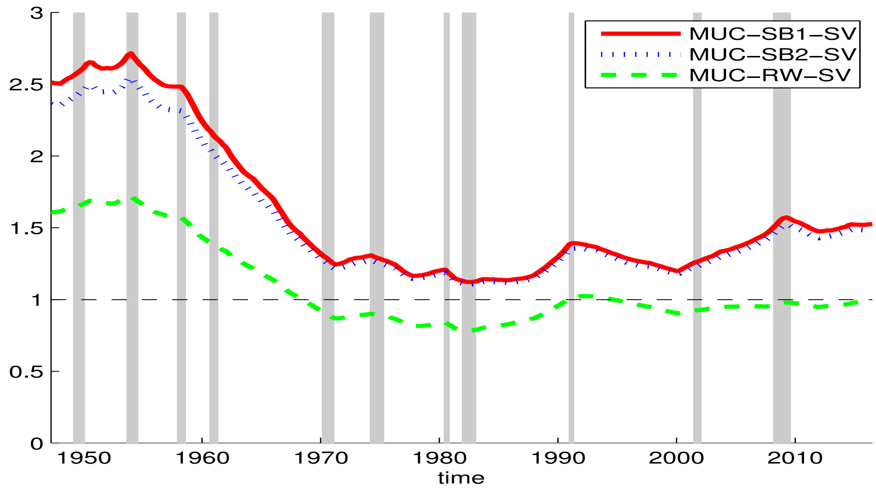
Total Iteration/Burn-in Iteration = 25,000/5,000; In *Algorithm 1* and *Algorithm 2*, the numbers of particles are set at $N = 400$ and $C = 200$, respectively; $A = 4 \times \begin{bmatrix} 1.5^2 & -0.5 \times 1.5^2 \\ -0.5 \times 1.5^2 & 1.5^2 \end{bmatrix}$; for UC0-SB1-SV, the prior distributions of $\sigma^2_{u,0}$ and $\sigma^2_{e,0}$ are $IG(4, A_{1,1})$ and $IG(4, A_{2,2})$ where $A_{i,i}$ is (i, i) element of the matrix A for $i = 1, 2$.



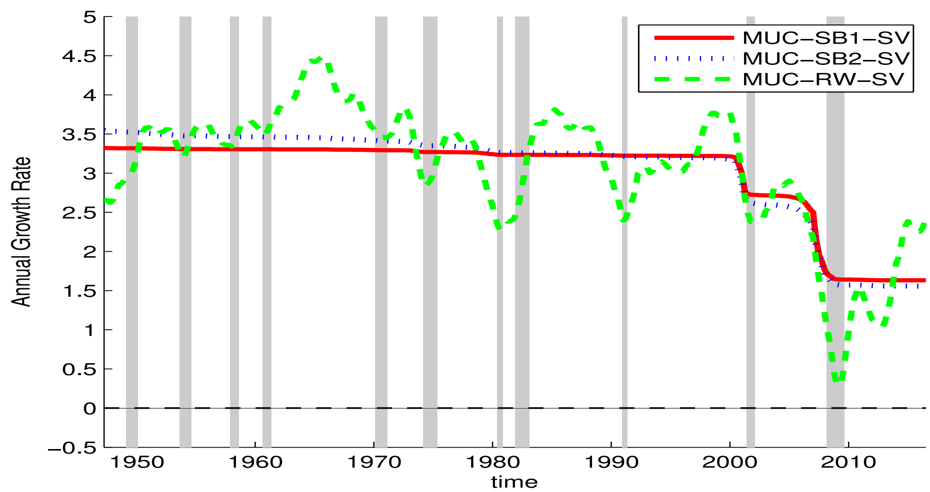
(a) Cyclical Component

(b) Trend Component

Figure 1: Posterior Mean Estimates: MUC-SB1-SV and MUC-SB2-SV stand for the multivariate UC models with one and two structural breaks in the long-run growth rate, respectively. MUC-RW-SV is the UC model in which the long-run growth rate follows a random walk process. The shaded areas present NBER recessions.

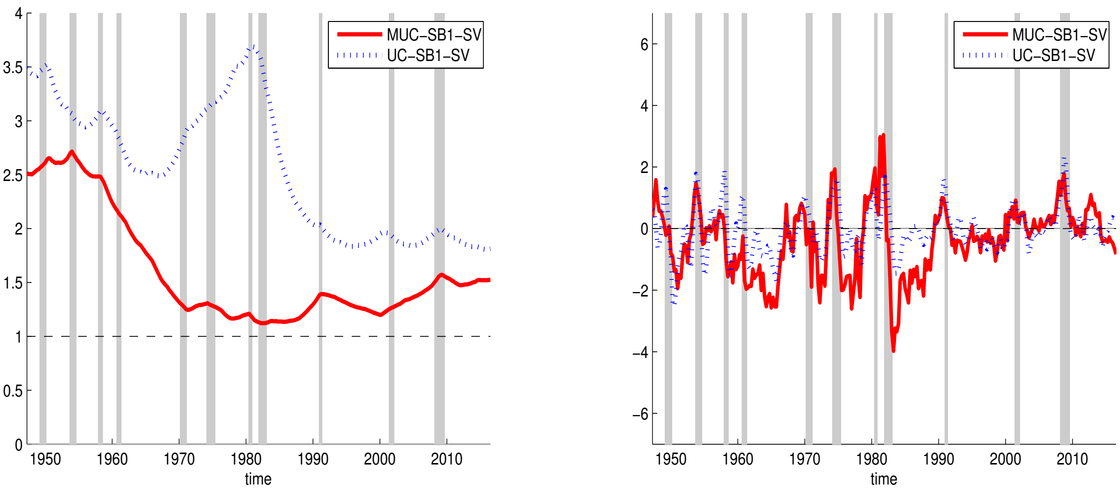


(a) Time-varying Noise-to-Signal Ratio



(b) Annual Long-run Growth Rate

Figure 2: Posterior Mean Estimates: MUC-SB1-SV and MUC-SB2-SV stand for the multivariate UC models with one and two structural breaks in the long-run growth rate, respectively. MUC-RW-SV is the UC model in which the long-run growth rate follows a random walk process. The shaded areas present NBER recessions.



(a) Noise-to-Signal Ratio

(b) Cycle Component

Figure 3: Posterior Mean Estimates: MUC-SB1-SV stands for the multivariate UC model with one structural break in the long-run growth rate. UC-SB1-SV stands for the univariate UC model with one structural break in the long-run growth rate.