A Bias-Corrected Method of Moments Approach to Estimation of Short-T Autoregressive Panels*

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Abstract

This paper proposes to use bias-corrected moment conditions to estimate autoregressive panels with short time dimension, in contrast to the IV/GMM approach which searchers for instruments that are uncorrelated with the errors. Our proposed approach is also to be contrasted with the use of bias corrections after fixed effects estimation. The advantage of the proposed approach lies in the fact that, by construction, the instruments have maximum correlation with the target variables. Two simple Bias-corrected Methods of Moments (BMM) estimators are proposed, depending on whether the underlying autoregressions have started from a distant past or from a finite period before the first observations. The unit root case is also considered. Monte Carlo experiments document their satisfactory small sample performance in comparison with a set of GMM estimators.

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1 Introduction

The IV/GMM literature utilizes instruments that are uncorrelated with the errors but are (ideally) highly correlated with the target variables (the included regressors). This approach have proved to be useful in different model setups, including the dynamic panel data models with the time dimension, denoted as T, short and the cross-section dimension, denoted as n, large ($n \to \infty$). Such as 'short-T' or micro panels constitute an important part of the applied microeconomic research. To this end, a number of well-known IV/GMM estimation methods have been proposed for short-T panels, including Anderson and Hsiao (1981 and 1982), Arellano and Bond (1991), Ahn and Schmidt (1995), Arellano and Bover (1995), Blundell and Bond (1998), and Hayakawa (2012), among others. Unlike the likelihood-based methods in the literature, the IV/GMM methods apply to autoregressive (AR) panels as well as AR panels augmented with strictly or weakly exogenous regressors (or ARX panels for short). However, the IV/GMM methods have also their drawbacks. Specifically, they could suffer from the well-known weak instrument problem, which could lead to unreliable performance in small samples.

This paper adds to the IV/GMM literature by using an idea of instrumenting the target variables by themselves, and correcting for the correlation of the errors and instruments at the population moment level. This idea differs from the bias-corrected estimation methods in the literature, which correct for the bias of the estimator, whereas we correct the bias of the moment conditions at the level of population moments. The usefulness of this idea is illustrated in the case of autoregressive panels with short time dimension, but the idea can be used in a variety of other settings, where model is sufficiently specified so that the correlation between the instruments and errors can be derived. The unit root case is also considered.

The beauty of such bias-corrected moment conditions (BMs) is that they circumvent the weak instrument problem, since, by construction, the correlation between the instruments and target variables is 100%. Two simple Bias-corrected Methods of Moments (BMM) estimators are proposed, depending on whether the underlying autoregressions have started from a distant past or from a finite period before the first observations. Monte Carlo experiments document their satisfactory small sample performance in comparison with a set of GMM estimators.

The remainder of this paper is organized as follows. Section 2 introduces the idea of BMs and develops two BMM estimators for autoregressive panel data models. Section 3 presents Monte

Carlo (MC) evidence and the last section concludes and outlines avenues for future research. The appendix presents proofs and additional results.

Notations: A generic positive finite (small and large) constants that do not depend on the sample size are denoted by c and K. They can take different values at different instances. All vectors are column vectors denoted by bold lowercase letters. Matrices are denoted by bold uppercase letters. The notation $\to_p$ denotes convergence in probability, and $\to_d$ denotes convergence in distribution.

2 Autoregressive Panel Data Models

Consider the following panel AR(1) model with individual effects:

$$y_{it} = \alpha_i + \phi y_{i,t-1} + u_{it}, \tag{1}$$

for $t = -m+1, -m+2..., 1, 2, ..., T$ and $i = 1, 2, ..., n$, with the starting values given by $y_{i,-m}$ for $m \geq 0$; where $-1 < \phi \leq 1$. It is assumed that a sample with T time periods and n cross-section units is available for estimation and inference throughout the paper.

ASSUMPTION 1 *(Available observations) Available observations are y_{it}, for $i = 1, 2, ..., n$, and $t = 1, 2, ..., T$; $T \geq 3$.*

Taking the first differences of (1), we obtain

$$\Delta y_{it} = \phi \Delta y_{i,t-1} + \Delta u_{it}, \tag{2}$$

for $t = 2, 3, ..., T$ and $i = 1, 2, ..., n,$ with $\Delta y_{i,1}$ given by

$$\Delta y_{i1} = \sum_{\ell=0}^{m-1} \phi^{\ell} \Delta u_{i,t-\ell} + \phi^{m} \Delta y_{i,-m+1}. \tag{3}$$

The contribution of the initial first differences, $\Delta y_{i,-m+1}$, to Δy_{i1} is $\phi^m \Delta y_{i,-m+1}$, and consequently the initialization of the process will be unimportant for $|\phi| < 1$ and m large. We consider both cases when m is large as well as m is small. In the former case we assume $|\phi| < 1$, and we cover the unit root case when m is finite.[1] Unless specified otherwise, m is treated as a fixed and finite integer. This paper aims for a minimal set of assumptions on the starting values and individual effects, since in practice such assumptions are difficult, if not impossible, to ascertain.

The assumption of the same starting point (m being common across units) as opposed to unit-specific starting time periods (m_i being different across i) is innocuous. This is because we can always set $m = \inf_i \{m_i\}$ and treat $y_{i,-m}$, for $i = 1, 2, ..., n$ as 'starting' values. Similarly, we could set $m = 0$ and treat y_{i0}, for $i = 1, 2, ..., n$, as 'starting' values without any consequences for the analysis below. The reason for explicitly allowing for the process to be initialized with the starting values $y_{i,-m-1}$, for $i = 1, 2, ..., n$, is for illustrative purposes with the aim of discussing the case with $|\phi| < 1$ and $m \to \infty$.

We impose the following assumptions on the errors u_{it}, and on the starting values $y_{i,-m}$, which will be sufficient for the consistency of the BMM estimator of ϕ. These assumptions will be refined further for inference.

ASSUMPTION 2 *(Errors) For each $i = 1, 2, ..., n$, the process $\{u_{it},\ t = -m+1, -m+2, ..., 1, 2, ..., T\}$ is distributed with mean 0, $E\left(u_{it}^2\right) = \sigma_i^2$, and there exist positive constants c and K such that $0 < c < \sigma_i^2 < K < \infty$ and $E\left(u_{it}^4\right) < K < \infty$ and for all i. In addition, u_{it} is independently distributed over t, and there exist constants $0 \leq \delta_\rho < 1$ and $0 \leq \delta_\varkappa < 1$ such that the following conditions hold:*

$$\sup_{i,t} \sum_{j=1}^{n} \left| E\left(u_{it} u_{jt}\right) \right| = O\left(n^{\delta_\rho}\right), \tag{4}$$

and

$$\sup_{i,t} \sum_{j=1}^{n} \left| E\left(\tilde{u}_{it}^2 \tilde{u}_{jt}^2\right) \right| = O\left(n^{\delta_\varkappa}\right), \tag{5}$$

where $\tilde{u}_{it}^2 = u_{it}^2 - \sigma_i^2$.

ASSUMPTION 3 *(Initialization and individual effects) Let $\eta_{i,m} = \alpha_i - (1-\phi)\, y_{i,-m}$. It is assumed that $E\left(\eta_{i,m}^4\right) < K < \infty$, for $i = 1, 2, ..., n$, and any $m \geq 0$, including $m \to \infty$. In addition, it is assumed either that:*

(a) $-1 < \phi \leq 1$, the process has started from a finite period in the past, namely m is a fixed

[1] Our approach can be developed also for a finite and fixed m and $|\phi| > 1$, but we do not consider explosive cases in this paper.

integer in the range $0 \leq m < K < \infty$*, and the following conditions hold:*

$$E\left(\Delta u_{it}\eta_{i,-m}\right) = 0, \text{ for } i = 1,2,...,n \text{ and } t = 3,4,...,T, \tag{6}$$

and there exist constants $0 \leq \delta_{\varphi} < 1$ *and* $0 \leq \delta_{\eta} < 1$ *such that*

$$\sup_{i,t} \sum_{j=1}^{n} \left| E\left(u_{it}u_{jt}\eta_{i,-m}\eta_{j,-m}\right)\right| = O\left(n^{\delta_{\varphi}}\right), \tag{7}$$

and

$$\sup_{i} \sum_{j=1}^{n} E\left(\tilde{\eta}^2_{i,-m}\tilde{\eta}^2_{j,-m}\right) = O\left(n^{\delta_{\eta}}\right), \tag{8}$$

where $\tilde{\eta}^2_{i,-m} = \eta^2_{i,-m} - \varsigma^2_{i,-m}$ *and* $\varsigma^2_{i,-m} = E\left(\eta^2_{i,m}\right)$.

(b) $|\phi| < 1$*, and* $\{y_{it}, i = 1,2,...,n\}$ *are initialized in a distant past, namely* $m \to \infty$.

Remark 1 *Assumption 2 allows errors to be heteroskedastic across i, but* unconditional *heteroskedasticity errors across t is ruled out.*

Remark 2 *Assumption 2 allows* u_{it} *to be weakly cross-sectionally correlated such that conditions (a) and (b) of Assumption 2 hold. These conditions rule out a presence of strong unobserved common factors in errors (strong in a sense that the cross-section arithmetic average of Euclidean norm of loadings is bounded away from zero as* $n \to \infty$*), but it is allowed for more general processes than commonly used spatial processes in the literature. For example, condition (a) allows for the largest eigenvalues of the* $n \times n$ *correlation matrices of error vectors* $\mathbf{u}_t = (u_{1t}, u_{2t}, ..., u_{nt})'$*, denoted as* $\mathbf{R}_t = E\left(\mathbf{u}_t\mathbf{u}_t'\right)$*, to diverge as* $n \to \infty$ *but at a rate slower than n, whereas commonly used spatial processes in the literature typically assume that these eigenvalues are all bounded. For further discussion, see Section 2 of Pesaran and Tosetti (2011).*

Remark 3 *Assumption 2 is sufficient for*

$$n^{-1}\sum_{i=1}^{n} u^2_{it} \to_p \bar{\sigma}^2, \tag{9}$$

as $n \to \infty$*, for any point in time* $t = -m+1, -m+2, ..., 1, 2, ..., T$*, as well as*

$$n^{-1}\sum_{i=1}^{n} u_{it}u_{it'} \to_p 0, \tag{10}$$

as $n \to \infty$, for any $t \neq t'$, $t, t' = -m+1, -m+2, ..., 1, 2, ..., T$, where $\bar{\sigma}^2 = \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} \sigma_i^2$. This is established in Lemma A.1 in Appendix. Conditions (9) and (10) are required for the consistency of the BMM estimator. Assumption 2 on errors could be generalized so long as the conditions (9) and (10) remain valid.

It is important to compare the assumptions on the starting values and individual effects with the literature. First, we model the first available observations on first differences, Δy_{i2} for $i = 1, 2, ..., n$, as opposed to a framework where it is conditioned on Δy_{i2} (or y_{i1}). When both T and m are finite and $n \to \infty$ then it is necessary to recognize that $P(y_{i1})$ and $P(\Delta y_{i2})$ both depend on individual effects and the parameter of interest ϕ for consistent estimation.

Assumption 3 part (a), which is for the case of a finite m, requires boundedness of the fourth moments of the starting values and conditions (6)-(8) to hold. The last two conditions, namely condition (7) and (6) are transparent conditions on the primitives of the model and they ensure that the cross-section averages $n^{-1} \sum_{i=1}^{n} \Delta y_{it}^2$ converge in probability to a real constant as $n \to \infty$, as opposed to a random variable. This is established in Lemma A.3 in Appendix. These two conditions could be generalised so long as Δy_{it}^2 remain weakly cross-sectionally dependent.

Condition (6) is a key condition in our framework, and it implies (together with Assumption 2)

$$E(\Delta y_{i,-m} \Delta u_{it}) = 0, \text{ for } i = 1, 2, ..., n \text{ and } t = 3, 4, ..., T, \tag{11}$$

which is the same condition that is required in the transformed MLE (TMLE) approach proposed by Hsiao et al. (2002) and extended by Hayakawa and Pesaran (2015). The difference between our setup on the individual effects and initial values and the setup of TMLE is that the latter imposes additional restrictions. In particular, TMLE requires $E(\Delta y_{i,-m})$ and $Var(\Delta y_{i,-m})$ to be invariant across i, whereas we do not impose such conditions in Assumption 3 part (a).

The GMM literature also require stronger conditions on the initial values and the individual effects compared with Assumption 3 part (a). The GMM conditions considered by Arellano and Bond (1991) assume $E(y_{is} \Delta u_{it}) = 0$ for $i = 1, 2, ..., n$, $s = 1, 2, ..., t-2$ and $t = 3, 4, ..., T$. To see why such conditions are stronger, it is illustrative to consider a special case where $|\phi| < 1$, $0 \leq m < K < \infty$, and the initial values are given by

$$y_{i,-m} = \mu_i + \upsilon_i, \tag{12}$$

where $\mu_i = \alpha_i / (1-\phi)$, $E(\upsilon_i) = 0$, and $E(\upsilon_i \Delta u_{it}) = 0$, for $i = 1, 2, ..., n$ and $t = 3, 4, ..., T$ (a setup which is in line with Assumptions G3 and F1 of Binder et al., 2005). Then, $E(y_{it}) = \mu_i$ for all $t = -m, -m+1, ..., 1, 2, ..., T$, and the GMM conditions $E(y_{is} \Delta u_{it}) = 0$ for $s < t-1$ preclude the possibility of the means μ_i to be correlated with Δu_{it}, for $t = 3, 4, ..., T$. Assumption 3 part (a) above allows for such a possibility since, in this special case, we have,

$$\begin{aligned}
\Delta y_{i,-m+1} &= y_{i,-m+1} - y_{i,-m} \\
&= \alpha_i + (1-\phi) y_{i,-m} + u_{i,-m}, \\
&= \alpha_i + (1-\phi) [\alpha_i / (1-\phi) + \upsilon_i] + u_{i,-m} \\
&= (1-\phi) \upsilon_i + u_{i,-m}
\end{aligned}$$

and

$$E(\Delta y_{i,-m+1} \Delta u_{it}) = E\{[(1-\phi) \upsilon_i + u_{i,-m}] \Delta u_{it}\} = 0,$$

for $i = 1, 2, ..., n$ and $t = 3, 4, ..., T$, and therefore Assumption 3 part (a) is not violated.

The system GMM approach considered by Arellano and Bover (1995) and Blundell and Bond (1998), requires further conditions on starting values and individual effects to hold, in addition to the conditions assumed by Arellano and Bond (1991). In particular, the system GMM approach also requires that $E[\Delta y_{i,t-1} (\alpha_i + u_{it})] = 0$, for $i = 1, 2, ..., n$ and $t = 3, 4, ..., T$. In view of (1), these additional conditions imply

$$E\left\{\left[\varepsilon_{i,t-1} + \phi^{t+m-2} \alpha_i - \phi^{t+m-2} (1-\phi) y_{i,-m}\right] (\alpha_i + u_{it})\right\} = 0, \tag{13}$$

for $i = 1, 2, ..., n$ and $t = 3, 4, ..., T$, where ε_{it} is given by

$$\varepsilon_{it} = u_{it} + \sum_{\ell=1}^{t+m-1} \phi^{\ell-1} (\phi - 1) u_{i,t-\ell}. \tag{14}$$

In a special case where α_i and $y_{i,-m-1}$ are independently distributed of errors u_{it} for $i = 1, 2, ..., n$ and $t = -m+1, -m+2, ..., 1, 2, ..., T$, m is fixed and finite, and $\phi \neq 0$, conditions in (13) reduce to

$$E\left(\alpha_i^2\right) = (1-\phi) E(y_{i,-m-1} \alpha_i), \tag{15}$$

for $i = 1, 2, ..., n$ and $t = 3, 4, ..., T$, which is rather strong in our view. For example, a possibility of zero starting values ($y_{i,-m-1} = 0$), nonzero individual effects ($\alpha_i \neq 0$), and $0 < |\phi| < 1$ is clearly ruled out by (15). These issues are discussed further in the Monte Carlo section.

Remark 4 *If it is a priori known that additional conditions on individual effects and starting values hold, in addition to Assumption 3 part (a) when $m < \infty$, then such conditions could be utilized for a more efficient estimation. The focus of this paper is on a minimal assumptions on the starting values and individual effects. In this respect, Assumption 3 part (a) is more general than conditions assumed in the existing GMM and TMLE literature.*

Our approach is closely related to the IV or GMM procedures used in the literature, with this important difference that we also use $\Delta y_{i,t-1}$ as an "instrument", which is regarded as being "invalid" in the GMM literature since it is correlated with Δu_{it}. By using $\Delta y_{i,t-1}$ as an instrument we then adjust the resultant moment condition for the non-zero correlation between $\Delta y_{i,t-1}$ and Δu_{it}. The advantage of using $\Delta y_{i,t-1}$ as an instrument lies in the fact that by construction it has a maximum correlation with the target variable (itself), so long as we are able to correct for the bias that arises due to $Cov\left(\Delta y_{i,t-1}, \Delta u_{it}\right) \neq 0$. To summarize, GMM searchers for instruments that are uncorrelated with the errors but are highly correlated with the target variables, the included regressors. In contrast, our approach uses the target variables as instruments but corrects for the non-zero correlations between the errors and the instruments. Both approaches employ method of moments, but differ in the way the moments are derived.

Using $\Delta y_{i,t-1}$ as an instrument, we note that

$$E\left[\Delta y_{it}\Delta y_{i,t-1} - \phi\left(\Delta y_{i,t-1}\right)^2\right] = E\left(\Delta u_{it}\Delta y_{i,t-1}\right), \tag{16}$$

for $i = 1, 2, ..., n$ and $t = 3, 4, ..., T$. To derive $E\left(\Delta u_{it}\Delta y_{i,t-1}\right)$, we use (1) and note that errors u_{it} are assumed to be serially uncorrelated by Assumption 2 as well as condition (11) hold under Assumption 3 part (a). It is then readily seen that

$$E\left(\Delta u_{it}\Delta y_{i,t-1}\right) = -E\left(u_{i,t-1}^2\right), \tag{17}$$

for $i=1,2,...,n$ and $t=3,4,...,T$. Under the assumption of unconditionally homoskedastic error variances across t, we have

$$E\left(u_{i,t-1}^2\right)=\sigma_i^2, \tag{18}$$

as well as

$$\frac{1}{2}E\left(\Delta u_{it}^2\right)=\sigma_i^2. \tag{19}$$

Substituting $\Delta u_{it}=\Delta y_{it}-\phi\Delta y_{i,t-1}$ in (19), we obtain

$$\sigma_i^2=\frac{1}{2}E\left[\left(\Delta y_{it}-\phi\Delta y_{i,t-1}\right)^2\right]. \tag{20}$$

Using results (18) and (20) on the right side of (17), we have

$$E\left(\Delta u_{it}\Delta y_{i,t-1}\right)=-\frac{1}{2}E\left[\left(\Delta y_{it}-\phi\Delta y_{i,t-1}\right)^2\right], \tag{21}$$

for $i=1,2,...,n$ and $t=3,4,...,T$, which, after substituting this expression to (16) yields the following bias-corrected moment (BM) condition for the estimation of ϕ,

$$E\left[\Delta y_{it}\Delta y_{i,t-1}-\phi\Delta y_{i,t-1}^2\right]=-\frac{1}{2}E\left[\left(\Delta y_{it}-\phi\Delta y_{i,t-1}\right)^2\right], \tag{22}$$

for $i=1,2,...,n$ and $t=3,4,...,T$.

The BMM estimator based on the BM condition (22) solves the sample counterpart of (22),

$$\frac{1}{n\left(T-2\right)}\sum_{t=3}^{T}\sum_{i=1}^{n}\left(\Delta y_{it}\Delta y_{i,t-1}-\hat{\phi}_{nT}\Delta y_{i,t-1}^2\right)=\frac{-1}{2n\left(T-2\right)}\sum_{t=3}^{T}\sum_{i=1}^{n}\left(\Delta y_{it}-\hat{\phi}_{nT}\Delta y_{i,t-1}\right)^2. \tag{23}$$

This is a quadratic equation in $\hat{\phi}_{nT}$ and consequently there are two solutions in general. To derive properties of the two solutions, we use ϕ_0 to denote the true value of ϕ, and we substitute (2) for Δy_{it} in (23), to obtain

$$\begin{aligned}&\frac{1}{n\left(T-2\right)}\sum_{t=3}^{T}\sum_{i=1}^{n}\left[\left(\phi_0\Delta y_{i,t-1}+\Delta u_{it}\right)\Delta y_{i,t-1}-\hat{\phi}_{nT}\Delta y_{i,t-1}^2\right]\\ =\ &\frac{-1}{2n\left(T-2\right)}\sum_{t=3}^{T}\sum_{i=1}^{n}\left(\phi_0\Delta y_{i,t-1}+\Delta u_{it}-\hat{\phi}_{nT}\Delta y_{i,t-1}\right)^2,\end{aligned}$$

which, after re-arranging, can be written as

$$\left(\hat{\phi}_{nT}-\phi_0\right)\left[\left(\hat{\phi}_{nT}-\phi_0\right)a_{nT}-b_{nT}\right]=\xi_{nT}, \tag{24}$$

where,

$$a_{nT}=\frac{1}{2n\left(T-2\right)}\sum_{t=3}^{T}\sum_{i=1}^{n}\Delta y_{it}^2, \tag{25}$$

$$b_{nT}=2a_{nT}-d_{nT},\ d_{nT}=\frac{-1}{n\left(T-2\right)}\sum_{t=3}^{T}\sum_{i=1}^{n}\Delta y_{i,t-1}\Delta u_{it}, \tag{26}$$

and

$$\xi_{nT}=\frac{-1}{n\left(T-2\right)}\sum_{t=3}^{T}\sum_{i=1}^{n}\left(\Delta u_{it}\Delta y_{i,t-1}+\frac{\Delta u_{it}^2}{2}\right). \tag{27}$$

We distinguish between the cases where m is fixed (Subsection 2.1) and $m\rightarrow\infty$ (Subsection 2.2). In both cases, we assume T is fixed and finite. To simplify notations we shall drop the subscript T below.

2.1 Initialization from a finite past

In this subsection we assume m and T are fixed and $n\rightarrow\infty$. Using Lemma A.3 in Appendix, we have, for a fixed T,

$$a_n=\bar{\omega}_{n,11}/2+O_p\left(n^{(\delta_\rho-1)/2}\right)+O_p\left(n^{(\delta_\varkappa-1)/2}\right)+O_p\left(\phi^{2m}n^{(1-\delta_\varphi)/2}\right), \tag{28}$$

where

$$\bar{\omega}_{nT,11}=\frac{\bar{\sigma}_n^2}{T-2}\left[1+\left(\phi-1\right)^2\sum_{\ell=1}^{t+m-2}\phi^{2\ell-2}\right]+\phi^{2m}\nu_n, \tag{29}$$

$\bar{\sigma}_n^2=n^{-1}\sum_{i=1}^n\sigma_i^2$, $\nu_n=\bar{\varsigma}_{n,-m}^2\left(T-2\right)^{-1}\sum_{t=3}^T\phi^{2t-4}$, and $\bar{\varsigma}_{n,-m}^2=n^{-1}\sum_{i=1}^n\varsigma_{i,-m}^2$. Hence,

$$a_n\rightarrow_p a>0, \tag{30}$$

where

$$a=\frac{\bar{\sigma}^2}{T-2}\left[1+\left(\phi-1\right)^2\sum_{\ell=1}^{t+m-2}\phi^{2\ell-2}\right]+\phi^{2m}\nu, \tag{31}$$

in which $\bar{\sigma}_n^2 \to_p \bar{\sigma}^2 > 0$,

$$\nu_n \to \nu = \frac{\bar{\varsigma}_{-m}^2}{T-2} \sum_{t=3}^{T} \phi^{2t-4} \geq 0,$$

and $\bar{\varsigma}_{n,-m}^2 \to \bar{\varsigma}_{-m}^2 \geq 0$. Using (28) and results (A.5)-(A.6) of Lemma A.2 in Appendix, we have under Assumptions 1, 2 and 3 part (a), and for a fixed T,

$$b_n = \bar{\omega}_{n,11} - \bar{\sigma}_n^2 + O_p\left(n^{(\delta_\rho - 1)/2}\right) + O_p\left(n^{(\delta_\varkappa - 1)/2}\right) + O_p\left(\phi^m n^{(1-\delta_\varphi)/2}\right), \tag{32}$$

and

$$\xi_n = O_p\left(n^{(\delta_\rho - 1)/2}\right) + O_p\left(n^{(\delta_\varkappa - 1)/2}\right) + O_p\left(\phi^m n^{(1-\delta_\varphi)/2}\right). \tag{33}$$

Hence,

$$b_n \to_p b = 2a - \bar{\sigma}^2. \tag{34}$$

Regardless of the value of $\bar{\varsigma}_{-m}^2$, we have

$$b > 0, \text{ for } -1 < \phi < 1. \tag{35}$$

In addition,

$$b > 0, \text{ for } \phi = 1, \text{ if and only if } \bar{\varsigma}_{-m}^2 > \bar{\sigma}^2. \tag{36}$$

The probability limits of the roots of (24) now follow from (30) and (33)-(36), and are summarized in the next proposition.

Proposition 1 *Suppose y_{it}, for $i = 1, 2, ..., n$, and $t = -m+1, -m+2, ..., 1, 2, ..., T$, are given by (1) with starting values $y_{i,-m}$, and let Assumptions 1, 2, and 3 part (a) hold. In addition, suppose $\bar{\varsigma}_{-m}^2 > \bar{\sigma}^2$ if $\phi_0 = 1$. Consider the two roots of (38), which we denote as $\hat{\phi}_{n,a}$ and $\hat{\phi}_{n,b}$. Let T and m be fixed and $n \to \infty$. Then for $T \geq 3$, we have*

$$\hat{\phi}_{n,a} \to_p \phi_0, \text{ and } \hat{\phi}_{n,b} \to_p \phi_0 + b/a, \tag{37}$$

where $\bar{\sigma}_n^2 \to_p \bar{\sigma}^2$, $a_n \to_p a > 0$ and $b_n \to_p b > 0$.

The smaller solution of (24) is therefore consistent. The smaller solution of (24) is given by

$$\hat{\phi}_{n,a} = \frac{\hat{\omega}_{n,01} + \hat{\omega}_{n,11} - \sqrt{\hat{D}_n}}{\hat{\omega}_{n,11}}, \tag{38}$$

where

$$\hat{D}_{nT} = \left(\hat{\omega}_{n,01} + \hat{\omega}_{n,11}\right)^2 - \hat{\omega}_{n,11}\left(\hat{\omega}_{n,00} + 2\hat{\omega}_{n,01}\right) = \hat{\omega}_{n,01}^2 + \hat{\omega}_{n,11}\left(\hat{\omega}_{n,11} - \hat{\omega}_{n,00}\right), \tag{39}$$

and

$$\hat{\omega}_{n,ps} = \frac{1}{n\left(T-2\right)} \sum_{t=3}^{T} \sum_{i=1}^{n} \Delta y_{i,t-p} \Delta y_{i,t-s}, \text{ for } p, s = 0, 1. \tag{40}$$

In practice, it is not guaranteed that $\hat{D}_n$ is positive, and therefore we propose the following estimator, which is always real

$$\hat{\phi}_n = \frac{\hat{\omega}_{n,01} + \hat{\omega}_{n,11} - r_n}{\hat{\omega}_{n,11}}, \tag{41}$$

where

$$r_n = \begin{cases} \sqrt{\hat{D}_n}, & \text{if } \hat{D}_n \geq 0, \\ 0, & \text{if } \hat{D}_n < 0, \end{cases} \tag{42}$$

and $\hat{D}_n$ is given by (39). The BMM estimator $\hat{\phi}_n$ can be equivalently written as

$$\hat{\phi}_n = I\left(\hat{D}_n \geq 0\right) \hat{\phi}_{n,a} + I\left(\hat{D}_n < 0\right) \frac{\hat{\omega}_{n,01} + \hat{\omega}_{n,11}}{\hat{\omega}_{n,11}}, \tag{43}$$

where $I(A)$ is an indicator function for an event A. For future reference, we denote $p_n = P\left(\hat{D}_n \geq 0\right) = E\left[I\left(\hat{D}_n \geq 0\right)\right]$.

The consistency of $\hat{\phi}_n$ is established in the following theorem.

Theorem 1 *Suppose y_{it}, for $i = 1, 2, ..., n$, and $t = -m+1, -m+2, ..., 1, 2, ..., T$, are given by (1) with starting values $y_{i,-m}$, and let Assumptions 1, 2, and 3 part (a) hold. In addition, suppose $\bar{\varsigma}_{-m}^2 > \bar{\sigma}^2$ if $\phi_0 = 1$. Consider $\hat{\phi}_n$ defined in (41), and let T be fixed and $n \to \infty$. Then for $T \geq 3$, we have $\hat{\phi}_n \to_p \phi_0$.*

To derive asymptotic distribution of $\hat{\phi}_n$, we strengthen Assumptions 2 and 3 part (a) to ensure cross-sectional independence.

ASSUMPTION 4 *(Cross-sectional independence) Errors u_{it} are independently distributed of $u_{jt'}$*

for all $i \neq j$, $i, j = 1, 2, ..., n$, and all $t, t' = -m+1, -m+2, ..., 1, 2, ..., T$. In addition, the starting values $y_{i,-m}$ are independently distributed of $y_{j,-m}$ and u_{jt} for all $i \neq j$, $i, j = 1, 2, ..., n$, and all $t = -m+1, -m+2, ..., 1, 2, ..., T$.

Remark 5 *Assumption 4 is sufficient for*

$$\vartheta_i = \frac{1}{(T-2)} \sum_{t=3}^{T} \left(\Delta u_{it} \Delta y_{i,t-1} + \frac{\Delta u_{it}^2}{2} \right), \tag{44}$$

to be independently distributed across i, which is convenient for the derivation of the asymptotic distribution of $\sqrt{n}\xi_{nT}$. Assumption 4 could be relaxed by assuming (i) $\{\vartheta_{iT}, \mathcal{I}_{i-1,T}\}$ is a martingale difference sequence, where $\mathcal{I}_{i,T} = \{\Delta y_{j,-m}; 1, \leq j \leq i\} \cup \{u_{jt}; -m \leq t \leq T, 1, \leq j \leq i\}$, and the conditions of central limit theorem for martingale difference arrays hold, namely (ii) $n^{-1}\sum_{i=1}^{n} \vartheta_i^2 / E\left(\vartheta_i^2\right) \rightarrow_p 1$, and (iii) $\max_{1 \leq i \leq n} |\vartheta_i| / \sqrt{n} \rightarrow_p 0$.[2]

Assumption 4 ensures $\delta_\rho = \delta_\varkappa = \delta_\varphi = 0$, and Lemma A.7 shows the rate of convergence of the BMM estimator $\hat{\phi}_n$ in this case is $\sqrt{n}$. The asymptotic distribution of $\hat{\phi}_n$ can be obtained from (24). Multiplying both sides of (24) by $\sqrt{n}$, we have

$$\sqrt{n}\left(\hat{\phi}_n - \phi_0\right)^2 a_{nT} - \sqrt{n}\left(\hat{\phi}_n - \phi_0\right) b_{nT} = \sqrt{n}\xi_n, \tag{45}$$

where the term $\sqrt{n}\xi_n$ can be written as

$$\sqrt{n}\xi_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \vartheta_i,$$

in which ϑ_i is given by (44). Since $\left(\hat{\phi} - \phi_0\right) = O_p\left(n^{-1/2}\right)$ by Lemma A.7, and noting that $a_n = O_p(1)$, we have

$$\sqrt{n}\left(\hat{\phi}_n - \phi_0\right)^2 a_n = O_p\left(n^{-1/2}\right). \tag{46}$$

Using, in addition, (35)-(36), it follows that the asymptotic distribution of $\sqrt{n}\left(\hat{\phi}_n - \phi_0\right)$ is given by $\sqrt{n}\xi_n / b$, where $b_n \rightarrow_p b > 0$. The term ϑ_{it} is independently distributed across i (under Assumption 4), $E(\vartheta_{it}) = \frac{1}{(T-2)}\sum_{t=3}^{T}\left[E\left(\Delta u_{it}\Delta y_{i,t-1}\right) + E\left(\Delta u_{it}^2/2\right)\right] = -\sigma_i^2 + \sigma_i^2 = 0$ (see (17)-(19)), and

[2] We note that $E\left(\vartheta_i^2\right) < K$ already follows from Assumptions 2 and 3 part (a).

$E\left(\vartheta_{it}^2\right) < K < \infty$. Hence,

$$\sqrt{n}\left(\hat{\phi}_n - \phi_0\right) \rightarrow_d N\left(0, \theta\right),$$

as $n \rightarrow \infty$, for $-1 < \phi \leq 1$, where

$$\theta = \frac{1}{b^2\left(T-2\right)^2} \sum_{t=3}^{T} \sum_{t'=3}^{T} \lim_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{i=1}^{n} E\left(\vartheta_{it}\vartheta_{it'}\right) \right). \quad (47)$$

These findings are summarized in the following theorem.

Theorem 2 *Suppose y_{it}, for $i = 1, 2, ..., n$, and $t = -m+1, -m+2, ..., 1, 2, ..., T$, are given by (1) with starting values $y_{i,-m}$, and let Assumptions 1, 2, 3 part (a), and 4 hold. In addition, suppose $\bar{\varsigma}_{-m}^2 > \bar{\sigma}^2$ if $\phi_0 = 1$. Consider $\hat{\phi}_n$ defined in (41), and let T be fixed and $n \rightarrow \infty$. Then for $T \geq 3$, we have $\sqrt{n}\left(\hat{\phi}_n - \phi_0\right) \rightarrow_d N\left(0, \theta\right)$, where θ is given by (47).*

For the estimation of the variance term θ, consider

$$\hat{\vartheta}_{it} = \Delta\hat{u}_{it}\Delta y_{i,t-1} + \frac{\Delta\hat{u}_{it}^2}{2}, \quad (48)$$

and $\hat{b}_n = 2a_n - \hat{d}_n$, in which a_n is given by (25)

$$\Delta\hat{u}_{it} = \Delta y_{it} - \hat{\phi}_n \Delta y_{i,t-1}, \quad (49)$$

and

$$\hat{d}_n = \frac{-1}{n\left(T-2\right)} \sum_{t=3}^{T} \sum_{i=1}^{n} \Delta y_{i,t-1}\Delta\hat{u}_{it}.$$

Then θ can be consistently estimated by

$$\hat{\theta}_n = \frac{1}{\hat{b}_n^2 n\left(T-2\right)^2} \sum_{t=3}^{T} \sum_{t'=3}^{T} \sum_{i=1}^{n} \hat{\vartheta}_{it}\hat{\vartheta}_{it'}. \quad (50)$$

A special case of the models in this subsection are autoregressive panel data models initialized long in the past, which are explored next.

2.2 Initialization from an infinite past

This subsection considers the case when $|\phi| < 1$, and $\{y_{it}, i = 1, 2, ..., n\}$ are initialized in a distant past, namely $m \to \infty$. We continue to assume that T is fixed and $n \to \infty$, and we continue to drop the subscript T to simplify the notations.

Whether m is finite or not has important consequences on the derivations above. Specifically, when m is finite, then the second moments

$$\begin{aligned} E\left(\Delta y_{it}^2\right) &= E\left[\left(\varepsilon_{it} + \phi^{t+m-1}\eta_{i,-m}\right)^2\right] \\ &= E\left(\varepsilon_{it}^2\right) + 2\phi^{t+m-1}E\left(\varepsilon_{it}\eta_{i,-m}\right) + \phi^{2t+2m-2}E\left(\eta_{i,-m}^2\right), \end{aligned}$$

can depend on t, due to the diminishing effects of the initial values, where ε_{it} is given by (14). This is no longer the case when $m \to \infty$, since

$$\lim_{m\to\infty} E\left(\varepsilon_{it}\right) = \sigma_i^2 + \left(\phi - 1\right)^2 \sigma_i^2 \lim_{m\to\infty} \sum_{\ell=1}^{t+m-1} \phi^{2\ell-2} = \frac{2}{1+\phi}\sigma_i^2,$$

$$\lim_{m\to\infty} \phi^{t+m-1}E\left(\varepsilon_{it}\eta_{i,-m}\right) = 0, \text{ and } \lim_{m\to\infty} \phi^{2t+2m-2}E\left(\eta_{i,-m}^2\right) = 0,$$

where $\left|E\left(\varepsilon_{it}\eta_{i,-m}\right)\right| < K$ and $E\left(\eta_{i,-m}^2\right) < K$ are both bounded. By taking into account that $E\left(\Delta y_{it}^2\right)$ does not depend on t when the process y_{it} is initialized in a distant past, the estimator $\hat{\phi}_n$ simplifies considerably. Specifically, when $m \to \infty$, we have,

$$\omega_{n,00} = \omega_{i,11} = \frac{2}{1+\phi}\sigma_i^2.$$

where

$$\begin{aligned} \omega_{n,00} &= \frac{1}{n\left(T-2\right)} \sum_{i=1}^{n} \sum_{t=3}^{T} \lim_{m\to\infty} E\left(\Delta y_{it}^2\right), \\ \omega_{n,11} &= \frac{1}{n\left(T-2\right)} \sum_{i=1}^{n} \sum_{t=3}^{T} \lim_{m\to\infty} E\left(\Delta y_{i,t-1}^2\right). \end{aligned}$$

Replacing $\hat{\omega}_{n,00}$ and $\hat{\omega}_{n,11}$ in the definition of the BMM estimator $\hat{\phi}_n$ in (41) by

$$\hat{\omega}_{n,0} = \frac{1}{n\left(T-1\right)} \sum_{i=1}^{n} \sum_{t=2}^{T} \Delta y_{it}^2, \tag{51}$$

then $\hat{D}_n$ in (39) simplifies to $\hat{D} = \hat{\omega}_{n,01}^2$, which is always nonnegative, and consequently the BMM estimator $\hat{\phi}_n$ simplifies to

$$\hat{\phi}_n = \frac{2\hat{\omega}_{n,01}}{\hat{\omega}_{n,0}} + 1 = \frac{2(T-1)\sum_{i=1}^{n}\sum_{t=3}^{T}\Delta y_{it}\Delta y_{i,t-1}}{(T-2)\sum_{i=1}^{n}\sum_{t=2}^{T}\Delta y_{it}^2} + 1, \tag{52}$$

which is more efficient than its counterpart given by (41), in the case when m is large. We shall refer to the estimator $\hat{\phi}_n$ given by (52), derived under the assumption of covariance stationarity of first differences, as an SBMM estimator.

The following theorem establishes consistency of SBMM estimator $\hat{\phi}_n$ given by (52) in the case of initialization in a distant past.

Theorem 3 *Suppose $m \to \infty$, and y_{it}, for all $i = 1, 2, ..., n$, and $t = ...1, 2, ..., T$, are given by (1) with $|\phi_0| < 1$. Let Assumption 1, 2, and 3 part (b) hold. Consider $\hat{\phi}_n$ defined in (52), and let T be fixed and $n \to \infty$. Then $\hat{\phi}_n \to_p \phi_0$.*

It is clear that $\hat{\phi}_n$ given by (52) can be inconsistent when m is finite, but it is possible that the bias due to the influence of starting values can be small. To illustrate the relevance of such bias, consider the difference between $E\left(\Delta y_{it}^2\right)$ and $E\left(\Delta y_{i,t-1}^2\right)$ when m is finite,

$$\begin{aligned} E\left(\Delta y_{it}^2\right) - E\left(\Delta y_{i,t-1}^2\right) &= \sigma_i^2(\phi-1)^2\phi^{2t+2m-4} \\ &\quad + 2\phi^{t+m-1}E\left(\phi\varepsilon_{it}\eta_{i,-m} - \varepsilon_{i,t-1}\eta_{i,-m}\right) \\ &\quad + \phi^{2t+2m-4}\left(\phi^2-1\right)E\left(\eta_{i,-m}^2\right), \end{aligned}$$

which confirms that initial values would not matter much when m is large. When m is finite, note that $\varepsilon_{it} = \Delta u_{it} + \phi\varepsilon_{i,t-1}$, and assuming condition (6) hold, then $E\left(\phi\varepsilon_{it}\eta_{i,-m} - \varepsilon_{i,t-1}\eta_{i,-m}\right) = 0$, and initial values would not matter when ϕ is close to one, and/or $E\left(\eta_{i,-m}^2\right)$ is close to the stationary value of $\sigma_i^2(1-\phi)/(1+\phi)$. But when neither of these conditions hold, then the relevance of the initial values for the SBMM estimator given by (52) is negligible only when m is large, and the bias emanating from the difference between $E\left(\Delta y_{it}^2\right)$ and $E\left(\Delta y_{i,t-1}^2\right)$ decays with m at an exponential rate.

Same as in the case of fixed and finite m, $\hat{\phi}_n$ given by (52) is asymptotically normal with mean ϕ_0 and the variance of order $O\left(n^{-1}\right)$ under cross-sectional independence. This is formally established in Theorem 6 in Appendix, which also provides an exact analytical expression for the asymptotic variance of $\hat{\phi}_n$. Alternatively, the asymptotic distribution can be established in an easier way by following the same steps as in the case of a finite past. In particular, the SBMM estimator $\hat{\phi}_n$ given by (52) satisfies

$$\frac{\sqrt{n}}{2}\left(\hat{\phi}_n-\phi_0\right)\hat{\omega}_{n,0}=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\zeta_i, \tag{53}$$

where

$$\zeta_i=\frac{1}{T-2}\sum_{t=3}^{T}\Delta u_{it}\Delta y_{i,t-1}+\frac{1+\phi_0}{2\left(T-1\right)}\sum_{t=2}^{T}\Delta y_{i,t-1}^2. \tag{54}$$

Assuming $m\to\infty$, and Assumptions 1, 2, 3 part (b), and 4 hold, then the term ζ_{iT} has following properties: $E\left(\zeta_{iT}\right)=0$, second moments of ζ_{iT} are bounded, and ζ_{iT} is independently distributed over i. Hence,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\zeta_i\to_d N\left(0,n^{-1}\sum_{i=1}^{n}E\left(\zeta_i^2\right)\right), \tag{55}$$

and consequently (since $\hat{\omega}_{n,0}\to_p\omega_0=2\sigma^2/\left(1+\phi\right)>0$ by Lemma A.4, where $n^{-1}\sum_{i=1}^{n}\sigma_i^2\to\sigma^2$), we have

$$\sqrt{n}\left(\hat{\phi}_n-\phi_0\right)\to_d N\left(0,\theta\right), \tag{56}$$

where

$$\theta=\frac{4}{\omega_0^2}\lim_{n\to\infty}n^{-1}\sum_{i=1}^{n}E\left(\zeta_i^2\right). \tag{57}$$

These findings are summarized in the following theorem.

Theorem 4 *Suppose $m\to\infty$, and y_{it}, for all $i=1,2,...,n$, and $t=...1,2,...,T$, are given by (1) with $|\phi_0|<1$. Let Assumption 1, 2, 3 part (b), and 4 hold. Consider $\hat{\phi}_n$ defined in (52), and let T be fixed and $n\to\infty$. Then $\sqrt{n}\left(\hat{\phi}_n-\phi_0\right)\to_d N\left(0,\theta\right)$, where θ is given by (57).*

A consistent estimator of the variance term θ is given by

$$\hat{\theta}_n=\frac{4}{\hat{\omega}_{n,0}^2 n}\sum_{i=1}^{n}\hat{\zeta}_{n,i}^2, \tag{58}$$

where $\hat{\omega}_{n,0}$ is given by (51), and

$$\hat{\zeta}_{n,i}=\frac{1}{T-2}\sum_{t=3}^{T}\Delta\hat{u}_{it}\Delta y_{i,t-1}+\frac{1+\hat{\phi}_n}{2\left(T-1\right)}\sum_{t=2}^{T}\Delta y_{it}^2,$$

in which $\Delta \hat{u}_{it} = \Delta y_{it} - \hat{\phi}_n \Delta y_{i,t-1}$.

An alternative consistent estimator of the variance of $\hat{\phi}_n$ can be obtained by applying the delta method directly to the expression (52), which defines the SBMM estimator. This is proposed in the next theorem.

Theorem 5 *Suppose $m \to \infty$, and y_{it}, for all $i = 1, 2, ..., n$, and $t = ...1, 2, ..., T$, are given by (1) with $|\phi_0| < 1$. Let Assumption 1, 2, 3 part (b), and 4 hold. Consider $\hat{\theta}_n$ given by*

$$\hat{\theta}_n = \hat{\mathbf{h}}_n' \hat{\mathbf{\Psi}}_n \hat{\mathbf{h}}_n, \tag{59}$$

where

$$\hat{\mathbf{h}}_n = \begin{pmatrix} -\frac{2\hat{\omega}_{n,01}}{\hat{\omega}_{n,0}^2} \\ \frac{2}{\hat{\omega}_{n,0}} \end{pmatrix}, \ \hat{\mathbf{\Psi}} = \begin{pmatrix} \hat{\psi}_{n,00} & \hat{\psi}_{n,01} \\ \hat{\psi}_{n,10} & \hat{\psi}_{n,11} \end{pmatrix}, \tag{60}$$

$\hat{\omega}_{n,0}$ is given by (51), $\hat{\omega}_{n,01}$ is given by (40), and the individual elements of $\hat{\mathbf{\Psi}}_n$ are given by

$$\hat{\psi}_{n,00} = \frac{1}{(T-1)^2} \sum_{t=2}^{T} \sum_{t'=2}^{T} \left[\frac{1}{n} \sum_{i=1}^{n} \left(\Delta y_{it}^2 - \hat{\omega}_{n,0} \right) \left(\Delta y_{it'}^2 - \hat{\omega}_{n,0} \right) \right],$$

$$\hat{\psi}_{n,01} = \hat{\psi}_{n,10} = \frac{1}{(T-1)(T-2)} \sum_{t=2}^{T} \sum_{t'=3}^{T} \left[\frac{1}{n} \sum_{i=1}^{n} \left(\Delta y_{it}^2 - \hat{\omega}_{n,0} \right) \left(\Delta y_{it'} \Delta y_{i,t'-1} - \hat{\omega}_{n,01} \right) \right], \text{ and}$$

$$\hat{\psi}_{n,11} = \frac{1}{(T-2)^2} \sum_{t=3}^{T} \sum_{t'=3}^{T} \left[\frac{1}{n} \sum_{i=1}^{n} \left(\Delta y_{it} \Delta y_{i,t-1} - \hat{\omega}_{n,01} \right) \left(\Delta y_{it'} \Delta y_{i,t'-1} - \hat{\omega}_{n,01} \right) \right].$$

Let T be fixed and $n \to \infty$. Then for $T \geq 3$, we have

$$\hat{\theta}_n \to_p \theta, \tag{61}$$

where $E\left[n \left(\hat{\phi}_n - \phi_0 \right)^2 \right] \to \theta$ and $\hat{\phi}_n$ is SBMM estimator given by (52).

3 Monte Carlo Evidence

This section investigates the small sample performance of the (S)BMM estimators given by (43) and (52), in comparison with the GMM methods commonly employed in the literature. We distinguish between the designs with a recent past and a distant past, and consider different options for the autoregressive coefficient ϕ.

3.1 Data generating process

The dependent variable is generated as

$$y_{it} = \alpha_i + \phi y_{i,t-1} + u_{it}, \tag{62}$$

for $i = 1, 2, ..., n$ and $t = -m_i+1, -m_i+2, ..., T$, with starting values $y_{i,-m_i}$. We consider $\phi = 0.5, 0.9$ and generate $m_i \sim IIDU\left[1, 4\right]$, and $m_i = 50$. The latter is considered to characterize the case where processes have started in a distant past.

Errors: We generate $\sigma_i^2 \sim IIDU\left[0.5, 1.5\right]$, and $u_{it} \sim IIDN\left(0, \sigma_i^2\right)$, for $t = -m_i+1, -m_i+2, ..., T$.

Individual effects and starting values: We generate $\alpha_i \sim IIDN\left(0, \sigma_\alpha^2\right)$, and $y_{i,-m_i} = \mu_i + v_{yi}$, where $\mu_i = \alpha_i / \left(1 - \phi\right)$, and $v_{yi} \sim IIDN\left(0, \sigma_{vy}^2\right)$, with $\sigma_{vy}^2 = 0.5/\left(1 - \phi^2\right)$. This ensures that the variance of $\left(y_{i,-m_i} - \mu_i\right)$, is, on average, half of the stationary value. We consider three options for $\sigma_\alpha = 1, 2, 5$, and we note that σ_α^2 is also equal to the variance ratio,

$$\sigma_\alpha^2 = \frac{Var\left(\alpha_i\right)}{n^{-1}\sum_{i=1}^{n} Var\left(u_{it}\right)},$$

since $n^{-1}\sum_{i=1}^{n} Var\left(u_{it}\right) = 1$.

3.1.1 Summary of experiments

There are 12 experiments overall, defined by two choices for the values of m_i, namely $m_i \sim IIDU\left[1, 4\right]$, and $m_i = 50$, two choices for the AR coefficient, $\phi = 0.5$, and 0.9, and three choices for the standard error of fixed effects, $\sigma_\alpha = 1, 2$, and 5. We consider $T = 5, 10, 15$, and $N = 150, 500, 1000$, and compute 2000 MC replications.

3.2 Estimation methods

3.2.1 BMM estimators

We implement BMM estimator given by (43) with its variance estimated by $\hat{\theta}_n$ given by (50). In addition, we also implement the SBMM estimator given by (52), and consider two options for the estimation of its variance, one is given by (58) and the other one given by (59). In all our simulation, inference based on (59) achieved marginally better performance, so we report only the findings using the variance estimator $\hat{\theta}_n$ given by (59).

3.2.2 GMM estimators

We implement the same set of GMM estimators as considered in Hayakawa and Pesaran (2015), which include both the first-difference GMM methods based on the moment conditions considered by Arellano and Bond (1991), and the system GMM methods based on the moment conditions considered by Arellano and Bover (1995) and Blundell and Bond (1998).

We consider two sets of moment conditions, which only exploit a subset of moment conditions considered by Arellano and Bond (1991). The first set of moment conditions, denoted as "DIF1" consists of

$$\text{DIF1: } E\left(y_{is}\Delta u_{y,it}\right) = 0, \text{ for } s = 1, 2, ..., t-2, \text{ and } t = 3, 4, ..., T, \tag{63}$$

which gives $m = 6, 36, 91$ moment conditions for $T = 5, 10, 15$, respectively. The second set of moment conditions, denoted as "DIF2", consists of

$$\text{DIF2: } E\left(y_{i,t-2-\ell}\Delta u_{y,it}\right) = 0, \text{ with } \ell = 0 \text{ for } t = 3, \text{ and } \ell = 0, 1 \text{ for } t = 4, 5, ..., T, \tag{64}$$

which gives $m = 5, 15, 25$ moment conditions for $T = 5, 10, 15$, respectively.

We also add conditions

$$E\left[\Delta y_{i,t-1}\left(\alpha_i + u_{y,it}\right)\right] = 0, \text{ for } t = 3, 4, ..., T, \tag{65}$$

in addition to DIF1 and DIF2, which are denoted as "SYS1" and "SYS2", respectively. For SYS1, we have $m = 9, 44, 104$ moment conditions for $T = 5, 10, 15$, respectively, while for SYS2 we have $m = 8, 23, 38$ moment conditions for $T = 5, 10, 15$, respectively.

We implement one-step, two-step and continuous updating (CU) GMM estimators, based on each of the four sets of moment conditions outlined above, as described in Section 4.1 of Hayakawa and Pesaran (2015).

GMM inference In addition to the conventional standard errors, we also compute Windmeijer (2005)'s standard errors with finite sample correction for the two-step GMM estimators and

Newey and Windmeijer (2009)'s alternative standard errors for the CU-GMM estimators. For the computation of optimal weighting matrix, a centered version is used except for the CU-GMM.

3.3 Monte Carlo findings

Findings are reported in Tables 1 to 8. The first four tables report findings for experiments with $m_i = 50$, where the initialization is unimportant. The last four tables report findings for experiments with a recent past, given by $m_i \sim IIDU[1, 4]$.

Tables 1-2 report findings for experiments with moderate value of the autoregressive coefficient ($\phi = 0.5$) and $m_i = 50$. In this case, ϕ is sufficiently low so that the GMM methods do not suffer from the weak instruments problem, and the inference based on these estimators has a correct size for a sufficiently large ratio n/T. Specifically, we observe serious size distortions for $n = 150$ and $T = 15$. BMM and SBMM estimators, on the other hand, achieve correct size, and their RMSE is in most cases similar to the GMM methods. As expected, SBMM performs better in terms of its RMSE compared with BMM estimator that does not impose covariance stationarity of first differences of the dependent variable.

Findings for experiments with $\phi = 0.9$ and $m_i = 50$ are reported in Tables 3-4. We see that the performance of the GMM methods have deteriorated compared with the experiments with $\phi = 0.5$. This is particularly important for the size, where all GMM estimators are quite oversized even for the largest value of n and the smallest value of T considered. We also see that the size worsens significantly with an increase in standard error of individual effects σ_α in the case of the system GMM methods. BMM and SBMM estimators are, on the other hand, unaffected by an increase in σ_α. In addition, there is barely any deterioration in performance of SBMM estimator compared with the experiments with $\phi = 0.5$ reported in Tables 1-2. The performance of the BMM estimator, on the other hand, has deteriorated somewhat, compared with the experiments with $\phi = 0.5$, but the size remains satisfactory.

Experiments with a recent past $m_i \sim IIDU[1, 4]$, and the variance of $(y_{i,-m_i} - \mu_i)$ equal, on average, half of the stationary value, are presented in Tables 5-8. Compared with the earlier experiments that are initialized in a distant past (with $m_i = 50$), the findings for the experiments with $m_i \sim IIDU[1, 4]$ are qualitatively similar for all estimators with the exception of SBMM estimator that assumes covariance stationarity of first differences of the dependent variable. We see SBMM estimator suffers from a small bias in the case $\phi = 0.9$. This bias has negative consequences for the reported size, which has increased and is now in the range 5-12%, depending on the sample size, with the over-rejection problem becoming more serious in the case of a larger values of n, as expected.

4 Conclusion

This paper proposed using target variables as instruments and correcting for the correlation of instruments and the error term at the population level. This idea was applied to the estimation of short-T autoregressive panel data models and it effectively circumvent the weak instruments problem. Two BMM estimators are proposed, one that relies on the covariance stationarity of first differences of the dependent variable, and one that does not. The performance of these estimators does not suffer when the variance of individual effects is increased, and, for the samples considered in the Monte Carlo experiments, the performance of these estimators is found satisfactory compared with a set of GMM methods. This evidence suggests that bias-corrected moment conditions are valuable additions to the GMM literature.

The idea of BMs opens new exciting research avenues. BMs could be considered in other settings, including spatial models, or large-n and large-T panel data setting, under various assumptions on the ratio T/n. BMs could be used to estimate unknown parameters of a known distributional functional form of slope coefficients in short-T autoregressive or vector autoregressive panels with heterogenous coefficients.

Table 1: Bias and RMSE findings for the estimation of ϕ in experiments with $m_i = 50$ and $\phi = 0.5$

	Bias (×100)			RMSE (×100)			Bias (×100)			RMSE (×100)			Bias (×100)			RMSE (×100)		
(N,T)	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**
	$\sigma_\alpha = 1$						$\sigma_\alpha = 2$						$\sigma_\alpha = 5$					
	BMM estimators																	
	BMM																	
150	2.67	0.77	0.06	14.33	7.14	4.67	2.67	0.77	0.06	14.33	7.14	4.67	2.67	0.77	0.06	14.33	7.14	4.67
500	0.88	-0.03	0.15	7.96	3.54	2.63	0.88	-0.03	0.15	7.96	3.54	2.63	0.88	-0.03	0.15	7.96	3.54	2.63
1000	0.36	0.13	0.02	5.30	2.46	1.78	0.36	0.13	0.02	5.30	2.46	1.78	0.36	0.13	0.02	5.30	2.46	1.78
	SBMM																	
150	0.09	0.18	-0.08	8.85	5.30	4.02	0.09	0.18	-0.08	8.85	5.30	4.02	0.09	0.18	-0.08	8.85	5.30	4.02
500	0.07	-0.10	0.10	4.70	2.94	2.32	0.07	-0.10	0.10	4.70	2.94	2.32	0.07	-0.10	0.10	4.70	2.94	2.32
1000	-0.11	0.10	0.03	3.26	2.03	1.54	-0.11	0.10	0.03	3.26	2.03	1.54	-0.11	0.10	0.03	3.26	2.03	1.54
	GMM estimators																	
	One-step first difference GMM estimator based on "DIF1"																	
150	-3.89	-2.53	-2.17	14.74	5.91	3.95	-6.18	-2.96	-2.32	18.81	6.39	4.17	-7.55	-3.07	-2.39	21.65	6.64	4.27
500	-1.05	-0.96	-0.65	7.87	3.02	2.01	-1.79	-1.13	-0.71	9.95	3.35	2.13	-2.06	-1.17	-0.74	11.64	3.45	2.16
1000	-0.69	-0.32	-0.35	5.49	2.07	1.35	-1.09	-0.43	-0.38	7.06	2.31	1.43	-1.27	-0.45	-0.39	8.09	2.37	1.45
	Two-step first difference GMM estimator based on "DIF1"																	
150	-3.58	-2.35	-2.11	15.59	6.88	5.32	-6.00	-2.83	-2.27	19.84	7.46	5.78	-7.77	-3.00	-2.35	22.95	7.71	5.88
500	-0.89	-0.91	-0.58	8.03	3.18	2.24	-1.72	-1.08	-0.67	10.04	3.52	2.37	-1.96	-1.12	-0.69	11.78	3.63	2.41
1000	-0.58	-0.30	-0.32	5.52	2.14	1.42	-1.01	-0.42	-0.36	7.13	2.39	1.52	-1.20	-0.45	-0.37	8.13	2.46	1.53
	Continuous-updating first difference GMM estimator based on "DIF1"																	
150	1.19	0.31	-0.34	16.64	7.33	6.62	1.99	0.41	-0.28	23.04	7.89	7.28	2.54	0.39	-0.42	27.59	8.26	7.47
500	0.49	-0.08	0.06	8.17	3.14	2.25	0.45	-0.10	0.04	10.28	3.45	2.38	0.84	-0.06	0.05	12.33	3.58	2.42
1000	0.12	0.12	0.01	5.57	2.16	1.41	0.12	0.09	0.00	7.23	2.39	1.49	0.24	0.09	0.01	8.31	2.46	1.52
	One-step first difference GMM estimator based on "DIF2"																	
150	-3.64	-1.59	-1.22	15.48	6.43	4.07	-6.85	-3.61	-2.15	21.93	9.22	5.62	-9.20	-7.04	-5.29	27.99	14.19	9.68
500	-1.04	-0.62	-0.28	8.20	3.23	2.22	-2.44	-1.37	-0.56	11.82	4.80	2.96	-3.13	-2.76	-1.50	15.24	7.61	4.96
1000	-0.62	-0.17	-0.17	5.68	2.27	1.52	-1.20	-0.59	-0.28	8.37	3.34	2.01	-1.56	-1.27	-0.77	10.84	5.37	3.39
	Two-step first difference GMM estimator based on "DIF2"																	
150	-3.68	-1.50	-1.05	16.36	6.88	4.47	-6.85	-3.57	-1.94	23.09	9.91	6.09	-9.50	-7.17	-5.23	29.81	15.44	10.79
500	-1.00	-0.57	-0.18	8.37	3.29	2.29	-2.46	-1.34	-0.44	12.03	4.95	3.03	-3.24	-2.81	-1.41	15.59	7.85	5.14
1000	-0.59	-0.16	-0.13	5.71	2.30	1.54	-1.23	-0.55	-0.23	8.47	3.39	2.04	-1.69	-1.27	-0.70	10.93	5.50	3.46
	Continuous-updating first difference GMM estimator based on "DIF2"																	
150	0.66	0.43	0.37	17.30	6.89	4.53	1.84	0.23	0.33	27.47	9.53	5.84	4.24	0.87	0.15	38.77	15.96	9.81
500	0.27	0.02	0.26	8.44	3.26	2.30	-0.06	-0.18	0.27	12.24	4.77	3.00	0.68	-0.20	0.41	16.75	7.62	4.92
1000	0.06	0.14	0.10	5.73	2.31	1.54	0.07	0.03	0.12	8.51	3.34	2.03	0.37	0.06	0.24	11.32	5.44	3.38
	One-step first difference GMM estimator based on "SYS1"																	
150	1.23	0.83	0.36	9.67	4.85	3.28	6.44	5.91	4.63	14.05	8.34	6.02	25.71	23.76	21.33	30.11	24.92	22.15
500	0.42	0.10	0.17	5.39	2.70	1.90	2.13	1.92	1.69	7.69	3.79	2.72	12.44	11.45	9.91	17.55	12.83	10.67
1000	0.13	0.19	0.08	3.82	1.88	1.28	1.13	1.21	0.81	5.40	2.57	1.66	7.30	6.94	5.66	12.35	8.03	6.24
	Two-step first difference GMM estimator based on "SYS1"																	
150	1.29	0.62	0.27	8.84	4.61	4.10	4.93	3.18	2.72	12.12	6.30	4.97	23.20	20.58	18.84	29.13	22.41	20.01
500	0.40	0.01	0.11	4.55	2.15	1.60	1.32	0.39	0.46	5.55	2.31	1.71	9.57	6.60	5.78	14.72	8.43	6.79
1000	0.16	0.11	0.06	3.24	1.42	1.01	0.61	0.25	0.14	3.80	1.50	1.05	4.53	2.72	2.23	8.64	3.88	2.89
	Continuous-updating first difference GMM estimator based on "SYS1"																	
150	0.72	0.34	-0.04	9.24	5.01	5.76	1.18	0.36	0.04	10.57	5.09	5.84	5.27	0.45	-0.14	17.66	5.32	5.86
500	0.17	-0.01	0.08	4.58	2.17	1.66	0.19	-0.01	0.07	5.06	2.18	1.64	0.56	-0.01	0.07	5.88	2.18	1.65
1000	0.06	0.10	0.05	3.26	1.42	1.02	0.10	0.12	0.05	3.62	1.43	1.02	0.21	0.11	0.06	3.81	1.43	1.02
	One-step first difference GMM estimator based on "SYS2"																	
150	1.40	1.66	1.34	9.66	5.28	3.85	6.77	8.12	7.30	14.36	10.41	8.58	26.59	28.96	28.88	31.16	29.94	29.50
500	0.45	0.36	0.51	5.39	2.80	2.13	2.20	2.75	2.70	7.88	4.55	3.71	12.81	15.42	15.58	18.34	16.86	16.38
1000	0.16	0.31	0.25	3.83	1.95	1.42	1.20	1.66	1.33	5.53	3.03	2.16	7.58	9.71	9.48	13.00	10.96	10.20
	Two-step first difference GMM estimator based on "SYS2"																	
150	1.01	0.76	0.36	8.75	5.00	4.01	4.87	3.86	3.28	12.08	7.08	5.52	23.78	23.77	23.33	29.59	25.50	24.45
500	0.29	0.03	0.19	4.54	2.51	1.96	1.27	0.56	0.64	5.62	2.73	2.16	10.28	8.49	8.45	15.43	10.48	9.64
1000	0.10	0.08	0.04	3.24	1.68	1.30	0.63	0.27	0.16	3.88	1.83	1.37	5.10	3.65	3.43	9.36	5.02	4.31
	Continuous-updating first difference GMM estimator based on "SYS2"																	
150	0.32	0.03	-0.39	9.14	5.16	4.30	0.77	0.02	-0.39	10.51	5.10	4.24	6.07	0.35	-0.38	19.36	5.97	4.23
500	0.02	-0.13	0.02	4.58	2.52	1.95	0.01	-0.12	0.02	5.14	2.54	1.97	0.62	-0.10	0.03	6.64	2.56	1.98
1000	-0.02	0.01	-0.04	3.26	1.68	1.30	0.03	0.02	-0.04	3.68	1.73	1.31	0.20	0.01	-0.03	4.04	1.73	1.31

Notes: The DGP is given by $y_{it} = \alpha_i + \phi y_{i,t-1} + u_{it}$, for $i = 1, 2, ..., n$ and $t = -m_i + 1, -m_i + 2, ..., T$, with starting values $y_{i,-m_i}$. Errors are generated as $u_{it} \sim IIDN\left(0, \sigma_i^2\right)$, $\sigma_i^2 \sim IIDU\left[0.5, 1.5\right]$. Individual effects are generated as $\alpha_i \sim IIDN\left(0, \sigma_\alpha^2\right)$, and starting values are given by $y_{i,-m_i} = \mu_i + v_{yi}$, where $\mu_i = \alpha_i / (1 - \phi)$, and $v_{yi} \sim IIDN\left(0, \sigma_{vy}^2\right)$, $\sigma_{vy}^2 = 2/\left(1 - \phi^2\right)$. BMM estimator is given by (43) and SBMM estimator is given by (52). See subsection 3.2 for description of individual estimation methods.

Table 2: Size (×100, $H_0 : \phi = 0.5$) and Power (×100, $H_1 : \phi = 0.6$) findings for the estimation of ϕ in experiments with $m_i = 50$ and $\phi = 0.5$

	Size (×100)			**Power (×100)**			**Size (×100)**			**Power (×100)**			**Size (×100)**			**Power (×100)**		
(N,T)	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**
	$\sigma_\alpha = 1$						$\sigma_\alpha = 2$						$\sigma_\alpha = 5$					
	BMM estimators																	
	BMM																	
150	2.45	3.25	5.10	14.85	36.25	58.80	2.45	3.25	5.10	14.85	36.25	58.80	2.45	3.25	5.10	14.85	36.25	58.80
500	2.50	4.65	5.55	32.35	80.30	95.95	2.50	4.65	5.55	32.35	80.30	95.95	2.50	4.65	5.55	32.35	80.30	95.95
1000	3.15	4.60	4.65	51.70	96.75	99.85	3.15	4.60	4.65	51.70	96.75	99.85	3.15	4.60	4.65	51.70	96.75	99.85
	SBMM																	
150	6.45	6.75	5.45	23.35	47.20	70.60	6.45	6.75	5.45	23.35	47.20	70.60	6.45	6.75	5.45	23.35	47.20	70.60
500	5.35	5.60	5.75	56.15	94.15	99.10	5.35	5.60	5.75	56.15	94.15	99.10	5.35	5.60	5.75	56.15	94.15	99.10
1000	4.45	5.30	4.80	86.40	99.85	100.00	4.45	5.30	4.80	86.40	99.85	100.00	4.45	5.30	4.80	86.40	99.85	100.00
	GMM estimators																	
	One-step first difference GMM estimator based on "DIF1"																	
150	7.15	9.00	9.65	22.25	68.05	96.10	7.80	8.15	9.75	20.90	63.55	94.70	8.90	8.75	10.05	20.75	61.40	93.85
500	5.60	6.10	6.50	32.05	97.00	100.00	5.60	5.85	6.50	25.80	93.95	99.95	6.10	6.05	6.55	23.00	92.35	100.00
1000	5.00	4.35	4.80	50.80	99.90	100.00	5.55	5.45	5.10	37.00	99.65	100.00	5.55	5.00	5.25	31.85	99.55	100.00
	Two-step first difference GMM estimator based on "DIF1"																	
150	9.95	24.15	52.90	23.80	74.50	96.75	11.05	24.90	54.60	23.25	72.45	95.30	13.15	24.00	54.95	24.20	72.15	94.90
500	6.50	9.00	16.15	32.25	96.95	99.95	6.05	9.70	16.10	25.60	94.45	99.95	6.65	8.55	16.05	23.20	92.75	100.00
1000	5.15	6.40	9.25	50.55	99.80	100.00	6.05	7.55	9.50	37.25	99.70	100.00	5.80	7.50	9.15	32.05	99.40	100.00
	Two-step first difference GMM estimator based on "DIF1" with Windmeijer standard errors																	
150	6.80	7.10	0.90	18.40	49.90	30.00	6.80	7.55	1.15	17.35	45.95	27.95	8.35	7.15	0.75	18.45	43.50	26.50
500	5.45	5.80	6.75	29.00	94.85	99.85	5.05	5.45	6.90	23.15	91.20	99.65	5.90	5.80	6.00	21.10	88.85	99.55
1000	4.85	5.25	4.85	49.05	99.80	100.00	5.20	5.75	5.15	35.35	99.50	100.00	5.40	5.50	5.25	30.70	99.15	100.00
	Continuous-updating first difference GMM estimator based on "DIF1"																	
150	10.25	28.40	60.20	18.20	60.00	87.95	10.70	26.60	61.65	17.15	55.90	86.95	12.20	27.85	60.85	18.65	54.35	86.70
500	6.70	9.00	16.50	26.15	94.15	99.90	5.65	9.20	17.35	20.85	89.05	99.80	6.05	9.45	17.55	18.30	86.65	99.70
1000	5.30	7.05	8.50	45.90	99.70	100.00	5.60	7.35	9.25	32.55	99.35	100.00	5.05	7.45	8.55	26.60	98.80	100.00
	Continuous-updating first difference GMM estimator based on "DIF1" with NW standard errors																	
150	8.55	31.65	76.85	15.70	63.35	93.80	8.05	29.75	77.45	14.10	57.80	94.05	9.05	30.00	76.10	14.55	55.85	92.40
500	5.95	9.70	22.05	24.85	93.85	99.65	5.30	9.35	22.00	18.40	88.60	99.75	5.75	9.70	22.55	17.00	86.45	99.75
1000	5.15	7.05	10.10	44.40	99.65	99.80	5.60	7.50	10.50	31.20	99.10	99.85	4.70	7.45	10.00	25.35	99.00	100.00
	One-step first difference GMM estimator based on "DIF2"																	
150	7.05	7.00	5.90	20.25	50.30	81.05	8.35	7.05	6.50	18.90	38.20	65.90	10.90	10.20	8.55	19.65	28.45	45.20
500	6.10	5.55	5.45	29.70	91.40	99.75	6.30	6.00	5.75	21.25	71.15	96.45	6.40	6.25	6.15	18.55	43.35	71.35
1000	5.00	4.65	4.45	46.85	99.50	100.00	6.45	4.90	5.35	29.10	90.80	99.95	5.25	5.80	5.35	23.30	58.25	91.60
	Two-step first difference GMM estimator based on "DIF2"																	
150	9.85	12.35	13.50	22.85	56.40	84.25	11.40	14.00	14.30	22.30	44.60	72.55	14.00	17.90	20.70	23.00	37.80	55.80
500	6.90	6.65	7.70	30.60	91.25	99.75	7.05	7.65	7.60	22.55	71.90	96.55	7.30	9.05	9.30	20.00	46.35	72.65
1000	5.40	5.20	4.90	47.40	99.30	100.00	6.75	6.00	6.05	29.30	90.80	99.95	5.90	7.00	7.10	24.40	59.90	92.15
	Two-step first difference GMM estimator based on "DIF2" with Windmeijer standard errors																	
150	7.00	7.45	5.35	17.90	44.75	71.75	8.90	6.80	6.30	17.70	32.50	54.85	10.70	8.85	7.95	18.00	23.20	33.85
500	5.80	5.05	5.80	28.35	90.05	99.65	6.20	5.85	5.80	20.45	68.05	95.70	6.40	7.00	6.40	18.20	40.20	65.35
1000	4.95	4.55	4.15	46.60	99.25	100.00	6.20	5.25	5.00	28.40	90.05	99.95	4.95	6.35	5.35	23.00	57.25	90.90
	Continuous-updating first difference GMM estimator based on "DIF2"																	
150	9.05	13.50	14.65	17.45	45.10	75.05	11.60	12.70	14.55	17.50	30.85	57.20	13.75	17.00	18.15	18.20	23.20	33.30
500	6.60	6.05	7.45	25.60	88.40	99.60	6.10	7.25	7.75	18.00	63.65	94.45	7.50	7.95	8.75	15.45	31.65	57.90
1000	4.95	5.35	5.10	42.85	99.10	100.00	6.00	6.70	6.50	25.10	87.40	99.95	5.60	7.20	6.25	19.65	49.10	86.40
	Continuous-updating first difference GMM estimator based on "DIF2" with NW standard errors																	
150	8.35	14.25	18.30	15.20	45.60	78.90	9.10	13.25	17.75	14.50	30.05	61.35	11.15	15.30	19.20	15.90	20.80	33.75
500	6.05	6.30	8.25	24.50	87.95	99.60	5.85	6.60	9.00	16.85	63.40	94.85	5.85	7.85	9.25	14.00	31.40	56.95
1000	4.70	5.40	5.30	42.00	98.85	99.75	5.70	6.20	6.65	24.05	87.00	99.75	5.00	5.95	7.05	17.70	48.60	85.80

Notes: See notes to Table 1.

Table 2 (Continued): Size (×100, $H_0 : \phi = 0.5$) and Power (×100, $H_1 : \phi = 0.6$) findings for the estimation of ϕ in experiments with $m_i = 50$ and $\phi = 0.5$

	Size (×100)			Power (×100)			Size (×100)			Power (×100)			Size (×100)			Power (×100)		
(N,T)	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**
	$\sigma_\alpha = 1$						$\sigma_\alpha = 2$						$\sigma_\alpha = 5$					
	GMM estimators																	
	One-step first difference GMM estimator based on "SYS1"																	
150	6.65	6.45	5.30	15.10	50.90	84.60	16.75	25.95	28.85	6.50	13.30	32.60	57.95	85.05	94.25	44.15	58.95	61.65
500	4.95	6.10	5.35	41.70	96.30	99.95	8.35	12.60	16.15	12.25	73.65	97.95	34.30	62.40	80.40	16.05	13.65	11.85
1000	4.65	4.95	4.80	73.70	99.85	100.00	7.05	9.40	8.60	36.80	98.05	100.00	24.45	50.35	67.55	5.55	17.55	50.70
	Two-step first difference GMM estimator based on "SYS1"																	
150	14.25	31.45	63.45	29.50	84.20	97.40	27.70	50.35	74.95	28.95	73.10	94.15	76.30	96.65	99.80	65.50	87.25	94.60
500	6.95	11.55	17.80	58.90	99.95	100.00	11.50	16.00	23.75	49.35	99.80	100.00	51.05	76.30	89.15	47.90	76.30	87.35
1000	6.35	7.20	9.45	86.95	100.00	100.00	8.50	9.40	11.35	75.45	100.00	100.00	36.70	53.45	66.95	57.00	96.95	99.80
	Two-step first difference GMM estimator based on "SYS1" with Windmeijer standard errors																	
150	8.00	3.80	0.15	19.80	47.90	4.55	10.90	5.05	0.25	14.20	21.55	1.85	46.30	56.50	31.45	33.90	29.30	14.70
500	5.15	5.90	4.80	54.80	99.75	100.00	5.60	6.45	4.45	40.20	98.25	99.90	22.15	38.40	45.55	18.15	30.45	38.45
1000	5.35	4.85	3.95	85.25	100.00	100.00	4.95	5.50	4.45	70.40	100.00	100.00	13.20	25.00	28.90	32.65	84.20	97.35
	Continuous-updating first difference GMM estimator based on "SYS1"																	
150	16.25	34.65	71.55	31.90	84.20	94.75	22.35	39.15	76.20	37.50	86.15	94.30	41.05	58.50	84.55	56.00	92.60	98.05
500	7.40	12.35	19.95	61.75	99.95	100.00	9.45	13.90	21.80	57.40	99.95	100.00	16.70	23.90	33.10	57.55	99.90	100.00
1000	6.65	7.05	9.25	87.15	100.00	100.00	7.35	7.60	10.15	79.50	100.00	100.00	11.55	12.30	15.75	75.05	100.00	100.00
	Continuous-updating first difference GMM estimator based on "SYS1" with NW standard errors																	
150	14.35	39.10	85.85	29.70	86.30	97.85	16.40	38.50	86.50	31.35	85.55	97.50	21.15	38.35	85.10	33.70	85.65	97.05
500	6.00	12.70	25.65	59.90	99.95	99.95	6.40	12.30	24.55	53.70	99.90	99.95	5.15	11.85	24.90	49.85	99.85	100.00
1000	6.20	7.25	11.20	86.10	99.95	99.85	6.20	7.25	11.30	78.95	100.00	100.00	5.15	7.15	11.00	73.75	100.00	100.00
	One-step first difference GMM estimator based on "SYS2"																	
150	6.50	7.90	7.75	14.30	41.70	67.70	17.80	33.60	42.10	6.55	8.90	11.90	59.15	90.00	98.15	45.80	74.65	85.35
500	4.95	5.75	6.05	40.85	94.40	99.60	9.20	15.30	23.15	11.30	57.55	84.80	35.50	70.50	90.40	17.25	27.35	34.50
1000	4.60	5.25	5.25	73.55	99.85	100.00	7.45	12.20	13.20	34.00	92.80	99.85	25.65	57.40	79.65	6.00	8.35	10.85
	Two-step first difference GMM estimator based on "SYS2"																	
150	13.60	19.65	24.85	29.30	69.70	88.60	27.30	39.30	47.00	28.05	55.30	72.80	76.20	96.50	99.55	65.65	85.35	92.15
500	7.05	9.00	9.85	59.70	98.30	100.00	12.25	13.90	15.50	47.50	97.00	99.75	52.80	77.50	90.05	47.60	63.70	65.15
1000	6.25	6.65	6.20	87.15	100.00	100.00	8.85	8.75	8.35	72.90	100.00	100.00	38.65	56.85	66.80	54.65	88.90	95.55
	Two-step first difference GMM estimator based on "SYS2" with Windmeijer standard errors																	
150	8.00	6.50	5.10	20.45	48.00	65.95	11.25	11.70	9.90	14.95	24.30	33.65	51.10	82.60	93.00	38.70	62.70	74.25
500	5.50	5.60	5.90	55.60	97.25	99.90	6.20	6.30	6.35	39.80	92.95	99.20	24.90	52.15	71.50	19.30	27.85	28.80
1000	5.25	5.00	4.55	85.95	100.00	100.00	5.50	5.40	5.05	69.10	99.95	100.00	14.55	32.35	45.75	30.00	71.30	85.90
	Continuous-updating first difference GMM estimator based on "SYS2"																	
150	16.10	20.70	26.80	32.35	74.60	90.05	21.40	24.50	32.60	37.40	77.60	92.95	42.00	43.45	52.30	57.55	87.30	96.15
500	7.05	9.35	9.35	62.40	98.75	100.00	9.70	10.85	11.30	56.65	98.65	99.90	17.05	20.05	20.80	58.60	99.25	99.95
1000	6.50	6.45	6.00	87.80	100.00	100.00	7.45	6.75	7.20	77.75	100.00	100.00	12.95	11.75	11.65	72.65	100.00	100.00
	Continuous-updating first difference GMM estimator based on "SYS2" with NW standard errors																	
150	13.45	19.45	31.00	29.95	73.30	90.85	15.95	18.80	30.75	32.75	72.60	91.65	24.20	18.70	31.35	37.40	71.45	91.10
500	6.15	8.80	10.40	60.40	98.70	99.80	7.00	8.80	10.10	54.80	98.20	99.90	6.70	8.95	10.00	49.40	98.05	99.90
1000	6.35	6.00	6.25	86.85	100.00	100.00	6.05	5.75	6.60	77.40	100.00	100.00	4.90	6.10	6.15	70.45	100.00	100.00

Notes: See notes to Table 1.

Table 3: Bias and RMSE findings for the estimation of ϕ in experiments with $m_i = 50$ and $\phi = 0.9$

	Bias (×100)			RMSE (×100)			Bias (×100)			RMSE (×100)			Bias (×100)			RMSE (×100)		
(N,T)	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**
	$\sigma_\alpha = 1$						$\sigma_\alpha = 2$						$\sigma_\alpha = 5$					
	BMM estimators																	
	BMM																	
150	-6.14	-2.05	-0.80	14.24	8.41	6.60	-6.14	-2.05	-0.80	14.24	8.41	6.60	-6.14	-2.05	-0.80	14.24	8.41	6.60
500	-2.64	-0.61	0.08	9.39	5.93	4.84	-2.64	-0.61	0.08	9.39	5.93	4.84	-2.64	-0.61	0.08	9.39	5.93	4.84
1000	-1.81	-0.10	0.33	7.79	4.89	4.03	-1.81	-0.10	0.33	7.79	4.89	4.03	-1.81	-0.10	0.33	7.79	4.89	4.03
	SBMM																	
150	-0.09	-0.04	0.09	9.50	5.74	4.50	-0.09	-0.04	0.09	9.50	5.74	4.50	-0.09	-0.04	0.09	9.50	5.74	4.50
500	0.11	0.03	-0.04	5.30	3.13	2.45	0.11	0.03	-0.04	5.30	3.13	2.45	0.11	0.03	-0.04	5.30	3.13	2.45
1000	0.05	0.00	-0.03	3.74	2.29	1.77	0.05	0.00	-0.03	3.74	2.29	1.77	0.05	0.00	-0.03	3.74	2.29	1.77
	GMM estimators																	
	One-step first difference GMM estimator based on "DIF1"																	
150	-46.78	-20.73	-12.48	63.01	24.35	14.20	-57.43	-23.45	-13.52	71.63	27.12	15.25	-60.74	-24.34	-13.84	75.34	27.93	15.56
500	-23.57	-9.27	-5.37	39.02	12.29	6.69	-34.68	-11.22	-6.18	51.03	14.37	7.54	-38.50	-11.99	-6.38	54.82	15.15	7.76
1000	-12.83	-5.07	-2.82	27.21	7.58	4.04	-19.88	-6.36	-3.31	35.57	8.90	4.54	-25.62	-6.97	-3.42	42.15	9.60	4.67
	Two-step first difference GMM estimator based on "DIF1"																	
150	-53.83	-25.63	-14.99	72.91	31.51	18.63	-65.57	-29.47	-16.51	83.39	35.37	20.24	-68.98	-30.40	-16.77	86.46	36.36	20.53
500	-26.16	-10.26	-5.79	44.12	13.96	7.49	-39.55	-12.63	-6.78	58.32	16.69	8.57	-43.81	-13.66	-7.05	62.51	17.66	8.84
1000	-13.64	-5.19	-2.94	28.72	8.00	4.37	-21.71	-6.67	-3.50	38.50	9.53	4.94	-28.47	-7.42	-3.62	46.32	10.41	5.08
	Continuous-updating first difference GMM estimator based on "DIF1"																	
150	-24.82	-8.33	-12.07	100.47	61.05	61.18	-41.16	-10.29	-14.44	114.94	70.49	67.44	-45.35	-12.37	-12.97	117.00	72.98	67.62
500	-1.17	0.13	-0.08	61.63	11.58	5.59	-6.51	0.43	-0.08	80.56	15.42	6.22	-8.94	0.44	-0.09	87.83	15.92	6.47
1000	3.04	0.36	0.05	36.21	6.66	3.40	5.43	0.48	0.02	52.62	8.07	3.73	2.28	0.45	0.10	63.23	8.74	3.82
	One-step first difference GMM estimator based on "DIF2"																	
150	-48.80	-26.16	-15.46	70.34	35.46	22.34	-63.44	-40.56	-28.95	82.01	49.27	36.22	-67.18	-47.44	-37.42	84.10	54.95	43.58
500	-24.37	-10.27	-4.92	43.80	16.88	8.69	-40.98	-24.68	-14.98	60.38	33.04	20.88	-46.74	-35.18	-26.87	65.90	41.75	32.31
1000	-13.36	-4.98	-2.33	30.47	10.17	5.31	-24.36	-14.04	-8.10	43.31	21.02	12.58	-34.82	-25.64	-19.87	53.40	31.73	25.13
	Two-step first difference GMM estimator based on "DIF2"																	
150	-54.96	-33.06	-20.36	79.49	45.47	30.14	-70.19	-51.17	-39.18	91.73	62.40	49.20	-74.07	-59.15	-50.63	94.02	68.81	58.99
500	-26.53	-12.15	-5.69	47.57	20.22	10.19	-44.87	-29.29	-19.13	66.19	39.41	26.82	-50.53	-41.71	-34.31	71.63	49.61	41.27
1000	-14.25	-5.34	-2.55	32.06	11.09	5.71	-26.02	-15.97	-9.53	45.66	24.06	14.95	-37.16	-28.76	-23.70	57.01	35.69	29.97
	Continuous-updating first difference GMM estimator based on "DIF2"																	
150	-28.20	-4.00	-1.20	108.04	71.17	47.58	-45.08	-12.94	-2.39	122.16	96.99	80.71	-50.52	-22.17	-12.28	125.39	108.22	96.90
500	0.84	0.73	-0.50	63.72	22.99	8.43	-9.58	6.10	3.79	90.55	51.49	33.13	-11.89	8.44	9.76	98.69	66.15	50.34
1000	2.31	0.09	-0.21	39.13	9.94	4.80	6.49	2.54	0.52	61.40	26.61	15.37	2.30	8.22	5.05	77.07	41.45	31.38
	One-step first difference GMM estimator based on "SYS1"																	
150	4.75	4.31	3.82	9.71	5.77	4.71	8.24	8.12	7.73	9.90	8.43	7.91	9.66	9.61	9.55	9.93	9.66	9.58
500	2.35	2.68	2.18	6.75	4.02	3.12	6.57	6.50	6.05	8.51	6.88	6.28	9.48	9.28	9.12	9.89	9.35	9.16
1000	1.12	1.65	1.31	5.27	3.01	2.20	4.78	4.97	4.60	6.98	5.45	4.89	8.81	8.73	8.52	9.51	8.82	8.57
	Two-step first difference GMM estimator based on "SYS1"																	
150	3.61	3.71	3.51	10.85	5.62	4.77	7.69	7.80	7.57	10.75	8.26	7.81	9.49	9.53	9.52	10.13	9.59	9.56
500	0.97	1.71	1.62	6.76	3.31	2.69	5.58	5.82	5.61	8.58	6.37	5.91	9.25	9.12	9.05	10.02	9.23	9.10
1000	0.13	0.92	0.78	4.71	2.41	1.75	3.61	4.16	4.00	6.87	4.81	4.39	8.46	8.49	8.39	9.66	8.64	8.46
	Continuous-updating first difference GMM estimator based on "SYS1"																	
150	1.04	0.33	0.04	13.58	6.54	6.93	5.29	3.08	1.51	15.46	8.87	8.06	7.03	5.59	4.45	16.48	10.51	9.66
500	-1.27	-0.04	0.09	7.46	3.06	2.11	1.90	0.91	0.27	10.14	4.55	2.49	7.17	4.18	2.05	13.18	7.94	5.36
1000	-1.08	0.02	0.02	5.27	2.17	1.39	0.44	0.28	0.10	7.76	2.84	1.57	5.25	2.60	0.81	11.32	6.15	3.40
	One-step first difference GMM estimator based on "SYS2"																	
150	4.98	5.49	5.74	9.90	6.63	6.33	8.34	8.64	8.62	9.98	8.92	8.77	9.68	9.71	9.73	9.95	9.76	9.76
500	2.47	3.33	3.33	6.73	4.42	3.99	6.67	6.97	6.99	8.57	7.32	7.16	9.52	9.41	9.40	9.98	9.47	9.43
1000	1.17	2.06	2.03	5.28	3.19	2.70	4.84	5.40	5.46	7.03	5.83	5.69	8.84	8.89	8.90	9.57	8.98	8.94
	Two-step first difference GMM estimator based on "SYS2"																	
150	3.71	4.12	4.30	10.99	6.20	5.49	7.78	8.02	7.95	10.83	8.61	8.27	9.50	9.57	9.60	10.19	9.67	9.66
500	1.03	1.70	1.70	6.83	3.50	2.86	5.65	5.74	5.62	8.66	6.40	5.95	9.28	9.11	9.05	10.09	9.25	9.13
1000	0.15	0.81	0.71	4.80	2.46	1.82	3.65	3.95	3.90	7.05	4.66	4.29	8.47	8.39	8.31	9.76	8.58	8.41
	Continuous-updating first difference GMM estimator based on "SYS2"																	
150	0.86	0.01	-0.20	14.18	6.77	5.19	4.75	2.83	1.27	15.70	9.11	6.82	7.13	5.22	4.52	16.88	10.61	8.99
500	-1.38	-0.21	-0.16	7.69	3.38	2.42	1.78	0.72	0.05	10.36	4.91	2.92	7.02	4.19	2.72	13.38	8.31	6.42
1000	-1.08	-0.11	-0.09	5.46	2.39	1.65	0.25	0.14	0.04	7.90	3.18	1.97	4.95	2.64	1.48	11.34	6.51	4.69

Notes: See notes to Table 1.

Table 4: Size (×100, $H_0 : \phi = 0.9$) and Power (×100, $H_1 : \phi = 1$) findings for the estimation of ϕ in experiments with $m_i = 50$ and $\phi = 0.9$

	Size (×100)			**Power (×100)**			**Size (×100)**			**Power (×100)**			**Size (×100)**			**Power (×100)**		
(N,T)	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**
	$\sigma_\alpha = 1$						$\sigma_\alpha = 2$						$\sigma_\alpha = 5$					
	BMM estimators																	
	BMM																	
150	6.55	5.85	5.70	16.90	24.75	29.95	6.55	5.85	5.70	16.90	24.75	29.95	6.55	5.85	5.70	16.90	24.75	29.95
500	6.05	5.80	5.75	19.80	31.30	40.95	6.05	5.80	5.75	19.80	31.30	40.95	6.05	5.80	5.75	19.80	31.30	40.95
1000	5.05	5.15	4.95	25.10	38.20	50.15	5.05	5.15	4.95	25.10	38.20	50.15	5.05	5.15	4.95	25.10	38.20	50.15
	SBMM																	
150	5.55	5.20	5.00	19.55	40.30	59.65	5.55	5.20	5.00	19.55	40.30	59.65	5.55	5.20	5.00	19.55	40.30	59.65
500	5.05	5.05	5.40	48.30	88.50	98.00	5.05	5.05	5.40	48.30	88.50	98.00	5.05	5.05	5.40	48.30	88.50	98.00
1000	5.45	5.60	4.85	76.45	99.30	100.00	5.45	5.60	4.85	76.45	99.30	100.00	5.45	5.60	4.85	76.45	99.30	100.00
	GMM estimators																	
	One-step first difference GMM estimator based on "DIF1"																	
150	22.40	39.15	50.85	32.05	77.10	96.75	28.05	42.95	53.10	37.05	78.85	96.65	29.10	45.55	55.10	39.05	78.75	97.00
500	13.95	21.00	26.75	23.30	70.60	97.75	19.60	23.70	29.55	29.05	71.15	97.35	20.95	25.40	30.30	30.35	70.80	97.40
1000	12.30	12.80	16.20	23.00	75.10	99.45	13.10	15.80	17.30	24.00	72.05	99.20	16.75	16.85	17.95	26.65	71.30	99.00
	Two-step first difference GMM estimator based on "DIF1"																	
150	36.30	65.90	82.10	45.05	86.90	97.15	42.05	70.05	83.05	49.40	89.00	97.75	45.35	71.00	83.55	53.45	89.30	97.40
500	20.15	30.85	42.45	29.20	75.05	98.10	27.65	36.15	46.70	36.05	75.60	98.50	28.85	37.75	46.70	37.90	75.40	98.20
1000	14.50	16.35	24.00	24.75	75.45	99.55	17.10	19.95	26.15	27.00	73.85	98.85	21.55	22.70	26.85	30.55	72.85	99.05
	Two-step first difference GMM estimator based on "DIF1" with Windmeijer standard errors																	
150	22.80	28.95	6.00	29.40	57.05	30.25	29.55	33.35	7.00	35.10	57.35	30.30	31.25	33.55	8.15	37.90	57.95	31.35
500	13.40	19.10	21.45	20.00	62.70	94.20	19.45	21.35	24.80	26.00	61.00	93.10	20.45	22.65	25.15	26.30	61.85	93.20
1000	10.55	11.50	14.55	19.65	69.70	98.75	11.95	14.15	17.45	19.10	67.00	98.20	14.85	16.15	17.75	23.00	64.60	97.90
	Continuous-updating first difference GMM estimator based on "DIF1"																	
150	27.30	50.65	77.30	31.90	57.45	83.00	35.25	55.75	79.20	38.85	61.65	82.30	36.60	58.65	78.20	41.65	63.10	84.45
500	13.60	15.30	23.00	17.45	35.45	77.45	19.25	18.80	25.50	24.05	33.70	71.05	20.70	18.95	24.60	24.30	33.95	70.15
1000	8.35	9.60	12.80	15.65	44.45	92.30	10.60	10.35	12.95	15.55	38.40	88.30	13.85	11.35	13.70	19.25	37.10	86.45
	Continuous-updating first difference GMM estimator based on "DIF1" with NW standard errors																	
150	29.80	40.90	84.20	33.05	47.15	87.55	39.65	43.90	83.75	42.20	48.90	86.55	40.65	52.25	85.50	44.00	56.20	87.65
500	13.65	9.15	20.85	16.75	27.65	73.75	22.75	9.60	22.00	25.45	22.70	67.20	24.25	14.55	30.40	26.55	25.05	66.50
1000	7.45	6.15	12.15	14.00	39.00	91.85	11.25	6.45	12.10	15.10	31.30	86.20	16.90	9.35	15.95	20.60	31.70	82.90
	One-step first difference GMM estimator based on "DIF2"																	
150	20.25	16.20	11.55	28.05	36.10	50.30	26.30	27.10	21.90	33.05	45.75	51.55	26.40	32.55	30.90	34.60	48.70	57.00
500	14.00	8.60	7.10	21.00	36.45	64.60	19.85	18.95	11.75	28.75	38.05	50.55	21.60	27.40	25.35	28.75	46.30	54.60
1000	11.05	6.35	6.15	21.75	42.70	79.85	14.60	11.20	8.15	22.75	35.75	57.95	20.70	22.95	21.85	29.20	44.10	54.20
	Two-step first difference GMM estimator based on "DIF2"																	
150	32.85	42.15	42.60	39.80	60.20	71.50	39.55	59.75	64.55	46.95	70.90	80.05	42.90	67.10	74.65	49.05	76.80	86.60
500	18.60	18.30	15.80	27.20	46.05	69.05	29.20	35.35	36.20	35.90	53.75	66.60	29.45	48.75	56.00	36.30	65.05	75.65
1000	13.80	10.10	9.85	24.00	45.40	81.50	18.30	20.30	19.70	26.20	44.25	63.45	24.80	35.95	42.35	32.70	53.90	66.75
	Two-step first difference GMM estimator based on "DIF2" with Windmeijer standard errors																	
150	22.00	21.05	17.05	27.70	34.75	44.00	27.90	31.25	29.20	33.00	42.60	47.30	31.10	35.70	36.20	36.65	44.75	51.75
500	13.60	10.15	9.50	18.90	35.65	62.15	20.75	21.50	19.80	27.10	35.65	50.15	22.35	27.20	29.75	27.45	39.80	52.75
1000	10.60	7.10	7.60	18.25	40.00	78.75	13.80	13.15	13.15	20.10	34.40	56.45	19.65	21.20	24.85	26.40	36.40	52.25
	Continuous-updating first difference GMM estimator based on "DIF2"																	
150	26.65	33.20	30.50	30.40	36.25	42.90	35.15	47.20	54.75	39.00	45.90	58.30	36.60	52.80	67.05	40.25	52.05	68.20
500	12.60	9.85	7.55	16.65	21.30	44.90	21.15	25.10	20.95	24.25	28.35	31.35	23.20	35.00	41.25	26.95	34.85	43.45
1000	8.10	5.25	6.40	14.60	25.65	66.65	11.75	12.10	9.55	16.20	19.60	30.05	17.20	21.20	23.85	22.05	25.20	28.10
	Continuous-updating first difference GMM estimator based on "DIF2" with NW standard errors																	
150	30.95	28.85	26.40	33.80	34.05	39.45	38.85	38.75	40.55	40.50	41.65	45.70	40.95	44.40	49.45	42.75	47.55	51.95
500	16.50	7.25	6.90	19.40	17.15	40.25	26.80	19.25	13.30	29.20	23.70	23.30	29.40	25.45	21.25	32.05	30.40	25.95
1000	8.85	4.00	5.85	13.85	23.85	65.10	16.40	7.40	6.40	19.65	13.75	25.60	24.40	12.75	11.15	28.55	18.15	17.75

Notes: See notes to Table 1.

Table 4 (Continued): Size (×100, $H_0 : \phi = 0.9$) and Power (×100, $H_1 : \phi = 1$) findings for the estimation of ϕ in experiments with $m_i = 50$ and $\phi = 0.9$

	Size (×100)			**Power (×100)**			**Size (×100)**			**Power (×100)**			**Size (×100)**			**Power (×100)**		
(N,T)	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**
	$\sigma_\alpha = 1$						$\sigma_\alpha = 2$						$\sigma_\alpha = 5$					
	GMM estimators																	
	One-step first difference GMM estimator based on "SYS1"																	
150	21.25	32.90	35.35	1.50	26.05	65.95	59.20	88.35	95.15	0.60	6.10	24.40	89.50	99.70	100.00	0.15	2.40	6.20
500	16.70	22.35	23.30	12.15	76.35	97.95	52.95	78.85	89.40	2.65	33.85	76.50	87.95	99.45	99.95	1.05	8.15	22.15
1000	13.05	16.75	14.60	42.90	97.10	99.95	40.70	65.90	78.60	12.05	75.55	98.30	82.10	98.95	99.90	2.90	23.70	54.70
	Two-step first difference GMM estimator based on "SYS1"																	
150	42.15	71.60	87.75	38.35	87.80	98.65	74.45	97.30	99.80	40.15	75.85	93.50	94.45	100.00	100.00	41.10	67.95	85.40
500	28.05	45.25	58.00	61.35	99.75	100.00	62.30	92.15	98.20	55.05	90.80	99.00	93.05	99.95	99.95	50.50	79.05	87.10
1000	18.95	32.95	39.15	84.10	100.00	100.00	50.35	82.90	92.45	71.95	98.20	99.95	90.80	99.90	100.00	61.85	87.35	94.45
	Two-step first difference GMM estimator based on "SYS1" with Windmeijer standard errors																	
150	17.75	12.00	1.10	7.85	10.30	1.75	44.25	61.55	23.20	4.65	3.55	4.55	72.65	94.40	74.50	3.50	5.25	3.55
500	13.40	12.70	7.55	29.60	80.85	93.40	36.45	57.05	64.30	12.45	31.95	39.10	74.65	96.95	99.60	4.60	10.85	8.00
1000	10.65	12.10	6.60	65.05	99.25	99.90	27.95	46.10	52.80	29.45	71.65	89.95	69.80	96.70	99.40	8.35	21.05	18.70
	Continuous-updating first difference GMM estimator based on "SYS1"																	
150	48.35	67.50	89.15	49.95	94.85	99.20	74.95	88.25	95.85	60.65	96.50	99.40	92.15	97.40	98.45	69.90	97.50	99.35
500	28.70	37.00	43.80	67.85	99.95	100.00	55.70	65.30	68.65	75.20	100.00	100.00	87.90	90.10	88.05	83.15	100.00	100.00
1000	21.15	24.75	25.75	87.75	100.00	100.00	43.65	47.85	48.90	86.20	100.00	100.00	81.55	82.75	78.40	90.45	100.00	100.00
	Continuous-updating first difference GMM estimator based on "SYS1" with NW standard errors																	
150	34.80	53.75	88.70	34.85	90.55	97.70	52.30	66.25	90.55	31.85	89.35	96.90	62.60	78.40	94.25	26.50	83.60	98.05
500	20.00	21.35	28.20	55.70	96.55	99.75	35.00	29.80	30.95	49.05	93.05	98.80	59.85	51.00	45.30	34.35	85.05	93.80
1000	14.35	15.30	14.45	79.60	99.40	99.90	26.20	17.75	14.70	69.10	96.15	99.35	51.25	33.95	22.60	50.30	91.45	96.65
	One-step first difference GMM estimator based on "SYS2"																	
150	21.55	44.10	59.85	1.40	16.50	33.55	59.75	92.20	98.50	0.70	4.25	9.15	89.95	99.80	100.00	0.15	2.05	4.05
500	17.20	29.30	39.30	11.95	67.30	91.85	53.45	83.60	95.25	2.45	26.15	51.95	88.10	99.65	99.95	1.05	7.25	14.10
1000	13.30	21.35	26.15	42.75	96.55	99.90	40.95	70.75	88.00	12.30	67.70	92.35	82.45	99.05	99.95	2.90	20.45	37.30
	Two-step first difference GMM estimator based on "SYS2"																	
150	39.25	57.85	70.00	36.00	68.95	81.70	71.70	94.85	99.35	37.20	53.55	65.60	94.05	100.00	100.00	39.15	50.05	55.15
500	25.60	33.75	38.90	57.15	96.60	99.75	60.50	87.85	95.60	50.60	81.60	93.90	92.75	99.90	100.00	47.20	61.80	68.45
1000	18.05	23.20	21.90	81.20	99.95	100.00	49.20	73.65	86.55	69.20	96.60	99.75	90.50	99.75	100.00	58.55	76.80	83.70
	Two-step first difference GMM estimator based on "SYS2" with Windmeijer standard errors																	
150	16.45	20.30	24.05	8.20	20.35	30.35	43.20	75.05	89.55	4.65	10.15	16.80	74.85	98.85	99.90	4.05	7.90	14.20
500	12.65	12.10	13.85	29.00	81.30	97.05	35.95	62.65	79.40	10.90	40.45	66.35	75.90	98.45	99.95	4.70	14.50	23.15
1000	10.15	10.20	7.55	63.95	99.20	100.00	26.35	48.30	62.05	27.55	81.15	97.00	70.55	97.60	99.90	8.30	30.55	49.85
	Continuous-updating first difference GMM estimator based on "SYS2"																	
150	47.70	53.40	60.45	49.80	87.20	94.35	74.35	80.40	82.00	59.00	91.00	97.20	92.15	95.25	95.20	68.50	94.95	98.15
500	27.75	27.65	27.00	64.60	99.20	100.00	55.45	56.85	53.65	72.60	99.55	100.00	87.45	87.40	83.90	81.40	99.90	100.00
1000	20.35	18.60	15.05	85.50	100.00	100.00	41.95	41.85	36.30	85.45	99.95	100.00	81.55	81.15	75.15	89.00	100.00	100.00
	Continuous-updating first difference GMM estimator based on "SYS2" with NW standard errors																	
150	32.30	31.15	39.45	33.35	68.10	87.20	50.80	48.90	50.00	30.15	66.55	88.35	63.45	64.90	66.70	25.50	59.75	85.55
500	19.20	14.50	13.40	52.30	93.50	98.80	35.00	24.70	18.60	45.15	86.25	96.35	59.70	48.65	36.85	32.70	80.65	92.20
1000	13.40	11.55	8.40	76.05	98.70	99.85	25.85	17.05	11.60	65.05	94.75	99.30	50.00	35.65	21.85	48.55	89.00	96.05

Notes: See notes to Table 1.

Table 5: Bias and RMSE findings for the estimation of ϕ in experiments with, $m_i \sim IIDU[1,4]$, and $\phi = 0.5$

	Bias (×100)			RMSE (×100)			Bias (×100)			RMSE (×100)			Bias (×100)			RMSE (×100)		
(N,T)	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**
	$\sigma_\alpha = 1$						$\sigma_\alpha = 2$						$\sigma_\alpha = 5$					
	BMM estimators																	
	BMM																	
150	2.04	0.37	0.12	13.93	6.81	4.77	2.04	0.37	0.12	13.93	6.81	4.77	2.04	0.37	0.12	13.93	6.81	4.77
500	0.93	0.03	0.03	8.42	3.48	2.56	0.93	0.03	0.03	8.42	3.48	2.56	0.93	0.03	0.03	8.42	3.48	2.56
1000	0.46	0.13	0.07	5.46	2.53	1.83	0.46	0.13	0.07	5.46	2.53	1.83	0.46	0.13	0.07	5.46	2.53	1.83
	SBMM																	
150	-0.57	-0.27	-0.08	8.46	5.18	4.15	-0.57	-0.27	-0.08	8.46	5.18	4.15	-0.57	-0.27	-0.08	8.46	5.18	4.15
500	-0.69	-0.30	-0.20	4.79	2.77	2.25	-0.69	-0.30	-0.20	4.79	2.77	2.25	-0.69	-0.30	-0.20	4.79	2.77	2.25
1000	-0.68	-0.20	-0.12	3.37	2.01	1.65	-0.68	-0.20	-0.12	3.37	2.01	1.65	-0.68	-0.20	-0.12	3.37	2.01	1.65
	GMM estimators																	
	One-step first difference GMM estimator based on "DIF1"																	
150	-3.91	-2.64	-2.10	14.78	5.88	3.97	-5.25	-3.15	-2.25	18.20	6.40	4.18	-6.46	-3.27	-2.31	20.37	6.62	4.28
500	-1.35	-0.89	-0.66	8.00	3.04	2.03	-1.77	-1.03	-0.69	10.06	3.27	2.13	-2.11	-1.09	-0.71	11.32	3.37	2.17
1000	-0.74	-0.39	-0.31	5.42	2.09	1.39	-0.98	-0.44	-0.34	6.85	2.21	1.46	-1.05	-0.46	-0.35	7.85	2.29	1.49
	Two-step first difference GMM estimator based on "DIF1"																	
150	-3.64	-2.43	-2.01	15.48	6.70	5.57	-5.37	-3.02	-2.18	19.13	7.42	5.92	-6.79	-3.10	-2.17	21.39	7.62	6.00
500	-1.22	-0.84	-0.62	8.13	3.20	2.29	-1.62	-1.00	-0.66	10.23	3.46	2.36	-2.07	-1.06	-0.68	11.44	3.56	2.42
1000	-0.66	-0.37	-0.31	5.49	2.16	1.47	-0.93	-0.43	-0.32	6.93	2.28	1.54	-0.99	-0.43	-0.33	7.92	2.37	1.58
	Continuous-updating first difference GMM estimator based on "DIF1"																	
150	1.06	0.18	-0.17	16.57	7.03	7.09	1.97	0.15	-0.25	22.11	7.69	7.71	2.74	0.22	-0.01	26.16	7.89	7.61
500	0.22	-0.01	0.03	8.24	3.14	2.30	0.61	-0.04	0.06	10.60	3.41	2.35	0.68	-0.02	0.04	11.97	3.51	2.42
1000	0.02	0.05	0.02	5.51	2.14	1.47	0.15	0.07	0.04	7.02	2.26	1.53	0.34	0.10	0.04	8.06	2.36	1.57
	One-step first difference GMM estimator based on "DIF2"																	
150	-3.80	-1.72	-1.10	15.53	6.27	4.14	-6.25	-3.78	-2.04	21.26	9.20	5.55	-8.49	-7.32	-5.34	26.11	14.21	9.83
500	-1.24	-0.61	-0.37	8.42	3.32	2.24	-1.92	-1.24	-0.60	12.05	4.81	2.93	-2.49	-2.52	-1.86	15.50	7.56	4.98
1000	-0.73	-0.26	-0.14	5.68	2.33	1.57	-1.13	-0.55	-0.31	8.14	3.26	2.02	-1.31	-1.15	-0.86	10.72	5.28	3.49
	Two-step first difference GMM estimator based on "DIF2"																	
150	-3.78	-1.60	-0.87	16.26	6.66	4.55	-6.63	-3.75	-1.78	22.32	9.97	5.99	-9.06	-7.57	-5.16	27.40	15.55	10.84
500	-1.23	-0.58	-0.29	8.52	3.40	2.30	-1.89	-1.22	-0.49	12.26	4.98	3.01	-2.76	-2.51	-1.75	15.76	7.84	5.19
1000	-0.72	-0.25	-0.09	5.75	2.37	1.58	-1.19	-0.53	-0.24	8.24	3.32	2.03	-1.44	-1.14	-0.80	10.82	5.40	3.58
	Continuous-updating first difference GMM estimator based on "DIF2"																	
150	0.68	0.40	0.56	17.07	6.63	4.59	1.52	0.02	0.47	25.89	9.47	5.82	4.82	0.39	0.37	36.14	16.13	9.77
500	0.12	0.03	0.15	8.56	3.34	2.29	0.71	-0.06	0.23	12.75	4.81	2.95	1.16	0.12	0.02	17.05	7.64	4.86
1000	-0.08	0.05	0.13	5.74	2.36	1.59	0.07	0.06	0.12	8.31	3.26	2.02	0.50	0.21	0.12	11.12	5.36	3.47
	One-step first difference GMM estimator based on "SYS1"																	
150	1.13	0.55	0.38	9.69	4.71	3.33	6.44	5.76	4.70	13.86	8.07	6.12	25.48	23.57	21.35	29.70	24.76	22.13
500	0.18	0.17	0.08	5.41	2.57	1.88	2.00	1.88	1.55	7.64	3.61	2.63	12.01	11.36	9.72	17.39	12.65	10.46
1000	0.04	0.16	0.09	3.85	1.86	1.35	1.08	1.07	0.86	5.41	2.47	1.73	6.55	6.68	5.63	11.80	7.79	6.22
	Two-step first difference GMM estimator based on "SYS1"																	
150	0.97	0.45	0.18	8.88	4.43	4.11	4.82	3.12	2.76	12.09	6.03	5.11	22.99	20.41	18.80	28.43	22.23	19.90
500	0.17	0.10	0.03	4.59	2.08	1.61	1.16	0.43	0.36	5.73	2.26	1.69	9.33	6.48	5.65	14.38	8.19	6.65
1000	0.05	0.06	0.03	3.27	1.44	1.06	0.51	0.17	0.14	3.85	1.50	1.10	4.05	2.56	2.23	8.23	3.78	2.89
	Continuous-updating first difference GMM estimator based on "SYS1"																	
150	0.35	0.28	-0.17	9.33	4.93	5.91	0.97	0.25	-0.10	10.41	5.10	5.93	3.78	0.38	-0.23	16.03	5.41	7.42
500	-0.07	0.07	0.00	4.62	2.10	1.65	0.11	0.05	0.00	5.18	2.13	1.64	0.34	0.06	0.00	5.91	2.13	1.65
1000	-0.05	0.05	0.02	3.28	1.44	1.07	0.03	0.06	0.03	3.66	1.45	1.07	0.05	0.06	0.03	3.80	1.46	1.07
	One-step first difference GMM estimator based on "SYS2"																	
150	1.29	1.36	1.39	9.68	5.05	3.90	6.75	8.02	7.41	14.15	10.21	8.74	26.31	28.85	28.98	30.76	29.84	29.58
500	0.25	0.41	0.37	5.41	2.69	2.08	2.14	2.70	2.49	7.83	4.40	3.54	12.45	15.36	15.31	18.27	16.71	16.11
1000	0.06	0.28	0.25	3.85	1.95	1.49	1.14	1.51	1.37	5.57	2.92	2.23	6.79	9.43	9.44	12.43	10.69	10.16
	Two-step first difference GMM estimator based on "SYS2"																	
150	0.75	0.40	0.43	8.78	4.77	3.96	4.83	3.81	3.39	12.08	6.81	5.63	23.77	23.65	23.49	29.26	25.35	24.55
500	0.08	0.09	0.06	4.56	2.37	1.93	1.17	0.56	0.50	5.78	2.65	2.14	10.16	8.41	8.21	15.20	10.30	9.43
1000	0.01	0.05	0.04	3.26	1.65	1.35	0.52	0.21	0.18	3.93	1.75	1.41	4.54	3.50	3.41	8.90	4.89	4.27
	Continuous-updating first difference GMM estimator based on "SYS2"																	
150	-0.03	-0.30	-0.32	9.31	4.99	4.22	0.55	-0.21	-0.33	10.41	5.02	4.32	4.71	0.15	-0.19	17.65	5.84	4.41
500	-0.19	-0.07	-0.11	4.61	2.38	1.95	-0.02	-0.10	-0.10	5.27	2.44	1.96	0.40	-0.07	-0.11	6.33	2.46	1.97
1000	-0.11	-0.03	-0.04	3.28	1.65	1.35	-0.03	-0.02	-0.02	3.73	1.68	1.35	0.05	-0.01	-0.03	4.04	1.70	1.35

Notes: See notes to Table 1.

Table 6: Size (×100, $H_0 : \phi = 0.5$) and Power (×100, $H_1 : \phi = 0.6$) findings for the estimation of ϕ in experiments with, $m_i \sim IIDU[1, 4]$, and $\phi = 0.5$

	Size (×100)			Power (×100)			Size (×100)			Power (×100)			Size (×100)			Power (×100)		
(N,T)	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**
	$\sigma_\alpha = 1$						$\sigma_\alpha = 2$						$\sigma_\alpha = 5$					
	BMM estimators																	
	BMM																	
150	2.35	3.30	5.70	15.25	37.20	57.10	2.35	3.30	5.70	15.25	37.20	57.10	2.35	3.30	5.70	15.25	37.20	57.10
500	3.15	4.60	4.80	34.00	80.60	96.60	3.15	4.60	4.80	34.00	80.60	96.60	3.15	4.60	4.80	34.00	80.60	96.60
1000	2.85	5.95	5.20	51.25	96.45	100.00	2.85	5.95	5.20	51.25	96.45	100.00	2.85	5.95	5.20	51.25	96.45	100.00
	SBMM																	
150	5.10	5.85	6.05	23.90	52.05	70.00	5.10	5.85	6.05	23.90	52.05	70.00	5.10	5.85	6.05	23.90	52.05	70.00
500	5.50	4.15	5.15	62.20	96.20	99.60	5.50	4.15	5.15	62.20	96.20	99.60	5.50	4.15	5.15	62.20	96.20	99.60
1000	5.05	4.80	5.75	89.75	99.65	100.00	5.05	4.80	5.75	89.75	99.65	100.00	5.05	4.80	5.75	89.75	99.65	100.00
	GMM estimators																	
	One-step first difference GMM estimator based on "DIF1"																	
150	6.85	8.45	9.75	20.50	68.50	95.70	7.50	8.45	9.65	19.15	65.05	94.10	8.45	8.65	10.40	18.85	63.75	93.40
500	5.85	6.00	7.15	33.35	96.35	100.00	6.15	5.85	6.75	26.50	94.05	100.00	6.35	5.30	7.15	23.75	92.40	100.00
1000	4.50	4.85	5.65	49.50	99.90	100.00	5.20	4.85	5.40	35.55	99.65	100.00	5.50	4.75	5.85	30.35	99.35	100.00
	Two-step first difference GMM estimator based on "DIF1"																	
150	9.40	24.40	53.35	24.25	76.40	95.50	10.50	23.65	55.75	21.75	73.90	94.20	10.65	24.40	54.55	22.65	72.65	95.05
500	6.40	9.90	17.05	34.70	96.40	100.00	6.90	9.20	16.00	26.50	94.75	99.95	7.10	9.50	16.65	23.80	93.45	99.90
1000	5.20	6.55	10.20	49.60	99.80	100.00	5.50	6.30	8.90	35.65	99.70	100.00	5.90	6.00	9.60	31.00	99.30	100.00
	Two-step first difference GMM estimator based on "DIF1" with Windmeijer standard errors																	
150	6.45	5.85	1.10	18.40	49.65	28.85	7.10	7.05	1.40	16.15	46.85	27.95	7.20	6.70	1.45	16.55	45.00	26.20
500	5.30	5.55	7.20	32.30	94.30	99.90	5.70	5.80	6.75	23.75	90.90	99.80	5.85	5.80	6.95	21.80	89.80	99.45
1000	4.55	4.75	5.70	48.25	99.70	100.00	4.80	4.85	5.30	34.50	99.40	100.00	5.20	4.80	5.60	29.50	99.20	100.00
	Continuous-updating first difference GMM estimator based on "DIF1"																	
150	9.35	24.20	63.00	17.95	60.75	87.60	10.30	27.05	63.90	16.20	56.75	85.20	10.95	24.95	64.10	16.60	55.25	83.40
500	6.85	9.25	18.40	29.20	93.40	99.95	6.55	8.95	17.15	21.35	89.90	99.90	6.85	8.45	16.75	19.65	87.95	99.50
1000	4.95	6.35	9.20	44.60	99.65	100.00	5.55	5.30	9.70	31.40	99.25	100.00	5.60	6.25	10.35	25.95	98.90	100.00
	Continuous-updating first difference GMM estimator based on "DIF1" with NW standard errors																	
150	8.25	28.00	79.70	15.90	62.75	93.70	8.20	29.05	77.90	13.55	58.75	92.80	7.85	28.25	80.00	14.00	57.35	90.95
500	5.70	9.55	22.50	27.80	92.90	99.90	6.00	9.45	21.95	19.55	89.70	99.85	5.60	8.65	20.10	17.90	87.05	99.55
1000	4.85	6.40	11.15	43.65	99.45	99.90	5.45	5.80	11.40	30.25	99.25	100.00	5.15	6.45	12.20	25.15	98.80	99.85
	One-step first difference GMM estimator based on "DIF2"																	
150	6.75	6.60	5.65	18.50	51.65	80.65	7.05	7.00	5.75	16.15	36.45	65.20	9.05	9.75	9.40	18.00	30.15	45.65
500	5.65	6.10	5.70	30.20	91.35	99.70	6.75	5.40	4.85	21.55	69.45	96.80	7.30	6.75	5.75	19.20	40.90	72.30
1000	5.15	4.75	5.60	46.00	99.35	100.00	5.35	4.95	4.85	28.50	91.50	99.90	5.55	5.75	5.35	21.45	58.00	91.20
	Two-step first difference GMM estimator based on "DIF2"																	
150	8.75	11.05	14.00	21.85	56.85	83.00	10.25	12.90	14.45	20.80	45.10	72.00	12.05	17.35	21.05	20.65	39.55	55.15
500	6.40	6.80	7.30	32.50	91.30	99.80	7.55	7.50	7.20	22.65	70.40	97.05	7.90	8.70	8.55	20.70	43.40	74.15
1000	5.60	5.90	6.70	46.05	99.30	100.00	5.65	5.40	5.80	29.35	91.25	99.95	6.15	6.75	7.15	22.25	58.80	91.35
	Two-step first difference GMM estimator based on "DIF2" with Windmeijer standard errors																	
150	6.50	6.15	5.35	16.90	45.75	68.95	6.75	7.10	5.85	15.45	32.10	53.80	8.70	9.65	7.65	16.50	24.95	34.05
500	5.15	5.55	5.55	30.55	89.55	99.35	6.35	5.60	5.05	20.10	66.10	95.80	6.65	6.70	6.15	18.65	38.20	67.55
1000	5.05	5.25	5.90	45.35	99.25	100.00	5.40	4.80	4.85	28.25	90.55	99.95	5.05	5.60	5.75	20.75	57.10	89.80
	Continuous-updating first difference GMM estimator based on "DIF2"																	
150	9.00	11.75	14.85	16.55	45.75	72.05	9.30	13.25	15.20	15.40	30.80	55.85	10.55	17.40	18.45	16.05	24.60	34.10
500	6.20	6.75	7.70	27.40	87.70	99.25	6.85	6.60	7.15	17.80	61.50	94.80	7.10	7.65	7.40	16.00	31.45	59.40
1000	5.40	5.80	6.50	42.50	99.05	100.00	4.85	5.35	6.55	25.45	88.25	99.90	5.05	7.45	6.65	18.15	48.95	86.05
	Continuous-updating first difference GMM estimator based on "DIF2" with NW standard errors																	
150	7.80	12.10	18.15	14.35	46.30	76.35	8.00	13.60	17.45	13.00	28.85	61.10	8.85	15.55	18.90	14.10	21.95	34.45
500	5.35	6.75	8.35	26.55	87.45	99.05	6.55	6.35	7.75	16.15	61.00	94.95	6.05	6.80	7.50	14.95	28.70	59.20
1000	5.10	5.95	6.95	40.55	98.90	99.85	4.90	5.25	6.80	24.40	87.65	99.75	4.60	6.55	7.40	16.60	48.05	84.75

Notes: See notes to Table 1.

Table 6 (Continued): Size (×100, $H_0 : \phi = 0.5$) and Power (×100, $H_1 : \phi = 0.6$) findings for the estimation of ϕ in experiments with, $m_i \sim IIDU\,[1, 4]$, and $\phi = 0.5$

	Size (×100)			Power (×100)			Size (×100)			Power (×100)			Size (×100)			Power (×100)		
(N,T)	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**
	$\sigma_\alpha = 1$						$\sigma_\alpha = 2$						$\sigma_\alpha = 5$					
	GMM estimators																	
	One-step first difference GMM estimator based on "SYS1"																	
150	7.25	6.40	6.70	15.15	52.70	84.45	16.90	23.70	28.10	5.70	12.40	31.10	58.95	84.75	95.20	44.35	57.05	63.35
500	5.85	4.05	5.40	43.75	96.65	100.00	9.10	10.15	13.55	14.00	75.80	98.40	34.25	62.00	79.40	14.25	12.45	12.30
1000	5.05	4.45	5.35	73.60	99.80	100.00	6.70	8.65	10.75	36.80	98.45	100.00	21.00	47.00	68.65	4.85	18.20	51.65
	Two-step first difference GMM estimator based on "SYS1"																	
150	14.20	26.65	61.70	30.80	86.75	97.45	28.90	48.55	74.55	29.60	72.10	93.25	75.95	96.80	99.80	64.20	85.85	94.75
500	7.25	10.70	18.40	61.20	99.85	100.00	12.05	14.35	24.25	49.75	99.75	100.00	50.30	75.45	88.30	47.95	76.30	88.65
1000	6.15	8.20	11.15	87.90	100.00	100.00	8.45	9.80	13.15	75.30	100.00	100.00	32.25	51.85	65.65	58.10	96.90	99.85
	Two-step first difference GMM estimator based on "SYS1" with Windmeijer standard errors																	
150	7.35	2.70	0.00	21.35	47.75	5.20	11.60	3.75	0.30	13.85	21.25	2.10	45.75	56.30	30.95	32.40	28.60	15.20
500	5.35	5.00	4.35	56.20	99.65	100.00	5.65	4.75	3.75	40.35	98.50	100.00	22.15	36.35	42.95	17.00	30.55	39.15
1000	5.35	5.55	5.40	86.40	100.00	100.00	5.70	5.30	6.05	70.85	100.00	100.00	12.20	21.50	29.70	35.70	85.75	97.30
	Continuous-updating first difference GMM estimator based on "SYS1"																	
150	16.00	32.50	73.20	33.95	84.70	94.30	20.80	39.65	75.85	37.75	86.40	95.65	39.80	57.40	83.90	56.15	92.80	97.30
500	8.05	10.95	20.25	63.05	99.85	100.00	9.60	12.00	22.40	55.65	99.90	100.00	15.75	21.15	33.50	60.55	99.85	100.00
1000	5.80	7.80	11.85	87.85	100.00	100.00	7.55	8.75	12.40	78.85	100.00	100.00	11.25	13.15	18.85	74.40	100.00	100.00
	Continuous-updating first difference GMM estimator based on "SYS1" with NW standard errors																	
150	13.30	37.30	87.80	32.00	86.35	97.80	14.55	38.05	87.40	31.55	85.80	97.95	18.80	37.45	85.85	34.80	85.65	96.90
500	7.30	11.45	26.10	60.70	99.85	99.95	7.30	10.75	25.15	53.50	99.85	100.00	7.00	11.45	25.20	49.65	99.75	100.00
1000	5.75	8.30	13.60	86.70	99.85	99.90	6.25	7.70	13.05	77.55	99.95	99.95	4.85	7.70	13.10	75.10	99.95	100.00
	One-step first difference GMM estimator based on "SYS2"																	
150	7.45	6.60	8.10	14.70	42.20	66.25	17.55	32.50	42.65	5.95	8.05	12.45	61.00	90.50	98.15	46.15	73.60	87.20
500	6.00	4.55	6.00	42.90	94.75	99.90	9.50	12.80	21.40	12.35	56.15	87.00	35.35	70.45	89.05	16.05	26.00	31.70
1000	5.10	4.65	6.55	73.65	99.75	100.00	6.95	11.30	13.75	34.30	93.80	99.85	21.85	53.50	80.40	5.40	8.45	10.70
	Two-step first difference GMM estimator based on "SYS2"																	
150	12.65	17.20	23.20	30.85	73.60	88.80	27.05	38.75	46.30	28.65	54.50	72.75	76.35	96.25	99.55	65.60	86.85	92.05
500	7.00	7.05	9.70	61.65	98.95	100.00	11.65	11.40	14.10	49.15	97.70	99.90	52.35	77.65	88.30	46.65	60.90	66.50
1000	5.95	5.90	7.90	87.95	99.95	100.00	8.85	7.25	9.45	74.05	99.95	100.00	34.25	53.95	67.75	55.70	89.55	95.10
	Two-step first difference GMM estimator based on "SYS2" with Windmeijer standard errors																	
150	7.20	5.35	5.20	22.00	51.05	66.25	12.10	11.50	10.70	14.25	22.70	32.20	51.35	83.80	94.35	38.40	61.10	76.70
500	5.50	4.35	5.55	57.40	98.25	99.95	5.80	5.05	6.60	39.45	94.40	99.40	24.55	52.15	68.15	18.95	28.40	31.20
1000	5.50	4.95	5.35	86.80	99.95	100.00	6.10	4.70	5.55	69.30	99.95	100.00	12.80	30.25	43.30	31.70	72.10	86.85
	Continuous-updating first difference GMM estimator based on "SYS2"																	
150	15.10	17.85	26.30	34.45	76.75	90.85	20.20	23.90	31.70	37.65	79.75	92.55	41.55	43.00	52.10	56.50	89.00	95.80
500	7.95	7.60	9.50	63.60	98.90	100.00	9.45	8.40	11.35	55.45	99.05	100.00	17.05	17.65	20.55	58.40	99.55	100.00
1000	6.00	5.80	7.15	88.30	99.95	100.00	7.65	6.35	7.70	78.30	100.00	100.00	11.55	10.75	13.10	72.35	99.95	100.00
	Continuous-updating first difference GMM estimator based on "SYS2" with NW standard errors																	
150	12.65	17.50	29.85	32.65	76.00	91.95	15.10	17.00	30.35	32.90	74.05	91.30	22.20	17.70	30.05	37.40	73.10	90.75
500	7.45	7.40	9.70	61.60	99.00	99.90	7.35	7.35	10.50	53.60	98.65	100.00	7.35	6.85	10.40	48.50	98.50	99.95
1000	5.90	5.55	7.40	87.15	99.90	99.85	6.35	5.60	7.25	76.85	99.95	99.95	5.50	5.35	7.10	72.95	99.95	100.00

Notes: See notes to Table 1.

Table 7: Bias and RMSE findings for the estimation of ϕ in experiments with distant past and $\phi = 0.9$

	Bias (×100)			**RMSE (×100)**			**Bias (×100)**			**RMSE (×100)**			**Bias (×100)**			**RMSE (×100)**		
(N,T)	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**
	$\sigma_\alpha = 1$						$\sigma_\alpha = 2$						$\sigma_\alpha = 5$					
	BMM estimators																	
	BMM																	
150	-6.49	-2.62	-1.14	14.28	8.41	6.49	-6.49	-2.62	-1.14	14.28	8.41	6.49	-6.49	-2.62	-1.14	14.28	8.41	6.49
500	-3.98	-0.96	-0.13	9.87	5.87	4.71	-3.98	-0.96	-0.13	9.87	5.87	4.71	-3.98	-0.96	-0.13	9.87	5.87	4.71
1000	-2.54	-0.37	0.27	8.01	4.94	4.01	-2.54	-0.37	0.27	8.01	4.94	4.01	-2.54	-0.37	0.27	8.01	4.94	4.01
	SBMM																	
150	-2.38	-1.60	-1.18	9.73	5.83	4.74	-2.38	-1.60	-1.18	9.73	5.83	4.74	-2.38	-1.60	-1.18	9.73	5.83	4.74
500	-2.47	-1.55	-1.01	5.56	3.52	2.65	-2.47	-1.55	-1.01	5.56	3.52	2.65	-2.47	-1.55	-1.01	5.56	3.52	2.65
1000	-2.40	-1.61	-1.08	4.26	2.76	2.07	-2.40	-1.61	-1.08	4.26	2.76	2.07	-2.40	-1.61	-1.08	4.26	2.76	2.07
	GMM estimators																	
	One-step first difference GMM estimator based on "DIF1"																	
150	-47.91	-19.37	-10.96	63.77	22.80	12.49	-51.23	-20.21	-11.49	66.40	23.86	13.06	-53.44	-20.69	-11.58	68.97	24.10	13.11
500	-23.76	-7.83	-4.29	40.86	10.85	5.51	-30.29	-8.71	-4.49	46.80	11.76	5.72	-32.28	-8.90	-4.60	48.30	11.97	5.83
1000	-13.06	-4.36	-2.38	28.15	6.76	3.47	-16.01	-4.85	-2.51	31.85	7.37	3.62	-17.98	-4.94	-2.56	33.10	7.49	3.67
	Two-step first difference GMM estimator based on "DIF1"																	
150	-53.78	-23.43	-12.64	72.64	28.87	15.85	-57.37	-24.83	-13.36	75.12	30.58	16.74	-59.65	-25.45	-13.49	78.89	30.90	16.90
500	-26.58	-8.52	-4.57	45.92	12.02	6.18	-34.18	-9.57	-4.79	52.66	13.13	6.44	-35.69	-9.86	-4.92	53.77	13.46	6.53
1000	-13.95	-4.43	-2.41	29.66	7.05	3.66	-17.19	-5.00	-2.57	33.65	7.70	3.85	-19.27	-5.10	-2.61	34.78	7.87	3.91
	Continuous-updating first difference GMM estimator based on "DIF1"																	
150	-24.57	-5.15	-7.50	100.69	51.85	48.26	-28.59	-8.95	-7.83	103.07	59.30	48.98	-32.49	-7.80	-8.27	110.95	58.39	51.01
500	1.92	0.70	0.10	60.20	10.24	4.69	-1.81	0.52	0.15	71.76	11.31	4.89	-2.96	0.54	0.10	74.13	11.57	4.89
1000	4.03	0.20	0.00	35.76	5.96	2.88	5.26	0.15	-0.01	43.01	6.38	3.00	5.14	0.27	-0.01	44.30	6.65	3.06
	One-step first difference GMM estimator based on "DIF2"																	
150	-52.85	-30.91	-17.65	71.76	39.69	24.52	-57.93	-41.23	-30.45	76.04	49.13	37.01	-61.79	-46.53	-37.62	80.87	53.36	43.18
500	-27.40	-12.94	-6.32	48.33	19.86	10.44	-39.45	-26.71	-15.41	58.21	34.07	21.33	-42.59	-33.72	-26.85	61.29	39.50	31.78
1000	-15.67	-6.91	-3.07	34.23	12.50	6.15	-22.53	-16.55	-9.00	41.02	22.83	13.43	-26.54	-23.98	-19.42	43.67	29.47	23.75
	Two-step first difference GMM estimator based on "DIF2"																	
150	-57.71	-38.44	-23.14	79.04	49.91	32.64	-63.46	-51.09	-40.70	83.87	61.34	49.60	-66.99	-56.61	-50.12	90.36	65.53	57.60
500	-29.95	-15.15	-7.49	52.20	23.44	12.53	-42.60	-30.83	-19.58	63.98	39.48	27.20	-45.15	-38.68	-33.47	65.42	45.67	39.74
1000	-16.64	-7.55	-3.31	35.59	13.75	6.63	-23.73	-18.33	-10.41	42.77	25.33	15.88	-27.49	-26.31	-22.49	45.22	32.46	27.54
	Continuous-updating first difference GMM estimator based on "DIF2"																	
150	-27.77	-3.32	-1.38	109.17	73.81	49.91	-34.97	-11.70	-1.62	115.84	91.99	77.89	-41.33	-13.60	-1.88	121.54	97.10	86.76
500	0.58	2.01	-0.63	67.35	26.31	11.08	-4.18	5.82	3.57	80.56	46.59	30.83	-5.01	9.77	8.93	85.87	55.26	46.17
1000	4.40	-0.57	-0.32	42.44	12.13	5.39	7.21	1.71	0.93	54.33	27.00	15.91	7.70	5.13	5.49	57.71	36.16	30.20
	One-step first difference GMM estimator based on "SYS1"																	
150	4.07	4.14	3.57	9.11	5.57	4.52	8.16	7.82	7.52	9.83	8.15	7.70	9.58	9.60	9.50	9.90	9.65	9.53
500	1.63	2.19	1.97	5.59	3.59	2.83	5.68	5.95	5.60	7.65	6.37	5.84	9.01	9.06	8.98	9.62	9.14	9.02
1000	0.83	1.27	1.18	4.09	2.56	2.00	3.61	4.37	4.11	6.02	4.83	4.40	8.48	8.36	8.26	9.29	8.47	8.31
	Two-step first difference GMM estimator based on "SYS1"																	
150	2.75	3.33	3.32	9.73	5.17	4.54	7.35	7.41	7.32	10.51	7.90	7.56	9.37	9.51	9.48	10.03	9.58	9.52
500	0.40	1.23	1.44	5.45	2.94	2.46	4.50	5.17	5.18	7.54	5.72	5.48	8.73	8.88	8.87	9.65	9.01	8.93
1000	-0.23	0.55	0.71	4.10	2.02	1.61	2.38	3.61	3.58	5.67	4.23	3.95	7.95	8.09	8.14	9.09	8.24	8.22
	Continuous-updating first difference GMM estimator based on "SYS1"																	
150	-0.01	-0.31	-0.20	12.94	6.04	6.44	4.05	2.12	1.40	15.47	8.25	7.40	6.79	4.98	3.49	16.25	10.19	9.66
500	-1.36	-0.17	0.10	6.60	2.85	2.11	0.56	0.44	0.27	9.49	4.02	2.45	5.66	3.60	1.77	12.88	7.60	4.88
1000	-0.84	-0.11	0.03	4.57	2.00	1.34	-0.96	0.09	0.11	6.56	2.59	1.59	4.13	1.97	0.53	10.85	5.57	2.73
	One-step first difference GMM estimator based on "SYS2"																	
150	4.25	5.24	5.49	9.17	6.31	6.11	8.28	8.34	8.47	10.04	8.64	8.62	9.61	9.71	9.71	9.93	9.76	9.73
500	1.71	2.76	3.07	5.58	3.89	3.69	5.76	6.43	6.63	7.72	6.78	6.81	9.03	9.22	9.31	9.64	9.29	9.35
1000	0.86	1.62	1.88	4.08	2.72	2.51	3.65	4.78	5.05	6.10	5.19	5.28	8.51	8.56	8.72	9.34	8.67	8.78
	Two-step first difference GMM estimator based on "SYS2"																	
150	2.82	3.67	3.96	9.74	5.67	5.16	7.47	7.64	7.66	10.70	8.27	7.98	9.43	9.55	9.54	10.11	9.66	9.60
500	0.43	1.14	1.51	5.51	3.06	2.59	4.62	5.02	5.13	7.78	5.65	5.47	8.74	8.84	8.87	9.73	9.00	8.96
1000	-0.21	0.44	0.60	4.13	2.12	1.70	2.32	3.36	3.43	5.66	4.03	3.81	7.98	7.97	8.04	9.16	8.16	8.14
	Continuous-updating first difference GMM estimator based on "SYS2"																	
150	-0.33	-0.52	-0.55	13.13	6.23	5.11	3.78	1.78	0.40	15.87	8.57	6.16	6.39	4.91	3.90	16.56	10.31	8.69
500	-1.56	-0.36	-0.10	6.91	3.17	2.39	0.27	0.04	-0.05	9.80	4.27	2.77	5.35	3.39	2.39	13.06	7.88	6.08
1000	-0.89	-0.21	-0.09	4.74	2.21	1.61	-1.20	-0.17	-0.06	6.85	2.76	1.91	3.81	1.93	1.07	11.03	5.94	4.10

Notes: See notes to Table 1.

Table 8: Size (×100, $H_0 : \phi = 0.9$) and Power (×100, $H_1 : \phi = 1$) findings for the estimation of ϕ in experiments with recnt past, $m_i \sim IIDU\,[1, 4]$, and $\phi = 0.9$

	Size (×100)			Power (×100)			Size (×100)			Power (×100)			Size (×100)			Power (×100)		
(N,T)	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**
	$\sigma_\alpha = 1$						$\sigma_\alpha = 2$						$\sigma_\alpha = 5$					
	BMM estimators																	
	BMM																	
150	6.65	6.30	5.30	15.50	25.35	29.05	6.65	6.30	5.30	15.50	25.35	29.05	6.65	6.30	5.30	15.50	25.35	29.05
500	7.55	6.40	6.35	23.00	31.25	39.10	7.55	6.40	6.35	23.00	31.25	39.10	7.55	6.40	6.35	23.00	31.25	39.10
1000	7.15	6.75	5.50	24.95	35.55	46.55	7.15	6.75	5.50	24.95	35.55	46.55	7.15	6.75	5.50	24.95	35.55	46.55
	SBMM																	
150	6.60	5.65	5.90	27.30	52.95	68.90	6.60	5.65	5.90	27.30	52.95	68.90	6.60	5.65	5.90	27.30	52.95	68.90
500	6.85	8.30	7.05	67.60	94.75	99.50	6.85	8.30	7.05	67.60	94.75	99.50	6.85	8.30	7.05	67.60	94.75	99.50
1000	9.65	11.60	8.80	92.50	99.95	100.00	9.65	11.60	8.80	92.50	99.95	100.00	9.65	11.60	8.80	92.50	99.95	100.00
	GMM estimators																	
	One-step first difference GMM estimator based on "DIF1"																	
150	24.65	38.80	47.55	34.45	76.15	96.50	26.05	39.45	49.25	35.10	76.20	96.95	25.45	40.70	48.85	35.55	77.20	96.85
500	16.85	18.90	22.00	26.65	70.60	98.65	19.65	21.50	21.85	30.45	70.75	98.40	19.00	21.40	22.65	29.60	70.15	98.65
1000	11.60	13.20	13.90	22.75	78.45	99.90	12.85	14.50	14.85	23.85	76.00	99.65	13.90	14.55	14.55	24.45	75.15	99.55
	Two-step first difference GMM estimator based on "DIF1"																	
150	36.45	63.55	77.80	44.15	85.75	97.65	37.90	65.00	79.50	45.65	85.70	97.65	39.50	66.30	77.55	46.45	87.40	97.85
500	21.45	27.35	36.80	30.60	74.50	98.35	26.15	29.85	38.00	35.20	75.05	98.45	24.90	31.15	38.35	34.75	75.00	98.50
1000	13.95	15.15	20.60	24.05	79.05	99.80	15.30	16.95	22.10	25.90	77.75	99.70	16.30	17.40	21.70	26.55	76.75	99.55
	Two-step first difference GMM estimator based on "DIF1" with Windmeijer standard errors																	
150	23.90	29.05	4.45	30.30	56.40	28.20	27.30	28.95	5.70	33.65	56.25	29.55	26.75	30.20	5.25	32.40	57.25	30.05
500	14.40	16.85	17.70	21.85	63.00	94.85	17.15	18.70	18.95	24.80	63.15	95.10	17.95	19.15	18.65	24.60	62.05	94.50
1000	10.15	10.85	13.00	18.80	72.60	99.60	11.15	13.10	13.95	19.80	71.50	99.35	11.60	12.35	13.95	20.35	70.10	99.40
	Continuous-updating first difference GMM estimator based on "DIF1"																	
150	28.75	51.05	73.40	32.25	57.30	81.30	31.00	52.05	73.80	35.10	56.75	81.35	31.60	54.85	76.40	35.20	58.50	80.95
500	14.50	15.60	21.35	18.30	36.85	81.70	16.70	16.50	22.55	21.45	36.50	78.45	16.70	17.70	22.40	21.60	37.15	78.95
1000	7.00	8.20	11.45	12.95	50.05	97.55	8.40	9.05	11.65	14.95	46.50	95.85	9.05	9.15	11.60	14.45	45.50	95.60
	Continuous-updating first difference GMM estimator based on "DIF1" with NW standard errors																	
150	31.60	40.55	81.80	34.35	46.95	86.25	34.80	41.05	82.85	37.75	47.00	86.50	35.95	47.75	84.35	38.55	52.50	87.30
500	14.50	9.75	20.80	18.05	30.05	80.55	18.85	10.50	22.65	22.95	27.60	76.45	20.45	13.00	29.30	23.50	29.80	76.25
1000	6.75	5.80	11.00	11.10	44.40	97.40	9.40	6.45	11.25	13.50	40.75	95.45	10.30	7.30	14.00	14.25	41.55	92.85
	One-step first difference GMM estimator based on "DIF2"																	
150	23.70	20.25	13.65	31.00	41.15	48.65	24.60	28.85	25.70	31.70	47.05	53.30	26.70	36.65	33.90	33.40	53.15	60.05
500	16.50	9.70	7.55	24.25	37.05	61.45	21.25	22.65	12.30	29.45	44.20	50.10	21.45	29.95	28.35	30.80	48.80	57.85
1000	11.85	8.15	6.85	20.40	41.40	74.75	14.40	15.75	8.75	24.45	40.25	55.90	17.20	23.75	24.00	24.95	45.15	55.75
	Two-step first difference GMM estimator based on "DIF2"																	
150	34.90	49.05	47.10	41.30	64.40	72.65	36.45	60.40	65.35	43.25	72.00	81.80	38.25	65.40	75.80	44.50	77.15	87.50
500	20.60	21.70	18.60	28.65	45.95	67.55	27.25	40.15	35.55	34.25	57.90	64.15	27.90	47.40	56.45	35.35	62.55	74.75
1000	13.70	12.45	10.55	22.50	44.20	75.95	18.00	24.90	20.60	26.80	47.70	61.40	19.20	33.95	40.25	27.25	52.50	66.60
	Two-step first difference GMM estimator based on "DIF2" with Windmeijer standard errors																	
150	25.30	23.75	18.15	30.30	38.25	42.75	26.30	31.65	31.65	31.90	41.70	48.45	28.20	35.45	37.40	33.20	45.25	51.80
500	16.60	12.45	10.20	21.40	33.85	59.45	21.00	22.95	20.85	26.15	37.75	49.15	21.25	25.75	32.40	27.45	39.65	52.80
1000	10.75	9.30	8.30	17.75	38.70	72.45	13.45	16.20	13.95	21.20	37.20	54.20	15.95	20.05	25.15	22.15	35.65	51.15
	Continuous-updating first difference GMM estimator based on "DIF2"																	
150	28.25	37.00	31.95	31.70	39.45	41.20	30.85	46.80	57.10	34.70	46.20	59.65	33.00	51.10	67.25	36.30	49.75	67.95
500	14.60	11.30	8.30	18.80	20.30	38.90	18.95	25.65	21.25	23.20	29.60	29.25	19.35	31.15	38.65	23.65	32.60	40.65
1000	8.65	5.80	5.75	12.95	24.05	57.85	11.05	13.35	9.60	16.10	20.70	26.65	11.85	19.80	23.95	17.10	24.80	29.60
	Continuous-updating first difference GMM estimator based on "DIF2" with NW standard errors																	
150	33.85	29.45	27.90	36.30	35.00	36.75	37.35	39.40	42.30	39.80	42.70	46.45	37.60	42.15	48.25	40.15	45.75	50.40
500	19.10	7.60	6.70	21.55	15.40	35.25	26.45	17.10	14.10	28.80	23.10	22.25	27.75	21.70	21.40	29.85	25.80	25.60
1000	8.85	3.95	5.35	12.40	21.15	55.20	15.10	8.00	6.10	19.35	14.90	21.20	17.70	10.85	11.35	21.25	16.80	17.70

Notes: See notes to Table 1.

Table 8 (Continued): Size (×100, $H_0: \phi = 0.9$) and Power (×100, $H_1: \phi = 1$) findings for the estimation of ϕ in experiments with recnt past, $m_i \sim IIDU[1, 4]$, and $\phi = 0.9$

	Size (×100)			Power (×100)			Size (×100)			Power (×100)			Size (×100)			Power (×100)		
(N,T)	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**	**5**	**10**	**15**
	$\sigma_\alpha = 1$						$\sigma_\alpha = 2$						$\sigma_\alpha = 5$					
	GMM estimators																	
	One-step first difference GMM estimator based on "SYS1"																	
150	21.30	31.90	35.40	3.35	32.20	72.85	58.80	86.45	95.35	0.60	8.85	29.85	87.90	99.85	100.00	0.60	2.60	8.10
500	11.45	19.50	20.80	27.70	87.20	99.30	47.75	73.95	86.05	7.15	49.20	86.45	84.30	99.35	99.95	1.50	11.55	29.95
1000	10.00	12.50	15.25	74.45	99.55	100.00	35.00	60.25	73.25	26.60	90.50	99.75	79.85	98.50	99.80	6.45	34.10	67.70
	Two-step first difference GMM estimator based on "SYS1"																	
150	40.60	68.95	87.25	43.55	92.45	99.15	72.25	96.75	99.90	41.80	79.90	94.75	94.00	100.00	100.00	42.15	70.85	87.25
500	21.15	42.35	55.70	76.45	99.95	100.00	56.55	90.55	97.50	66.30	95.50	99.60	92.10	99.90	100.00	59.95	81.25	90.65
1000	15.70	25.40	38.00	96.05	100.00	100.00	42.65	80.65	91.15	84.50	99.45	100.00	90.05	99.80	100.00	73.15	90.55	96.45
	Two-step first difference GMM estimator based on "SYS1" with Windmeijer standard errors																	
150	16.85	10.40	1.20	11.15	14.30	2.65	43.25	58.15	21.80	5.40	5.10	5.45	71.30	94.20	71.50	3.50	5.30	3.20
500	10.15	11.80	6.85	50.10	91.05	96.75	31.30	56.00	59.60	21.45	47.80	53.50	69.85	97.80	99.35	7.00	14.90	10.70
1000	9.15	8.85	6.90	87.05	99.85	100.00	23.05	41.80	49.75	47.00	85.70	97.00	67.10	96.20	99.35	16.45	28.40	30.35
	Continuous-updating first difference GMM estimator based on "SYS1"																	
150	47.50	68.15	88.85	54.90	98.35	99.85	70.80	87.30	94.95	65.55	98.90	99.95	90.25	96.20	98.30	73.60	99.40	99.90
500	24.50	34.30	44.15	81.05	100.00	100.00	50.40	64.05	68.20	85.00	100.00	100.00	85.35	89.70	87.70	89.95	100.00	100.00
1000	16.60	20.45	25.25	97.75	100.00	100.00	38.85	49.35	50.60	93.30	100.00	100.00	80.25	82.15	79.70	95.45	100.00	100.00
	Continuous-updating first difference GMM estimator based on "SYS1" with NW standard errors																	
150	33.80	52.75	89.10	37.55	94.30	98.35	49.90	62.50	90.65	35.80	93.30	97.45	63.25	73.30	94.20	27.80	86.95	97.90
500	16.40	19.40	29.95	68.25	98.00	99.75	31.55	27.00	31.10	56.75	94.30	97.85	56.15	48.40	45.15	42.40	88.10	93.55
1000	11.45	13.55	14.35	91.65	99.65	100.00	20.80	17.90	15.45	80.20	97.10	99.70	47.80	30.00	22.20	60.05	91.25	95.40
	One-step first difference GMM estimator based on "SYS2"																	
150	22.00	42.35	58.10	3.15	21.45	40.00	59.50	90.35	98.15	0.60	6.20	12.55	88.15	99.90	100.00	0.55	2.30	4.65
500	12.00	25.55	36.45	27.65	82.85	95.45	48.20	79.40	93.25	7.15	40.10	63.25	84.30	99.55	100.00	1.50	10.45	17.65
1000	10.35	16.55	25.75	74.55	99.60	100.00	35.70	66.55	85.15	26.55	85.20	97.35	79.95	98.80	99.90	6.35	29.30	46.75
	Two-step first difference GMM estimator based on "SYS2"																	
150	37.50	54.20	68.15	40.15	76.25	86.40	70.55	94.10	99.00	38.80	58.65	68.05	93.85	100.00	100.00	40.55	50.90	58.00
500	19.70	29.30	35.70	72.75	99.20	99.95	54.60	83.65	93.00	61.10	89.30	96.95	91.25	99.85	100.00	54.90	68.55	71.00
1000	14.90	17.25	20.45	95.05	100.00	100.00	40.90	71.35	83.95	82.70	98.85	99.90	89.05	99.70	99.95	69.75	81.10	87.30
	Two-step first difference GMM estimator based on "SYS2" with Windmeijer standard errors																	
150	15.45	19.20	23.20	10.30	27.35	37.50	43.15	73.25	89.55	5.45	10.95	19.20	72.45	98.90	100.00	3.70	9.70	15.10
500	9.50	11.55	13.20	47.05	92.45	98.85	31.80	58.80	75.15	20.60	58.35	78.45	71.25	98.50	99.95	6.25	20.00	31.25
1000	8.60	7.30	7.85	85.15	99.95	100.00	23.20	42.10	57.70	46.30	92.50	99.20	67.75	97.40	99.80	15.70	41.85	63.10
	Continuous-updating first difference GMM estimator based on "SYS2"																	
150	44.05	51.50	59.35	52.20	91.15	97.35	70.60	79.65	80.70	63.75	93.15	97.95	89.60	95.45	95.25	71.75	96.75	99.35
500	24.85	25.25	28.20	78.35	99.85	100.00	50.10	53.65	53.65	82.60	99.90	100.00	84.35	88.20	85.90	88.20	100.00	100.00
1000	16.70	16.35	15.05	96.70	100.00	100.00	39.15	41.15	36.15	92.50	100.00	100.00	79.25	80.60	73.75	94.40	100.00	100.00
	Continuous-updating first difference GMM estimator based on "SYS2" with NW standard errors																	
150	31.30	29.90	39.70	35.30	74.40	91.85	48.60	45.85	48.35	34.70	73.10	92.00	61.75	65.10	66.15	26.15	66.85	89.40
500	16.15	13.55	14.50	64.45	95.75	99.30	31.80	22.80	18.05	54.35	91.45	97.40	55.80	43.75	37.30	41.85	84.65	92.80
1000	11.40	9.65	8.40	91.15	99.80	100.00	21.15	14.80	11.85	79.10	95.90	99.30	47.55	30.00	21.10	58.35	90.75	95.85

Notes: See notes to Table 1.

A Lemmas

Lemma A.1 *Under Assumption 2, we have*

$$n^{-1}\sum_{i=1}^{n}u_{it}^{2}=\bar{\sigma}_{n}^{2}+O_{p}\left(n^{(\delta_{\varkappa}-1)/2}\right), \tag{A.1}$$

for $t=-m+1,-m+2,...,1,2,...,T,$ *where* $\bar{\sigma}_{n}^{2}=n^{-1}\sum_{i=1}^{n}\sigma_{i}^{2}$, *and*

$$n^{-1}\sum_{i=1}^{n}u_{it}u_{it'}=O_{p}\left(n^{(\delta_{\rho}-1)/2}\right), \tag{A.2}$$

for $t\neq t'$, $t,t'=-m+1,-m+2,...,1,2,...,T$.

Proof. Note that

$$n^{-1}\sum_{i=1}^{n}u_{it}^{2}=n^{-1}\sum_{i=1}^{n}\tilde{u}_{it}^{2}+\bar{\sigma}_{n}^{2}, \tag{A.3}$$

where $\bar{\sigma}_{n}^{2}=n^{-1}\sum_{i=1}^{n}\sigma_{i}^{2}$, $\tilde{u}_{it}^{2}=u_{it}^{2}-\sigma_{i}^{2}$, and $E\left(\tilde{u}_{it}^{2}\right)=E\left(u_{it}^{2}\right)-\sigma_{i}^{2}=0$ by construction. Taking variance of the first term on the right side of (A.3), and using condition (5) of Assumption 2, we have

$$\begin{aligned} Var\left(n^{-1}\sum_{i=1}^{n}\tilde{u}_{it}^{2}\right) &= n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}E\left(\tilde{u}_{it}^{2}\tilde{u}_{jt}^{2}\right), \\ &\leq n^{-1}\sup_{i}\sum_{j=1}^{n}\left|E\left(\tilde{u}_{it}^{2}\tilde{u}_{jt}^{2}\right)\right|, \\ &= O\left(n^{\delta_{\varkappa}-1}\right), \end{aligned}$$

for $t=-m+1,-m+2,...,1,2,...,T$. Hence

$$n^{-1}\sum_{i=1}^{n}\tilde{u}_{it}^{2}=O_{p}\left(n^{(\delta_{\varkappa}-1)/2}\right),$$

for $t=-m+1,-m+2,...,1,2,...,T$, and result (A.1) follows.

Consider next $n^{-1}\sum_{i=1}^{n}u_{it}u_{it'}$ for any $t\neq t'$, $t,t'=-m+1,-m+2,...,1,2,...,T$. We have $E\left(u_{it}u_{it'}\right)=0$ for $t\neq t'$, and

$$\begin{aligned} Var\left(n^{-1}\sum_{i=1}^{n}u_{it}u_{it'}\right) &= n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}E\left(u_{it}u_{it'}u_{jt}u_{jt'}\right), \\ &= n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}E\left(u_{it}u_{jt}\right)E\left(u_{it'}u_{jt'}\right), \end{aligned}$$

where $E\left(u_{it}u_{it'}u_{jt}u_{jt'}\right)=E\left(u_{it}u_{jt}\right)E\left(u_{it'}u_{jt'}\right)$ follows from the independence of u_{it} and $u_{it'}$ for

$t \neq t'$. Using condition (4) of Assumption 2, and the boundedness of variances σ_i^2, we obtain

$$
\begin{aligned}
n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n} E\left(u_{it}u_{jt}\right)E\left(u_{it'}u_{jt'}\right) &\leq K_0 n^{-1}\sup_i \sum_{j=1}^{n}\left|E\left(u_{it}u_{jt}\right)E\left(u_{it'}u_{jt'}\right)\right|, \\
&= O\left(n^{\delta_\rho - 1}\right), \qquad \text{(A.4)}
\end{aligned}
$$

for $t \neq t'$, $t, t' = -m+1, -m+2, ..., 1, 2, ..., T$, where by Cauchy-Schwarz inequality

$$
\begin{aligned}
\sum_{j=1}^{n}\left|E\left(u_{it}u_{jt}\right)E\left(u_{it'}u_{jt'}\right)\right| &\leq \left(\sum_{j=1}^{n}\left[E\left(u_{it}u_{jt}\right)\right]^2\right)^{1/2}\left(\sum_{j=1}^{n}\left[E\left(u_{it'}u_{jt'}\right)\right]^2\right)^{1/2} \\
&\leq K\left(\sum_{j=1}^{n}\rho_{ijt}^2\right)^{1/2}\left(\sum_{j=1}^{n}\rho_{ijt'}^2\right)^{1/2} \\
&= O\left(n^{\delta_\rho}\right),
\end{aligned}
$$

$\rho_{ij} = E\left(u_{it}u_{jt}\right)/\left(\sigma_i\sigma_j\right)$ is the correlation of u_{it} and u_{jt}, $\sigma_i^2 < K < \infty$, for $i = 1, 2, ..., n$ by Assumption 2, $\left|\rho_{ij}\right| \leq 1$ by definition, and therefore $\sup_{it}\sum_{j=1}^{n}\rho_{ijt}^2 \leq \sup_{it}\sum_{j=1}^{n}\rho_{ijt}$, but (due to bounded error variances), $\sup_{i,t}\sum_{j=1}^{n}\rho_{ijt} = O\left(n^{\delta_\rho}\right)$ is implied by condition (4) of Assumption 2. Result (A.2) now follows from (A.4). This completes the proof. ■

Lemma A.2 *Suppose y_{it}, for $i = 1, 2, ..., n$, and $t = -m+1, -m+2, ..., 1, 2, ..., T$, are given by (1) with starting values $y_{i,-m}$, and let Assumptions 1, 2, and 3 part (a) hold. Consider*

$$d_{nT} = \frac{-1}{n(T-2)}\sum_{t=3}^{T}\sum_{i=1}^{n}\Delta y_{i,t-1}\Delta u_{it},$$

and

$$\xi_{nT} = \frac{-1}{n(T-2)}\sum_{t=3}^{T}\sum_{i=1}^{n}\left(\Delta u_{it}\Delta y_{i,t-1} + \frac{\Delta u_{it}^2}{2}\right).$$

Then, we have

$$d_{nT} = \bar{\sigma}_n^2 + O_p\left(T^{-1/2}n^{(\delta_\varkappa - 1)/2}\right) + O_p\left(T^{-1/2}n^{(\delta_\rho - 1)/2}\right) + O_p\left(\phi^m n^{(1-\delta_\varphi)/2}\right), \qquad \text{(A.5)}$$

and

$$\xi_{nT} = O_p\left(T^{-1/2}n^{(\delta_\rho - 1)/2}\right) + O_p\left(T^{-1/2}n^{(\delta_\varkappa - 1)/2}\right) + O_p\left(\phi^m n^{(1-\delta_\varphi)/2}\right). \qquad \text{(A.6)}$$

Proof. Result (A.5) is established first. Substituting

$$\Delta y_{i,t-1} = \varepsilon_{i,t-1} + \phi^{t+m-2}\eta_{i,-m}, \qquad \text{(A.7)}$$

where ε_{it} is given by (14), in the expression for d_{nT}, we have

$$\begin{aligned} d_{nT} &= \frac{-1}{n(T-2)} \sum_{t=3}^{T} \sum_{i=1}^{n} \left[\left(\varepsilon_{i,t-1} + \phi^{t+m-2} \eta_{i,-m} \right) \Delta u_{it} \right], \\ &= A_{nT} + B_{nT}, \end{aligned} \tag{A.8}$$

where

$$A_{nT} = \frac{-1}{n(T-2)} \sum_{t=3}^{T} \sum_{i=1}^{n} \varepsilon_{i,t-1} \Delta u_{it},$$

and

$$B_{nT} = -\frac{\phi^m}{n(T-2)} \sum_{t=3}^{T} \sum_{i=1}^{n} \phi^{t-2} \eta_{i,-m} \Delta u_{it}.$$

We consider each of the two terms on the right side of (A.8) in turn. It is convenient to write the term A_{nT} as

$$A_{nT} = A_{1nT} + A_{2nT} + A_{3nT}, \tag{A.9}$$

where,

$$\begin{aligned} A_{1nT} &= \frac{1}{n(T-2)} \sum_{t=3}^{T} \sum_{i=1}^{n} u_{i,t-1}^2, \\ A_{2nT} &= \frac{-1}{n(T-2)} \sum_{t=3}^{T} \sum_{i=1}^{n} u_{it} u_{i,t-1}, \end{aligned}$$

and

$$A_{3nT} = \frac{-1}{n(T-2)} \sum_{t=3}^{T} \sum_{i=1}^{n} \sum_{\ell=1}^{t+m-2} \phi^{\ell-1} (\phi - 1) u_{i,t-1-\ell} \Delta u_{it}.$$

We establish stochastic bounds for each these three terms next. For the first term, we have

$$A_{1nT} = \left(\frac{1}{n(T-2)} \sum_{t=3}^{T} \sum_{i=1}^{n} \tilde{u}_{i,t-1}^2 \right) + \bar{\sigma}_n^2,$$

where $\bar{\sigma}_n^2 = n^{-1} \sum_{i=1}^{n} \sigma_i^2$, and $\tilde{u}_{it}^2 = u_{it}^2 - \sigma_i^2$. The mean of A_{1nT} is $\bar{\sigma}_n^2$ and the variance is

$$\begin{aligned} \frac{1}{n^2 (T-2)^2} \sum_{t=3}^{T} \sum_{t'=3}^{T} \sum_{i=1}^{n} \sum_{j=1}^{n} E\left(\tilde{u}_{i,t-1}^2 \tilde{u}_{j,t'-1}^2 \right) &\leq \frac{1}{n(T-2)^2} \sum_{t=3}^{T} \sup_i \sum_{j=1}^{n} \left| E\left(\tilde{u}_{i,t-1}^2 \tilde{u}_{j,t-1}^2 \right) \right|, \\ &= O\left(T^{-1} n^{\delta_{\varkappa} - 1} \right), \end{aligned}$$

where we used $E\left(\tilde{u}_{i,t-1}^2 \tilde{u}_{j,t'-1}^2 \right) = 0$ for any $t \neq t'$, $t, t' = 3, 4, ..., T$, and condition (5) of Assumption 2. It follows that

$$A_{1nT} = \bar{\sigma}_n^2 + O_p\left(T^{-1/2} n^{(\delta_{\varkappa} - 1)/2} \right). \tag{A.10}$$

The second term on the right side of (A.9) features products $u_{it} u_{i,t-1}$. The mean of the second term is zero and its variance is

$$Var\left(A_{2nT}\right)=\frac{1}{n^{2}\left(T-2\right)^{2}}\sum_{t=3}^{T}\sum_{t'=3}^{T}\sum_{i=1}^{n}\sum_{j=1}^{n}E\left(u_{it}u_{i,t-1}u_{jt'}u_{j,t'-1}\right).$$

But u_{it} and $u_{it'}$ are independent for any i and any $t\neq t'$, and therefore $E\left(u_{it}u_{i,t-1}u_{jt'}u_{j,t'-1}\right)=0$ for $t\neq t'$, $t,t'=3,4,...,T$, and, using the same arguments as in the proof of (A.4),

$$\begin{aligned}Var\left(A_{2nT}\right) &= \frac{1}{\left(T-2\right)^{2}}\sum_{t=3}^{T}\left(\frac{1}{n^{2}}\sum_{i=1}^{n}\sum_{j=1}^{n}E\left(u_{it}u_{jt}\right)E\left(u_{i,t-1}u_{j,t-1}\right)\right),\\ &= \frac{1}{\left(T-2\right)^{2}}\sum_{t=3}^{T}O\left(n^{\delta_{\rho}-1}\right)\\ &= O\left(T^{-1}n^{\delta_{\rho}-1}\right).\end{aligned}$$

It now follows that

$$A_{2nT}=O_{p}\left(T^{-1/2}n^{(\delta_{\rho}-1)/2}\right). \tag{A.11}$$

It is convenient to write the third term as

$$A_{3nT}=C_{nT,1}+C_{nT,2},$$

where

$$C_{nT,1}=-\frac{1}{n\left(T-2\right)}\sum_{t=3}^{T}\sum_{i=1}^{n}\sum_{\ell=1}^{t+m-2}\phi^{\ell-1}\left(\phi-1\right)u_{i,t-1-\ell}u_{it},$$

and

$$C_{nT,2}=\frac{1}{n\left(T-2\right)}\sum_{t=3}^{T}\sum_{i=1}^{n}\sum_{\ell=1}^{t+m-2}\phi^{\ell-1}\left(\phi-1\right)u_{i,t-1-\ell}u_{i,t-1}.$$

Consider $C_{nT,1}$ first. We have $E\left(C_{nT,1}\right)=0$ and the variance of $C_{nT,1}$ is

$$Var\left(C_{nT,1}\right)=\frac{1}{n^{2}\left(T-2\right)^{2}}\sum_{t=3}^{T}\sum_{t'=3}^{T}\sum_{\ell=1}^{t+m-2}\sum_{\ell'=1}^{t+m-2}\sum_{i=1}^{n}\sum_{j=1}^{n}\phi^{\ell+\ell'-2}\left(\phi-1\right)^{2}E\left(u_{i,t-1-\ell}u_{it}u_{j,t'-1-\ell'}u_{jt'}\right).$$

Using again the independence of u_{it} and $u_{jt'}$ for any i,j and any $t\neq t'$, the expectation term $E\left(u_{i,t-1-\ell}u_{it}u_{j,t'-1-\ell'}u_{jt'}\right)$ is zero for $t\neq t'$. When $t=t'$, then $E\left(u_{i,t-1-\ell}u_{it}u_{j,t-1-\ell'}u_{jt}\right)$ is zero for $\ell\neq\ell'$. Hence $E\left(u_{i,t-1-\ell}u_{it}u_{j,t'-1-\ell'}u_{jt'}\right)$ can be nonzero only when $t=t'$ and $\ell=\ell'$, in which case we have (noting that $\ell>1$) $E\left(u_{i,t-1-\ell}u_{it}u_{j,t-1-\ell}u_{jt}\right)=E\left(u_{it}u_{jt}\right)E\left(u_{i,t-1-\ell}u_{j,t-1-\ell}\right)$. Utilizing these results, we obtain

$$Var\left(C_{nT,1}\right)=\frac{1}{n^{2}\left(T-2\right)^{2}}\sum_{t=3}^{T}\sum_{\ell=1}^{t+m-2}\sum_{i=1}^{n}\sum_{j=1}^{n}\phi^{2\ell-2}\left(\phi-1\right)^{2}E\left(u_{it}u_{jt}\right)E\left(u_{i,t-1-\ell}u_{j,t-1-\ell}\right),$$

and using the same arguments as in the proof of (A.4), we have

$$\begin{aligned} Var\left(C_{nT,1}\right) &\leq \frac{Kn^{\delta_\rho-1}}{T-2}\left(\frac{1}{T-2}\sum_{t=3}^{T}\sum_{\ell=1}^{t+m-2}\phi^{2\ell-2}\left(\phi-1\right)^2\right), \\ &= O\left(T^{-1}n^{\delta_\rho-1}\Theta_T\right), \end{aligned}$$

where

$$\Theta_T = \frac{1}{T-2}\sum_{t=3}^{T}\sum_{\ell=1}^{t+m-2}\phi^{2\ell-2}\left(\phi-1\right)^2,$$

is bounded for $-1 < \phi \leq 1$. It follows that $C_{nT,1} = O_p\left(T^{-1/2}n^{(\delta_\rho-1)/2}\right)$. Following the same steps, we obtain $C_{nT,2} = O_p\left(T^{-1/2}n^{(\delta_\rho-1)/2}\right)$, which in turn implies

$$A_{3nT} = O_p\left(T^{-1/2}n^{(\delta_\rho-1)/2}\right). \tag{A.12}$$

Finally, using (A.10)-(A.12), we have

$$A_{nT} = \bar{\sigma}_n^2 + O_p\left(T^{-1/2}n^{(\delta_\varkappa-1)/2}\right) + O_p\left(T^{-1/2}n^{(\delta_\rho-1)/2}\right) \tag{A.13}$$

Consider next the second term on the right side of (A.8). It is convenient to write B_{nT} as

$$B_{nT} = \frac{1}{T-2}\sum_{t=3}^{T}\phi^{t-2}\left(B_{nt,1}+B_{nt,2}\right),$$

where

$$B_{nt,1} = -\frac{\phi^m}{n}\sum_{i=1}^{n}\eta_{i,-m}u_{it},$$

and

$$B_{nt,2} = -\frac{\phi^m}{n}\sum_{i=1}^{n}\eta_{i,-m}u_{i,t-1}.$$

We derive stochastic bounds for each of these terms. Consider $B_{nt,1}$, for $t = 3,4,...,T$, first. The mean of $B_{nt,1}$ is zero and its variance is

$$\begin{aligned} Var\left(B_{nt,1}\right) &= \frac{\phi^{2m}}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}E\left(u_{it}u_{jt}\eta_{i,-m}\eta_{j,-m}\right), \\ &\leq \frac{\phi^{2m}}{n}\sup_{i,m,t}\sum_{j=1}^{n}\left|E\left(u_{it}u_{jt}\eta_{i,-m}\eta_{j,-m}\right)\right| \\ &= O\left(\phi^{2m}n^{\delta_\varphi-1}\right), \end{aligned}$$

where we have used condition (7) of Assumption 3. It follows that $B_{nt,1} = O_p\left(\phi^m n^{(\delta_\varphi-1)/2}\right)$ for $t = 3,4,...,T$. Using the same steps, we also obtain $B_{nt,2} = O_p\left(\phi^m n^{(\delta_\varphi-1)/2}\right)$ for $t = 3,4,...,T$, and therefore (noting that $-1 < \phi \leq 1$)

$$B_{nT} = \frac{1}{T-2}\sum_{t=3}^{T}\phi^{t-2}O_p\left(\phi^m n^{(\delta_\varphi-1)/2}\right) = O_p\left(\phi^m n^{(\delta_\varphi-1)/2}\right). \tag{A.14}$$

Finally, using (A.13) and (A.14) in (A.8), we obtain

$$d_{nT} = \bar{\sigma}_n^2 + O_p\left(T^{-1/2}n^{(\delta_\varkappa-1)/2}\right) + O_p\left(T^{-1/2}n^{(\delta_\rho-1)/2}\right) + O_p\left(\phi^m n^{(1-\delta_\varphi)/2}\right),$$

which completes the proof of (A.5).

Consider (A.6) next, and note that ξ_{nT} can be written as

$$\xi_{nT} = d_{nT} - \frac{1}{n(T-2)}\sum_{t=3}^{T}\sum_{i=1}^{n}\frac{\Delta u_{it}^2}{2}. \tag{A.15}$$

Consider the second term on the right side of (A.15), which can be written as

$$\frac{1}{n(T-2)}\sum_{t=3}^{T}\sum_{i=1}^{n}\frac{\Delta u_{it}^2}{2} = F_{nT} + G_{nT} + A_{2nT} + \bar{\sigma}_n^2, \tag{A.16}$$

where

$$\begin{aligned} F_{nT} &= \frac{1}{n(T-2)}\sum_{t=3}^{T}\sum_{i=1}^{n}\frac{\tilde{u}_{it}^2}{2}, \\ G_{nT} &= \frac{1}{n(T-2)}\sum_{t=3}^{T}\sum_{i=1}^{n}\frac{\tilde{u}_{i,t-1}^2}{2}, \end{aligned}$$

$\tilde{u}_{it}^2 = u_{it}^2 - \sigma_i^2$, and A_{2nT} is defined below (A.8). The terms F_{nT} and G_{nT} are established using the same arguments as in the proof of A_{1nT}. In particular we obtain $E(F_{nT}) = E(G_{nT})$, and the variances satisfy

$$Var(F_{nT}) = O\left(T^{-1}n^{\delta_\varkappa-1}\right), \text{ and } Var(G_{nT}) = O\left(T^{-1}n^{\delta_\varkappa-1}\right),$$

which implies $F_{nT} = O_p\left(T^{-1/2}n^{(\delta_\varkappa-1)/2}\right)$ and $G_{nT} = O_p\left(T^{-1/2}n^{(\delta_\varkappa-1)/2}\right)$. A stochastic bound for A_{2nT} is given by (A.11). Using these results in (A.16) yields

$$\frac{1}{n(T-2)}\sum_{t=3}^{T}\sum_{i=1}^{n}\frac{\Delta u_{it}^2}{2} = \bar{\sigma}_n^2 + O_p\left(T^{-1/2}n^{(\delta_\varkappa-1)/2}\right) + O_p\left(T^{-1/2}n^{(\delta_\rho-1)/2}\right),$$

and using this result together with the stochastic bound for d_{nT} given by (A.5), we obtain (A.6), as required. This completes the proof. ■

Lemma A.3 *Suppose y_{it}, for $i = 1, 2, ..., n$, and $t = -m+1, -m+2, ..., 1, 2, ..., T$, are given by*

(1) with starting values $y_{i,-m}$, and let Assumptions 1, 2, and 3 part (a) hold. Consider

$$a_{nT} = \frac{1}{2n\left(T-2\right)} \sum_{t=3}^{T} \sum_{i=1}^{n} \Delta y_{i,t-1}^{2}.$$

Then, we have

$$a_{nT} = \bar{\omega}_{nT,11}/2 + O_p\left(T^{-1/2} n^{(\delta_\rho - 1)/2}\right) + O_p\left(T^{-1/2} n^{(\delta_\varkappa - 1)/2}\right) + O_p\left(\phi^{2m} n^{(1-\delta_\varphi)/2}\right), \qquad \text{(A.17)}$$

where $\bar{\omega}_{nT,11}$ is given by (29), namely

$$\bar{\omega}_{nT,11} = \frac{\bar{\sigma}_n^2}{T-2}\left[1 + (\phi - 1)^2 \sum_{\ell=1}^{t+m-2} \phi^{2\ell-2}\right] + \phi^{2m} \nu_{nT},$$

$\bar{\omega}_{nT,11}$ has a strictly positive support, $\bar{\sigma}_n^2 = n^{-1}\sum_{i=1}^{n} \sigma_i^2$,

$$\nu_{nT} = \frac{\bar{\varsigma}_{n,-m}^2}{(T-2)} \sum_{t=3}^{T} \phi^{2t-4} = O\left(1\right),$$

and $\bar{\varsigma}_{n,-m}^2 = n^{-1}\sum_{i=1}^{n} \varsigma_{i,-m}^2$, and $\varsigma_{i,-m}^2 = E\left(\eta_{i,-m}^2\right)$. In addition, assume also that T is fixed and $n \to \infty$. Then,

$$\bar{\omega}_{nT,11} \to_p \bar{\omega}_{T,11} > 0, \qquad \text{(A.18)}$$

where

$$\omega_{T,11} = \frac{\bar{\sigma}^2}{T-2}\left[1 + (\phi - 1)^2 \sum_{\ell=1}^{t+m-2} \phi^{2\ell-2}\right] + \phi^{2m} \nu_T, \qquad \text{(A.19)}$$

in which $n^{-1}\sum_{i=1}^{n} \sigma_i^2 \to_p \bar{\sigma}^2$,

$$\bar{\nu}_{nT} \to_p \frac{\bar{\varsigma}_{-m}^2}{T-2} \sum_{t=3}^{T} \phi^{2t-4} \equiv \nu_T,$$

and $n^{-1}\sum_{i=1}^{n} \varsigma_{i,-m}^2 \to_p \bar{\varsigma}_{-m}^2$.

Proof. Substituting (A.7) for $t \geq 4$ in the definition of a_{nT} yields

$$\begin{aligned} a_{nT} &= \frac{1}{2n\left(T-2\right)} \sum_{t=3}^{T} \sum_{i=1}^{n} \left(\varepsilon_{i,t-1} + \phi^{t+m-2} \eta_{i,-m}\right)^2 \\ &= \frac{1}{2n\left(T-2\right)} \sum_{t=3}^{T} \sum_{i=1}^{n} \varepsilon_{i,t-1}^2 \\ &\quad + \frac{\phi^m}{n\left(T-2\right)} \sum_{t=3}^{T} \sum_{i=1}^{n} \phi^{t-2} \varepsilon_{i,t-1} \eta_{i,-m} \\ &\quad + \frac{\phi^{2m}}{2n\left(T-2\right)} \sum_{t=3}^{T} \sum_{i=1}^{n} \phi^{2t-4} \eta_{i,-m}^2 \end{aligned} \qquad \text{(A.20)}$$

where ε_{it} is given by (14). We focus on each of the three terms on the right side of (A.20) in turn. The first term can be written as

$$\frac{1}{2n\left(T-2\right)}\sum_{i=1}^{n}\sum_{t=3}^{T}\varepsilon_{i,t-1}^{2}=\left(A_{nT}^{\dagger}+B_{nT}^{\dagger}+C_{nT}^{\dagger}+D_{nT}^{\dagger}\right)/2,$$

where

$$\begin{aligned}
A_{nT}^{\dagger} &= \frac{1}{\left(T-2\right)}\sum_{t=3}^{T}\frac{1}{n}\sum_{i=1}^{n}\left(u_{i,t-1}^{2}+\sum_{\ell=1}^{t+m-2}\phi^{2\ell-2}\left(\phi-1\right)^{2}u_{i,t-1-\ell}^{2}\right),\\
B_{nT}^{\dagger} &= \frac{2}{\left(T-2\right)}\sum_{t=3}^{T}\frac{1}{n}\sum_{i=1}^{n}\sum_{\ell=1}^{t+m-2}\phi^{\ell-1}\left(\phi-1\right)u_{i,t-1-\ell}u_{i,t-1},\\
C_{nT}^{\dagger} &= \frac{2}{\left(T-2\right)}\sum_{t=3}^{T}\frac{1}{n}\sum_{i=1}^{n}\sum_{\ell=1}^{t+m-2}\sum_{\ell'=1,\ell\neq\ell'}^{t+m-2}\phi^{\ell+\ell'-2}\left(\phi-1\right)^{2}u_{i,t-1-\ell}u_{i,t-1-\ell'},
\end{aligned}$$

Using the same arguments as in the proof of Lemma A.2, we obtain, by computing the variances of the terms $B_{nT}^{\dagger}$, $C_{nT}^{\dagger}$, and $D_{nT}^{\dagger}$, and noting that $E\left(B_{nT}^{\dagger}\right)=E\left(C_{nT}^{\dagger}\right)=E\left(D_{nT}^{\dagger}\right)=0$,

$$\begin{aligned}
B_{nT}^{\dagger} &= O_{p}\left(T^{-1/2}n^{(\delta_{\rho}-1)/2}\right),\\
C_{nT}^{\dagger} &= O_{p}\left(T^{-1/2}n^{(\delta_{\rho}-1)/2}\right),\\
D_{nT}^{\dagger} &= O_{p}\left(T^{-1/2}n^{(\delta_{\rho}-1)/2}\right).
\end{aligned}$$

Consider the term $A_{nT}^{\dagger}$ next, which can be written as

$$A_{nT}^{\dagger}=A_{nT,1}^{\dagger}+A_{nT,2}^{\dagger},$$

where

$$\begin{aligned}
A_{nT,1}^{\dagger} &= \frac{1}{n\left(T-2\right)}\sum_{t=3}^{T}\sum_{i=1}^{n}\left[\left(u_{i,t-1}^{2}-\sigma_{i}^{2}\right)+\sum_{\ell=1}^{t+m-2}\phi^{2\ell-2}\left(\phi-1\right)^{2}\left(u_{i,t-1-\ell}^{2}-\sigma_{i}^{2}\right)\right],\\
A_{nT,2}^{\dagger} &= \frac{\bar{\sigma}_{n}^{2}}{T-2}\left[1+\left(\phi-1\right)^{2}\sum_{\ell=1}^{t+m-2}\phi^{2\ell-2}\right].
\end{aligned}$$

where $\bar{\sigma}_{n}^{2}=n^{-1}\sum_{i=1}^{n}\sigma_{i}^{2}$. Using the same arguments as in the proof of Lemma A.2, we obtain

$$A_{nT,1}^{\dagger}=O_{p}\left(T^{-1/2}n^{(\delta_{\varkappa}-1)/2}\right).$$

Consider next the second term on the right side of (A.20), which can be written as

$$\frac{\phi^{m}}{n\left(T-2\right)}\sum_{t=3}^{T}\sum_{i=1}^{n}\phi^{t-2}\varepsilon_{i,t-1}\eta_{i,-m}=\frac{1}{T-2}\sum_{t=3}^{T}\phi^{t-2}F_{nt}^{\dagger},$$

where

$$F_{nt}^{\dagger} = \frac{\phi^m}{n} \sum_{i=1}^{n} \varepsilon_{i,t-1} \eta_{i,-m}.$$

Consider next the last term on the right side of (A.20). We have

$$\frac{\phi^{2m}}{2n(T-2)} \sum_{t=3}^{T} \sum_{i=1}^{n} \phi^{2t-4} \eta_{i,-m}^2 = \frac{\phi^{2m}}{2n(T-2)} \sum_{t=3}^{T} \sum_{i=1}^{n} \phi^{2t-4} \tilde{\eta}_{i,-m}^2 + \phi^{2m} \frac{\nu_{nT}}{2},$$

where

$$\nu_{nT} = \frac{1}{n(T-2)} \sum_{t=3}^{T} \sum_{i=1}^{n} \phi^{2t-4} E\left(\eta_{i,-m}^2\right).$$

Since $\tilde{\eta}_{i,-m}^2 = \eta_{i,-m}^2 - E\left(\eta_{i,-m}^2\right)$, we have $E\left(\tilde{\eta}_{i,-m}^2\right) = 0$ by construction, and using also $-1 < \phi < 1$, we obtain

$$\begin{aligned} Var\left(\frac{\phi^{2m}}{2n(T-2)} \sum_{t=3}^{T} \sum_{i=1}^{n} \phi^{2t-4} \tilde{\eta}_{i,-m}^2\right) &\leq \phi^{4m} \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} E\left(\tilde{\eta}_{i,-m}^2 \tilde{\eta}_{j,-m}^2\right) \\ &\leq \frac{\phi^{4m}}{n} \sup_{i,m} \sum_{j=1}^{n} E\left(\tilde{\eta}_{i,-m}^2 \tilde{\eta}_{j,-m}^2\right). \end{aligned}$$

Using next condition (8) of Assumption 3 yields

$$Var\left(\frac{\phi^{2m}}{2n(T-2)} \sum_{t=3}^{T} \sum_{i=1}^{n} \phi^{2t-4} \tilde{\eta}_{i,-m}^2\right) = O\left(\phi^{4m} n^{\delta_\eta - 1}\right),$$

and therefore

$$\frac{\phi^{2m}}{2n(T-2)} \sum_{t=3}^{T} \sum_{i=1}^{n} \phi^{2t-4} \eta_{i,-m}^2 = \phi^{2m} \nu_{nT} + O_p\left(\phi^{2m} n^{(\delta_\eta - 1)/2}\right),$$

where $\nu_{nT} = O(1)$ since $E\left(\eta_{i,-m}^2\right) < \left[E\left(\eta_{i,-m}^4\right)\right]^{1/2} < K < \infty$ under Assumption 3 and $-1 < \phi \leq 1$.

Finally, noting that

$$\bar{\omega}_{nT,11} = A_{nT,2}^{\dagger} + \phi^{2m} \nu_{nT},$$

we obtain

$$a_{nT} = \bar{\omega}_{nT,11}/2 + O_p\left(T^{-1/2} n^{(\delta_\rho - 1)/2}\right) + O_p\left(T^{-1/2} n^{(\delta_\varkappa - 1)/2}\right) + O_p\left(\phi^{2m} n^{(1-\delta_\varphi)/2}\right).$$

Result (A.18) directly follows from (A.17). This completes the proof. ■

Lemma A.4 *Let $m \to \infty$ and suppose y_{it}, for all $i = 1, 2, ..., n$, and $t = ... - 1, 0, 1, 2, ..., T$, are* *given by (2) with $|\phi| < 1$, and let Assumptions 1 and 2 hold. Consider $\hat{\omega}_s$, for $s = 0, 1$, defined by*

$$\hat{\omega}_{nT,0} = \frac{1}{n(T-1)} \sum_{i=1}^{n} \sum_{t=2}^{T} \Delta y_{it}^2, \text{ and } \hat{\omega}_{nT,1} = \frac{1}{n(T-2)} \sum_{i=1}^{n} \sum_{t=3}^{T} \Delta y_{it} \Delta y_{i,t-1}.$$

Then, we have

$$\hat{\omega}_{nT,s} = \omega_{n,s} + O_p\left(T^{-1/2} n^{(\delta_\rho - 1)/2}\right) + O_p\left(T^{-1/2} n^{(\delta_\varkappa - 1)/2}\right), \tag{A.21}$$

for $s = 0, 1$, where

$$\omega_{n,0} = \frac{2}{1+\phi} \bar{\sigma}_n^2, \tag{A.22}$$

and

$$\omega_{n,1} = \frac{\phi - 1}{1+\phi} \bar{\sigma}_n^2. \tag{A.23}$$

Proof. Under the conditions of the lemma, Δy_{it} has the following MA(∞) representation,

$$\Delta y_{it} = d(L) u_{it}, \tag{A.24}$$

where

$$d(L) = \sum_{\ell=0}^{\infty} d_\ell L^\ell = 1 + \sum_{\ell=1}^{\infty} \phi^{\ell-1} (\phi - 1) L^\ell.$$

Let

$$\hat{\gamma}_n(t, t-s) = \frac{1}{n} \sum_{i=1}^{n} \Delta y_{it} \Delta y_{i,t-s}, \tag{A.25}$$

and note that $\hat{\omega}_s$ can be written as

$$\hat{\omega}_{nT,s} = \frac{1}{T-2-s} \sum_{t=2+s}^{T} \hat{\gamma}_n(t, t-s),$$

for $s = 0, 1$, where

$$\begin{aligned} \hat{\gamma}_n(t, t-s) &= \frac{1}{n} \sum_{i=1}^{n} d(L) u_{it} d(L) u_{i,t-s} \\ &= a_n(t, t-s) + b_n(t, t-s), \end{aligned}$$

in which

$$a_n(t, t-s) = \sum_{\ell=0}^{\infty} d_{\ell+s} d_\ell \left(\frac{1}{n} \sum_{i=1}^{n} u_{i,t-\ell-s}^2 \right),$$

and

$$b_n(t, t-s) = \sum_{\ell=0}^{\infty} \sum_{\ell'=0, \ell' \neq \ell+s}^{\infty} d_\ell d_{\ell'} \left(\frac{1}{n} \sum_{i=1}^{n} u_{i,t-\ell} u_{i,t-\ell'-s} \right).$$

Using the same arguments as in the proof of Lemma A.3, we obtain

$$\frac{1}{T-2-s}\sum_{t=2+s}^{T} a_n(t,t-s) = \omega_{n,s} + O_p\left(T^{-1/2}n^{(\delta_{\varkappa}-1)/2}\right),$$

and

$$\frac{1}{T-2-s}\sum_{t=2+s}^{T} b_n(t,t-s) = O_p\left(T^{-1/2}n^{(\delta_{\rho}-1)/2}\right),$$

for $s=0,1$, where

$$\omega_{n,0} = \bar{\sigma}_n^2 \sum_{\ell=0}^{\infty} d_\ell^2 = \frac{2}{1+\phi}\bar{\sigma}_n^2,$$

and

$$\omega_{n,0} = \bar{\sigma}_n^2 \sum_{\ell=1}^{\infty} d_{\ell+1}d_\ell = \frac{\phi-1}{1+\phi}\bar{\sigma}_n^2.$$

The result (A.21) now follows. This completes the proof. ■

Lemma A.5 *Let $m\to\infty$ and suppose y_{it}, for all $i=1,2,...,n$, and $t=...-1,0,1,2,...,T$, are given by (1) with $|\phi|<1$. Let Assumptions 1, 2 and 4 hold. Consider*

$$g_{s_1s_2}(q) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left[E\left(\Delta y_{it}\Delta y_{i,t-s_1}y_{i,t-q}\Delta y_{i,t-q-s_2}\right) - E\left(\Delta y_{it}\Delta y_{i,t-s_1}\right)E\left(y_{i,t-q}\Delta y_{i,t-q-s_2}\right)\right], \tag{A.26}$$

for $q\geq 0$ and $(s_1,s_2)=(0,0),(0,1),(1,1)$. Then

$$g_{s_1s_2}(q) = \sum_{\ell=1}^{4} A_\ell(q,s_1,s_2) - \omega_{s_1}\omega_{s_2}, \tag{A.27}$$

where for $(s_1,s_2)=(0,0),(0,1),(1,1)$,

$$A_1(q,s_1,s_2) = \begin{cases} \delta\left[(\phi-1)^{3s_2-s_1} + \frac{\phi^{2q+3s_2-s_1}(1-\phi)^3}{(1+\phi)(1+\phi^2)}\right], & \text{for } q=0 \\ \delta\left[\phi^{2q+2s_2-s_1-2}(\phi-1)^{2+s_2} + \frac{\phi^{2q+3s_2-s_1}(1-\phi)^3}{(1+\phi)(1+\phi^2)}\right], & \text{for } q>0 \end{cases}, \tag{A.28}$$

$$A_2(q,s_1,s_2) = \omega_{s_1}\omega_{s_2} - \sigma^4\left[(\phi-1)^{s_1+s_2} + \frac{(1-\phi)^3\phi^{s_1+s_2}}{(1+\phi)(1+\phi^2)}\right], \text{ for } q\geq 0, \tag{A.29}$$

$$A_3(q,s_1,s_2) = \begin{cases} \omega_q\omega_{s_2-s_1+q} - \sigma^4\left[(\phi-1)^{s_2-s_1} + \phi^q\phi^{s_2-s_1+q}\frac{(1-\phi)^3}{(1+\phi)(1+\phi^2)}\right], & \text{for } q=0 \\ \omega_q\omega_{s_2-s_1+q} - \sigma^4\left[\phi^{2q+s_2-s_1-2}(\phi-1)^2 + \phi^q\phi^{s_2-s_1+q}\frac{(1-\phi)^3}{(1+\phi)(1+\phi^2)}\right], & \text{for } q>0 \end{cases}, \tag{A.30}$$

$$A_4(q,s_1,s_2)=\begin{cases}\omega_{s_1}^2\omega_{s_2}^2-\sigma^4\left[(\phi-1)^{s_1+s_2}+\frac{\phi^{s_1+s_2}(1-\phi)^3}{(1+\phi)(1+\phi^2)}\right], & \textit{for } q=0\\ \omega_{q+s_2}\omega_{q-s_1}-\sigma^4\left[\phi^{s_2}(\phi-1)^{2-s_1}+\phi^{q+s_2}\phi^{q-s_1}\frac{(1-\phi)^3}{(1+\phi)(1+\phi^2)}\right], & \textit{for } q=1\\ \omega_{q+s_2}\omega_{q-s_1}-\sigma^4\left[\phi^{2q+s_2-s_1-2}(\phi-1)^2+\phi^{q+s_2}\phi^{q-s_1}\frac{(1-\phi)^3}{(1+\phi)(1+\phi^2)}\right], & \textit{for } q>1\end{cases}, \tag{A.31}$$

and ω_s*, for* $s=0,1,2,...,$ *are given by*

$$\omega_0=\frac{2}{1+\phi}\sigma^2,\ \textit{and}\ \omega_s=\frac{\phi^{q-1}(\phi-1)}{1+\phi}\sigma^2,\ \textit{for}\ s>0.$$

Proof. Consider $\lim_{n\to\infty} n^{-1}\sum_{i=1}^n E(\Delta y_{i,t-q}\Delta y_{i,t-q-s})$, for $q\geq 0$ and $s=0,1$, first. Substituting (A.24) for $\Delta y_{i,t-q}$ and $\Delta y_{i,t-q-s}$ yields

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n E(\Delta y_{i,t-q}\Delta y_{i,t-q-s}) &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n E[d(L)u_{i,t-q}d(L)u_{i,t-q-s}]\\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n\sum_{\ell=0}^\infty\sum_{\ell'=0}^\infty d_\ell d_{\ell'}E\left(u_{i,t-q-\ell}u_{i,t-q-s-\ell'}\right),\end{aligned}$$

where $d(L)=\sum_{\ell=0}^\infty d_\ell L^\ell$ is defined below (A.24). Noting that $E\left(u_{i,t-q-\ell}u_{i,t-q-s-\ell'}\right)=0$, for $q+\ell\neq q-s-\ell'$, under Assumption 4, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n E(\Delta y_{i,t-q}\Delta y_{i,t-q-s})=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n\sum_{\ell=0}^\infty d_{\ell+s}d_\ell E\left(u_{i,t-q-\ell-s}^2\right).$$

But $\lim_{n\to\infty} n^{-1}\sum_{i=1}^n E\left(u_{i,t-q-\ell-s}^2\right)=\sigma^2$ under Assumption 2, and therefore

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n E(\Delta y_{i,t-q}\Delta y_{i,t-q-s})=\sigma^2\sum_{\ell=0}^\infty d_{\ell+s}d_\ell=\omega_s, \tag{A.32}$$

for $s=0,1$ and any $q\geq 0$, where ω_s, for $s=0,1$, are defined by (A.22) and (A.23), respectively. Hence,

$$\begin{aligned}g_{s_1,s_2}(q) &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n\left[E(\Delta y_{i,t}\Delta y_{i,t-s_1}y_{i,t-q}\Delta y_{i,t-q-s_2})-E(\Delta y_{i,t}\Delta y_{i,t-s_1})E(y_{i,t-q}\Delta y_{i,t-q-s_2})\right]\\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n E(\Delta y_{i,t}\Delta y_{i,t-s_1}y_{i,t-q}\Delta y_{i,t-q-s_2})-\omega_{s_1}\omega_{s_2}.\end{aligned} \tag{A.33}$$

Consider next $\lim_{n\to\infty} n^{-1}\sum_{i=1}^n E(\Delta y_{it}\Delta y_{i,t-s_1}y_{i,t-q}\Delta y_{i,t-q-s_2})$ for $q\geq 0$ and $(s_1,s_2)=(0,0),(0,1),(1,1)$. Substituting (A.24), the expectation term can be written as

$$E(\Delta y_{it}\Delta y_{i,t-s_1}y_{i,t-q}\Delta y_{i,t-q-s_2})=\sum_{\ell_1=0}^\infty\sum_{\ell_2=0}^\infty\sum_{\ell_3=0}^\infty\sum_{\ell_4=0}^\infty d_{\ell_1}d_{\ell_2}d_{\ell_3}d_{\ell_4}E(u_{i,t-\ell_1}u_{i,t-s_1-\ell_2}u_{i,t-q-\ell_3}u_{i,t-q-s_2-\ell_4}).$$

But $E\left(u_{i,t-\ell_1}u_{i,t-s_1-\ell_2}u_{i,t-q-\ell_3}u_{i,t-q-s_2-\ell_4}\right)$ is nonzero only in the following four exclusive cases:

(i) $\ell_1 = s_1+\ell_2 = q+\ell_3 = q+s_2+\ell_4$;

(ii) $\ell_1 = s_1+\ell_2$, $q+\ell_3 = q+s_2+\ell_4$, and $\ell_1 \neq q+\ell_3$;

(iii) $\ell_1 = q+\ell_3$, $s_1+\ell_2 = q+s_2+\ell_4$, and $\ell_1 \neq s_1+\ell_2$; and

(iv) $\ell_1 = q+s_2+\ell_4$, $s_1+\ell_2 = q+\ell_3$, and $\ell_1 \neq s_1+\ell_2$.

We examine each of the four cases next. For case (i), we obtain

$$\begin{aligned} A_1(q,s_1,s_2) &\equiv \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sum_{\ell=0}^{\infty} d_{\ell+q+s_2}d_{\ell+q+s_2-s_1}d_{\ell+s_2}d_\ell E\left(u_{i,t-q-s_2-\ell_4}^4\right) \\ &= \delta\sum_{\ell=0}^{\infty} d_{\ell+q+s_2}d_{\ell+q+s_2-s_1}d_{\ell+s_2}d_\ell \\ &= \delta\left(d_{q+s_2}d_{q+s_2-s_1}d_{s_2} + \sum_{\ell=1}^{\infty} d_{\ell+q+s_2}d_{\ell+q+s_2-s_1}d_{\ell+s_2}d_\ell\right) \end{aligned}$$

where we have used condition $E\left(u_{it}^4\right) = \delta_i < C_0$, which ensures $\lim_{n\to\infty} n^{-1}\sum_{i=1}^{n} E\left(u_{i,t-q-s_2-\ell_4}^4\right) = \delta$, where $\delta = \lim_{n\to\infty} n^{-1}\sum_{i=1}^{n}\delta_i$. Noting that $d_0 = 1$ and $d_\ell = \phi^{\ell-1}(\phi-1)$ for $\ell > 1$ (see the definition of $d(L)$ below (A.24)), we obtain

$$\sum_{\ell=1}^{\infty} d_{\ell+q+s_2}d_{\ell+q+s_2-s_1}d_{\ell+s_2}d_\ell = \phi^{2q+3s_2-s_1}\frac{(1-\phi)^3}{(1+\phi)\left(1+\phi^2\right)},$$

and

$$d_{q+s_2}d_{q+s_2-s_1}d_{s_2} = \begin{cases} 1, & \text{for } q=s_1=s_2=0 \\ (\phi-1)^3, & \text{for } q=0,\ (s_1,s_2)=(0,1) \\ (\phi-1)^2, & \text{for } q=0,\ (s_1,s_2)=(1,1) \\ \phi^{2q-2}(\phi-1)^2, & \text{for } q>0,\ (s_1,s_2)=(0,0) \\ \phi^{2q+2s_2-s_1-2}(\phi-1)^3, & \text{for } q>0,\ (s_1,s_2)=(0,1),(1,1) \end{cases}.$$

The latter term can be equivalently (and more conveniently) written as

$$d_{q+s_2}d_{q+s_2-s_1}d_{s_2} = \begin{cases} (\phi-1)^{3s_2-s_1}, & \text{for } q=0, \text{ and } (s_1,s_2)=(0,0),(0,1),(1,1) \\ \phi^{2q+2s_2-s_1-2}(\phi-1)^{2+s_2}, & \text{for } q>0, \text{ and } (s_1,s_2)=(0,0),(0,1),(1,1) \end{cases}.$$

Hence (A.28) follows.

For case (ii), we obtain

$$\begin{aligned}
A_2(q,s_1,s_2) &\equiv \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sum_{\ell=0}^{\infty}\sum_{\ell'=0,\ell'\neq\ell}^{\infty} d_{\ell+s_1}d_\ell d_{\ell'+s_2}d_{\ell'}E\left(u_{i,t-\ell-s}\right)E\left(u_{i,t-q-\ell'-s_2}\right)\\
&= \left(\sigma^2\right)^2\sum_{\ell=0}^{\infty}\sum_{\ell'=0,\ell'\neq\ell}^{\infty} d_{\ell+s_1}d_\ell d_{\ell'+s_2}d_{\ell'}\\
&= \sigma^4\left(\sum_{\ell=0}^{\infty} d_{\ell+s_1}d_\ell\sum_{\ell'=0}^{\infty} d_{\ell'+s_2}d_{\ell'}-\sum_{\ell=0}^{\infty} d_{\ell+s_1}d_{\ell+s_2}d_\ell^2\right)\\
&= \omega_{s_1}\omega_{s_2}-\sigma^4\sum_{\ell=0}^{\infty} d_{\ell+s_1}d_{\ell+s_2}d_\ell^2,
\end{aligned}$$

where we have used similar arguments as in the derivation of case (i). Noting that

$$\begin{aligned}
\sum_{\ell=0}^{\infty} d_{\ell+s_1}d_{\ell+s_2}d_\ell^2 &= d_{s_1}d_{s_2}+\sum_{\ell=1}^{\infty} d_{\ell+s_1}d_{\ell+s_2}d_\ell^2\\
&= d_{s_1}d_{s_2}+\frac{(\phi-1)^4\,\phi^{s_1+s_2}}{1-\phi^4}\\
&= d_{s_1}d_{s_2}+\frac{(1-\phi)^3\,\phi^{s_1+s_2}}{(1+\phi)\left(1+\phi^2\right)}\\
&= (\phi-1)^{s_1+s_2}+\frac{(1-\phi)^3\,\phi^{s_1+s_2}}{(1+\phi)\left(1+\phi^2\right)},
\end{aligned}$$

we obtain (A.29).

Similarly, we obtain for case (iii),

$$\begin{aligned}
A_3(q,s_1,s_2) &\equiv \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sum_{\ell=0}^{\infty}\sum_{\ell'=0,\ell'\neq\ell}^{\infty} d_{\ell+q}d_\ell d_{\ell'+s_2-s_1+q}d_{\ell'}E\left(u_{i,t-q-\ell}^2\right)E\left(u_{i,t-q-s_2-\ell'}^2\right)\\
&= \sigma^4\sum_{\ell=0}^{\infty} d_{\ell+q}d_\ell\sum_{\ell'=0,\ell'\neq\ell}^{\infty} d_{\ell'+s_2-s_1+q}d_{\ell'}\\
&= \sigma^4\sum_{\ell=0}^{\infty} d_{\ell+q}d_\ell\sum_{\ell'=0}^{\infty} d_{\ell'+s_2-s_1+q}d_{\ell'}-\sigma^4\sum_{\ell=0}^{\infty} d_{\ell+q}d_{\ell+s_2-s_1+q}d_\ell^2.
\end{aligned}$$

But

$$\sigma^2\sum_{\ell=0}^{\infty} d_{\ell+q}d_\ell=\omega_q,\text{ and }\sigma^2\sum_{\ell'=0}^{\infty} d_{\ell'+s_2-s_1+q}d_{\ell'}=\omega_{s_2-s_1+q},$$

for $q\geq 0$, where ω_0 is defined by (A.22), ω_1 is defined in (A.23), and ω_q for $q>1$ is defined by

$$\omega_q=\sigma^2\frac{\phi^{q-1}(\phi-1)}{1+\phi}.$$

In addition,

$$\begin{aligned}
\sum_{\ell=0}^{\infty} d_{\ell+q} d_{\ell+s_2-s_1+q} d_\ell^2 &= d_q d_{s_2-s_1+q} + \sum_{\ell=1}^{\infty} d_{\ell+q} d_{\ell+s_2-s_1+q} d_\ell^2 \\
&= d_q d_{s_2-s_1+q} + \phi^q \phi^{s_2-s_1+q} \frac{(\phi-1)^4}{1-\phi^4} \\
&= d_q d_{s_2-s_1+q} + \phi^q \phi^{s_2-s_1+q} \frac{(1-\phi)^3}{(1+\phi)\left(1+\phi^2\right)},
\end{aligned}$$

where

$$d_q d_{s_2-s_1+q} = \begin{cases} (\phi-1)^{s_2-s_1}, & \text{for } q=0 \text{ and } (s_1,s_2)=(0,0),(0,1),(1,1) \\ \phi^{2q+s_2-s_1-2}(\phi-1)^2, & \text{for } q\geq 0, \text{ and } (s_1,s_2)=(0,0),(0,1),(1,1) \end{cases}.$$

Hence, (A.30) follows.

The last case to consider is case (iv). For $q=0$ and $(s_1,s_2)=(1,1)$, we have

$$A_4(0,1,1) \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sum_{\ell=0}^{\infty} \sum_{\ell'=0,\ell'\neq\ell}^{\infty} d_{\ell+1} d_\ell d_{\ell'+1} d_{\ell'} E\left(u_{i,t-1-\ell}^2\right) E\left(u_{i,t-1-\ell'}^2\right),$$

which, using the same arguments as before, reduces to

$$\begin{aligned}
A_4(0,1,1) &= \omega_1^2 - \sigma^4 \sum_{\ell=0}^{\infty} d_{\ell+1} d_{\ell+1} d_\ell^2 \\
&= \omega_1^2 - \sigma^4 \left[(\phi-1)^2 + \frac{\phi^2(1-\phi)^3}{(1+\phi)\left(1+\phi^2\right)}\right].
\end{aligned}$$

For the remaining possibilities, $q>0$ and $(s_1,s_2)=(0,0),(0,1),(1,1)$ or $q=0$ and $(s_1,s_2)=(0,0),(0,1)$, we have

$$\begin{aligned}
A_4(q,s_1,s_2) &\equiv \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sum_{\ell=0}^{\infty} \sum_{\ell'=0,\ell'\neq\ell}^{\infty} d_{\ell+q+s_2} d_\ell d_{\ell'+q-s_1} d_{\ell'} E\left(u_{i,t-q-s_2-\ell}^2\right) E\left(u_{i,t-q-\ell'}^2\right) \\
&= \omega_{q+s_2}\omega_{q-s_1} - \sigma^4 \sum_{\ell=0}^{\infty} d_{\ell+q+s_2} d_{\ell+q-s_1} d_\ell^2,
\end{aligned}$$

where (noting that $q\geq s_1$)

$$\begin{aligned}
\sum_{\ell=0}^{\infty} d_{\ell+q+s_2} d_{\ell+q-s_1} d_\ell^2 &= d_{q+s_2} d_{q-s_1} + \sum_{\ell=1}^{\infty} d_{\ell+q+s_2} d_{\ell+q-s_1} d_\ell^2 \\
&= d_{q+s_2} d_{q-s_1} + \phi^{q+s_2}\phi^{q-s_1} \frac{(\phi-1)^4}{1-\phi^4} \\
&= d_{q+s_2} d_{q-s_1} + \phi^{q+s_2}\phi^{q-s_1} \frac{(1-\phi)^3}{(1+\phi)\left(1+\phi^2\right)},
\end{aligned}$$

in which

$$d_{q+s_2}d_{q-s_1} = \begin{cases} 1, & \text{for } q=0, \text{ and } (s_1,s_2)=(0,0) \\ \phi-1 & \text{for } q=0, \text{ and } (s_1,s_2)=(0,1) \\ \phi^{s_2}(\phi-1)^{2-s_1}, & \text{for } q=1, \text{ and } (s_1,s_2)=(0,0),(0,1),(1,1) \\ \phi^{2q+s_2-s_1-2}(\phi-1)^2, & \text{for } q>1, \text{ and } (s_1,s_2)=(0,0),(0,1),(1,1) \end{cases}.$$

Hence, we obtain (A.31).

Overall, we have

$$\lim_{n\to\infty}\frac{1}{n}E\left(\Delta y_{it}\Delta y_{i,t-s_1}y_{i,t-q}\Delta y_{i,t-q-s_2}\right)=\sum_{\ell=1}^{4}A_\ell\left(q,s_1,s_2\right),$$

for $q \geq 0$ and $(s_1,s_2)=(0,0),(0,1),(1,1)$, and substituting this expression in (A.33) yields (A.27), as required. ∎

Lemma A.6 *Let $m\to\infty$ and suppose y_{it}, for all $i=1,2,...,n$, and $t=...-1,0,1,2,...,T$, are given by (1) with $|\phi|<1$. Let Assumptions 1, 2 and 4 hold. Consider $\hat{\boldsymbol{\omega}}_{nT}=(\hat{\omega}_{nT,0},\hat{\omega}_{nT,1})'$, where $\hat{\omega}_{nT,s}$, for $s=0,1$, are defined by*

$$\hat{\omega}_{nT,0}=\frac{1}{n(T-1)}\sum_{i=1}^{n}\sum_{t=2}^{T}\Delta y_{it}^2, \text{ and } \hat{\omega}_{nT,1}=\frac{1}{n(T-2)}\sum_{i=1}^{n}\sum_{t=3}^{T}\Delta y_{it}\Delta y_{i,t-1}. \tag{A.34}$$

Furthermore, let T be fixed and $n\to\infty$. Then, we have

$$\sqrt{n}\left(\hat{\boldsymbol{\omega}}_{nT}-\boldsymbol{\omega}_n\right)\to_d N\left(\mathbf{0},\boldsymbol{\Psi}_T\right), \tag{A.35}$$

where $\boldsymbol{\omega}_n=(\omega_{n,0},\omega_{n,1})'$, $\omega_{n,s}$, for $s=0,1$, are given by

$$\omega_{n,0}=\frac{2}{1+\phi}\bar{\sigma}_n^2 \text{ and } \omega_{n,1}=\frac{\phi-1}{1+\phi}\bar{\sigma}_n^2,$$

respectively, and

$$\boldsymbol{\Psi}=\begin{pmatrix}\psi_{00} & \psi_{01}\\ \psi_{10} & \psi_{11}\end{pmatrix}, \tag{A.36}$$

with its elements given by

$$\begin{aligned}
\psi_{00} &= \frac{1}{T-1}\left[g_{00}(0)+2\sum_{q=1}^{T-2}\left(1-\frac{q}{T-1}\right)g_{00}(q)\right],\\
\psi_{01} &= \psi_{10}=\frac{1}{T-1}\left[g_{01}(0)+\sum_{q=1}^{T-3}\left(2-\frac{2q+1}{T-2}\right)g_{01}(q)+g_{01}(T-2)\right], \text{ and}\\
\psi_{11} &= \frac{1}{T-2}\left[g_{11}(0)+2\sum_{q=1}^{T-3}\left(1-\frac{q}{T-2}\right)g_{11}(q)\right],
\end{aligned}$$

in which $g_{00}(q)$*,* $g_{01}(q)$*, and* $g_{11}(q)$ *are defined in (A.26), namely*

$$g_{s_1 s_2}(q) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\left[E\left(\Delta y_{it}\Delta y_{i,t-s_1} y_{i,t-q}\Delta y_{i,t-q-s_2}\right) - E\left(\Delta y_{it}\Delta y_{i,t-s_1}\right)E\left(y_{i,t-q}\Delta y_{i,t-q-s_2}\right)\right],$$

for $(s_1, s_2) \in (0,0), (0,1), (1,1)$ *and* $q \geq 0$*.*

Proof. Using the definitions of the vectors $\hat{\boldsymbol{\omega}}_{nT} = (\hat{\omega}_{nT,0}, \hat{\omega}_{nT,1})'$ and $\boldsymbol{\omega}_n = (\omega_{n,0}, \omega_{n,1})'$, the vector $(\hat{\boldsymbol{\omega}}_{nT} - \boldsymbol{\omega}_{nT})$ can be written as

$$\hat{\boldsymbol{\omega}}_{nT} - \boldsymbol{\omega}_{nT} = n^{-1}\sum_{i=1}^{n}\mathbf{S}_T'\hat{\mathbf{v}}_{T,i}, \tag{A.37}$$

where

$$\underset{2\times(2T-3)}{\mathbf{S}_T} = \begin{pmatrix} \boldsymbol{\tau}_{T-1}'/(T-1) & \mathbf{0}_{1\times(T-2)} \\ \mathbf{0}_{1\times(T-1)} & \boldsymbol{\tau}_{T-2}'/(T-2) \end{pmatrix}, \tag{A.38}$$

$\mathbf{0}_{m\times n}$ is an $m\times n$ zero matrix, $\boldsymbol{\tau}_T$ is a $T\times 1$ vector of ones, $\tilde{\mathbf{v}}_{T,i} = \left(\tilde{\mathbf{v}}_{T,i0}', \tilde{\mathbf{v}}_{T,i1}'\right)'$,

$$\underset{(T-1)\times 1}{\tilde{\mathbf{v}}_{T,i0}} = \begin{pmatrix} \Delta y_{i,2}^2 - E\left(\Delta y_{i,2}^2\right) \\ \Delta y_{i,3}^2 - E\left(\Delta y_{i,3}^2\right) \\ \vdots \\ \Delta y_{i,T}^2 - E\left(\Delta y_{i,T}^2\right) \end{pmatrix}, \text{ and } \underset{(T-2)\times 1}{\tilde{\mathbf{v}}_{T,i1}} = \begin{pmatrix} \Delta y_{i,3}\Delta y_{i,2} - E\left(\Delta y_{i,3}\Delta y_{i,2}\right) \\ \Delta y_{i,4}\Delta y_{i,3} - E\left(\Delta y_{i,4}\Delta y_{i,3}\right) \\ \vdots \\ \Delta y_{i,T}\Delta y_{i,T-1} - E\left(\Delta y_{i,T}\Delta y_{i,T-1}\right) \end{pmatrix}.$$

Under Assumption 4, $\tilde{\mathbf{v}}_{T,i}$ is independently distributed of $\tilde{\mathbf{v}}_{T,j}$ for any $i \neq j$. In addition, $E\left(\tilde{\mathbf{v}}_{T,i}\right) = \mathbf{0}$, and $\left\|E\left(\tilde{\mathbf{v}}_{T,i}\tilde{\mathbf{v}}_{T,i}'\right)\right\| < K$. Hence,

$$n^{-1/2}\sum_{i=1}^{n}\tilde{\mathbf{v}}_{T,i} \to_d N\left(\mathbf{0}_{2T-3\times 1}, \boldsymbol{\Upsilon}_T\right), \tag{A.39}$$

where

$$\boldsymbol{\Upsilon}_T = \lim_{n\to\infty} E\left(\hat{\mathbf{v}}_{T,i}\hat{\mathbf{v}}_{T,i}'\right) = \begin{pmatrix} \boldsymbol{\Upsilon}_{T,00} & \boldsymbol{\Upsilon}_{T,01} \\ \boldsymbol{\Upsilon}_{T,10} & \boldsymbol{\Upsilon}_{T,11} \end{pmatrix}, \tag{A.40}$$

$$\underset{(T-1)\times(T-1)}{\boldsymbol{\Upsilon}_{T,00}} = \begin{pmatrix} g_{00}(0) & g_{00}(1) & g_{00}(2) & \cdots & g_{00}(T-2) \\ g_{00}(1) & g_{00}(0) & g_{00}(1) & \cdots & g_{00}(T-3) \\ g_{00}(2) & g_{00}(1) & g_{00}(0) & \cdots & g_{00}(T-4) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{00}(T-2) & g_{00}(T-3) & g_{00}(T-4) & \cdots & g_{00}(0) \end{pmatrix},$$

$$\underset{(T-1)\times(T-2)}{\boldsymbol{\Upsilon}_{T,01}} = \boldsymbol{\Upsilon}'_{T,10} = \begin{pmatrix} g_{01}(0) & g_{01}(1) & g_{01}(2) & \cdots & g_{01}(T-3) \\ g_{01}(1) & g_{01}(0) & g_{01}(1) & \cdots & g_{01}(T-4) \\ g_{01}(2) & g_{01}(1) & g_{01}(0) & & g_{01}(T-5) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{01}(T-3) & g_{01}(T-4) & g_{01}(T-5) & \cdots & g_{01}(0) \\ g_{01}(T-2) & g_{01}(T-3) & g_{01}(T-4) & & g_{01}(1) \end{pmatrix},$$

and

$$\boldsymbol{\Upsilon}_{T,11} = \begin{pmatrix} g_{11}(0) & g_{11}(1) & g_{11}(2) & \cdots & g_{11}(T-3) \\ g_{11}(1) & g_{11}(0) & g_{11}(1) & \cdots & g_{11}(T-4) \\ g_{11}(2) & g_{11}(1) & g_{11}(0) & \cdots & g_{11}(T-5) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{11}(T-3) & g_{11}(T-4) & g_{11}(T-5) & \cdots & g_{11}(0) \end{pmatrix},$$

in which $g_{00}(q), g_{01}(q)$, and $g_{11}(q)$ are defined by (A.26). Using (A.37) and (A.39), we have

$$\sqrt{n}\left(\hat{\boldsymbol{\omega}}_{nT} - \boldsymbol{\omega}_n\right) \rightarrow_d N\left(\mathbf{0}, \boldsymbol{\Psi}_T\right) \text{ with } \boldsymbol{\Psi}_T = \mathbf{S}'_T \boldsymbol{\Upsilon}_T \mathbf{S}_T,$$

and, after substituting (A.40) for $\boldsymbol{\Upsilon}_T$ and (A.38) for $\mathbf{S}_T$, the expression for $\boldsymbol{\Psi}_T$ reduces to

$$\boldsymbol{\Psi}_T = \begin{pmatrix} \psi_{T,00} & \psi_{T,01} \\ \psi_{T,10} & \psi_{T,11} \end{pmatrix},$$

with

$$\begin{aligned} \psi_{T,00} &= \frac{1}{T-1}\left[g_{00}(0) + 2\sum_{q=1}^{T-2}\left(1 - \frac{q}{T-1}\right) g_{00}(q)\right], \\ \psi_{T,01} &= \psi_{T,10} = \frac{1}{T-1}\left[g_{01}(0) + \sum_{q=1}^{T-3}\left(2 - \frac{2q+1}{T-2}\right) g_{01}(q) + g_{01}(T-2)\right], \text{ and} \\ \psi_{T,11} &= \frac{1}{T-2}\left[g_{11}(0) + 2\sum_{q=1}^{T-3}\left(1 - \frac{q}{T-2}\right) g_{11}(q)\right], \end{aligned}$$

where $g_{00}(q), g_{01}(q)$, and $g_{11}(q)$ are defined by (A.26). This completes the proof. ■

The subscript T is dropped in the following lemma to simplify the notations.

Lemma A.7 *Suppose y_{it}, for $i = 1, 2, ..., n$, and $t = -m+1, -m+2, ..., 1, 2, ..., T$, are given by (1) with starting values $y_{i,-m}$, and let Assumptions 1, 2, 3 part (a), and 4 hold. In addition, suppose $\bar{\varsigma}^2_{-m} > \bar{\sigma}^2$ if $\phi_0 = 1$. Consider $\hat{\phi}_n$ defined in (41). Let T be fixed and $n \rightarrow \infty$. Then for $T \geq 3$, we have $\hat{\phi}_n - \phi_0 = O_p\left(n^{-1/2}\right)$.*

Proof. Let $\delta = \max\{\delta_\rho, \delta_\varkappa m, \delta_\varphi\}$ and recall that $\delta < 1$ by Assumptions 2 and 3 part (a). Then, using (28), (32), and (33) in (24) yields

$$\left(\hat{\phi}_n-\phi_0\right)\left[\left(\hat{\phi}_n-\phi_0\right)\frac{\bar{\omega}_{n,11}}{2}-\left(\bar{\omega}_{n,11}-\bar{\sigma}_n^2\right)+O_p\left(n^{(\delta-1)/2}\right)\right]=O_p\left(n^{(\delta-1)/2}\right). \tag{A.41}$$

Note that for $-1<\phi<1$, $\left(\bar{\omega}_{n,11}-\bar{\sigma}_n^2\right)\rightarrow_p b>0$, regardless of $\bar{\varsigma}_{-m}^2$. For $\phi=1$, $\left(\bar{\omega}_{n,11}-\bar{\sigma}_n^2\right)\rightarrow_p b>0$ if and only if $\bar{\varsigma}_{-m}^2>\bar{\sigma}^2$. Consider the terms in square brackets on the left side of equation (A.41) and note that by Proposition 1, we have $\hat{\phi}_n\rightarrow_p\phi_0$. Note, in addition, that $\bar{\omega}_{n,11}=O_p(1)$ (see (29)), and therefore

$$\left(\hat{\phi}_n-\phi_0\right)\frac{\bar{\omega}_{n,11}}{2}\rightarrow_p 0.$$

Moreover, since $(\bar{\omega}_{n,11}-\bar{\sigma}_n^2)\rightarrow_p b>0$, we have

$$\left[\left(\hat{\phi}_n-\phi_0\right)\frac{\bar{\omega}_{n,11}}{2}-\left(\bar{\omega}_{n,11}-\bar{\sigma}_n^2\right)+O_p\left(n^{(\delta-1)/2}\right)\right]\rightarrow_p b>0.$$

Using this result in (A.41), and setting $\delta=0$, we obtain

$$\left(\hat{\phi}_n-\phi_0\right)\left[b+o_p(1)\right]=O_p\left(n^{-1/2}\right),$$

and therefore $\left(\hat{\phi}_n-\phi_0\right)=O_p\left(n^{-1/2}\right)$. ■

B Proofs of Propositions and Theorems

This section of the Appendix provides proofs of theorems and propositions for the case of fixed and finite T. The subscript T is omitted to simplify the notations.

Proof of Proposition 1. Let $\left(\hat{\phi}_n-\phi_0\right)\rightarrow_p\tilde{\phi}$. It follows from (24), (30), and (33)-(36) that $\tilde{\phi}$ satisfies the following quadratic equation.

$$\tilde{\phi}\left(\tilde{\phi}a-b\right)=0,$$

where $a_n\rightarrow_p a>0$, and $b_n\rightarrow_p b>0$. Hence, the two roots are

$$\tilde{\phi}=0 \text{ and } \tilde{\phi}=b/a,$$

and the result (37) follows. ■

Proof of Theorem 1. It follows from (31) that $b=2a-\bar{\sigma}^2>0$ under the assumptions of the theorem. Hence, $b/a>0$ and the probability limit of $\hat{D}_n$ as $n\rightarrow\infty$ must be nonzero and positive. Consequently, $p_a=P\left(\hat{D}_n\geq 0\right)\rightarrow 1$ as $n\rightarrow\infty$. Using this result and noting that $\left(\hat{\omega}_{n,01}+\hat{\omega}_{n,11}\right)/\hat{\omega}_{n,11}=O_p(1)$, it follows from (43) that $\hat{\phi}_n$ and $\hat{\phi}_{n,a}$ have the same probability limits. But, using Proposition 1, $\hat{\phi}_{n,a}$ is consistent. This completes the proof. ■

Proof of Theorem 2. Using (46) in (45), and noting that $b_n \to_p b > 0$, we have

$$\sqrt{n}\left(\hat{\phi}_n - \phi_0\right) = -\sqrt{n}\frac{\xi_n}{b} + O_p\left(n^{-1/2}\right).$$

Noting that the variance of $\sqrt{n}\xi_n$ is bounded away from 0, we obtain

$$\sqrt{n}\left(\hat{\phi}_n - \phi_0\right) \to_D \sqrt{n}\frac{\xi_n}{b}.$$

But $\sqrt{n}\xi_n = n^{-1/2}\sum_{i=1}^{n}\vartheta_i$, where ϑ_i is independently distributed of ϑ_j for all $i \neq j$, $i, j = 1, 2, ..., n$, $E(\vartheta_i) = 0$, and $E\left(\vartheta_i^2\right) < K < \infty$. Hence

$$\sqrt{n}\xi_n \to_D N\left(0, n^{-1}\sum_{i=1}^{n} E\left(\vartheta_i^2\right)\right),$$

and $\sqrt{n}\left(\hat{\phi}_n - \phi_0\right) \to_D N(0, \theta)$ with $\theta = b^{-2}n^{-1}\sum_{i=1}^{n} E\left(\vartheta_i^2\right)$. This completes the proof. ■

Proof of Theorem 3. By Lemma A.4 we have $\hat{\omega}_{n,s} - \omega_{n,s} \to_p 0$, for $s = 0, 1$ under the conditions stated in Theorem 3. Noting that $\bar{\sigma}_n^2 \to_p \sigma^2$, it follows that $\hat{\omega}_{n,s} \to_p \omega_s$ for $s = 0, 1$, where

$$\omega_0 = \frac{2}{1+\phi}\sigma^2, \text{ and } \omega_{n,1} = \frac{\phi - 1}{1+\phi}\sigma^2.$$

Hence $\hat{\phi}_n \to_p 2\omega_1/\omega_0 + 1$, where $\hat{\phi}_n$ is given by (52). But $2\omega_1/\omega_0 + 1 = \phi$, as required, which completes the proof. ■

Proof of Theorem 4. This theorem has been established in the main text. Specifically, using (53) and (55), and noting that $\hat{\omega}_{n,0} \to_p \omega_0 = 2\sigma^2/(1+\phi) > 0$ by Lemma A.4, where $n^{-1}\sum_{i=1}^{n}\sigma_i^2 \to \sigma^2$ we obtain (56) with θ given by (57). This completes the proof. ■

Proof of Theorem 5. Noting that $\hat{\theta}_n = \hat{\mathbf{h}}_n'\hat{\mathbf{\Psi}}_n\hat{\mathbf{h}}_n$ and $\theta = \lim_{n\to\infty} E\left[n\left(\hat{\phi}_a - \phi\right)^2\right] = \mathbf{h}'\mathbf{\Psi}\mathbf{h}$ (see C.2), where $\hat{\mathbf{h}}_n$ and $\hat{\mathbf{\Psi}}_n$ are defined in (60), $\mathbf{h}$ is defined in (C.3), and $\mathbf{\Psi}$ is given by (A.36), sufficient conditions for (61) are:

$$\hat{\mathbf{h}} \to_p \mathbf{h}, \tag{B.1}$$

and

$$\hat{\mathbf{\Psi}} \to_p \mathbf{\Psi}. \tag{B.2}$$

Consider condition (B.1) first, and note that $\hat{\mathbf{h}}_n = (-2\hat{\omega}_{n,1}/\hat{\omega}_{n,0}, 2/\hat{\omega}_{n,0})$ and $\mathbf{h} = (-2\omega_1/\omega_0, 2/\omega_0)$. Hence, sufficient conditions for (B.1) to hold are

$$\hat{\omega}_{n,s} - \omega_s \to_p 0, \text{ for } s = 0, 1, \tag{B.3}$$

and ω_0 is bounded away from 0. Using Lemma A.4 and noting that $\bar{\sigma}_n^2 \to_p \sigma^2$ establishes that (B.3) hold. Moreover, $\omega_0 = 2\sigma^2/(1+\phi) > 0$ and ω_0 is bounded away from 0 for any given $\sigma^2 > 0$ and any given $\phi > -1$. Therefore (B.1) holds.

Consider (B.2) next, where $\mathbf{\Psi}$ is given by (A.36) and $\hat{\mathbf{\Psi}}_n$ is defined in the second part of (60). Thus, sufficient conditions for (B.2) to hold are

$$\frac{1}{n}\sum_{i=1}^{n}\left(\Delta y_{it}\Delta y_{i,t-s_1}-\hat{\omega}_{n,s_1}\right)\left(\Delta y_{i,t-q}\Delta y_{i,t-q-s_2}-\hat{\omega}_{n,s_2}\right)-g_{s_1,s_2}\left(q\right)\rightarrow_p 0, \tag{B.4}$$

for $q\geq 0$, and $(s_1,s_2)=(0,0),(0,1),(1,1)$, where $g_{s_1,s_2}(q)$ is defined in (A.26). But,

$$\hat{\omega}_{n,s_1}\hat{\omega}_{n,s_2}-\omega_{s_1}\omega_{s_2}\rightarrow_p 0 \text{ for any } s_1,s_2=0,1,$$

is implied by (B.3). Moreover, by independence of Δy_{it} and $\Delta y_{jt'}$ for any $i\neq j$ and any t,t', and noting that $E\left|\Delta y_{i,t}\Delta y_{i,t-s_1}y_{i,t-q}\Delta y_{i,t-q-s_2}\right|$ is finite, we have

$$\frac{1}{n}\sum_{i=1}^{n}\left[\Delta y_{it}\Delta y_{i,t-s_1}y_{i,t-q}\Delta y_{i,t-q-s_2}-E\left(\Delta y_{i,t}\Delta y_{i,t-s_1}y_{i,t-q}\Delta y_{i,t-q-s_2}\right)\right]\rightarrow_p 0,$$

It now follows that (B.4) holds, which in turn establishes the validity of (B.2). This completes the proof. ■

C Additional Theorems and Proofs

The asymptotic distribution of SBMM estimator $\hat{\phi}_n$ defined in (52) under Assumptions 1, 2, 3 part (b), and 4, is established in the next theorem. We continue to assume T is fixed and we omit the subscript T in this section in Appendix to simplify the notations.

Theorem 6 *Let $m\rightarrow\infty$ and suppose y_{it}, for all $i=1,2,...,n$, and $t=...-2,-1,0,1,2,...,T$, are given by (1) with $|\phi|<1$. Let Assumptions 1, 2, 3 part (b), and 4 hold. Consider $\hat{\phi}_n$ defined in (52). Let T be fixed and $n\rightarrow\infty$. Then, we have*

$$\sqrt{n}\left(\hat{\phi}_n-\phi\right)\rightarrow_d N\left(0,\theta\right), \tag{C.1}$$

where θ is given by equation (C.4), and it depends on T, ϕ, and the ratio δ/σ^4, where $\delta=\lim_{n\rightarrow\infty}n^{-1}\sum_{i=1}^{\infty}\delta_i$ and $\sigma^2=\lim_{n\rightarrow\infty}n^{-1}\sum_{i=1}^{n}\sigma_i^2$.

Proof of Theorem 6. The estimator $\hat{\phi}_n$, defined in (52), can be written as

$$\hat{\phi}_n=f\left(\hat{\boldsymbol{\omega}}\right)=\frac{2\hat{\omega}_{n,1}+\hat{\omega}_{n,0}}{\hat{\omega}_{n,0}},$$

where $\hat{\boldsymbol{\omega}}_n=(\hat{\omega}_{n,0},\hat{\omega}_{n,1})'$ and the function $f\left(\hat{\boldsymbol{\omega}}_n\right)$ is continuously differentiable in the neighborhood of $\boldsymbol{\omega}=(\omega_0,\omega_1)'$, in which $\omega_0>0$. Using the delta method and Lemma A.6, we obtain

$$\sqrt{n}\left(\hat{\phi}_n-\phi\right)=\sqrt{n}\left[f\left(\hat{\boldsymbol{\omega}}_n\right)-f\left(\boldsymbol{\omega}\right)\right]\rightarrow_d N\left(0,\theta\right),$$

where

$$\theta = \mathbf{h}'\mathbf{\Psi}\mathbf{h}, \tag{C.2}$$

$$\mathbf{h} = \left.\frac{\partial f\left(\hat{\boldsymbol{\omega}}\right)}{\partial \hat{\boldsymbol{\omega}}}\right|_{\hat{\boldsymbol{\omega}}=\boldsymbol{\omega}} = \left(-\frac{2\omega_1}{\omega_0^2}, \frac{2}{\omega_0}\right)' = \left(\frac{1-\phi}{2\left(1+\phi\right)\sigma^2}, \frac{1+\phi}{\sigma^2}\right)', \tag{C.3}$$

and $\mathbf{\Psi}$ is given by (A.36) of Lemma A.6. Using the following definitions:

$$\tilde{\mathbf{h}} = \sigma^2\mathbf{h} \text{ and } \tilde{\mathbf{\Psi}} = \sigma^{-4}\mathbf{\Psi},$$

and substituting the expression for $g_{s_1 s_2}(q)$ given by result (A.27) of Lemma A.5 in the expression (A.36) for $\mathbf{\Psi}$, we obtain the following expression for θ,

$$\theta = \left(\sigma^2\mathbf{h}\right)'\left(\sigma^{-4}\mathbf{\Psi}\right)\left(\sigma^2\mathbf{h}\right) = \tilde{\mathbf{h}}'\tilde{\mathbf{\Psi}}\tilde{\mathbf{h}}, \tag{C.4}$$

where

$$\tilde{\mathbf{h}} = \begin{pmatrix} \frac{1-\phi}{2(1+\phi)} \\ 1+\phi \end{pmatrix}, \; \tilde{\mathbf{\Psi}} = \begin{pmatrix} \tilde{\psi}_{00} & \tilde{\psi}_{01} \\ \tilde{\psi}_{10} & \tilde{\psi}_{11} \end{pmatrix},$$

$$\begin{aligned}
\tilde{\psi}_{00} &= \frac{1}{T-1}\left[\tilde{g}_{00}(0) + 2\sum_{q=1}^{T-2}\left(1-\frac{q}{T-1}\right)\tilde{g}_{00}(q)\right], \\
\tilde{\psi}_{01} &= \tilde{\psi}_{10} = \frac{1}{T-1}\left[\tilde{g}_{01}(0) + \sum_{q=1}^{T-3}\left(2-\frac{2q+1}{T-2}\right)\tilde{g}_{01}(q) + \tilde{g}_{01}(T-2)\right], \\
\tilde{\psi}_{11} &= \frac{1}{T-2}\left[\tilde{g}_{11}(0) + 2\sum_{q=1}^{T-3}\left(1-\frac{q}{T-2}\right)\tilde{g}_{11}(q)\right],
\end{aligned}$$

$$\tilde{g}_{s_1,s_2}(q) = \sum_{\ell=1}^{4}\tilde{A}_\ell(q,s_1,s_2) - \tilde{\omega}_{s_1}\tilde{\omega}_{s_2}, \text{ for } q \geq 0 \text{ and } (s_1,s_2) = (0,0),(0,1),(1,1),$$

$$\tilde{A}_1(q,s_1,s_2) = \begin{cases} \frac{\delta}{\sigma^4}\left[(\phi-1)^{3s_2-s_1} + \frac{\phi^{2q+3s_2-s_1}(1-\phi)^3}{(1+\phi)\left(1+\phi^2\right)}\right], & \text{for } q=0, \text{ and } (s_1,s_2)=(0,0),(0,1),(1,1) \\ \frac{\delta}{\sigma^4}\left[\phi^{2q+2s_2-s_1-2}(\phi-1)^{2+s_2} + \frac{\phi^{2q+3s_2-s_1}(1-\phi)^3}{(1+\phi)\left(1+\phi^2\right)}\right], & \text{for } q>0, \text{ and } (s_1,s_2)=(0,0),(0,1),(1,1) \end{cases}$$

$$\tilde{A}_2(q,s_1,s_2) = \tilde{\omega}_{s_1}\tilde{\omega}_{s_2} - (\phi-1)^{s_1+s_2} - \frac{(1-\phi)^3\phi^{s_1+s_2}}{(1+\phi)\left(1+\phi^2\right)},$$

$$\tilde{A}_3(q,s_1,s_2) = \begin{cases} \tilde{\omega}_q\tilde{\omega}_{s_2-s_1+q} - (\phi-1)^{s_2-s_1} - \phi^q\phi^{s_2-s_1+q}\frac{(1-\phi)^3}{(1+\phi)\left(1+\phi^2\right)}, & \text{for } q=0 \\ \tilde{\omega}_q\tilde{\omega}_{s_2-s_1+q} - \phi^{2q+s_2-s_1-2}(\phi-1)^2 - \phi^q\phi^{s_2-s_1+q}\frac{(1-\phi)^3}{(1+\phi)\left(1+\phi^2\right)}, & \text{for } q>0 \end{cases},$$

$$\tilde{A}_4\left(q,s_1,s_2\right)=\begin{cases}\tilde{\omega}_{s_1}^2\tilde{\omega}_{s_2}^2-\left(\phi-1\right)^{s_1+s_2}-\frac{\phi^{s_1+s_2}(1-\phi)^3}{(1+\phi)\left(1+\phi^2\right)}, & \text{for } q=0\\ \tilde{\omega}_{q+s_2}\tilde{\omega}_{q-s_1}-\phi^{s_2}\left(\phi-1\right)^{2-s_1}-\phi^{q+s_2}\phi^{q-s_1}\frac{(1-\phi)^3}{(1+\phi)\left(1+\phi^2\right)}, & \text{for } q=1\\ \tilde{\omega}_{q+s_2}\tilde{\omega}_{q-s_1}-\phi^{2q+s_2-s_1-2}\left(\phi-1\right)^2-\phi^{q+s_2}\phi^{q-s_1}\frac{(1-\phi)^3}{(1+\phi)\left(1+\phi^2\right)}, & \text{for } q>1\end{cases},$$

and $\tilde{\omega}_s$ for $s=0,1,2,...$, is given by

$$\tilde{\omega}_0=\frac{2}{1+\phi}, \text{ and } \omega_s=\frac{\phi^{s-1}\left(\phi-1\right)}{1+\phi}, \text{ for } s>0.$$

■

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