Unbiased estimation of autoregressive models for bounded stochastic processes*

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Abstract

The paper investigates the estimation bias of autoregressive models for bounded stochastic processes, and the performance of the standard procedures in the literature that aim at correcting the estimation bias. It is shown that in some cases the bounded nature of the stochastic processes worse the estimation bias effect, which suggests the design of bound-specific bias correction methods. The paper focuses on two popular autoregressive estimation bias correction procedures, which are extended to cover bounded stochastic processes. Finite sample performance of the new proposal is carried out using Monte Carlo simulations, which reveal that accounting for the bounded nature of the stochastic processes leads to improve the estimation of autoregressive models. Finally, an illustration is given using the current account balance of some developed countries, for which shocks persistence measures are computed.

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1 Introduction

Since the seminal paper of Nelson and Plosser (1982), any analysis that consider the use of time series data always begin with the study of the time properties of the variables. Most of the times, this implies the use of some unit root tests and the statistical inference that is drawn from their application is relevant for subsequent analyses. For instance, a quite popular practice is to determine the degree of shocks persistence by means of estimating autoregressive models.

This latter analysis provides very interesting insights about the evolution of the variable being studied. For instance, this is the case of the analysis of the persistence in variables such as the real exchange rates, where some practitioners have studied the number of periods that a shock takes to vanish – see Balli et al. (2014), among others. Similarly, Watson (2014) studies the effect of the Great Recession on the inflation persistence. This type of analyses, however, is not straightforward given that we should take into account that the OLS estimator is consistent, but biased in finite samples, and this bias must be removed in order to appropriately measure the degree of persistence. There exist different proposals in the literature which try to correct this finite sample bias. In this regard, we can cite here the contributions of Andrews (1993), Andrews and Chen (1994), Kilian (1998), Hansen (1999), Rossi (2005) and Perron and Yabu (2009a), among others, which develop different valid techniques to remove the mentioned bias.

However, some commonly employed variables in this type of works may be affected by the presence of some bounds. There are some important macroeconomic variables such as nominal interest rates, unemployment rates, exchange rates or the great ratios, among others, that are bounded by definition, preventing these variables from exhibiting a large variance. This feature generates tension on the statistical inference associated to standard unit root tests.

The standard order of integration analysis of time series consider that an I(1) non-stationary stochastic process can vary freely in the limit, that is, they ignore the constraints that impose the existence of bounds. This fact is relevant in the sense that the behavior of this type of variables might look like as if they were stationary, when in fact they are non-stationary. In this regard, Cavaliere (2005) and Cavaliere and Xu (2014) show that standard unit root tests might lead to misleading conclusions if the bounded nature of the time series is not accounted for. Therefore, it seems to be sensible to analyze the influence of these bounds on the determination of the time series properties of the variables.

The goal of this paper is to assess whether the use of bias-corrected autoregressive parameters lead to obtain statistics such as shock persistence measures, the long-run variance (LRV) estimates and unit root tests statistics for bounded time series with good finite sample performance. To address this issue, this paper investigates the performance of some popular bias correction methods mentioned above when they are applied to bounded stochastic processes. The first stage of the analysis focuses on some of these standard bias correction procedures, showing that, in general, the amount of estimation bias that is corrected is small when the bounded nature of the time series is ignored. This gives ground to extend standard bias correction procedures that incorporate the effect that bounds might have on the estimation of autoregressive models for (possibly persistent) time series.

The paper proceeds as follows. Section 2 describes the model for bounded stochastic processes and investigates the consistence and finite sample bias of the OLS estimation procedure. In addition, we essay the performance of some of the most relevant standard methods proposed in the literature that correct the finite sample bias. Section 3 propose an extension of two bias correction procedures for bounded stochastic processes. Section 4 analyzes the finite sample performance of the suggested approaches in finite samples. Section 5 conducts an empirical illustration focusing on current account balance. Finally, Section 6 concludes.

2 The model

Let x_t be a stochastic process with data generating process (DGP) given by:

$$x_t = \mu + y_t \tag{1}$$

$$y_t = \alpha y_{t-1} + u_t, \tag{2}$$

$t = 1, \ldots, T$, where $x_t \in [\underline{b}, \overline{b}]$ almost surely for all t and $y_0 = O_p(1)$. The presence of bounds requires that Δx_t lies within the interval $[\underline{b} - x_{t-1}, \overline{b} - x_{t-1}]$, where $[\underline{b}, \overline{b}]$ denote the boundaries that affect the time series. The disturbance term u_t is assumed to admit the following decomposition:

$$u_t = \varepsilon_t + \underline{\xi}_t - \overline{\xi}_t, \tag{3}$$

with $\varepsilon_t = C(L) \upsilon_t$, where $C(L) = \sum_{j=0}^{\infty} c_j L^j$ with $\sum_{j=0}^{\infty} j |c_j| < \infty$, and υ_t is a martingale difference sequence adapted to the filtration $F_t = \sigma - field \{\upsilon_{t-j}; j \geq 0\}$. The LRV of ε_t is given by $\sigma^2 = \lim_{T\to\infty} E[T^{-1}(\sum_{t=1}^{T} \varepsilon_t)^2]$. The variables $\underline{\xi}_t$ and $\overline{\xi}_t$ are non-negative processes (regulators) such that $\underline{\xi}_t > 0$ if and only if $y_{t-1} + \varepsilon_t < \underline{b} - \mu$ and $\overline{\xi}_t > 0$ if and only if $y_{t-1} + \varepsilon_t > \overline{b} - \mu$. The stochastic processes involved in (3) satisfy the Assumptions A and B in Cavaliere and Xu (2012), so that $(\underline{b} - \mu) = \underline{c}\sigma T^{1/2}$ and $(\overline{b} - \mu) = \overline{c}\sigma T^{1/2}$, with $\underline{c} \leq 0 \leq \overline{c}$, $\underline{c} \neq \overline{c}$. This representation can be particularized to the cases of stochastic processes that are only limited below – i.e., $x_t \in [\underline{b}, \infty]$ – or only limited above – i.e., $x_t \in [-\infty, \overline{b}]$ – but also covers the case of unbounded processes – i.e., $x_t \in [-\infty, \infty]$.

Estimation of autoregressive models is at the heart of popular practices in empirical economics such as order of integration analysis and computation of shock persistence measures. However, it is well known that their estimation provides biased estimates in finite samples, although the bias disappears asymptotically. In this regard, it is of interest to study whether dealing with bounded stochastic processes defined by (1) to (3) presents any different features compared to the unbounded situation. The following theorem shows, for a simple model specification, that the estimation of autoregressive models for bounded processes is consistent.

Theorem 1 *Let $\{y_t\}_{t=1}^{T}$ be the stochastic process given by (2) where $|\alpha| < 1$ and $\varepsilon_t \sim iid\,(0, \sigma^2)$ with $E\left(\varepsilon_t^4\right) < \infty$. Then, as $T \to \infty$ the ordinary least-square (OLS) estimator:*

$$\hat{\alpha} \xrightarrow{p} \alpha.$$

The proof is given in the appendix. Although the estimation of the autoregressive parameter is consistent, there might be some estimation bias in finite samples. To show the extent of the additional estimation bias effect introduced by bounds, we have conducted a small simulation experiment using a symmetric bounded stochastic process defined by (1) to (3) with α taking values between 0 and 1 in steps of 0.05 size – i.e., $\alpha = \{0, 0.05, 0.1, \ldots, 0.95, 1\}$ – $\varepsilon_t \sim iid\,(0, 1)$, $\overline{c} = \{0.1, 0.3, 0.5, 0.7, 1\}$ and $T = \{50, 100, 200, 500\}$ – 1,000 replications are conducted. Figure 1 reveals that the estimation bias depends not only on T and α, but also on $\overline{c}$. Regardless of T and α, the bias is bigger the narrower the rank of variation defined by the bounds. As expected, the magnitude of the bias reduces as T increases, but even for large T we observe non-negligible bias for small $\overline{c}$. Finally, note that for large values of $\overline{c}$, for which the time series is near-unbounded, the estimation bias tends to increase as α approaches one.

We essay different standard approaches in the literature that try to correct the estimation bias of autoregressive models. First, we focus on the median-unbiased (MU) estimation procedure for AR(1) models in Andrews (1993), which requires the computation of look-up tables to obtain a correspondence between the value of the OLS estimation of the autoregressive parameter ($\hat{\alpha}$) and the median of the empirical distribution that is obtained assuming that $\alpha = \hat{\alpha}$, which defines $\hat{\alpha}_{MU}$, i.e., the median-unbiased autoregressive estimator of α. Andrews (1993) suggests to use $\hat{\alpha}_{MU}$ instead of $\hat{\alpha}$. Other alternatives have been proposed in the literature – see, for instance, Kilian (1998), Hansen (1999), Rossi (2005)) – although initial simulations, not reported here to save space, reveal that they are beaten by Andrews (1993) MU estimator.

Figures 2 to 5 compare the mean of the estimation bias of OLS and Andrews MU procedures for $\overline{c} = \{0.1, 0.3, 0.7, 1\}$, $T = \{50, 100, 200, 500\}$ and $\alpha = \{0, 0.05, 0.1, \ldots, 0.95, 1\}$. In general, Andrews (1993) MU estimation procedure gives more accurate estimates of the autoregressive parameter, regardless of the values of $\overline{c}$, α and T. However, the MU estimation bias is still important for small values of $\overline{c}$, although it clearly reduces as $\overline{c}$

increases. It is worth noticing that these improvements are obtained using an estimation procedure designed for unbounded processes, but we have shown that the amount of bias correction depends on $\bar{c}$. This feature leads us to hypothesize that better results are to be expected if bound-specific MU estimates are used, which implies extending Andrews (1993) proposal to bounded time series.

3 Bias correction methods for bounded stochastic processes

The estimators mentioned above ignore the bounded nature of x_t, an important feature that might be taken into account in order to improve the estimates of the autoregressive parameters. To address this issue, we have proceeded to modify two bias correction estimation procedures considering that $x_t \in \left[\underline{b}, \overline{b}\right]$.

3.1 Andrews MU estimation procedure

Andrews MU estimator requires the computation of look-up tables that establish a correspondence between the OLS and the MU estimate of the autoregressive parameter. In our case, this is an intensive computational problem since these look-up tables have to be obtained for different combinations of $[\underline{c}, \bar{c}]$ values. As an example, Table 1 presents the asymptotic look-up table for the AR(1) symmetric bounds case – a MATLAB code is available to compute the tables for any values of the bounds. It can be seen that the procedure works well in almost all cases, although for the small values of $\bar{c}$ we observe that there is a mild bias estimation error of the autoregressive parameter as α approaches one. The computation of look-up tables for AR(p) processes can be done following the proposal in Andrews and Chen (1994), although it would represent a higher computational cost since it requires the use of bootstrapping.

3.2 Weighted symmetric least-squares estimation procedure

Following Roy and Fuller (2001), Roy, Falk and Fuller (2004) and Perron and Yabu (2009a), we suggest the use of the modified estimator given by:

$$\hat{\alpha}_{TW} = \hat{\alpha}_W + C\left(\hat{\tau}_W\right)\hat{\sigma}_W, \tag{4}$$

where $\hat{\alpha}_W$ denotes the weighted symmetric least-squares (WSLS) estimate of the autoregressive parameter for AR(1) models proposed in Fuller (1996):

$$\hat{\alpha}_W = \frac{\sum_{t=2}^{T}\hat{y}_t\hat{y}_{t-1}}{\sum_{t=2}^{T-1}\hat{y}_t^2 + T^{-1}\sum_{t=1}^{T}\hat{y}_t^2},$$

with $\hat{y}_t$ the OLS estimated residuals in (1),

$$\hat{\sigma}_W^2 = \frac{\sum_{t=2}^{T} \left(\hat{y}_t - \hat{\alpha}_W \hat{y}_{t-1}\right)^2}{(T-2)\left[\sum_{t=2}^{T-1} \hat{y}_t^2 + T^{-1} \sum_{t=1}^{T} \hat{y}_t^2\right]},$$

and $\hat{\tau}_W = \left(\hat{\alpha}_W - 1\right)/\hat{\sigma}_W$ the pseudo t-ratio statistic to the null hypothesis that $\alpha = 1$. The modification in (4) requires the definition of $C\left(\hat{\tau}_W\right)$ that, following Roy and Fuller (2001) and Perron and Yabu (2009a), is given by the following discontinuous function:

$$C\left(\hat{\tau}_W\right) = \begin{cases} -\hat{\tau}_W & \text{if } \hat{\tau}_W > \tau_{pct} \\ I_p T^{-1} \hat{\tau}_W - 2\left[\hat{\tau}_W + K\left(\hat{\tau}_W + A\right)\right]^{-1} & \text{if } -A < \hat{\tau}_W \leq \tau_{pct} \\ I_p T^{-1} \hat{\tau}_W - 2\left[\hat{\tau}_W\right]^{-1} & \text{if } -\left(2T\right)^{1/2} < \hat{\tau}_W \leq -A \\ 0 & \text{if } \hat{\tau}_W \leq -\left(2T\right)^{1/2} \end{cases}, \quad (5)$$

with $K = \left[\left(1 + I_p T^{-1}\right) \tau_{pct} \left(\tau_{pct} + A\right)\right]^{-1} \left[2 - I_p T^{-1} \tau_{pct}^2\right]$, $I_p = \lfloor (p+1)/2 \rfloor$, being $\lfloor \cdot \rfloor$ the integer part, p denotes the order of the autoregressive model – $p = 1$ in this case – and τ_{pct} is a percentile of the limiting distribution of $\hat{\tau}_W$ when $\alpha = 1$. The percentile τ_{pct} is either set at the median ($\tau_{0.5}$) or at the 85th percentile ($\tau_{0.85}$) of the distribution of $\hat{\tau}_W$. Finally, the function K depends on the deterministic specification that is used in (1).[1] The value of the constant A is empirically chosen in Roy and Fuller (2001) after conducting simulation experiments, who set it at $A = 5$ for unbounded stochastic processes.[2]

Table 2 summarizes selected percentiles of the distribution of $\hat{\tau}_W$ for different values of the (symmetric) bound parameters – the last row shows the percentiles for unbounded stochastic processes. As it can be seen, the limiting distribution of $\hat{\tau}_W$ depends on the bounds, with a limiting distribution more shifted to the left the narrower the rank of variation defined by the bounds.

Let us focus on the median of the distribution as the percentile used in the bias correction. First, note that the use of $A = 5$ for the unbounded stochastic process case does not pose incongruences for the definition of the function in (5), since $-A < \tau_{0.5}$.[3] However, we can see that the median of the distribution $\hat{\tau}_W$ moves away from -1.21 as the rank of variation defined by the bounds decreases, which might produce poor

[1] See Roy and Fuller (2001) for the function that corresponds with the linear time trend. It is worth noticing that Perron and Yabu (2009b) use the same function when testing for multiple shifts in the trend.

[2] Roy and Fuller (2001) also set $A = 5$ for the linear time trends, whereas Perron and Yabu (2009b) specify $A = 10$.

[3] This is also valid for the linear time trend case, for which Roy and Fuller (2001) estimated $\tau_{0.5} = -1.96$ and $A = 5$, as mentioned above. Note that the consideration of slope trend shifts in Perron and Yabu (2009b) lead them to specify $A = 10$ for the one break case – it is well known that the limiting distribution of $\hat{\tau}_W$ shifts to the left as the number of slope trend shifts increases.

performance of the correction when $\overline{c} < 0.5$.[4] In this regards, an extensive simulation experiment has been conducted to assess the sensitivity of the modified estimator to the specification of the constant $A = \{5, 6, \ldots, 15\}$. Results available upon request indicate that the modified estimator shows good performance when $A = 5$ and $\overline{c} > 0.1$, although marginal differences are found for the other values of A. Besides, for the smallest value of the bound parameter ($\overline{c} > 0.1$), we find that $A = 10$ gives good results.

Finally, for AR(p) models the estimation procedure is similar but now the autoregressive parameter α is estimated from:

$$\hat{y}_t = \alpha \hat{y}_{t-1} + \sum_{j=1}^{k} \psi_j \Delta \hat{y}_{t-j} + \varepsilon_t, \tag{6}$$

with $\hat{y}_t$ the OLS estimated residuals in (1). In this case, the WSLS estimate of α ($\hat{\alpha}_W$) can be obtained as described in Fuller (1996) and its truncated version $\hat{\alpha}_{TW}$ is computed as defined in (5).

3.3 Iterative estimation of the bounds

The empirical implementation of the method of bias correction in bounded series that we propose requires some additional steps. Given a time series with known theoretical limits $\underline{b}$ and $\overline{b}$, we can estimate the bounds as:

$$[\hat{\underline{c}}, \hat{\overline{c}}] = \left[\frac{\underline{b} - \hat{D}_t}{\hat{\sigma} T^{1/2}}, \frac{\overline{b} - \hat{D}_t}{\hat{\sigma} T^{1/2}} \right],$$

which requires an estimation of the deterministic component (D_t) and the long-run variance (σ^2). In our case $D_t = \mu$ and it can be easily estimated by regressing the series to a constant. However, the estimation of the long-run variance requires further attention because its estimation also suffers from bias estimation problems of the autoregressive parameters. To address this issue we suggest essaying the following iterative method:

1. Estimate the LRV ignoring the bounds. In this regard, we can use the parametric estimation method proposed in Ng and Perron (2001) and Perron and Qu (2007), which also allows us to select the optimal lag if we have modelled the series as an autoregressive process.

[4] Our guess bases on the fact that Roy and Fuller (2001) define $A = 5$ for the linear time trend case, for which the median of the distribution of $\hat{\tau}_W$ is $\tau_{0.5} = -1.96$. Consequently, we might expect that $A = 5$ is also valid for cases where $\overline{c} \geq 0.5$, although it should be born in mind that the K function involved in the correction depends on the deterministic specification.

2. Compute an initial educated estimation of the bounds:

$$\left[\hat{\underline{c}}^0, \hat{\overline{c}}^0\right] = \left[\frac{\left(\underline{b} - \hat{D}_t\right)}{\hat{\sigma}_0 T^{1/2}}, \frac{\left(\overline{b} - \hat{D}_t\right)}{\hat{\sigma}_0 T^{1/2}}\right].$$

3. Estimation of α

   (a) For the MU-based procedure, compute the look-up tables corresponding to $[\hat{\underline{c}}^0, \hat{\overline{c}}^0]$ by simulation and obtain $\hat{\alpha}_{MU}$

   (b) For the truncated WSLS-based procedure, compute the percentiles of the $\hat{\tau}_W$ distribution corresponding to $[\hat{\underline{c}}^0, \hat{\overline{c}}^0]$ by simulation and obtain $\hat{\alpha}_{TW}$ as defined in (5).

4. Use $\hat{\alpha}$ from the previous step to estimate the LRV again as follows,

$$y_t - \hat{\alpha} y_{t-1} = \mu + \sum_{j=1}^{k} \psi_j \Delta y_{t-j} + \varepsilon_t$$

$$\hat{\sigma}_1^2 = \frac{\sum_{t=1}^{T} \hat{\varepsilon}_t^2}{(1-\hat{\alpha})^2}.$$

5. Obtain the new bounds as:

$$\left[\hat{\underline{c}}^1, \hat{\overline{c}}^1\right] = \left[\frac{\left(\underline{b} - \hat{D}_t\right)}{\hat{\sigma}_1 T^{1/2}}, \frac{\left(\overline{b} - \hat{D}_t\right)}{\hat{\sigma}_1 T^{1/2}}\right].$$

6. Iterate the procedure until $\left|\sum_{t=1}^{T} \hat{\varepsilon}_{t,l}^2 - \sum_{t=1}^{T} \hat{\varepsilon}_{t,l-1}^2\right| < Tol$, being Tol the desired level of tolerance and l the step of iteration.

4 Finite sample performance

4.1 The AR(1) case

In this section we analyze different bias correction methods that are available in the literature when estimating autogressive processes. The DGP is given by (1) to (3) with $\mu = 0$ and $\varepsilon_t \sim iid\ N(0,1)$. The symmetric bounds are defined by $[\underline{c}, \overline{c}] = [-\overline{c}, \overline{c}]$, with $\overline{c} = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1, 1.5\}$, $\alpha = \{0, 0.1, 0.2, ..., 1\}$, $T = 100$ and 1,000 replications are used. We consider three different cases depending on the method that is used to estimate the order of the autoregressive process. First, we focus on the situation in which p is known. Second, we treat p as an additional unknown parameter, which is estimated using both the MAIC information criterion in Ng and Perron (2001) and the BIC information criterion, specifying a maximum of $p_{\max} = \lfloor 12(T/100)^{1/4} \rfloor$ lags.

Tables 3 and 4 present the mean of the empirical distribution of the $\hat{\alpha}_{MU}$ and $\hat{\alpha}_{TW}$ estimators, respectively, for different values of $\bar{c}$ and α, assuming that the order of the autoregressive correction in (6) is known – i.e., $k = p - 1$ – the modified $\hat{\alpha}_{TW}$ estimator bases on the use of either the asymptotic $\tau_{pct} = \tau_{50}$ or $\tau_{pct} = \tau_{85}$ percentiles. Let us first focus on the $\hat{\alpha}_{MU}$ estimator. As can be seen from Table 3, for $T = 50$ and $\bar{c} > 0.1$ the $\hat{\alpha}_{MU}$ estimator tends to under-estimate α, being under-estimation distortion more pronounced as α approaches one. Note that the bias correction does not provide good results for $\bar{c} = 0.1$. The performance improves as T increases, showing that the estimation bias is almost fully corrected in most cases – the exception is found for $\alpha > 0.8$ and $\bar{c} = 0.1$, in which case the estimated value of the autoregressive parameter is below the populational one. It is worth noticing that there is a mild under-estimation distortion when $\alpha = 1$, something that is not found for the I(0) stationary cases.

Table 4 reports two versions of the $\hat{\alpha}_{TW}$ estimator, depending on the τ_{pct} value that is used in the truncation. In general and for a given values of T, α and $\bar{c}$, the estimator $\hat{\alpha}_{TW}$ that bases on τ_{50} – henceforth, $\hat{\alpha}_{TW}^{\tau_{50}}$ – outperforms the one that relies on τ_{85} – henceforth, $\hat{\alpha}_{TW}^{\tau_{85}}$. It is worth noticing that in most cases both estimators $\hat{\alpha}_{TW}^{\tau_{50}}$ and $\hat{\alpha}_{TW}^{\tau_{85}}$ produce the same result, but when they differ, $\hat{\alpha}_{TW}^{\tau_{50}}$ does better than $\hat{\alpha}_{TW}^{\tau_{85}}$. As expected, the bias correction improves as T increases, regardless of α and $\bar{c}$.

When the analysis comes to compare $\hat{\alpha}_{MU}$ and $\hat{\alpha}_{TW}$ estimators, we observe that for $T = 50$, $\alpha \geq 0.2$ and $\bar{c} < 0.3$, $\hat{\alpha}_{MU}$ outperforms $\hat{\alpha}_{TW}^{\tau_{50}}$. However, the converse situation is found when $\bar{c} \geq 0.3$, in which case a minimal dominance of $\hat{\alpha}_{TW}^{\tau_{50}}$ is observed regardless of α – note that, in general, differences are small. The picture is qualitatively similar for $T = 200$. Considering these elements, we can conclude that $\hat{\alpha}_{MU}$ offers an overall compromise in terms of bias estimation correction compared to $\hat{\alpha}_{TW}$, since although $\hat{\alpha}_{TW}^{\tau_{50}}$ outperforms $\hat{\alpha}_{MU}$ in some cases, the difference is not large.

The analysis so far considers that the true order of the autoregressive model is used, something that is not realistic from an empirical point of view. Tables 5 and 6 summarize the performance of $\hat{\alpha}_{TW}$ when k in (6) is selected using the MAIC and BIC informaction criteria, respectively. In general, the use of MAIC leads $\hat{\alpha}_{TW}$ to over-estimate α compared to the estimates that rely on BIC. Consequently, BIC-based estimates would be preferred to MAIC-based ones. If we compare the results in Tables 4 and 6, we can see that the estimation of p improves the performance of $\hat{\alpha}_{TW}^{\tau_{50}}$ in those cases in which $\hat{\alpha}_{MU}$ outperformed $\hat{\alpha}_{TW}^{\tau_{50}}$ with known p. Further, if the comparison is established between $\hat{\alpha}_{TW}^{\tau_{50}}(BIC)$ and $\hat{\alpha}_{MU}$, we can observe that there are some cases in which $\hat{\alpha}_{MU}$ outperforms $\hat{\alpha}_{TW}^{\tau_{50}}(BIC)$ – for instance, $T = 50$ with $\bar{c} < 0.3$, or $T = 200$ with $\bar{c} < 0.3$ and $\alpha < 0.7$ – although there are many cases in which the converse situation is found – for instance, for $T = 50$, $\bar{c} \geq 0.3$ and $\alpha > 0.3$.

Taking all these elements together, we can conclude that there is not a clear predominance of one estimator over the other, but considering that p is not known in practice, we believe that the use of $\hat{\alpha}_{TW}^{\tau_{50}}$ with k in (6) selected using BIC provides a good compromise in terms of bias estimation correction for empirical applications. Finally, it is worth mentioning that the computation of $\hat{\alpha}_{TW}^{\tau_{50}}$ bases on the percentile of the limiting distribution of $\hat{\tau}_W$, whereas $\hat{\alpha}_{MU}$ uses look-up tables that have been simulated for the specific empirical size. This feature might explain the superiority of $\hat{\alpha}_{MU}$ over $\hat{\alpha}_{TW}^{\tau_{50}}$ in some cases, so that it might be the case that using the percentiles of the finite sample distribution of $\hat{\tau}_W$ would lead to improvements. Notwithstanding, an important advantage is that the computational cost to obtain $\hat{\alpha}_{TW}^{\tau_{50}}$ (BIC) is smaller than getting sample size-specific look-up tables that deliver $\hat{\alpha}_{MU}$.

4.2 The AR(p) case

This section generalizes the analysis considering the DGP defined by an AR(2) process:

$$\begin{aligned} x_t &= \mu + y_t \\ y_t &= \alpha y_{t-1} + \psi \Delta y_{t-1} + u_t, \end{aligned}$$

with $\mu = 0$, $\alpha = \{0.8, 0.9, 0.95, 1\}$ and $\psi = 0.5$. The disturbance term u_t is assumed to be decomposed as defined in (3) with $\varepsilon_t \sim iid\ N(0,1)$. In this case, we only report simulation results for $\hat{\alpha}_{TW}$ since Andrews (1993) estimator was proposed just for the AR(1) case. It would be possible to compute the generalization of the MU estimator for AR(p) cases suggested in Andrews and Chen (1994), but the high computational cost that this will imply prevent us to do so.[5] Table 7 presents the mean of the distribution of $\hat{\alpha}_{TW}$ using either $\tau_{pct} = \tau_{50}$ and $\tau_{pct} = \tau_{85}$, when p is known or estimated.

Let us first focus on the case where k in (6) is known. Some remarks are in order. First, note that for a given T the results for $\hat{\alpha}_{TW}^{\tau_{50}}$ and $\hat{\alpha}_{TW}^{\tau_{85}}$ are almost equivalent when $\bar{c} \geq 0.3$, taking values around the true α when $T = 200$. It is worth noticing that a mild under-estimation is produced when $\alpha = 1$, which might lead to suggest that the super-efficient estimator of Perron and Yabu (2009a) could be used in this case. Second, $\hat{\alpha}_{TW}^{\tau_{50}}$ provides better estimates than $\hat{\alpha}_{TW}^{\tau_{85}}$ for $\bar{c} = 0.2$, although in both cases the estimated values are below the true α and, more interestingly, these estimates collapse around a similar value, regardless of the true α. For instance, when $\bar{c} = 0.2$ the mean of $\hat{\alpha}_{TW}^{\tau_{50}}$ is estimated around 0.5 for $T = 50$ and 0.74 for $T = 200$, regardless of α. Finally, a strange phenomenon appears for $\bar{c} = 0.1$, since the estimation bias correction gets worse as T

[5] Andrews and Chen (1994) procedure would require the use of bootstrap for each replication of the Monte Carlo simulation experiment.

increases – as before, the mean of the estimators seems to collapse around the same value for all range of α parameters that has been specified.

Results based on MAIC selection of k in (6) provide over-estimation of α when $\alpha = 0.8$, but the results are reasonably good for $T = 200$ with $\bar{c} \geq 0.5$. Similar to the previous case, $\hat{\alpha}_{TW}^{\tau_{50}}$ and $\hat{\alpha}_{TW}^{\tau_{85}}$ tend to collapse around a given value when $\bar{c} \leq 0.3$, providing useless estimates. For instance, when $\bar{c} = 0.1$ the mean of $\hat{\alpha}_{TW}^{\tau_{50}}$ is placed around 0.94 ($T = 50$) and 0.96 ($T = 200$) for the different values of α. Better results are obtained when k is chosen using the BIC. In this case, the estimates are located around the true α for $T = 200$ and $\bar{c} \geq 0.4$, although mild under-estimation is produced when $\alpha = 1$ – this might be solved with the super-efficient estimator of Perron and Yabu (2009a). The estimation procedure does not provide satisfactory results for $\bar{c} < 0.4$. Thus, a similar picture as for the known k case is obtained for $\bar{c} = 0.2$ or $\bar{c} = 0.3$, that is, estimates clearly below the true α, which collapse around a given value regardless of the true α. Finally, we also observe the same behavior for $\bar{c} = 0.1$, although the value to which the estimators collapse tends to increase as T does.

In all, results based on BIC seem to provide an overall better performance, although the performance of the estimators is not satisfactory for small values of $\bar{c}$, say $\bar{c} < 0.4$. Further, we have observed that the use of the super-efficient estimator of Perron and Yabu (2009a) might help to correct mild under-estimation distortions in the neighborhood of I(1) non-stationarity.

5 Empirical illustration

The persistence of the current account balance (CAB) disequilibrium is a crucial issue for assessing the long-term solvency and sustainability of the external debt of a country. Different conditions related to the order of integration of the ratio of CAB over GDP have been proposed in the literature to test external sustainability and, consequently, the unit root approach has been extensively applied.[6] Previous empirical literature considers that the current account is an unbounded variable. If this feature is not accounted for when testing the order of integration of the variables or computing their degree of persistence, the conclusions drawn from the unit root test statistics or the autoregressive parameter estimates can be misleading.

The ratio of CAB over GDP is not theoretically bounded, since flows of goods, services, income and transfers could exceed the total value of GDP. However, economic

[6]The economic theory underpinning this empirical literature stems from the inter-temporal approach to the current account, which was initially proposed by Sachs (1981) and Buiter (1981) and later extended by Obstfeld and Rogoff (1995) and by Gourinchas and Rey (2007). This approach considers a country's inter-temporal budget constraint that links the net foreign asset position and the future dynamics of the current account. Recently, Camarero et al. (2015) contribute to the discussion in the context of the European monetary integration process.

prudence does not advise maintaining large imbalances, which are a reflection of serious dysfunctions in the internal macroeconomic fundamentals, in the current account balance. Policy makers can introduce the control of CAB into their macro-prudential objectives more or less explicitly. Recently, and due to the aftermath of the Great Recession in Europe and, especially, in the European Monetary Union, a special system for monitoring macroeconomic imbalances was introduced for countries belonging to the euro area. This system called, Macroeconomic Imbalance Procedure (MIP) was introduced in 2011 and aims to identify and prevent potentially harmful macroeconomic imbalances that could adversely affect economic stability in a particular Member State, the euro area, or the EU as a whole. It controls a total of fourteen indicators, covering the major sources of macroeconomic imbalances and setting indicative thresholds for each of them. Among them, we can find several related to the health of the external foreign sector and, most interestingly for our case, the thresholds for the ratio of CAB over GDP, which are +6% and -4%, with a dynamic of a 3-year backward moving average.

We work with a sample of sixteen countries and annual data from 1980 to 2016. The source is the International Monetary Fund, World Economic Outlook Database, October 2016 and the evolution of the series is displayed in Figure 6. This figure includes the upper and lower thresholds in red, although some countries, such as Norway, Sweden, Switzerland and the United Kingdom, are not monitored by the MIP. The model estimated for each country (i) is as follows:

$$CAB_{t,i} = \mu_i + \rho_i CAB_{i,t-1} + \sum_{j=1}^{k_i} \psi_{i,j} \Delta CAB_{i,t-j} + \varepsilon_{i,t}.$$

Results in Table 8 show that the degree of persistence, measured by $\hat{\rho}_i$ is relatively low and, on the whole, far from the unit root area when standard OLS or bias-corrected OLS are used. Nevertheless, when the bounded nature is considered, most countries show a unit root, emphasizing the insufficiency of the market to promote effective adjustments to offset external disequilibria, and supporting the surveillance measures proposed by the European Commission, as Camarero et al. (20015) highlight.

6 Conclusions

This paper analyzes the behavior of the first order autoregressive estimator when the stochastic process being studied is influenced by the presence of bounds that regulate its evolution. We consider both the standard OLS estimator as well as the techniques proposed in order to correct the finite sample bias of this estimator.

We first show that the presence of bounds clearly distorts the performance of both types of estimators. The more limited the stochastic process – i.e., the narrower the fluctuation bands – the higher the distortion effect. This is especially harmful when the autoregressive parameter takes values close to 1, given that the estimated values tend to take values close to 0. This clearly alters the interpretation of the results, leading practitioners to observe a scarce level of persistence, when the variable is indeed extremely persistent.

In order to remove this effect, we have proposed some modifications of the methods proposed by Andrews (1993) and Perron and Yabu (2009a). Based on simulation exercises, we have found that these extensions are quite helpful in order to appropriately determine shocks persistence for bounded stochastic processes. However, it seems that more work in this line is required, given that both methods still exhibit some distortions when the variable is highly regulated.

Finally, we have applied these new methods to the analysis of current account balance for a sample of developed countries. Our results show that the use of the proposed methods improve our knowledge about the stochastic properties of the variables under study. In particular, this allows us to carry out shock persistence analysis in a more properly way.

A Mathematical appendix

Lemma 1 *Let $\{y_t\}_{t=1}^T$ be the stochastic process given by (2)-(3). Then, as $T \to \infty$*
a) $T^{-2} \sum_{t=1}^{T} y_t^2 \to \frac{\sigma_\varepsilon^2}{1-\alpha^2}$
b) $T^{-2} \sum_{t=1}^{T} y_t y_{t-1} \to \frac{\alpha\sigma_\varepsilon^2}{1-\alpha^2}$.

Proof. In order to prove the previous results, we have to take into account that ■

$$\begin{aligned} y_t &= \alpha y_{t-1} + u_t = \alpha \; y_{t-1} + \varepsilon_t + \xi_t - \bar{\xi}_t \\ &= \alpha \; y_{t-1} + \varepsilon_t + r_t \end{aligned}$$

For statement (a):

$$\begin{aligned} T^{-2} \sum_{t=1}^{T} y_t^2 &= T^{-2} \sum_{t=1}^{T} \sum_{i=1}^{t} \left(\alpha^i u_{t-i}\right)^2 = T^{-2} \sum_{t=1}^{T} \sum_{i=1}^{t} \alpha^{2i} (\varepsilon_{t-i} + r_{t-i})^2 \\ &= T^{-2} \sum_{t=1}^{T} \sum_{i=1}^{t} \alpha^{2i} \varepsilon_{t-i}^2 + T^{-2} \sum_{t=1}^{T} \sum_{i=1}^{t} \alpha^{2i} (r_{t-i}\varepsilon_{t-i} + r_t^2) \\ &= T^{-1} \sum_{t=1}^{T} T^{-1} \sum_{i=1}^{t} \alpha^{2i} \varepsilon_{t-i}^2 + o_p(1) \\ &= T^{-1} \sum_{t=1}^{T} \frac{\sigma_\varepsilon^2}{1-\alpha^2} \\ &= \frac{\sigma_\varepsilon^2}{1-\alpha^2}, \end{aligned}$$

given that by assumption A1(b) of Cavaliere and Xu (2014) we have that:

$$T^{-1} \sum_{i=1}^{T} (r_{t-i}\varepsilon_{t-i} + r_t^2) = o_p(1).$$

For statement (b):

$$\begin{aligned} T^{-1} \sum_{t=1}^{T} y_t y_{t-1} &= T^{-1} \sum_{t=1}^{T} \left[\alpha y_{t-1}^2 + (\varepsilon_t + r_t)\, y_{t-1}\right] \\ &= T^{-1} \sum_{t=1}^{T} \alpha y_{t-1}^2 + T^{-1} \sum_{t=1}^{T} (\varepsilon_t + r_t)\, y_{t-1} \\ &= \frac{\alpha\sigma_\varepsilon^2}{1-\alpha^2} + o_p(1), \end{aligned}$$

given that $\sum_{t=1}^{T}(\varepsilon_t + r_t)\, y_{t-1} = o_p(T)$ – see Lemma A.4 of Cavaliere and Xu (2014).

A.1 Proof of Theorem 1

Let us consider that the variable y_t is generated by (2) and (3) with $|\alpha| < 1$ and let the first order autoregressive parameter be defined as $r = N/D$, where

$$N = \frac{1}{T-1}\sum_{t=1}^{T-1} y_t \; y_{t-1} \tag{7}$$

$$D = \frac{1}{T}\sum_{t=1}^{T} y_t^2 \tag{8}$$

To obtain the asymptotic behavior of the first-order autoregressive estimator we need to know the convergence of N and D. This is straightforward if we use the results of Lemma 1 so that

$$r = \frac{\frac{\alpha\sigma_\varepsilon^2}{1-\alpha^2}}{\frac{\sigma_\varepsilon^2}{1-\alpha^2}} + o_p(1),$$

that is, we obtain that $r \xrightarrow{p} \alpha$.

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Table 1: Andrews MU estimates for symmetric bounded stochastic processes

	α	α_{MU} $\bar{c} = 0.1$	$\bar{c} = 0.3$	$\bar{c} = 0.5$	$\bar{c} = 0.7$	$\bar{c} = 1$
0	0.00	0.000	-0.000	-0.000	0.000	0.000
0.1	0.10	0.099	0.100	0.100	0.100	0.100
0.2	0.20	0.200	0.200	0.199	0.200	0.200
0.3	0.30	0.299	0.299	0.300	0.299	0.300
0.4	0.40	0.399	0.399	0.400	0.399	0.399
0.5	0.50	0.498	0.499	0.500	0.499	0.499
0.6	0.60	0.596	0.599	0.599	0.599	0.599
0.7	0.70	0.691	0.699	0.699	0.699	0.699
0.8	0.80	0.779	0.799	0.799	0.799	0.799
0.9	0.90	0.851	0.898	0.898	0.898	0.898
1	1.00	0.875	0.983	0.993	0.996	0.997

Table 2: Percentiles of the limiting distribution of $\hat{\tau}_W$ for different (symmetric) bounds

$(\underline{c}, \bar{c})$	1%	2.5%	5%	7%	7.5%	10%	15%	50%	85%
$(-0.1, 0.1)$	-9.16	-9.01	-8.88	-8.82	-8.80	-8.74	-8.64	-8.25	-7.89
$(-0.2, 0.2)$	-5.39	-5.18	-5.02	-4.94	-4.93	-4.86	-4.76	-4.38	-4.07
$(-0.3, 0.3)$	-4.58	-4.21	-3.94	-3.82	-3.79	-3.70	-3.56	-3.11	-2.80
$(-0.4, 0.4)$	-4.17	-3.85	-3.58	-3.44	-3.41	-3.28	-3.09	-2.52	-2.17
$(-0.5, 0.5)$	-3.75	-3.49	-3.27	-3.15	-3.13	-3.02	-2.85	-2.22	-1.79
$(-0.6, 0.6)$	-3.37	-3.14	-2.95	-2.86	-2.84	-2.74	-2.60	-2.04	-1.56
$(-0.7, 0.7)$	-3.15	-2.89	-2.70	-2.61	-2.59	-2.50	-2.38	-1.89	-1.42
$(-0.8, 0.8)$	-3.11	-2.81	-2.56	-2.45	-2.43	-2.33	-2.20	-1.74	-1.32
$(-0.9, 0.9)$	-3.10	-2.79	-2.54	-2.40	-2.38	-2.26	-2.08	-1.59	-1.20
$(-1.0, 1.0)$	-3.14	-2.81	-2.55	-2.40	-2.38	-2.25	-2.06	-1.46	-1.08
$(-1.5, 1.5)$	-3.13	-2.81	-2.53	-2.39	-2.36	-2.23	-2.03	-1.20	-0.49
$(-\infty, \infty)$	-3.12	-2.80	-2.53	-2.39	-2.37	-2.24	-2.04	-1.21	-0.24

Table 3: Bias corrected estimator using Andrews median-unbiased estimator

T	$\alpha\backslash\bar{c}$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.5
50	0	0.33	0.07	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
	0.1	0.35	0.12	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11
	0.2	0.36	0.20	0.19	0.19	0.19	0.19	0.19	0.19	0.19	0.19	0.19
	0.3	0.40	0.28	0.28	0.28	0.28	0.27	0.28	0.28	0.28	0.28	0.27
	0.4	0.43	0.38	0.37	0.37	0.37	0.37	0.37	0.37	0.37	0.37	0.37
	0.5	0.46	0.48	0.46	0.46	0.47	0.46	0.47	0.46	0.46	0.46	0.46
	0.6	0.48	0.57	0.56	0.56	0.56	0.56	0.56	0.56	0.56	0.56	0.56
	0.7	0.49	0.64	0.66	0.66	0.66	0.66	0.66	0.66	0.66	0.66	0.66
	0.8	0.52	0.70	0.74	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75
	0.85	0.53	0.72	0.76	0.78	0.79	0.80	0.80	0.80	0.80	0.80	0.80
	0.9	0.53	0.75	0.78	0.80	0.82	0.83	0.84	0.84	0.84	0.84	0.84
	0.95	0.55	0.76	0.79	0.81	0.83	0.85	0.86	0.87	0.88	0.88	0.89
	1	0.57	0.77	0.83	0.86	0.87	0.87	0.88	0.89	0.90	0.91	0.93
200	0	0.03	0.02	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03
	0.1	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
	0.2	0.19	0.19	0.19	0.19	0.19	0.19	0.19	0.19	0.19	0.19	0.19
	0.3	0.29	0.29	0.29	0.29	0.29	0.29	0.29	0.29	0.29	0.29	0.29
	0.4	0.39	0.39	0.39	0.39	0.39	0.39	0.39	0.39	0.39	0.39	0.39
	0.5	0.49	0.49	0.49	0.49	0.49	0.49	0.49	0.49	0.49	0.49	0.49
	0.6	0.60	0.59	0.59	0.59	0.59	0.59	0.59	0.59	0.59	0.59	0.59
	0.7	0.71	0.69	0.69	0.69	0.69	0.69	0.69	0.69	0.69	0.69	0.69
	0.8	0.78	0.79	0.79	0.79	0.79	0.79	0.79	0.79	0.79	0.79	0.79
	0.85	0.81	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84
	0.9	0.84	0.88	0.89	0.89	0.89	0.89	0.89	0.89	0.89	0.89	0.89
	0.95	0.86	0.91	0.93	0.94	0.94	0.94	0.94	0.94	0.94	0.94	0.94
	1	0.88	0.94	0.95	0.96	0.97	0.97	0.97	0.97	0.97	0.98	0.98

Table 4: Mean of the distribution of $\hat{\alpha}_{TW}$. The AR(1) case with k known

		τ_{50}											τ_{85}										
T	$\alpha\backslash\bar{c}$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.5	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.5
50	0	0.22	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.03	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
	0.1	0.24	0.09	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.04	0.09	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11
	0.2	0.25	0.17	0.20	0.21	0.21	0.21	0.21	0.21	0.21	0.21	0.21	0.06	0.17	0.20	0.21	0.21	0.21	0.21	0.21	0.21	0.21	0.21
	0.3	0.30	0.24	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.07	0.24	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30
	0.4	0.34	0.33	0.39	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.08	0.31	0.39	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40
	0.5	0.37	0.42	0.48	0.49	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.11	0.37	0.47	0.49	0.50	0.50	0.50	0.49	0.49	0.49	0.49
	0.6	0.39	0.50	0.57	0.59	0.59	0.59	0.59	0.59	0.59	0.59	0.59	0.12	0.43	0.56	0.58	0.59	0.59	0.59	0.59	0.59	0.59	0.58
	0.7	0.41	0.57	0.66	0.68	0.68	0.68	0.68	0.68	0.68	0.68	0.68	0.14	0.48	0.63	0.67	0.68	0.68	0.68	0.68	0.68	0.68	0.67
	0.8	0.44	0.63	0.74	0.76	0.77	0.77	0.77	0.77	0.77	0.77	0.77	0.16	0.52	0.69	0.75	0.76	0.77	0.77	0.77	0.77	0.76	0.76
	0.85	0.45	0.66	0.76	0.80	0.82	0.82	0.82	0.82	0.81	0.81	0.81	0.17	0.54	0.71	0.77	0.80	0.81	0.81	0.81	0.81	0.81	0.80
	0.9	0.45	0.69	0.78	0.83	0.85	0.86	0.86	0.86	0.86	0.86	0.85	0.17	0.55	0.72	0.79	0.83	0.84	0.85	0.85	0.85	0.85	0.84
	0.95	0.47	0.70	0.80	0.83	0.86	0.89	0.90	0.90	0.91	0.90	0.90	0.19	0.57	0.73	0.80	0.83	0.86	0.88	0.89	0.89	0.89	0.88
	1	0.49	0.72	0.84	0.88	0.90	0.91	0.92	0.93	0.94	0.94	0.95	0.18	0.58	0.76	0.83	0.86	0.88	0.90	0.91	0.92	0.93	0.93
200	0	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00
	0.1	0.08	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.08	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
	0.2	0.15	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.15	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20
	0.3	0.23	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.23	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30
	0.4	0.30	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.30	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40
	0.5	0.37	0.49	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.37	0.49	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50
	0.6	0.43	0.59	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.42	0.59	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.60
	0.7	0.51	0.68	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.47	0.68	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70
	0.8	0.61	0.77	0.79	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.51	0.77	0.79	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
	0.85	0.65	0.81	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.53	0.80	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84
	0.9	0.68	0.86	0.88	0.89	0.89	0.89	0.89	0.89	0.89	0.89	0.89	0.55	0.83	0.88	0.89	0.89	0.89	0.89	0.89	0.89	0.89	0.89
	0.95	0.71	0.89	0.93	0.94	0.94	0.94	0.94	0.94	0.94	0.94	0.94	0.57	0.85	0.91	0.93	0.94	0.94	0.94	0.94	0.94	0.94	0.94
	1	0.74	0.92	0.95	0.97	0.97	0.98	0.98	0.98	0.99	0.99	0.99	0.59	0.87	0.93	0.95	0.96	0.97	0.97	0.98	0.98	0.98	0.98

Table 5: Mean of the distribution of $\hat{\alpha}_{TW}$. The AR(1) case with k estimated using MAIC

		τ_{50}											τ_{85}										
T	$\alpha\backslash\bar{c}$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.5	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.5
50	0	0.94	0.88	0.72	0.51	0.39	0.34	0.30	0.26	0.24	0.23	0.20	0.92	0.85	0.53	0.33	0.25	0.22	0.21	0.20	0.18	0.17	0.12
	0.1	0.94	0.88	0.73	0.56	0.45	0.41	0.38	0.35	0.33	0.31	0.29	0.92	0.84	0.57	0.40	0.33	0.31	0.29	0.28	0.27	0.26	0.22
	0.2	0.94	0.87	0.75	0.61	0.52	0.48	0.45	0.43	0.41	0.40	0.37	0.91	0.84	0.62	0.47	0.41	0.39	0.38	0.37	0.36	0.35	0.31
	0.3	0.94	0.88	0.77	0.66	0.59	0.55	0.52	0.50	0.49	0.48	0.45	0.91	0.85	0.66	0.54	0.49	0.47	0.46	0.45	0.44	0.43	0.40
	0.4	0.94	0.90	0.79	0.70	0.64	0.62	0.59	0.57	0.56	0.55	0.53	0.92	0.86	0.70	0.61	0.56	0.55	0.54	0.53	0.52	0.52	0.49
	0.5	0.95	0.90	0.81	0.73	0.69	0.67	0.65	0.64	0.63	0.62	0.61	0.93	0.87	0.73	0.66	0.63	0.62	0.61	0.60	0.60	0.59	0.57
	0.6	0.94	0.92	0.83	0.76	0.74	0.72	0.71	0.70	0.69	0.69	0.67	0.92	0.88	0.76	0.71	0.69	0.68	0.68	0.67	0.67	0.66	0.64
	0.7	0.95	0.94	0.87	0.81	0.78	0.77	0.76	0.76	0.75	0.75	0.74	0.93	0.89	0.81	0.77	0.75	0.74	0.74	0.74	0.73	0.73	0.71
	0.8	0.94	0.95	0.90	0.87	0.85	0.84	0.83	0.82	0.82	0.81	0.81	0.92	0.90	0.85	0.83	0.82	0.81	0.81	0.81	0.80	0.80	0.79
	0.85	0.95	0.95	0.92	0.89	0.88	0.87	0.86	0.86	0.85	0.85	0.84	0.92	0.91	0.85	0.85	0.85	0.84	0.84	0.84	0.84	0.83	0.82
	0.9	0.95	0.94	0.92	0.91	0.90	0.90	0.90	0.89	0.89	0.88	0.87	0.92	0.89	0.86	0.86	0.87	0.87	0.87	0.87	0.87	0.87	0.86
	0.95	0.96	0.95	0.92	0.91	0.91	0.92	0.92	0.93	0.92	0.92	0.91	0.92	0.89	0.86	0.86	0.88	0.89	0.90	0.90	0.90	0.90	0.89
	1	0.95	0.94	0.93	0.93	0.93	0.93	0.94	0.95	0.95	0.95	0.96	0.92	0.89	0.87	0.88	0.89	0.90	0.91	0.92	0.93	0.94	0.94
200	0	0.97	0.71	0.18	0.12	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.97	0.50	0.13	0.11	0.11	0.11	0.11	0.11	0.11	0.10	0.09
	0.1	0.97	0.74	0.27	0.21	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.97	0.57	0.22	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.18
	0.2	0.97	0.78	0.36	0.30	0.29	0.29	0.29	0.29	0.29	0.29	0.29	0.97	0.62	0.31	0.29	0.29	0.29	0.29	0.29	0.29	0.29	0.28
	0.3	0.97	0.81	0.44	0.39	0.39	0.38	0.38	0.38	0.38	0.38	0.38	0.97	0.67	0.40	0.39	0.38	0.38	0.38	0.38	0.38	0.38	0.37
	0.4	0.98	0.84	0.54	0.48	0.48	0.48	0.48	0.48	0.48	0.48	0.47	0.98	0.73	0.49	0.48	0.48	0.48	0.47	0.47	0.47	0.47	0.46
	0.5	0.99	0.87	0.62	0.57	0.57	0.57	0.57	0.57	0.57	0.56	0.56	0.99	0.78	0.58	0.57	0.57	0.57	0.56	0.56	0.56	0.56	0.56
	0.6	0.99	0.89	0.71	0.66	0.66	0.66	0.66	0.65	0.65	0.65	0.65	0.99	0.83	0.67	0.66	0.66	0.65	0.65	0.65	0.65	0.65	0.65
	0.7	1.00	0.91	0.79	0.75	0.75	0.74	0.74	0.74	0.74	0.74	0.74	0.99	0.87	0.76	0.74	0.74	0.74	0.74	0.74	0.74	0.74	0.74
	0.8	1.00	0.94	0.86	0.84	0.83	0.83	0.83	0.83	0.83	0.83	0.82	1.00	0.91	0.84	0.83	0.83	0.83	0.83	0.82	0.82	0.82	0.82
	0.85	1.00	0.96	0.89	0.87	0.87	0.87	0.87	0.87	0.87	0.86	0.86	0.99	0.93	0.88	0.87	0.87	0.87	0.86	0.86	0.86	0.86	0.86
	0.9	0.99	0.98	0.93	0.91	0.91	0.91	0.90	0.90	0.90	0.90	0.90	0.99	0.96	0.91	0.91	0.90	0.90	0.90	0.90	0.90	0.90	0.90
	0.95	0.99	0.98	0.96	0.95	0.95	0.95	0.95	0.95	0.94	0.94	0.94	0.98	0.96	0.94	0.94	0.94	0.94	0.94	0.94	0.94	0.94	0.94
	1	0.99	0.98	0.98	0.97	0.98	0.98	0.98	0.99	0.99	0.99	0.99	0.98	0.95	0.96	0.96	0.97	0.97	0.98	0.98	0.98	0.98	0.98

Table 6: Mean of the distribution of $\hat{\alpha}_{TW}$. The AR(1) case with k estimated using BIC

		τ_{50}											τ_{85}										
T	$\alpha\backslash\bar{c}$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.5	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.5
50	0	0.41	0.13	0.06	0.05	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.24	0.10	0.05	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.03
	0.1	0.42	0.18	0.14	0.14	0.13	0.13	0.13	0.13	0.13	0.13	0.13	0.23	0.15	0.13	0.13	0.13	0.13	0.13	0.13	0.13	0.13	0.12
	0.2	0.41	0.25	0.23	0.23	0.22	0.22	0.22	0.22	0.22	0.22	0.22	0.23	0.23	0.22	0.22	0.22	0.22	0.22	0.22	0.22	0.22	0.22
	0.3	0.46	0.31	0.31	0.32	0.31	0.31	0.31	0.31	0.31	0.31	0.31	0.25	0.29	0.31	0.31	0.31	0.31	0.31	0.31	0.31	0.31	0.31
	0.4	0.48	0.38	0.41	0.41	0.41	0.41	0.41	0.41	0.41	0.41	0.41	0.26	0.35	0.40	0.41	0.41	0.41	0.41	0.41	0.41	0.41	0.40
	0.5	0.50	0.47	0.49	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.27	0.41	0.49	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.49
	0.6	0.51	0.55	0.58	0.59	0.59	0.59	0.59	0.59	0.59	0.59	0.59	0.28	0.47	0.57	0.59	0.59	0.59	0.59	0.59	0.59	0.59	0.58
	0.7	0.53	0.62	0.67	0.68	0.68	0.68	0.68	0.68	0.68	0.68	0.68	0.30	0.53	0.64	0.67	0.68	0.68	0.68	0.68	0.68	0.68	0.67
	0.8	0.56	0.66	0.75	0.77	0.77	0.77	0.77	0.77	0.77	0.77	0.76	0.31	0.56	0.70	0.75	0.76	0.77	0.76	0.76	0.76	0.76	0.75
	0.85	0.57	0.69	0.77	0.81	0.82	0.82	0.82	0.81	0.81	0.81	0.81	0.33	0.57	0.72	0.78	0.80	0.81	0.81	0.81	0.81	0.80	0.80
	0.9	0.58	0.70	0.79	0.83	0.85	0.86	0.86	0.86	0.86	0.85	0.85	0.32	0.58	0.73	0.79	0.83	0.84	0.85	0.85	0.85	0.85	0.84
	0.95	0.57	0.71	0.81	0.83	0.86	0.88	0.90	0.90	0.90	0.90	0.90	0.33	0.59	0.74	0.80	0.84	0.86	0.88	0.89	0.89	0.89	0.88
	1	0.59	0.72	0.83	0.88	0.90	0.91	0.92	0.93	0.94	0.94	0.95	0.32	0.60	0.77	0.83	0.86	0.88	0.90	0.91	0.92	0.92	0.93
200	0	0.57	0.07	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.52	0.06	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
	0.1	0.58	0.15	0.14	0.14	0.14	0.14	0.14	0.14	0.14	0.14	0.14	0.53	0.14	0.14	0.14	0.14	0.14	0.14	0.14	0.14	0.14	0.14
	0.2	0.58	0.24	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.53	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23	0.23
	0.3	0.59	0.33	0.32	0.32	0.32	0.32	0.32	0.32	0.32	0.32	0.32	0.54	0.32	0.32	0.32	0.32	0.32	0.32	0.32	0.32	0.32	0.32
	0.4	0.62	0.42	0.42	0.42	0.42	0.42	0.42	0.42	0.42	0.42	0.42	0.57	0.41	0.42	0.42	0.42	0.42	0.42	0.42	0.42	0.42	0.42
	0.5	0.65	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.61	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51
	0.6	0.69	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.66	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.60	0.60
	0.7	0.73	0.69	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.69	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70
	0.8	0.78	0.77	0.79	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.71	0.77	0.79	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
	0.85	0.78	0.82	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.71	0.81	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84
	0.9	0.79	0.87	0.88	0.89	0.89	0.89	0.89	0.89	0.89	0.89	0.89	0.69	0.84	0.88	0.89	0.89	0.89	0.89	0.89	0.89	0.89	0.89
	0.95	0.80	0.89	0.93	0.94	0.94	0.94	0.94	0.94	0.94	0.94	0.94	0.69	0.86	0.91	0.93	0.94	0.94	0.94	0.94	0.94	0.94	0.94
	1	0.82	0.92	0.95	0.97	0.97	0.98	0.98	0.98	0.99	0.99	0.99	0.70	0.87	0.93	0.95	0.96	0.97	0.97	0.98	0.98	0.98	0.98

Table 7: Mean of the distribution of $\hat{\alpha}_{TW}$. The AR(2) case

k	τ_{pct}	T	$\alpha\backslash\bar{c}$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.5
known	τ_{50}	50	0.8	0.94	0.50	0.61	0.71	0.76	0.77	0.78	0.78	0.78	0.78	0.78
			0.9	0.95	0.50	0.62	0.74	0.80	0.84	0.86	0.87	0.87	0.87	0.88
			0.95	0.93	0.51	0.62	0.74	0.81	0.85	0.87	0.89	0.90	0.91	0.92
			1	0.92	0.49	0.60	0.73	0.81	0.86	0.88	0.90	0.91	0.92	0.95
		200	0.8	0.45	0.71	0.78	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
			0.9	0.45	0.74	0.85	0.88	0.89	0.90	0.90	0.90	0.90	0.90	0.90
			0.95	0.45	0.74	0.87	0.91	0.93	0.94	0.94	0.94	0.95	0.95	0.95
			1	0.44	0.74	0.87	0.92	0.95	0.96	0.97	0.97	0.98	0.98	0.99
	τ_{85}	50	0.8	0.73	0.38	0.60	0.71	0.75	0.77	0.78	0.78	0.78	0.78	0.78
			0.9	0.70	0.40	0.60	0.73	0.80	0.83	0.85	0.86	0.87	0.87	0.87
			0.95	0.70	0.40	0.60	0.73	0.80	0.84	0.86	0.88	0.89	0.90	0.91
			1	0.70	0.39	0.59	0.72	0.80	0.85	0.88	0.90	0.91	0.92	0.94
		200	0.8	0.37	0.71	0.78	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
			0.9	0.38	0.74	0.85	0.88	0.89	0.90	0.90	0.90	0.90	0.90	0.90
			0.95	0.38	0.74	0.87	0.91	0.93	0.94	0.94	0.94	0.94	0.95	0.94
			1	0.38	0.74	0.87	0.92	0.95	0.96	0.97	0.97	0.98	0.98	0.99
MAIC	τ_{50}	50	0.8	0.94	0.88	0.84	0.86	0.88	0.89	0.88	0.88	0.88	0.87	0.87
			0.9	0.96	0.88	0.86	0.89	0.91	0.92	0.92	0.92	0.91	0.91	0.91
			0.95	0.94	0.87	0.86	0.89	0.92	0.93	0.93	0.93	0.94	0.94	0.94
			1	0.94	0.88	0.86	0.89	0.92	0.94	0.94	0.94	0.94	0.94	0.96
		200	0.8	0.96	0.89	0.86	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84
			0.9	0.96	0.93	0.91	0.90	0.91	0.91	0.91	0.91	0.91	0.91	0.91
			0.95	0.96	0.93	0.93	0.93	0.94	0.95	0.95	0.95	0.95	0.95	0.95
			1	0.96	0.93	0.91	0.94	0.96	0.97	0.97	0.98	0.98	0.98	0.99
	τ_{85}	50	0.8	0.92	0.84	0.79	0.82	0.85	0.86	0.87	0.87	0.87	0.86	0.85
			0.9	0.94	0.84	0.80	0.85	0.88	0.90	0.90	0.90	0.90	0.90	0.90
			0.95	0.91	0.83	0.80	0.85	0.88	0.90	0.91	0.92	0.92	0.93	0.93
			1	0.92	0.83	0.80	0.85	0.89	0.91	0.92	0.93	0.93	0.93	0.95
		200	0.8	0.96	0.87	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84	0.84
			0.9	0.95	0.90	0.89	0.90	0.91	0.91	0.91	0.91	0.91	0.91	0.91
			0.95	0.95	0.90	0.91	0.93	0.94	0.95	0.95	0.95	0.95	0.95	0.95
			1	0.95	0.90	0.90	0.93	0.95	0.96	0.97	0.98	0.98	0.98	0.99
BIC	τ_{50}	50	0.8	0.49	0.51	0.68	0.75	0.78	0.78	0.79	0.79	0.79	0.79	0.79
			0.9	0.50	0.52	0.70	0.78	0.83	0.86	0.87	0.87	0.88	0.88	0.88
			0.95	0.51	0.54	0.70	0.78	0.83	0.86	0.89	0.90	0.91	0.92	0.92
			1	0.49	0.53	0.68	0.77	0.83	0.87	0.90	0.91	0.92	0.93	0.95
		200	0.8	0.59	0.72	0.78	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
			0.9	0.60	0.75	0.85	0.88	0.89	0.90	0.90	0.90	0.90	0.90	0.90
			0.95	0.61	0.75	0.87	0.91	0.93	0.94	0.94	0.94	0.95	0.95	0.95
			1	0.61	0.75	0.87	0.92	0.95	0.96	0.97	0.97	0.98	0.98	0.99
	τ_{85}	50	0.8	0.28	0.45	0.64	0.74	0.77	0.78	0.78	0.79	0.79	0.79	0.78
			0.9	0.28	0.45	0.66	0.76	0.82	0.84	0.86	0.87	0.87	0.87	0.87
			0.95	0.26	0.46	0.66	0.76	0.81	0.85	0.88	0.89	0.90	0.91	0.91
			1	0.26	0.45	0.64	0.75	0.82	0.86	0.89	0.90	0.91	0.92	0.94
		200	0.8	0.57	0.72	0.78	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
			0.9	0.57	0.75	0.85	0.88	0.89	0.90	0.90	0.90	0.90	0.90	0.90
			0.95	0.58	0.75	0.87	0.91	0.93	0.94	0.94	0.94	0.95	0.95	0.94
			1	0.58	0.75	0.87	0.92	0.95	0.96	0.97	0.97	0.98	0.98	0.99

Table 8: Persistence of Current Account Balance

	$\hat{\alpha}_{OLS}$	$\hat{\alpha}_W$	$\hat{\alpha}_{TW}$ (τ_{50})	$\hat{\alpha}_{TW}$ (τ_{85})	k
Austria	0.7815	0.8946	1.0000	0.9486	1
Belgium	0.7997	0.9406	1.0000	1.0000	2
Finland	0.8973	0.8958	1.0000	0.9398	1
France	0.8831	0.8685	0.9185	0.9058	1
Germany	0.9492	0.9785	1.0000	1.0000	1
Greece	0.9054	0.8825	1.0000	0.9340	1
Ireland	0.8780	0.9773	1.0000	1.0000	1
Italy	0.7857	0.8261	0.8968	0.8756	1
Luxembourg	0.7508	0.8355	1.0000	1.0000	2
Netherlands	0.8319	0.9065	1.0000	1.0000	1
Norway	0.8269	0.8213	1.0000	1.0000	1
Portugal	0.8235	0.8034	1.0000	1.0000	1
Spain	0.9160	0.8905	1.0000	0.9360	1
Sweden	0.9186	0.9667	1.0000	1.0000	1
Switzerland	0.8153	0.9240	1.0000	1.0000	5
United Kingdom	0.9120	0.9458	1.0000	1.0000	1

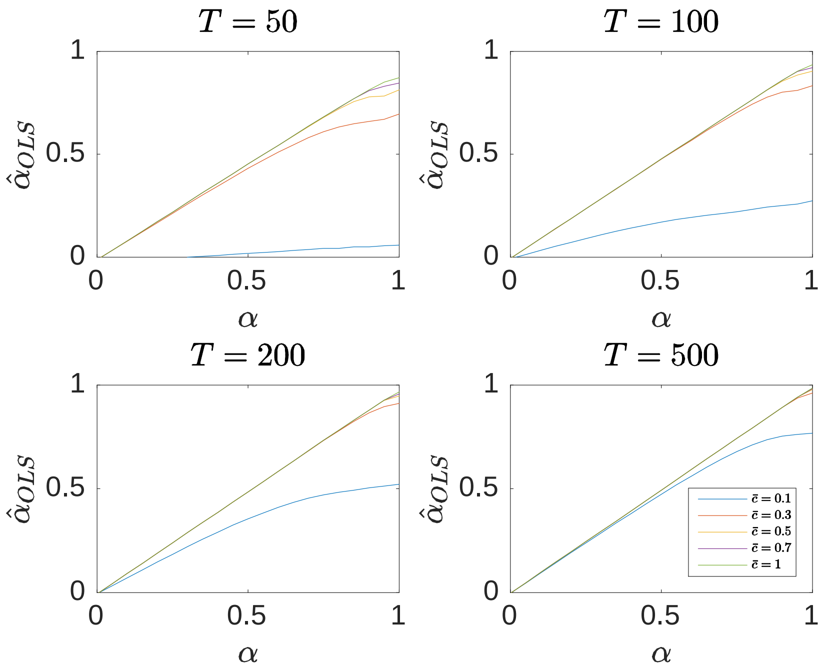

Figure 1: Mean of the OLS α estimate for different (symmetric) bounded time series

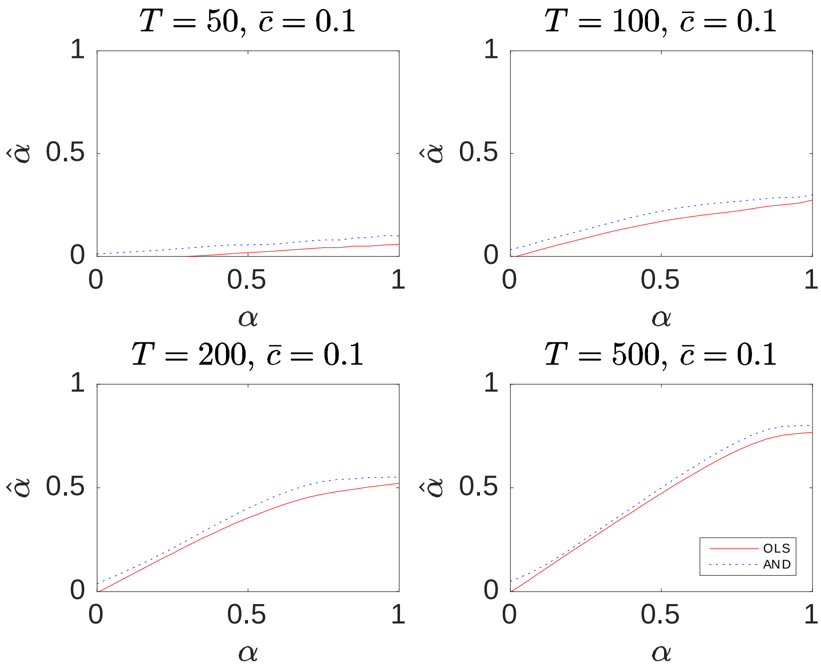

Figure 2: OLS and Andrews MU estimates for symmetric bounded processes with $\bar{c} = 0.1$

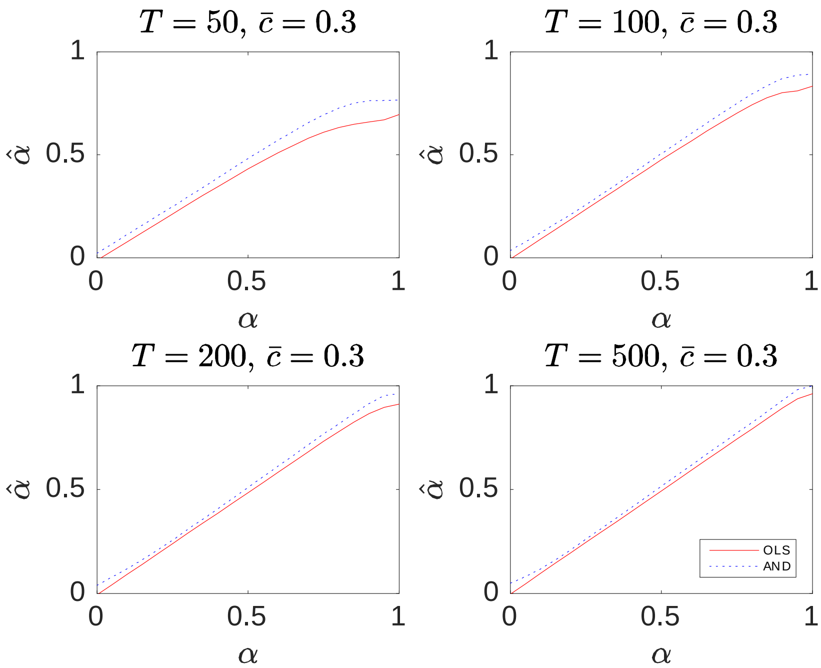

Figure 3: OLS and Andrews MU estimates for symmetric bounded processes with $\bar{c} = 0.3$

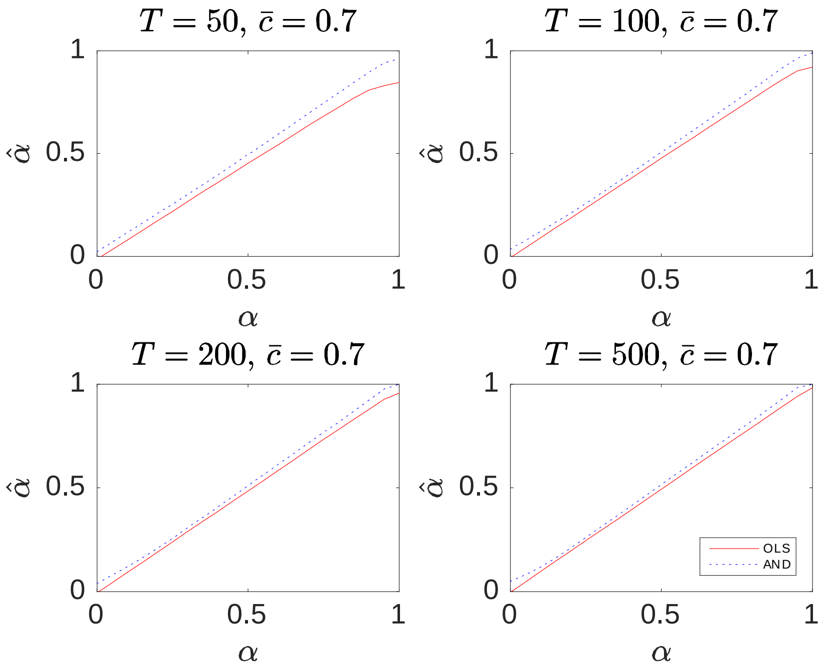

Figure 4: OLS and Andrews MU estimates for symmetric bounded processes with $\bar{c} = 0.7$

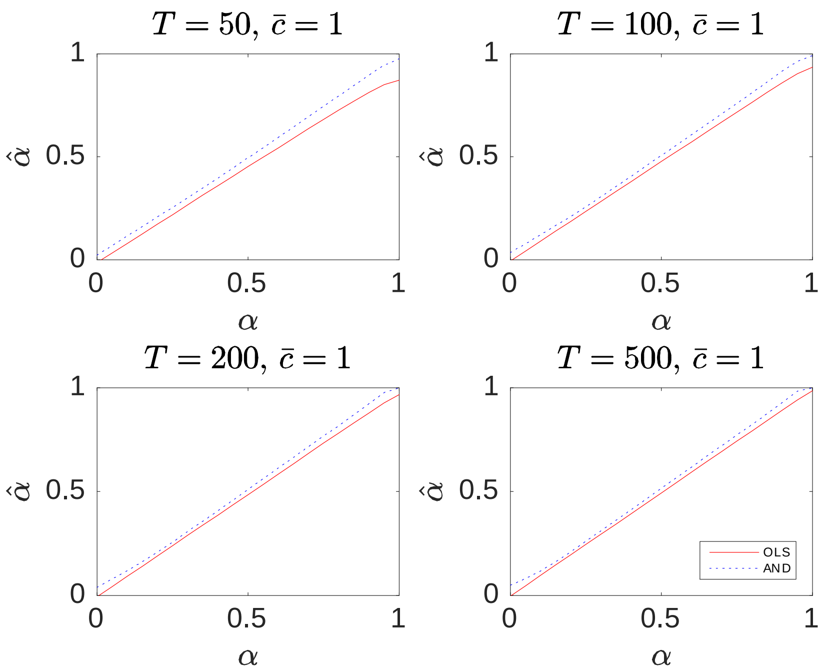

Figure 5: OLS and Andrews MU estimates for symmetric bounded processes with $\bar{c} = 1$

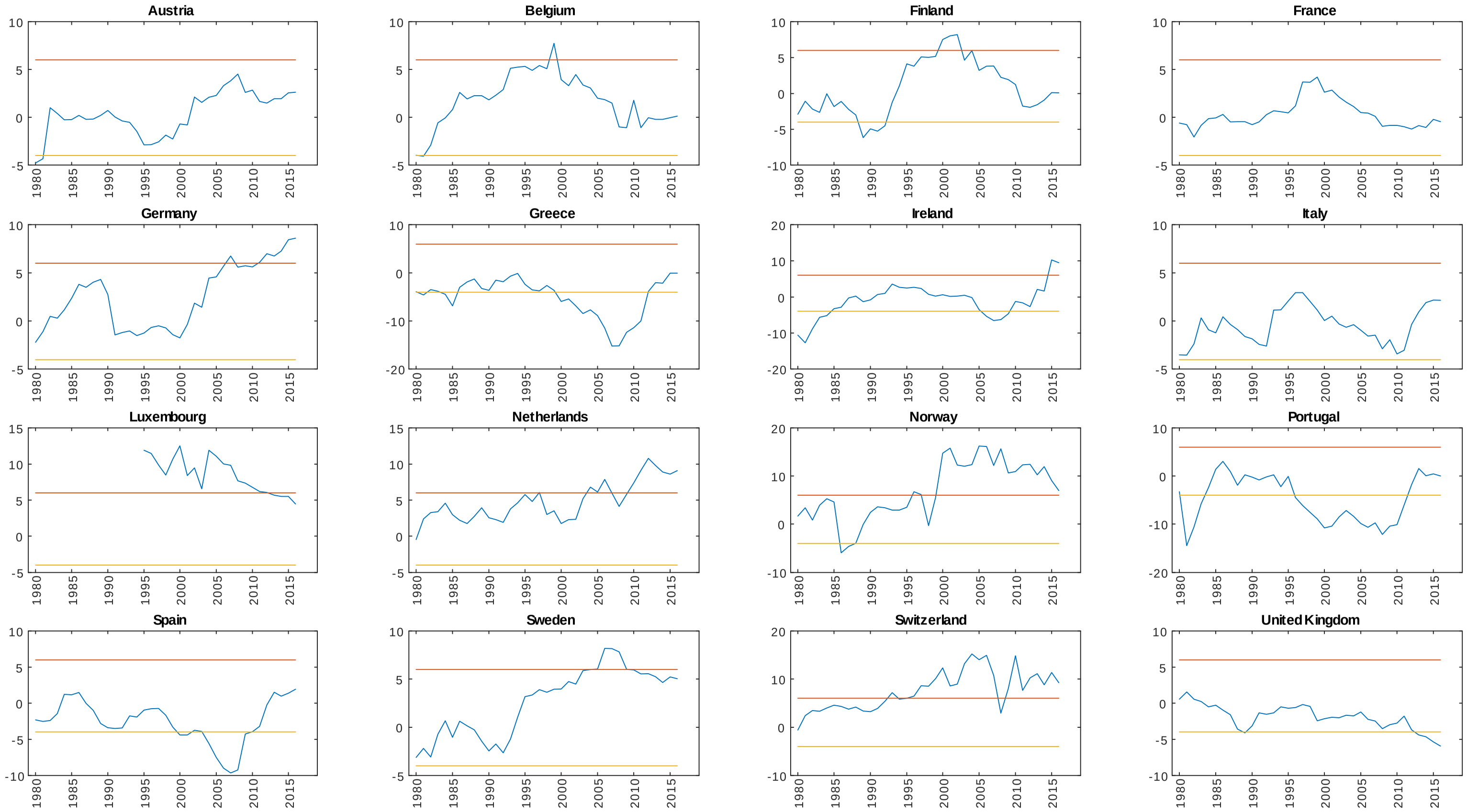

Figure 6: Current account over GDP ratio for selected countries