Contagion effects of U.S. Dollar and Chinese Yuan in Forward and Spot Foreign Exchange Markets

Abstract

Financial contagion in forex markets is modeled by the application of a bivariate Hawkes stochastic jump process. The self-exciting and mutually exciting properties of the jump-clustering model allow for illustrating internal and cross-sectional transmission processes. The results obtained suggest stronger effects from US to mutual markets than in the reverse case. Cross-sectional excitation dynamics in the spot markets are larger than in the forward markets. As a central result, we can observe that the results for the Hawkes-model parameters are more significant in the forward markets. Transmission dynamics beyond volatility determine the likelihood of contagion occurrence. The significance of the decay parameters towards the long term jump intensities supports the importance of abrupt fluctuations in the contagion discourse.

Keywords: Financial Contagion, Jump Clustering, Hawkes Process

1. Introduction

Empirically founded analysis of financial contagion is comprehensively studied and various techniques are presented in the literature (Grubel and Fadner, 1971; King and Wadhwani, 1990; Eichengreen et al., 1994). Empirical research by Baig and Goldjain (1999), Forbes and Rigobon (2002), and Favero and Giavazzi (2002) identify the conditions for rejecting parameter stability upon financial transmission processes mainly by using vector autoregressive models (VAR). The related literature can be analyzed under currency comovements, stochastic volatility, and exchange rate variance properties.

In terms of volatility and correlation in exchange rates various results are identified. Amira et al. (2011) quantify the relationship between return, volatility, and correlation using the generalized impulse response function and GARCH models; they tested for the asymmetries in the return-correlation and volatility-correlation relationships. For short and intermediate horizons, they find that the impact of volatility on correlation is asymmetric: volatility seems to have a greater impact on correlation during market downturn periods than during market upturn periods. Further, the increase in the correlation is more related to the market direction than the level of volatility. During downturn markets the level of correlation increases and the association between large volatilities and high correlations is mainly due to the simultaneous effect of bad news on both variables. Chiang et al. (2007) find higher correlation during crisis periods for Asian markets with the help of multivariate GARCH. Ang and Bekaert (2002) find international evidence for high-volatility correlation regimes during bear markets; they observe higher volatility than correlation. Multiple interpretations for exchange rate volatility; such as purchasing power parity (PPP) and regime explanations are given in Frenkel and Goldstein (1989).

Pioneering studies of stochastic volatility models are made by Bates (1996) and Heston (1993), relying on currency option pricing. Melino and Turnbull (1990) derive a stochastic volatility model for foreign exchange rate options and achieve a better fit to the data than empirical methods. Andersen (2003) constructs a GMM estimator for a jump diffusion model and derives accurate and reliable results. A summary for FX options models is given in Wystup (2006). Carr and Wu (2007) developed and estimated subclasses of models, which capture stochastic skew behavior of currency options outperforming traditional jump-diffusion models. Clark (2011) supports that stochastic volatility improves accuracy of forecasts. Ait-Sahalia et al. (2014) tested policy interventions credit default swaps (CDS) by the contagion jump feedback model.

The related explanations of exchange rate volatility are analyzed by various streams in the literature. Caporale and Pittis (1995) explain the basis for the link between exchange rate volatility to exchange rate regimes; he observes the rise of volatility when a country moves from a fixed to a flexible exchange rate regime, especially because of nominal exchange rate movements. Baxter and Stockman (1989) identified the variability of output, trade variables, and both private and government consumption under alternative real exchange rate regimes using different detrending techniques; and find evidence that the real exchange rate depends on real exchange-rate regimes. VAR and variance decomposition models to estimate relative contribution of real and nominal shocks to real exchange fluctuations are presented by Clarida and Gali (1994), Enders and Lee (1999), and Rogers (1999). A common focus is given on the fundamental determinants of long-run equilibrium real exchange rate fluctuations. Long run real exchange rate dynamics and fundamentals are analyzed by Ricci et al. (2008) and deviations from PPP are given in Mendoza (1995) and Rogoff (1996). Explicit time-varying nature of market data is captured in Aboura and Chevallier (2015). Increasing mutual dependencies and comovements in the financial markets lead to models of connectedness (Diebold and Yilmaz, 2014, 2015) and mutual excitements (Ait-Sahalia et al., 2014, 2015) in recent studies.

This study gains new insight into the propagation dynamics of spillover effects in international forex markets. We apply the stochastic jump diffusion model proposed by Ait-Sahalia et. al. (2015) to spot and forward forex markets. In this study, we apply a Hawkes (1971) diffusion model to contagious effects in bilateral exchange rates. The Hawkes process is a mutually dependent and self-exciting process, which allows for the simulation of cross-sectional and serial dependence clustering. This property enables us to model lagged transmissions of financial turbulence. The detailed specifications of the spillover dynamics are aimed to give evidence about the role of the transmission of contagion shocks. This study proceeds as follows. Section 2 gives a description and analysis of the data. In section 3, we pursue the risk evaluation of the currencies. Section 4 introduces the contagion model. The results for spot and forward exchange rate model specifications are documented in section 5. Finally, section 6 concludes and summarizes the results.

2. Data Description

2.1. Data Sample

The following exchange rate returns are used in the model implementation: Australian Dollar (AUD), Brazilian Real (BRL), Canadian Dollar (CAD), Chinese Yuan Renminbi (CNY), Danish Krone (DKK), Euro (EUR), Japanese Yen (JPY), Mexican Peso (MXN), British Pound (GBP). In sequel, the exchange rates are denominated as domestic exchanges against the U.S. Dollar (USD) rate. We use logged differences of the exchange rates ($\log e_t - \log e_{t-1}$). The data sample spans the period from 04/2004 to 04/2011. The data sample contains several periods of economic volatility regimes; such as shrinkage in the oil production, the Iraq war, the mortgage crisis and the subsequent macroeconomic recovery period. The aim of the data sample choice is to achieve an international diversified representation of different country market sizes. The data are obtained from the Thomson Reuters Datastream database, FRED and BIS. In the spot market, we use the U.S. Dollar and The Chinese Renminbi Yuan, expressed as broad trade-weighted bilateral exchange rates and use them to build a benchmark against the remaining currencies in our models. Since we are aware of the bilateral nature of exchange rates, we intend to achieve a filtered unilateral effect by introducing some exogenous notion in the applied time series. Therefore we do not test, i.e. CNY/USD on JPY/USD, we test CNY (and USD respectively), expressed as broad trade-weighted bilateral exchange rate and use them to build a benchmark against the remaining currencies in our models. The resulting effect will show filtered effect of CNY (and USD respectively) on each single exchange rates, accounting for their specific dynamics in the international forex markets. To achieve an exogenous variable series role of benchmark we calculate a trade weighted exchange rate in the forward currency market. The trade weighted exchange rate is calculated as the percentage weight of the trade volume (import + export) multiplied by the bilateral exchange rate e. The benchmark forward exchange rates (USD and CNY) are calculated according to trade percentage weights collected from BIS for CNY and FRED for USD.[1]

[1] We use the following calculation method: $E = \sum_{i=1}^{n} w_i e_i$ where w_i, is the percentage weight of trade volume and e, is the bilateral exchange rate. In the case of the Chinese Yuan, the weighted forward exchange rate is normalized through the division by the CNY/USD forward exchange rate: $E = (\sum_{i=1}^{n} w_i e_i)/e_{\text{norm}}$, where e_{norm} is the CNY/USD forward rate.

USD-EUR USD-MXN USD-CAD USD-CNY USD-JPY

Fig. 1. Exceedance Correlations Spot market (USD originated).

2.2. Descriptive Statistics

The descriptive statistics for the spot market exchange rates are given in Table 1. The highest skewness values are observed for CAD, DKK, and MXN in the spot exchange markets. The highest kurtosis values are given for CAD, CNY, DKK, and MXN. In the forward exchange rates the largest skewness values are observed for BRL, CNY, and MXN. The largest kurtosis values are observed for CAD, CNY, MXN and USD. The results for the Jarque-Bera test reveal that we find significant skewness and excess kurtosis for all currencies. A more accurate illustration of contagion dynamics can give the exceedance correlation and copula with higher moments generating dynamics which are described in the next section.

3. Risk Evaluation

3.1. Exceedance Correlations

In this section, we document the presence of nonlinear dependence by using exceedance correlations as proposed by Longin and Solnik (2001) and Ang and Chen (2002). We assume that we have two exchange rate returns X and Y which have been standardized with mean zero and variance one. Exceedance correlation measures the correlations of two stocks as being conditional on exceeding some threshold, that is:

$$\tilde{\rho}(\mathrm{p}) = \begin{cases} \mathrm{Corr}\left[X, Y | X \leq Q_x(\mathrm{p}) \text{ and } Y \leq Q_y(\mathrm{p})\right], \text{ for } \mathrm{p} \leq 0.5 \\ \mathrm{Corr}\left[X, Y | X > Q_x(\mathrm{p}) \text{ and } Y > Q_Y(\mathrm{p})\right], \text{ for } \mathrm{p} > 0.5, \end{cases} \tag{1}$$

In Fig. 1 exceedance correlation is illustrated as follows. The vertical axis gives the correlation between two assets given that both exceed that quantile. The horizontal axis shows the probability distribution in a given interval from (0 - 0.5) and (0.5 - 1). The exceedance correlation is estimated from the underlying data distribution. We implement the risk evaluation section for the major currencies, which are assumed to be of central importance in the contagion discourse. The results with regard to the distribution of exceedance correlation are presented subsequently. The results for the USD spot market are given as follows. The results for EUR and JPY show a nonlinear shape but are symmetric. MXN and CAD have almost linear shape and symmetric distribution for upper and lower quantiles. The results for CNY fall somewhat apart, right skewed with stronger exceedance towards the upper quantiles (0.7 - 0.8).

In the CNY spot markets (Fig. 4), we observe a right skewed and nonlinear shape of the exceedance correlation towards the right extreme for all currencies. The results for the USD forward markets reveal a nonlinear shape for CNY, Euro, and CAD. CNY is right skewed towards the right extreme quantiles; CAD is left skewed and EUR is symmetric. In the CNY forward markets, the EUR is slightly right tailed and nonlinear, JPY is peaked and right tailed, MXN lower peaked, right tailed and nonlinear, and CAD has a nonlinear shape and is left skewed.

In sum, right extreme high exceedance correlations for CNY markets can be observed. USD and MXN are linear and symmetric in the spot, as well as, in the forward markets. In general, spot markets exhibit higher exceedance correlation values.

3.2. Copulas

More formally, we can express nonlinear dependence in the form of copulas. Copulas support the shape and direction of the exceedance correlations and can be expressed as follows:

$$C(u, v, \rho, \upsilon) = \Phi_\rho\left(\Phi^{-1}(u), \Phi^{-1}(v); \rho, \nu\right) =$$

Table 1. Descriptive Statistics of Currency Exchange Rates

	Mean	Median	Maximum	Minimum	Std. Deviation	Skewness	Kurtosis	Probability	Jarque-Bera
	Spot Exchange Rates								
AUD/USD	0.000203	0.0006	0.0770	-0.0821	0.00996	-0.751	15.270	0.000	11754.16
BRL/USD	-0.000320	-0.0007	0.0755	-0.0967	0.01050	0.015	17.053	0.000	15192.07
CAD/USD	-0.003390	0.0000	0.4091	-0.8909	0.08120	-2.908	34.469	0.000	78776.11
CNY/USD	-0.000088	0.0000	0.0161	-0.0202	0.00159	-0.361	46.711	0.000	147005.80
DKK/USD	0.003280	0.0000	0.7123	-0.4561	0.07810	2.038	22.792	0.000	31408.84
EUR/USD	0.000062	0.0000	0.0462	-0.0300	0.00664	0.191	6.016	0.000	711.28
JPY/USD	-0.000149	-0.0001	0.0306	-0.0522	0.00702	-0.504	7.488	0.000	1627.89
MXN/USD	0.000073	-0.0003	0.0811	-0.0596	0.00687	1.039	23.986	0.000	34209.79
GBP/USD	-0.000035	0.0001	0.0443	-0.0497	0.00690	-0.344	8.333	0.000	2224.17
USD	-0.000118	-0.0001	0.0216	-0.0411	0.00521	-0.398	6.849	0.000	1188.44
	Forward Exchange Rates								
AUD/USD	0.000161	0.0008	0.0815	-0.0755	0.0101	-0.548	14.640	0.000	10514.42
BRL/USD	-0.000280	-0.0006	0.0814	-0.0703	0.0099	0.839	13.534	0.000	8753.14
CAD/USD	-0.000360	0.0000	0.5261	-0.6371	0.0775	-0.087	25.112	0.000	37610.14
CNY/USD	-0.000118	0.0000	0.0153	-0.0227	0.0016	-1.338	33.748	0.000	73274.60
DKK/USD	-0.000043	0.0000	0.0376	-0.0462	0.0065	-0.237	6.575	0.000	1000.80
EUR/USD	-0.000047	0.0000	0.0385	-0.0459	0.0064	-0.197	6.639	0.000	1031.01
JPY/USD	-0.000156	0.0001	0.0629	-0.0365	0.0070	0.063	8.723	0.000	2520.64
MXN/USD	0.000061	-0.0005	0.0753	-0.0488	0.0068	0.924	20.589	0.000	24061.02
GBP/USD	0.000055	0.0000	0.0390	-0.0434	0.0067	0.081	7.128	0.000	1312.92
USD	-0.000118	-0.0001	0.1285	-0.1587	0.0195	-0.136	24.563	0.000	35769.65

Note: USD is expressed as trade-weighted exchange rate.

USD-CNY
USD-EUR USD-JPY
USD-MXN USD-CAD

Fig. 2. Copula Probability Densities in Spot Markets (USA originated).

$$= \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \left(1 + \frac{x^2+y^2-2\rho xy}{\nu(1-\rho^2)}\right)^{-\frac{\nu+2}{2}} \mathrm{dy\ dx}.$$

where, u, v are the exchange rates, Φ^{-1} is the inverse cumulative distribution function of a standard univariate Student-t distribution with ν is the degrees of freedom, and Φ_ρ is the joint cumulative distribution of a multivariate Student-t distribution with zero mean vector and covariance matrix equal to the correlation matrix ρ.

In the USD spot market, we observe similar results for CAD, MXN, and the EUR: correlation at the extremes, lower correlation for the middle quantiles, and more correlation for the right extreme. Very high correlation at the extremes is given for JPY (Fig. 2). A constant correlation level but no extreme correlation is given for CNY. In the USD forward market results (Fig. 6) high extreme correlation is given for the EUR, MXN, and CAD. Moderate correlation for CNY, and moderate correlation for JPY for middle quantiles with higher values for correlation at the extremes.

In the CNY spot exchange market, in the case of EUR, JPY, and MXN moderate correlation is given, where more higher correlation at the extremes can be observed. CAD has a general low correlation level and a low extreme correlation level. In the CNY forward exchange rate markets (Fig. 7), generally low correlation and low correlation at the extremes can be observed for EUR, JPY and MXN. The CAD exhibits lower correlation at the extremes and lower correlation in general. Forward and spot markets show almost the same dynamics, whereas MXN spot exchange markets have more extreme correlation.

The results suggest that the USD creates strong extreme correlation effects, especially in the forward markets. The results for CNY are more moderate; however, some extreme correlation effects can be observed.

3.3. Backtesting

Assessment of the accuracy of VaR forecasts should ideally be done by tracking the performance of a model in the future using operational criteria. Under this objective, backtesting evaluates VaR forecasts by checking how a VaR forecast model performs over a period in the past.

We estimate GARCH-models to implement the VaR approach. We use a rudimentary GARCH(1,1) model specification:

$$\sigma_{t+1}^2 = \omega + \alpha Y_t^2 + \beta \sigma_t^2 . \tag{2}$$

The GARCH specifications for each currency are given respectively in Table 2. Based on the VaR model, we are able to derive a violation rate and the VaR volatility. The relative number of violations is named as violation ratios which measures the quality of VaR forecasts. Violation ratios is the actual number of VaR violations compared with the expected value of number of violations. The violation rate is given as:

$$\mathrm{VR} = \frac{\nu_1}{p \times \mathrm{W}_T}$$

$$\eta_t = \begin{cases} 1 & \text{if } y_t \le -\mathrm{Va}R_t \\ 0 & \text{if } y_t > -\mathrm{Va}R_t \end{cases}$$

where, the estimation window W_T is the number of observations used to forecast risk, ν is the number of instances, $\nu_i, i = 0,1$ number of violations $(i = 1)$ and no violations $(i = 0)$ observed in $\{\eta_t\}$, $\nu_1 = \sum \eta_t$, $\nu_0 = \mathrm{W}_T - \nu_1$, p is the probability level of the VaR estimation, $\eta_t = 0,1$ indicates whether a VaR violation occurs, (for

Table 2. Backtesting

Spot Exchange Market

	CAD/USD	CNY/USD	Euro/USD	JPY/USD	MXN/USD	USD
ω	8.8212e-07*	1.7227e-07**	4.0597e-07	1.7837e-06**	4.613e-07**	2e-07
	(4.9068e-07)	(4.2279e-08)	(2.9003e-07)	(9.3271e-07)	(2.1823e-07)	(1.4371e-07)
GARCH(1)	0.92**	0.43**	0.94**	0.93**	0.88**	0.94**
	(0.017)	(0.049)	(0.015)	(0.024)	(0.019)	(0.013)
ARCH(1)	0.057**	0.56**	0.052**	0.041**	0.11**	0.046**
	(0.014)	(0.13)	(2.97)	(0.015)	(0.021)	(0.012)
log likelihood	3457.35	5681.07	3568.33	3482.73	3716.13	3831.48
Violation Ratio	1.063	2.83	1.41	1.53	0.94	1.30
VaR volatility	0.0066	0.0016	0.0051	0.0052	0.011	0.0041
independence test	reject	reject	reject	reject	reject	reject
coverage test	reject	reject	reject	reject	reject	reject

Forward Exchange Market

	CAD/USD	CNY/USD	Euro/USD	JPY/USD	MXN/USD	USD
ω	8.8212e-07*	2e-07**	2e-07	9.131e-07*	4.1063e-07**	4.7062e-06**
	(4.90e-07)	(5.363e-08)	(1.651e-07)	(1.781)	(1.87e-07)	(1.071e-06)
GARCH(1)	0.92**	0.63**	0.94**	0.91**	0.87**	0.58**
	(0.017)	(0.044)	(0.013)	(41.17)	(0.019)	(0.023)
ARCH(1)	0.057**	0.36**	0.052**	0.085**	0.12**	0.41**
	(0.014)	(0.091)	(0.014)	(4.081)	(0.022)	(0.078)
log likelihood	3457.35	5241.42	3634.58	3506.93	3738.99	3246.22
Violation Ratio	4.61	1.41	0.000	1.53	0.95	4.13
VaR volatility	0.19	0.0051	5.29	0.0052	0.011	0.044
independence test	reject	reject	reject	reject	reject	reject
coverage test	reject	reject	reject	reject	reject	reject

Notes: Standard errors are in parentheses. * and ** indicate significance at the 95% and 97.5% confidence levels, respectively. USD is expressed as trade-weighted exchange rate.

violation $\eta_t = 1$). If the actual return on a particular day exceeds the VaR forecast the VaR limit is violated. We can distinguish between the cases where model underforecasts risk VR>1 and model overforecasts risk VR<1.

The VaR-GARCH(1,1) model, generally underforecasts risk with the exception of the EUR in the forward markets and the MXN in the spot and forward markets. In the forward market kurtosis, VaR volatility and the VaR violation rate have a more apparent positive correlation. Generally, currencies which present higher values for kurtosis reveal higher a VaR violation rate and VaR volatilities with some exceptions (See Table 2). Backtesting results show that the GARCH component for CNY is much lower than for the other currencies. The results for independence and coverage tests, namely rejection, show that the violations are not independently distributed. On the contrary, the violations are clustered and occur in a sequence. The VaR model systematically underestimates the risk. Kurtosis and VaR violation rate are supported by copula results, we observe correlation in the extremes. Based on previous research (i.e., Ait-Sahalia et al. (2015), Favero and Giavazzi (2002)) it has been observed that abrupt fluctuations coupled with clustering causes contagion (compare independence tests for supportive results). Conventional VaR, GARCH, and moving average models cannot account for these sudden fluctuations exhaustively.

Instead models that allow jump clustering, mean reversion, and mutually excitation dynamics (cross-excitation, self-excitation) should be considered to deal with these dynamics. The model in the following section intends to shed light on these issues.

4. Model Estimation Procedure

In this study, closed-form expressions for observable moments of the log-returns are implemented in the case when the volatility follows a Heston model. The Heston model is able to capture the following properties: variability of jump activity, clustering of jumps, and exponential decay or mean reversion. These properties allow for simulating propagation of adverse shocks, which assumes that a shock in a domestic market increases the likelihood of shocks in other international markets. The stochastic jump volatility model is given as in Eq.(3):

$$dX_{i,t} = \mu_i dt + \sqrt{V_{i,t}} dW^X_{i,t} + Z_{i,t} dN_{i,t} \,. \tag{3}$$

The instantaneous variance follows the Heston (1993) modeling in Eq.(4):

$$dV_{i,t} = \kappa_i(\theta_i - V_{i,t})dt + \eta_i \sqrt{V_{i,t}} dW^V_{i,t} \,. \tag{4}$$

4.1. Asset Return Dynamics

The log-return asset follows the semi-martingale dynamics as in Eq.(5):

$$dX_{i,t} = \mu_i dt + \sum_{j=1}^{m} \sigma_i dW_{i,t} + Z_i dN_{i,t} \quad , \quad i = 1, ..., m \tag{5}$$

where $W_t := [W_{1,t}, ..., W_{m,t}]'$ is a m-dimensional standard Brownian motion, $Z_t := [Z_1, ..., Z_m]'$ is the vector of jump sizes, independently distributed with F_{Z_i} for $(-\infty, +\infty)$, and $N_t := [N_{1,t}, ..., N_{m,t}]'$ is the vector of Hawkes processes.

Hawkes processes as mutually exciting processes are a special case of path-dependent point processes. Hawkes processes are defined by the intensity process $\lambda_{i,t}$, which is described upon the jump size distribution $\mathscr{F}_t$ and can be considered as the conditional mean jump rate per unit of time.

We identify spillover effects in the form of jumps that go beyond the standard volatility definition. To measure the occurrence of a jump process (Hawkes process), we observe the dynamics within intervals. The probability of a jump event in a given interval is defined as in Eq.(6):

$$\begin{cases} \mathbb{P}\left[N_{i,t+\triangle} - N_{i,t} = 0 | \mathscr{F}_t\right] = 1 - \lambda_{i,t}\Delta + o(\Delta) \\ \mathbb{P}\left[N_{i,t+\triangle} - N_{i,t} = 1 | \mathscr{F}_t\right] = \lambda_{i,t}\Delta + o(\Delta) \end{cases} . \tag{6}$$

4.2. Jump intensity properties

Then, the jump intensity can be written as in Eq.(7):

$$\lambda_i = \lambda_{i,\infty} + \sum_{j=1}^{m} \lambda_j \int_{-\infty}^{t} g_{i,j}(t-s)ds = \lambda_{i,\infty} + \sum_{j=1}^{m} \left(\int_{0}^{\infty} g_{i,j}(u)du \right) \lambda_j \,. \tag{7}$$

Eq.(8) shows how future jump intensities are linked to past jumps in form of the function $g(\cdot)$:

$$\lambda_{i,t} = \lambda_{i,\infty} + \sum_{j=1}^{m} \int_{-\infty}^{t} g_{i,j}(t-s) dN_{j,s} \quad , \quad i = 1, ..., m \,. \tag{8}$$

In other terms, the equations (7) and (8) express the distribution of the jumps N, which determines the jump intensity λ_{ij}. The mean reversion characteristic in jump intensities is given by Eq. (9):

$$g_{i,j}(t-s)=\beta_{i,j}\, e^{-\alpha(t-s)},\;\; s<t,\;\; i,j=1,...,m \tag{9}$$

where, β can be perceived as the jump amplitude and α the mean reversion rate for a certain interval. Each jump intensity written as a differential equation follows the mean reverting dynamics in Eq.(10)

$$d\lambda_{i,t}=\alpha_i(\lambda_{i,\infty}-\lambda_{i,t})dt+\sum_{j=1}^{m}\beta_{i,j}dN_{j,t}\,. \tag{10}$$

The equation shows that the jump intensities within a certain interval depend on the partial adjustment dynamic of the jump intensities along the long-term jump intensity value. A partial decay is achieved by the mean reverting parameter alpha. The jump numbers or events, measured by N is summed up over each currency exchange rate. The jumps are multiplied by β, the jump amplitude, to illustrate the real effect of the jump events with respect to the jump intensity. Indices i and j determine the direction of the jump amplitudes according to their source and target currencies. If stochastic volatilities and stochastic jump intensities are added to the state vector, the model can be restricted to be part of the class of generalized affine jump-diffusion processes:

$$dA_t=\mu^A(A_t)dt+\sigma^A(A_t)dW_t^A+\sum_{j=1}^{m}dJ_{j,t}\,. \tag{11}$$

In Eq.(11) it is assumed that the jump size has a fixed restriction. The jump size distribution has the following cumulative probability distribution:

$$F_{Z_i}(x)=\begin{cases}\mathbb{P}\left[N_{i,t+\triangle}-N_{i,t}=0|\mathscr{F}_t\right]=1-\lambda_{i,t}\Delta+o(\Delta)\\ \mathbb{P}\left[N_{i,t+\triangle}-N_{i,t}=1|\mathscr{F}_t\right]=\lambda_{i,t}\Delta+o(\Delta)\end{cases}. \tag{12}$$

The corresponding probability density function was used by Kou (2002) for option pricing:

$$\mathbb{E}[Z_i^k]=(-1)^k\frac{k!p_i}{\gamma_{i,1}^k}+\frac{k!(1-p_i)}{\gamma_{i,2}^k},\;\; k=1,2,... \tag{13}$$

4.3. *Inference Procedure*

4.3.1. *GMM Estimation*

To accomplish the model implementation and to estimate a specific parameter that depends on stochastic volatility, we need to calculate the empirical moments. GMM estimations are applied in discrete-time approximations of the unconditional moments for auto- and cross-covariances[2] as in Eq. (14)

$$\begin{array}{ll}\mathbb{E}[dX_{i,t}] & \\ \mathbb{E}[(dX_{i,t}-\mathbb{E}[dX_{i,t}])^r] & r=2,...,4\\ \mathbb{E}[dX_{i,t}dX_{j,t}-\mathbb{E}[dX_{i,t}]\mathbb{E}[dX_{j,t}]] & i\neq j\\ \mathbb{E}[dX_{i,t+\tau}dX_{j,t}-\mathbb{E}[dX_{i,t}]\mathbb{E}[dX_{j,t}]], & \tau>0\end{array}. \tag{14}$$

4.3.2. *Moments Computation - Univariate Case*

To improve the accuracy and reduce the number of the moment conditions to be used, compared with the discrete case, based on the GMM-estimation in section 4.3.1, unconditional interval-based moments are calculated. The specification in Eq. (15) enforces the GMM-objective function to become minimal with less effort:

[2]For exact derivation of the GMM moments see Ait-Sahalia et al. (2015)

$$\begin{cases} \mathbb{E}\left[\int_{s1}^{s2} dX_{i,t}\right] \\ \mathbb{E}\left[\left(\int_{s1}^{s2} dX_{i,t} - \mathbb{E}\left[\int_{s1}^{s2} dX_{i,t}\right]\right)^r\right] \\ \mathbb{E}\left[\int_{s1}^{s2} dX_{i,t} \int_{s1}^{s2} dX_{j,u} - \mathbb{E}\left[\int_{s1}^{s2} dX_{i,t}\right]\mathbb{E}\left[\int_{s1}^{s2} dX_{j,u}\right]\right] \\ \mathbb{E}\left[\int_{s3}^{s4} dX_{i,u} \int_{s1}^{s2} dX_{j,t} - \mathbb{E}\left[\int_{s3}^{s4} dX_{i,u}\right]\mathbb{E}\left[\int_{s1}^{s2} dX_{j,t}\right]\right] \end{cases} \tag{15}$$

with $(i, j = 1,2; 0 \leq s_1 \leq s_2 \leq s_3 \leq s_4)$. The expressions of these moments in the univariate case are given in Eq.(16), where jump intensities depend upon the history of their own past jumps:

$$\begin{cases} dX_t = \mu dt + \sqrt{V_t} dW_t^x + Z_t dN_t \\ dV_t = \kappa(\theta - V_t)dt + \eta\sqrt{V_t} dW_t^v \\ d\lambda_t = \alpha(\lambda_\infty - \lambda_t)dt + \beta dN_t \end{cases} . \tag{16}$$

4.4. Bivariate Case

4.4.1. Model formulation

We specify the default intensities $\lambda_{i,t}$ and the associated counting processes $N_{i,t}$, $i = 1,...,m$ as a multivariate Hawkes process (mutually exciting jump process) with exponential decay. In this case, where there are any further jumps, there is a mean reversion with the jump intensity decaying back to $\lambda_{i,\infty}$ at rate α_i. The following parameter restrictions are imposed: $0 \leq \gamma_i \leq 1$, $\lambda_{i,t} \geq \lambda_{i,\infty} \geq 0$, and $\alpha_i > \beta_{i,j} \geq 0$, $i, j = 1,...,m$.

In the optimal weight matrix of the GMM estimation, more weight is put on the third and fourth moments. For the purpose of identification, we impose the following restrictions, $\alpha_1 = \alpha_2 =: \alpha$ and $\lambda_{1,\infty} = \lambda_{2,\infty} =: \lambda_\infty$. We use the following bivariate Hawkes diffusion model for implementation of our contagion model:

$$\begin{cases} dX_{1,t} = \mu_1 dt + \sqrt{V_{1,t}} dW_{1,t}^X + Z_{1,t} dN_{1,t} \\ dX_{2,t} = \mu_2 dt + \sqrt{V_{2,t}} dW_{2,t}^X + Z_{2,t} dN_{2,t} \\ dV_{1,t} = \kappa(\theta_1 - V_{1,t})dt + \eta_1\sqrt{V_{1,t}} dW_t^V \\ dV_{2,t} = d\left(\frac{\theta_1}{\theta_2}\right) V_{1,t} \\ d\lambda_{1,t} = \alpha_1(\lambda_{1,\infty} - \lambda_{1,t})dt + \beta_{11} dN_{1,t} + \beta_{12} dN_{2,t} \\ d\lambda_{2,t} = \alpha_2(\lambda_{2,\infty} - \lambda_{2,t})dt + \beta_{21} dN_{1,t} + \beta_{22} dN_{2,t} \end{cases} \tag{17}$$

with $\mathbb{E}\left[dW_{1,t}^X dW_{2,t}^X\right] =: \rho\, dt$ and $\mathbb{E}\left[dW_{i,t}^X dW_t^V\right] =: \rho_i^v\, dt$, $i = 1,2$. The corresponding integral equation for $\lambda_{i,t}$ is defined as

$$\lambda_{i,t} = \lambda_{\infty,i} + \int_{-\infty}^{t} \beta_{i,1} e^{-\alpha_i(t-s)} dN_{1,s} + \int_{-\infty}^{t} \beta_{i,2} e^{-\alpha_i(t-s)} dN_{2,s}, \quad i = 1,2.$$

The estimated equations system is composed by the domestic and foreign asset return dynamics $dX_{1,t}$ and $dX_{2,t}$ and the stochastic volatilities $dV_{1,t}$ and $dV_{2,t}$. The stochastic volatilities are interconnected with the correlation coefficient $\rho = dW_1 dW_2$. The domestic jump intensity is driven by the domestic market jump amplitude, β_{11}, and the foreign market transmission jump amplitude, β_{12}, which can be considered as the contagious spillover process. The precise effect of a jump in currency j on the jump intensity of currency i, is determined by the parameter $\beta_{i,j}$, $i = 1,...,m$. The foreign jump intensity is driven by the domestic transmission jump amplitude, β_{21}, and the internal foreign counterpart, β_{22}, respectively.

4.4.2. Parameters

The related parameters are given as follows: μ_1, μ_2, rate of return of the asset, β_{ij}, jump amplitude are responsible for mutually exciting process, $\alpha = \alpha_1 = \alpha_2$, speed of jump mean reversion, λ_1, λ_2, jump intensity, $\lambda_{1,\infty} = \lambda_{2,\infty}$, long term jump intensity, $\sqrt{\theta_1}$, $\sqrt{\theta_2}$, volatility, ρ, correlation coefficient, and $1/\gamma_1$, $1/\gamma_2$, jump size parameters. Identification is achieved by equalizing the adjustment parameters as $\alpha = \alpha_1 = \alpha_2$ and the long-term jump intensities,

$\lambda_\infty = \lambda_{1,\infty} = \lambda_{2,\infty}$. The country specific jump intensities, λ_1,λ_2, are estimated via endogenous simulation.[3]In case of self- excitation and mutually excitation, jump excitation parameters α, β are estimated using the maximum likelihood, while λ_∞ is estimated such that the unconditional expected jump intensity $E[\lambda]$ is equal to the average jump occurrences per year.

5. Model Implementation

5.1. Testing for Contagion

The mutually exciting jump diffusion model accounts for some kind of lead-lag relationships of international returns. Thus, the implemented model is able to test asymmetric jump excitation. To test the hypothesis of financial contagion, we need to recover its definition: A significant increase in cross-market linkages after a shock to one country or group of countries.[4] Thus, the hypothesis can be formulated with respect to Eq.(17) as follows. The hypothesis of cross-sectional contagion is tested as

$$H_0^I : \beta_{i,j} = 0, i \neq j\; i,j = 1,2.$$

We test the following null hypotheses to identify further excitation jump dynamics: $H_0^{II} : \beta_{i,j} = 0,\ i,j = 1,2$, $H_0^{III} : \beta_{i,i} = 0,\ i = 1,2$.

Thus, we test whether each jump intensity, domestic or foreign, is amplified by the cross-boarder jump amplitude under the null hypothesis for absence of cross-border transmission processes. Under the construction of these hypotheses, it is possible to define some considerations to identify certain extensions and potential realizations of these hypotheses, such as, larger cross section excites, as source jump components or as target jump components. In sequel, we analyze some further implications based on the model results. Since our model accounts for some special jump dynamics, we investigate jump intensities and clustering. In this context, differences between forward and spot markets are of further interest.

5.2. Main Results

In this section, we document the model results for each market and for each bilateral model specification separately. The following subsections report the results for the most central model parameters.

5.2.1. Spot markets

We start to present our results with Table 3, which gives the results for USD related models in the spot market. As a key parameter correlation ρ is significant (at a level of 5%) for the following cases: AUD (0.40), CNY (0.70), EUR (2.31), JPY (0.59), MXN (0.40), and GBP (1.00). We observe a very high correlation for EUR and the GBP. The internal excitation parameter $\beta_{1,1}$ for USD is significant in the following cases: CNY (7.13), EUR (17,12), and MXN (15.13). In the case of BRL, the internal excitation parameter $\beta_{1,1}$ of USD is very high. The mutually excitation parameter, originating from other spot markets towards USD, $\beta_{1,2}$, is significant in the cases of EUR (1.14), MXN (1.14), and GBP (13.30). GBP strongly affects the USD market. The mutually excitation parameter, originating from USD towards the other spot markets, $\beta_{2,1}$, is significant for EUR (13.00), JPY (1.28), and MXN (23.00). Interestingly, we see that the impact of the US excitation parameter on MXN and EUR is much higher than in the reserve case. We see an asymmetric behavior in terms of magnitude of the mutually excitation parameter from the USD spot market towards the other spot markets.[5] The internal excitation parameter $\beta_{2,2}$ for other spot markets than USD, is significant in the following cases: CNY (17.23), EUR (7.26), JPY (26.36), MXN (7.27). In the cases for CNY, EUR, and MXN, $\beta_{2,2}$ is significant, as well as $\beta_{1,1}$. The long term jump intensity parameter λ_∞,

[3] See Ait-Sahalia for the detailed presentation of model identification.

[4] See Forbes and Rigobon (2002) for the use of the contagion definition.

[5] These findings are supported by Wang and Yang (2009). They find that a depreciation against USD leads to significantly greater volatility than an appreciation for AUD and GBP.

Table 3. Bivariate Models with US Spot Exchange Rate Pairs

1	USD	USD	USD	USD	USD	USD	USD	USD	USD
2	AUD/USD	BRL/USD	CAD/USD	CNY/USD	DKK/USD	EUR/USD	JPY/USD	MXN/USD	GBP/USD
α	4.47	25.35	20.05	19.27***	33.35	40.27***	35.47***	20.27***	54.76***
	(11.83)	(62.07)	(78.30)	(4.95)	(60.84)	(1.30)	(0.07)	(0.19)	(5.46)
β_{11}	17.13	23.51	16.58	7.13***	3.88	17.12***	0.00	15.13***	0.00
	(32.73)	(67.42)	(84.20)	(0.12)	(19.03)	(0.02)	(0.25)	(0.04)	(6.00)
β_{12}	15.14	1.21	1.72	1.14	2.29	1.14***	0.01	1.14***	13.30***
	(49.63)	(29.22)	(35.78)	(0.78)	(38.85)	(0.42)	(0.01)	(0.07)	(4.83)
β_{21}	9.00	9.55	12.04	3.00	16.87	13.00***	1.28**	23.00***	19.46
	(12.38)	(10.20)	(85.77)	(3.20)	(58.26)	(0.17)	(0.55)	(0.03)	(11.24)
β_{22}	0.43	4.58	8.01	17.27***	16.48	7.26***	26.63***	7.27***	11.27
	(15.59)	(30.27)	(30.47)	(3.80)	(94.61)	(0.15)	(0.07)	(0.15)	(8.85)
λ_∞	0.44	1.03	0.75	1.00***	0.62***	1.059***	0.00	0.70***	1.40***
	(0.98)	(0.78)	(1.81)	(0.02)	(0.10)	(0.00)	(0.00)	(0.00)	(0.04)
λ_1	-0.20	21.65	9.78	2.89	0.87	1.94	0.00	4.91	2.05
λ_2	0.04	11.22	11.02	13.91	2.09	2.05	0.00	9.77	2.67
$\sqrt{\theta_1}$	0.15***	0.12***	0.12***	0.65***	0.12***	1.14***	0.13***	0.15***	0.14***
	(0.00)	(0.00)	(0.02)	(0.08)	(0.00)	(0.026)	(0.00)	(0.01)	(0.00)
$\sqrt{\theta_2}$	0.18**	0.19***	0.14***	0.18	0.14***	1.17***	0.16***	0.18***	0.28***
	(0.02)	(0.06)	(0.06)	(0.61)	(0.01)	(0.00)	(0.01)	(0.02)	(0.01)
ρ	0.40***	0.17	0.28	0.70	0.30	2.31***	0.59***	0.40**	1.00***
	(0.14)	(0.24)	(0.26)	(7.53)	(0.27)	(0.036)	(0.18)	(0.16)	(0.08)
μ_1	0.21***	0.08	0.05	0.21	0.02	0.21***	0.00	0.21***	0.08***
	(0.02)	(0.07)	(0.16)	(0.21)	(0.03)	(0.02)	(0.01)	(0.01)	(0.01)
μ_2	0.20***	0.36	0.16	0.20	0.04	0.19**	0.00	0.20***	0.06***
	(0.05)	(0.48)	(0.69)	(0.40)	(0.09)	(0.09)	(0.02)	(0.02)	(0.02)
$1/\gamma_1$	0.03***	0.00	0.01	1.03***	0.02	2.03***	0.35**	0.31***	0.04***
	(0.01)	(0.00)	(0.04)	(0.00)	(0.01)	(0.00)	(0.08)	(0.00)	(0.00)
$1/\gamma_2$	0.03***	0.03**	0.01	1.02***	0.02	1.03***	0.07	0.28***	0.02***
	(0.97)	(0.01)	(0.02)	(0.01)	(0.02)	(0.00)	(1.75)	(0.00)	(0.00)

Notes: Parameter Estimates for the Bivariate Hawkes Currency Spot Exchange Rate Model: Pairs with USD. This table reports the GMM estimates for the 13 parameters of the bivariate Hawkes currency exchange rate model with daily-based exchange rates measured in logarithmic returns; asymptotic standard errors are in parentheses. *, **, and *** indicate significance at the 95%, 97.5%, and 99.5% confidence levels, respectively. We consider nine series: AUD; BRL; CAD; CNY; DKK; EUR; JPY; MXN; and GBP.

is significant for the following markets: DKK (0.62) MXN (0.70), and GBP (1.40). Long term jump intensities have the highest values for JPY, GBP, and CNY. The α decay parameter, indicating speed of mean reversion towards the long term jump intensity, has the following values: CNY (19.27), EUR (40.27), JPY (35.47), MXN (20.27), and GBP (54.76). It can be seen that the α-parameter is significant for the cases where the long-term intensity λ_∞ or as in the case of JPY the correlation parameter ρ is significant. Therefore, we conjecture that the jump intensity underlies the mean reverting dynamics as given in the model by Eq. (17).

In Table 4 the results for CNY related models in the spot market are presented. The correlation parameter ρ is significant for the following cases: AUD (0.40), BRL (0.41), CAD (0.40), EUR (1.00), MXN (0.40), and GBP (0.70). We observe a very high correlation for EUR. The internal excitation parameter $\beta_{1,1}$ for CNY is significant in the following cases AUD (12.13), BRL (15.98), CAD (47.13), EUR (44.64), MXN (15.13), GBP (12.91), and USD (110.31). The excitation parameter $\beta_{1,1}$ of CNY is very high for EUR and USD. The mutually excitation parameter, originating from complementary spot markets towards CNY, $\beta_{1,2}$, is significant in the cases of BRL (9.30), EUR (0.49), JPY (2.33), MXN (34.14), GBP (30.14), USD (7.20). MXN and GBP strongly affects the CNY market. The mutually excitation parameter, originating from CNY towards the other spot markets, $\beta_{2,1}$, is significant for AUD (9.00), CAD (11.00), EUR (10.33), MXN (13.00), and GBP (30.00). Comparing the impact and direction of the mutual excitation parameters, we observe mixed results. EUR's contagion effects on CNY are limited, whereas CNY's effects on EUR are larger. Effects of MXN and USD on CNY are larger. An indirect effect of USD on MXN should be kept in mind. The contagion effects concerning GBP are nearly the same. In the case of AUD and CAD the direction goes only from CNY to the other markets, $\beta_{2,1}$.[6] In the cases of BRL, JPY, and USD the contagion effect, $\beta_{1,2}$, goes solely towards CNY. The internal excitation parameter $\beta_{2,2}$ for other spot markets than CNY, is significant in the following cases: BRL (37.99), CAD (17.27), EUR (17.92), JPY (14.53), MXN (7.27), GBP (7.27), and USD (15.89). In the cases for CNY in all cases, except for DKK, JPY and AUD, we find that both internal excitation parameters $\beta_{1,1}$, $\beta_{2,2}$ are significant at the same time. The α-decay parameters are significant except for DKK, CNY (19.27), EUR (97.07), JPY (35.47), MXN (20.27), and GBP (54.76). It can be seen that the α-parameter is significant for the cases where the long-term intensity λ_∞ or as in the case of JPY the correlation parameter ρ is significant.

5.2.2. *Forward markets*

Table 5 gives the results for USD related models in the forward market. The correlation parameter ρ is significant for the following cases CAD (0.19), CNY (0.14), and GBP (0.26). CNY and GBP are the cases where the correlation parameter is significant in both market types, spot and forward markets. Compared with the results of the spot exchange rates, the ρ-parameter values are lower for the forward exchange rates. The internal excitation parameter $\beta_{1,1}$ for USD is significant in the following cases: BRL (15.13), CNY (7.13), EUR (27.12), and MXN (15.13). The mutually excitation parameter, originating from reciprocal forward markets towards USD, $\beta_{1,2}$, is significant in the cases of BRL (2.214), CNY (1.14), JPY (2.51), MXN (1.14), and GBP (45.96). As in the spot exchange rate market, GBP strongly affects the USD market. The mutually excitation parameter, originating from USD towards the reciprocal forward markets, $\beta_{2,1}$, is significant for AUD (13.00), BRL (20.55), CAD (20.00), CNY (3.00), JPY (1.20), and MXN (23.00). Comparing the impact and direction of the mutual excitation parameters, we observe notable results. BRL's and MXN's contagion effects on USD are limited, whereas USD's effects on BRL and MXN are larger. A similar asymmetric effect is given for CAD, USD originated contagion is significantly large, whereas there is no significant effect for the reverse case. In the cases of CAD and AUD, the direction of transmission, $\beta_{2,1}$, goes only from USD to the reciprocal markets. In the case of GBP, the contagion effect $\beta_{1,2}$, goes solely towards USD. The internal excitation parameter $\beta_{2,2}$ for reciprocal spot markets, is significant in the following cases: BRL (14.58), CAD (1.26), CNY (17.27), DKK (24.51), JPY (12.37), and MXN (7.27). In the cases of BRL, CNY, and MXN, we find that the internal excitation parameters $\beta_{1,1}$, $\beta_{2,2}$ are significant at

[6] Australia has large banking sector claims in China, Canada is a big trade partner.

Table 4. Bivariate Models with CNY Spot Exchange Rate Pairs

1	CNY/USD	CNY/USD	CNY/USD	CNY/USD	CNY/USD	CNY/USD	CNY/USD	CNY/USD	CNY/USD
2	AUD/USD	BRL/USD	CAD/USD	DKK/USD	EUR/USD	JPY/USD	MXN/USD	GBP/USD	USD
α	12.47***	46.55***	44.27***	33.35	75.86***	19.03***	20.27***	29.03***	151.72***
	(0.85)	(12.46)	(1.97)	(60.84)	(0.00)	(2.51)	(0.94)	(1.38)	(36.36)
β_{11}	12.13***	15.98***	47.13***	3.88	44.64***	2.47	15.13***	12.91***	110.31***
	(2.45)	(1.35)	(1.88)	(19.03)	(0.00)	(2.12)	(1.14)	(0.45)	(28.44)
β_{12}	3.14	9.30*	1.14	2.29	0.49***	2.33***	34.14***	30.14***	7.20***
	(3.06)	(4.47)	(1.03)	(38.85)	(0.00)	(0.31)	(2.65)	(2.28)	(1.79)
β_{21}	9.00***	8.56	11.00***	16.87	10.33***	0.00	13.00***	30.00***	0.00***
	(3.22)	(27.01)	(0.37)	(58.26)	(0.00)	(4.42)	(0.86)	(0.69)	(31.52)
β_{22}	1.94	37.99***	17.27***	16.48	17.92***	14.53***	7.27***	7.27***	15.89***
	(5.06)	(0.16)	(1.91)	(94.61)	(0.00)	(1.89)	(2.28)	(0.95)	(0.25)
λ_∞	0.44***	0.39***	0.40***	0.62***	0.00	1.35***	0.70	1.70***	1.07***
	(0.07)	(0.06)	(0.01)	(0.10)	(0.00)	(0.02)	(0.01)	(0.02)	(0.01)
λ_1	-3.04	1.80	-5.50	0.87	0.00	2.35	-1.76	-4.61	4.14
λ_2	-2.08	3.93	-1.59	2.09	0.00	5.71	-0.68	-4.10	1.20
$\sqrt{\theta_1}$	0.15***	0.03***	0.15***	0.12***	0.13***	0.02***	0.15***	0.15**	0.00
	(0.01)	(0.00)	(0.01)	(0.00)	(0.00)	(0.00)	(0.01)	(0.05)	(0.05)
$\sqrt{\theta_2}$	0.18***	0.21***	0.18***	0.14***	0.18***	0.15***	0.18***	0.18	0.12**
	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.00)	(0.02)	(0.15)	(0.00)
ρ	0.40***	0.41***	0.40***	0.30	1.00***	0.23	0.40***	0.70***	0.57
	(0.05)	(0.14)	(0.04)	(0.27)	(0.06)	(0.16)	(0.05)	(0.18)	(21.79)
μ_1	0.21***	0.02***	0.21***	0.02	0.05***	0.02***	0.21***	0.21***	0.03
	(0.02)	(0.00)	(0.03)	(0.03)	(0.00)	(0.00)	(0.03)	(0.01)	(0.01)
μ_2	0.20***	0.18**	0.20***	0.04	-0.07***	0.09***	0.20***	0.20***	0.02***
	(0.06)	(0.08)	(0.03)	(0.09)	(0.02)	(0.03)	(0.03)	(0.05)	(0.01)
$1/\gamma_1$	0.03***	0.01***	0.03***	0.02	-1.26**	0.01***	0.31***	1.03***	0.01***
	(0.00)	(0.00)	(0.00)	(0.01)	(0.64)	(0.00)	(0.00)	(0.00)	(0.00)
$1/\gamma_2$	0.03	0.05***	0.03***	0.02	1.19	0.02***	0.28***	1.03***	0.02***
	(0.02)	(0.00)	(0.00)	(0.02)	(0.97)	(0.00)	(0.00)	(0.00)	(0.00)

Notes: Parameter Estimates for the Bivariate Hawkes Currency Spot Exchange Rate Model: Pairs with CNY. This table reports the GMM estimates for the 13 parameters of the bivariate Hawkes currency exchange rate model with daily-based exchange rates measured in logarithmic returns; asymptotic standard errors are in parentheses. *, **, and *** indicate significance at the 95%, 97.5%, and 99.5% confidence levels, respectively. We consider nine series: AUD; BRL; CAD; CNY; DKK; EUR; JPY; MXN; and GBP.

the same time. The decay parameter α is given as BRL (24.27), CAD (2.27), CNY (1.27), DKK (121.92), JPY (32.39), MXN (20.27), and GBP (183.40). Similar to the spot exchange rates, it can be seen that the α-parameter is significant for the cases, where the long-term intensity λ_∞ or ρ (for JPY) are significant.

We present the results for CNY related models in the forward market in Table 6. The correlation parameter ρ is significant for DKK, EUR, and USD. Compared with results of the spot exchange rate market (Table 4) the ρ-parameter value is higher for the forward exchange rate. The internal excitation parameter $\beta_{1,1}$ for CNY is significant in the following cases AUD (12.13), BRL (17.13), CAD (2.12), DKK (1.12), CNY (7.13), EUR (1.13), JPY (17.13), and GBP (17.13); exceptions are MXN and USD. The mutually excitation parameter, originating from reciprocal markets towards CNY, $\beta_{1,2}$, is significant in the cases of AUD (17.14), BRL (11.14), CAD (1.14), DKK (1.14), JPY (32.14), and GBP (17.14). Direct effects of JPY on CNY are predominant. The mutually excitation parameter, originating from CNY towards the reciprocal markets, $\beta_{2,1}$, is significant for all cases except for EUR and USD. Mutually excitation in forward exchange markets for EUR and USD disappear. The internal excitation parameter $\beta_{2,2}$ for other spot markets than CNY, is significant except for MXN and USD. We observe that the internal excitation parameters $\beta_{1,1}$ and $\beta_{2,2}$ are significant at the same time. The α-decay parameter is significant in several cases, BRL (20.27), CAD (1.12), DKK (1.77), JPY (25.27), GBP (20.27), and USD (47.95), exceptions are AUD, EUR, and MXN.

5.2.3. Summary of the results

In the spot exchange markets we observe stronger contagion effects from US to other markets than in the reverse case. That means that a jump in the USD, or in other terms, an extreme deviation in the US markets increases the jump intensity in the mutual currency. In nearly all cases the reverse effect is prevalent. There is a significant reversal effect on the jump intensity of the USD from other markets, however in weaker form.[7] The only exception is JPY, where no significant effect can be seen.[8]

In the case of US originated contagion, parameter values for spot exchange rate returns are higher than parameter values for forward exchange rate returns. In the case of CNY originated contagion, parameter values for internal excitation parameters (β_{11},β_{22}) are higher for the forward market and the parameters are higher for crossover excitations (β_{12},β_{21}) in the spot exchange rate market. Moreover, a mutually cross-contagion is prevalent.

The order of the models according to the parameter significance is given as CNY spot (Table 4), CNY forward (Table 6), USD forward (Table 5), and USD spot (Table 3). We observe that uncertainty in USD spot markets is predominant. The widely significant parameters across the models are θ_1, θ_2, α, λ_∞. In cases of mutually exciting contagion, the role of USD is predominant, in this relation the value and the significance of β_{21}, as in the case of MXN, is exemplary (See Table 3 and 5). The values for β_{21} are greater than for the reverse case. However, an exception is given for GBP in the USD forward market. Here, the contagion occurs in the opposite direction; the GBP has a very large impact on the USD.

6. Summary and conclusions

We implemented a Hawkes jump diffusion model in the spot and forward exchange markets which is accurate for analyzing jump clustering in the forex markets. We tested the dynamics of mutually jump excitement processes in the forex markets. The contagion parameters β_{21} and β_{12} and the copula models make contribution to the contagion discourse in the same direction, respectively. The results reveal stronger effects from the US Dollar to mutual markets than in the reverse case. Cross-sectional excitation in the spot markets are higher than in the forward markets. As a central result we can observe that the results for the Hawkes-model parameters are more significant in the forward markets. The spot markets are governed by more uncertainty on the contrary, the currency risk is

[7] See Kenourgios (2011) et al. for stronger effects from US to other FX markets.

[8] Note that Orlov (2009) finds that contagion during the Asian crisis is to be manifested in greater comovements along high-frequency components.

more likely to spread around, therefore contagion parameters are more dominant. It should be remembered that by definition the forward markets are agreed contracts among financial intermediaries, sudden dynamic changes, and therefore, are moderated.[9] Jump clustering is supported by the rejected independence and coverage tests (See Table 2). Generally, currencies which present higher values for kurtosis reveal higher VaR violation rate and VaR volatilities. The results for the exceedance correlations show that, in general, spot markets exhibit higher exceedance correlation values. The copula models show strong correlation at the extremes, for USD and CNY for both type of markets. The correlation for CNY is more moderate. In forward markets, kurtosis, VaR Volatility and VaR violation rate exhibit similar outcomes.

To understand the contagion discourse in a better way, it should be stated that the contagion occurs in most cases beyond volatility. In terms of expectations of future exchange rate dynamics, we should emphasis the unexpected part in these dynamics. In this regard, asymmetry in these expectations is involved. The asymmetry depends on each currency pair. Internal market dynamics, as well as the transmission of country-specific dynamics are important features in determining the exact impact of the asymmetry on the evolution of these parameters. The contagion dynamics do not evolve constantly. Being far from a continuous process, contagion occurs in the case when we observe abrupt dynamics. Therefore, it depends on the joint occurrence of specific market conditions, which our model parameters try to mimic.

Mean reversion in the contagion debate is a further aspect that needs to be paid attention to. As contagion occurs according to specific market conditions, it is of transitory nature, whenever these conditions are no longer given. The decay parameter α, gives some indication about the mean reversion dynamics in our model. For high values of the α-parameter, we observe rapid decay of the jump intensity. This means that the abrupt fluctuation in the exchange rate dynamic is slowed down.

A further aspect, is the long-term jump intensity, that can be seen as an equilibrium dynamic in the jump intensity. High volatile markets such as the GBP prevail significant volatility terms $(\sqrt{\theta_1}, \sqrt{\theta_2})$ and long term jump intensities and high mean version parameters in all model specification results. However, the interpretation of the long term jump intensity should be done carefully, because divergent effects may be involved.[10] Further research might include the interpretation of potential reasons within a larger macroeconomic equilibrium model, that accounts for stochastic jump diffusion and contagion dynamics of FX markets.

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[9] Compare Chu-Sheng (2009) who observes that the conditional forward bias is due to a time-varying risk premium, in particular the currency risk premium, and not to irrationality among market participants.

[10] Speculative and other trading motives can shape the exchange rate market dynamics. Narayana et al. (2015) find that currency profits were maximized during the crisis period, and were lowest during the post-crisis period.

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Appendix

Table 5. Bivariate Models with US 3-M Forward Exchange Rate Pairs

1 2	USD AUD/USD	USD BRL/USD	USD CAD/USD	USD CNY/USD	USD DKK/USD	USD EUR/USD	USD JPY/USD	USD MXN/USD	USD GBP/USD
α	10.27	24.27***	2.27***	1.27***	121.92***	21.27	32.39***	20.27***	183.40***
	(19.07)	(2.32)	(0.60)	(0.10)	(11.07)	(1.39)	(0.04)	(1.29)	(17.60)
β_{11}	10.13	15.13***	1.12	7.13***	0.00	27.12**	0.00	15.13***	1.81
	(20.15)	(3.57)	(0.75)	(0.01)	(8.52)	(2.7667)	(0.11)	(0.26)	(73.77)
β_{12}	1.14	2.21**	1.14	1.14***	0.22	9.14	2.51***	1.14***	45.96**
	(3.10)	(0.94)	(0.99)	(0.05)	(8.67)	(2.50)	(0.00)	(0.46)	(0.55)
β_{21}	13.00***	20.55***	20.003***	3.00***	9.91	24.0031	1.20***	23.00***	0.00
	(3.72)	(0.86)	(1.26)	(0.97)	(11.53)	(1.817)	(0.07)	(0.23)	(14.23)
β_{22}	7.27	14.58***	1.26**	17.27***	24.51	25.26	12.37***	7.27***	12.66
	(18.07)	(3.01)	(0.24)	(0.09)	(14.30)	(3.11)	(0.04)	(1.00)	(12.86)
λ_∞	0.40	2.03***	1.39	1.00***	0.88***	2.05***	0.00	0.70***	2.86***
	(0.52)	(0.053)	(0.75)	(0.02)	(0.04)	(0.0095)	(0.00)	(0.01)	(0.31)
λ_1	-1.17	13.68	-0.31	-0.21	0.88	-1.14	0.00	4.91	3.67
λ_2	-3.71	34.14	-3.091	-0.04	1.18	-4.05	0.00	9.77	3.07
$\sqrt{\theta_1}$	0.15	0.15	1.14***	0.65***	0.12***	0.14***	0.12***	0.15**	0.11***
	(0.10)	(0.11)	(0.042)	(0.01)	(0.00)	(0.22)	(0.00)	(0.07)	(0.01)
$\sqrt{\theta_2}$	0.18***	0.19***	1.17***	0.18***	0.14***	0.17	0.16***	0.18**	0.15***
	(0.03)	(0.018)	(0.38)	(0.03)	(0.00)	(0.064)	(0.01)	(0.09)	(0.01)
ρ	0.11	0.45	0.19***	0.14***	-0.03	0.89	-0.05	0.16	0.26*
	(0.92)	(1.078)	(0.073)	(0.04)	(0.26)	(1.37)	(0.28)	(0.21)	(0.13)
μ_1	0.21	0.21	0.209***	0.21***	0.02	0.90***	0.00	0.21***	0.05***
	(0.35)	(0.21)	(0.062)	(0.02)	(0.01)	(0.11)	(0.01)	(0.04)	(0.02)
μ_2	0.20**	0.2***	0.19	0.20***	0.03	0.91***	0.00	0.20***	0.06***
	(0.09)	(0.064)	(0.22)	(0.01)	(0.02)	(0.22)	(0.02)	(0.06)	(0.02)
$1/\gamma_1$	0.03	0.04**	0.31***	1.03***	0.02***	0.73***	-0.29	0.31***	0.01***
	(0.35)	(0.017)	(0.036)	(0.03)	(0.00)	(0.0095)	(0.10)	(0.00)	(0.00)
$1/\gamma_2$	0.03	0.037***	0.27***	1.00***	0.02***	0.27***	0.20	0.28***	0.02***
	(0.03)	(0.00096)	(0.067)	(0.07)	(0.00)	(0.0093)	(0.18)	(0.00)	(0.00)

Notes: Parameter Estimates for the Bivariate Hawkes Currency Forward Exchange Rate Model: Pairs with USD. This table reports the GMM estimates for the 13 parameters of the bivariate Hawkes currency exchange rate model with daily-based exchange rates measured in logarithmic returns; asymptotic standard errors are in parentheses. *, **, and *** indicate significance at the 95%, 97.5%, and 99.5% confidence levels, respectively. We consider nine series: AUD; BRL; CAD; CNY; DKK; EUR; JPY; MXN; and GBP.

Table 6. Bivariate Models with CNY 3-M Forward Exchange Rate Pairs

1 2	CNY/USD AUD/USD	CNY/USD BRL/USD	CNY/USD CAD/USD	CNY/USD DKK/USD	CNY/USD EUR/USD	CNY/USD JPY/USD	CNY/USD MXN/USD	CNY/USD GBP/USD	CNY/USD USD
α	4.27	20.27*	1.12***	1.77***	1.27	25.27***	1.27	20.27***	47.95***
	(4.43)	(10.37)	(0.023)	(0.13)	(1.87)	(2.17)	(1.48)	(2.46)	(9.63)
β_{11}	12.13***	17.13**	2.12***	1.12***	1.13***	17.13***	1.12	17.13***	0.00
	(3.00)	(7.10)	(0.015)	(0.14)	(0.42)	(0.16)	(1.97)	(1.91)	(1.22)
β_{12}	17.14***	11.14	1.14***	1.14***	2.14	32.14***	1.14	17.14***	0.012
	(1.31)	(5.99)	(0.45)	(0.10)	(2.66)	(11.43)	(1.55)	(2.35)	(10.76)
β_{21}	15.00***	13.00***	21.00***	23.00***	4.00	33.00***	21.00***	13.00***	2.36
	(1.58)	(3.57)	(0.029)	(0.017)	(6.85)	(0.22)	(8.83)	(1.80)	(21.27)
β_{22}	9.27	7.27***	5.26***	3.26***	5.27***	7.27***	5.26	17.27***	18.62
	(5.69)	(3.23)	(0.029)	(0.12)	(1.95)	(1.91)	(3.13)	(1.65)	(16.64)
λ_∞	0.40***	0.40	1.84***	1.69***	3.06***	0.70***	1.79***	1.70***	2.09***
	(0.09)	(0.22)	(0.70)	(0.020)	(0.01)	(0.04)	(0.19)	(0.00)	(0.036)
λ_1	-0.09	-1.86	0.31	0.039	0.79	-0.96	0.26	-3.24	2.094
λ_2	-0.06	-1.25	-2.09	-2.60	-1.76	-0.79	-1.96	-2.60	3.59
$\sqrt{\theta_1}$	0.15***	0.15***	2.09***	0.14***	0.15***	0.47***	0.14	0.15***	0.040***
	(0.01)	(0.01)	(0.11)	(0.0071)	(0.02)	(0.01)	(0.093)	(0.04)	(0.014)
$\sqrt{\theta_2}$	0.18***	0.18***	17.47***	0.37***	0.18***	0.42***	0.17	0.18***	0.46***
	(0.02)	(0.02)	(0.91)	(0.0055)	(0.01)	(0.02)	(0.17)	(0.07)	(0.035)
ρ	-0.02	0.02	-0.010	0.10***	0.20***	-0.01	0.40	-0.02	0.80*
	(0.05)	(0.06)	(0.00)	(0.041)	(0.07)	(0.01)	(0.31)	(0.66)	(0.36)
μ_1	0.21***	0.21***	0.21***	0.21***	0.91***	0.21***	0.20***	0.21***	-0.10***
	(0.02)	(0.02)	(0.081)	(0.0077)	(0.01)	(0.03)	(0.038)	(0.02)	(0.027)
μ_2	0.20***	0.20***	0.19	0.19***	0.91***	0.20***	0.19	0.20***	0.31
	(0.03)	(0.05)	(0.73)	(0.038)	(0.02)	(0.02)	(0.25)	(0.02)	(0.18)
$1/\gamma_1$	0.03***	0.03***	0.13***	0.31***	0.73***	0.31***	0.31***	1.03***	-0.049***
	(0.01)	(0.00)	(0.049)	(0.028)	(0.00)	(0.01)	(0.060)	(0.00)	(0.0059)
$1/\gamma_2$	0.03	0.03***	0.42***	0.27***	0.28***	0.28***	0.27***	1.03***	0.095***
	(0.39)	(0.04)	(0.095)	(0.0032)	(0.00)	(0.01)	(0.028)	(0.00)	(0.011)

Notes: Parameter Estimates for the Bivariate Hawkes Currency Forward Exchange Rate Model: Pairs with CNY. This table reports the GMM estimates for the 13 parameters of the bivariate Hawkes currency exchange rate model with daily-based exchange rates measured in logarithmic returns; asymptotic standard errors are in parentheses. *, **, and *** indicate significance at the 95%, 97.5%, and 99.5% confidence levels, respectively. We consider nine series: AUD; BRL; CAD; CNY; DKK; EUR; JPY; MXN; and GBP.

CNY-EUR CNY-JPY CNY-MXN CNY-CAD

Fig. 3. Exceedance correlations in Spot market (CNY Originated).

USD-CNY USD-EUR USD-JPY USD-MXN
USD-CAD

Fig. 4. Exceedance Correlations in Forward market (USD originated).

CNY-EUR CNY-JPY
CNY-MXN CNY-CAD

Fig. 5. Copula Probability Densities in Spot Markets (CNY originated).

USD-CNY USD-EUR
USD-JPY USD-MXN
USD-CAD

Fig. 6. Copula Probability Densities in Forward Markets (USA originated).

CNY-EUR CNY-JPY
CNY-MXN CNY-CAD

Fig. 7. Copula Probability Densities in Forward Markets (CNY originated).

Significance Tests of Violation Rates

Unconditional Coverage Test . The unconditional coverage property ensures that the theoretical confidence level p matches the empirical probability of violation.[11]

Consider the indicator sequence, $\{I\}_{t=1}^{T}$, which is constructed from a given interval forecast. To test the unconditional coverage, the hypothesis that $E[I_t] = p$ should be tested against the alternative $E[I_t] \neq p$. The likelihood under the null hypothesis is simply

$$L(p; I_1, I_2, ...I_T) = (1-p)^{n_0} p^{n1},$$

and under the alternative hypothesis:

$$L(\pi; I_1, I_2, ...I_T) = (1-\pi)^{n_0} \pi^{n1}.$$

The unconditional coverage test is formulated according to Eq. (A1)

$$LR_{UC} = -2\log\left[L(p; I_1, I_2, ..., I_T)/L(\hat{\pi}; I_1, I_2, ..., I_T)\right] \sim \chi^2(s-1) = \chi^2(1) \,(\mathrm{A1})$$

,where $\hat{\pi} = n_1/(n_0+n_1)$ is the maximum likelihood estimate of π, and $s = 2$ is the number of possible outcomes of the sequence.

Independence Test . The independence property is subtler, requiring any two observations in the hit sequence to be independent of each other. Intuitively, the fact that a violation has been observed today should not convey any information about the likelihood of observing a violation tomorrow. To test the Likelihood Ratio test of independence, independence will be tested against an explicit first-order Markov alternative. Consider a binary first-order Markov chain, $\{I_t\}$, with the transition probability matrix

$$\Pi_1 = \begin{bmatrix} 1-\pi_{01} & \pi_{01} \\ 1-\pi_{11} & \pi_{11} \end{bmatrix},$$

[11] See Christoffersen (1998) for the details and derivation of violation ratios.

where $\pi_{ij} = Pr(I_t = j \,|\, I_{t-1} = i)$. The approximate likelihood function for this process is

$$L(\hat{\Pi}_2; I_1, I_2, ..., I_T) = (1-\pi_{01})^{n_{00}} \pi_{01}^{n_{01}} (1-\pi_{11})^{n_{10}} \pi_{11}^{n_{11}}$$

, where n_{ij} is the number of observations with value i followed by j. Maximizing the log-likelihood function and doing the parameter estimation

$$\hat{\Pi}_1 = \begin{bmatrix} \frac{n_{00}}{n_{00}+n_{01}} & \frac{n_{01}}{n_{00}+n_{01}} \\ \frac{n_{10}}{n_{10}+n_{11}} & \frac{n_{11}}{n_{10}+n_{11}} \end{bmatrix}.$$

In a first-order Markov model the hypothesis that the sequence is independent is tested by noting that

$$\Pi_2 = \begin{bmatrix} 1-\pi_2 & \pi_2 \\ 1-\pi_2 & \pi_2 \end{bmatrix}.$$

The Likelihood under the null hypothesis becomes

$$L(\Pi_2; I_1, I_2, ..., I_T) = (1-\pi_2)^{(n_{00}+n_{10})} \pi_2^{(n_{01}+n_{11})}$$

and the Maximum Likelihood estimate is

$$\hat{\Pi}_2 = \frac{(n_{01}+n_{11})}{(n_{00}+n_{10}+n_{01}+n_{11})}.$$

The Likelihood Ratio test of independence is asymptotically distributed as a χ^2 with $(s-1)^2$ degrees of freedom, that is

$$LR_{ind} = -2\log\left[L(\hat{\Pi}_2; I_1, I_2, ..., I_T)/L(\hat{\Pi}_1; I_1, I_2, ..., I_T)\right] \overset{\text{asy}}{\sim} \chi^2\left((s-1)^2\right) = \chi^2(1)\,(\text{A2}).$$