A Quantile Approach for Ascending Auctions under Asymmetry

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Abstract

Asymmetry is a relevant question when bidders of different size and technological capability are competing at the same auction. Gimenes (2017) observed a screening level associated with the optimal reservation price policy fairly high when analysing USFS timber auctions, resulting in a low probability of trading. That intriguing result was due to a sharp increase in the private value conditional quantile functions, which may be a consequence of not taking into account asymmetry among the bidders. In this paper, we suggest an identification and estimation approach to recover the private value conditional quantile function under asymmetric bidders. The estimation strategy is done in two steps: in the first-step, the asymmetric parameters are estimated; and then used in the second-step to estimate the private value conditional quantile functions of each type. Simulations show a good performance of the estimator even in small samples. We also investigate the impact of asymmetry on the optimal screening levels.

JEL: C14, D44

Keywords: Private values; asymmetry; ascending auctions; seller expected revenue; quantile regression identification; quantile regression estimation.

1 Introduction and Motivation

The private value distribution is the main primitive of the auction theoretical game since it characterizes the bidder's demand function and bidding behaviour. Bidder's private values are usually not observed preventing the estimation of the private value distribution. In the last thirty years, the structural research in auctions have proposed several identification and estimation approaches based on restrictions imposed on equilibrium bids. See e.g. Paarsch (1992), Donald and Paarsch (1993, 1996), Laffont, Ossard and Vuong (1995), Guerre, Perrigne and Vuong (2000), Haile and Tamer (2003), Li and Zheng (2009), Aradillas-Lopez, Gandhi and Quint (2013), Marmer and Shneyerov (2012) and Gimenes (2017).

The references mentioned above are built under the symmetric framework, which assumes that bidders draw their private value from the same symmetric distribution of private values. Such assumption allows researchers to focus on a representative agent's decision rule when describing the equilibrium bidding behaviour. However, factors as firm's size, capacity constraint, technology and distance to the market of interest may cause asymmetry among the bidders, leading to different bidding behaviours. For instance, Athey, Levin and Seira (2011) identify different entry behaviour between bidders with and without manufacturing capabilities in USFS timber auctions when comparing auction formats. They also found that "mills" (bidders with manufacturing capabilities) behave more cooperatively in open oral auctions than in first-price sealed-bid auctions (FPSB). Indeed, Baldwin, Marshall and Richard (1997) as well as Marshall, Meurer, Richard and Stromquist (1994) suggest that coalition among bidders is easier to be sustained in second/oral ascending auctions than in FPSB auctions. Developing a strategy to recover the primitives of each type in an auction with asymmetric bidders is the first step to understand bidder's behaviour when facing bidders of different types and also to identify a pattern of collusive behaviour. Some papers have proposed identification and estimation approaches allowing for asymmetry, see e.g. Pesendorfer (2000), Campo, Perrigne and Vuong (2003), Hong and Shum (2003), Athey, Levin and Seira (2011), Bendstrup and Paarsch (2006) and Feir, Marmer, Shneyerov and Kaplan (2014).

Bendstrup and Paarsch (2006) have developed a nonparametric identification strategy to recover the private value distribution of asymmetric bidders in English auctions from the distribution of winning bids when the identity of the winner is known. The identification strategy is based on a system of Pfaffian integral equations that may be very difficult to solve. This paper suggests an easier identification and estimation approach based on quantile regression given knowledge of the bidder's identity. The identification strategy is built using the distribution of winning bids conditional on bidder i winning the auction. The estimation approach is given in two-steps: in the first step, the parameters of asymmetry are estimated by maximum likelihood; in the second step, the estimated asymmetric parameters are used to estimate the type-private value conditional quantile functions. The setup developed to estimate the asymmetric parameters allows to test the null hypothesis of symmetry among the bidders via a Likelihood Ratio test. Simulations show that the two-step estimator has good performance even in small samples. We also investigate the effects of the degree of asymmetry on optimal reservation price policies.

Quantiles and quantile regression have been considered in the auction setup by Marmer and Shneyerov (2012), Guerre and Sabbah (2012), Gimenes (2017) and Gimenes and Guerre (2015). One of the motivations for the use of quantiles to analyse auctions is the fact that auctions are fundamentally related with ranks, which is the argument of the quantile function. In addition, it allows an interesting interpretation of the optimal screening level. As shown in Gimenes and Guerre (2015), the quantile regression approach addresses two important issues of the auction estimation literature: the curse of dimensionality and the estimation of the private values closer to the boundaries. Addressing the former issue is relevant because allows us to use a large number of auction characteristics in the analysis. Kernel estimation and standard quantile regression approaches may lead to poor estimation of the private value distribution close to the boundary. The former issue is due to the fact that the bids belong to a compact set, whereas the latter is due to a flat objective function. Since the winner very likely belongs to the upper part of the private value distribution, estimating his private value properly is indeed a relevant matter. Our estimator can be improved by using a suitable version of the augmented quantile regression proposed in Gimenes and Guerre

(2015).

The paper is organized as follows. Section 2 presents the model and identification approach. Section 3 studies the optimal reservation price policy under asymmetric bidders. Section 4 proposes a two-step estimation approach to recover the parameters of asymmetry and the private value conditional quantile function of the types. Section 5 presents some finite sample performance of the estimator as well as Monte Carlo simulations of the effects of the degree of asymmetry on the optimal reservation price policy.

2 Identification

A single and indivisible object with some characteristics $x \in \mathbb{R}^d$ is auctioned to $N \geq 2$ bidders through an ascending auction. The seller sets a reservation price r prior to the auction that is the minimum price that he would be willing to accept. Both the set of auction covariates x and the number of actual bidders N participating in the auction are common knowledge. The object is sold to the highest bidder for the price of his last bid, provided that it is at least as high as the reservation price r. Within the IPV paradigm, each bidder $i = 1, \cdots, N$ is assumed to have a private value v_i for the auctioned good, which is not observed by other bidders. The bidder only knows his own private value, but it is common knowledge for bidders and sellers that each private value has been independently drawn from a c.d.f. $F_k(\cdot|X)$ conditional upon $X = (1, x')'$, or equivalently, with a conditional quantile function $V_k(\alpha|X)$, $\alpha \in [0, 1]$, defined as

$$V_k(\alpha|X) := \inf \{v : F_k(v|X) \geq \alpha\}, \tag{2.1}$$

$k = 1, \cdots, K$, where k denotes the bidder's type, $K \leq N$. In other words, bidders are asymmetric and characterized by K different classes of distributions. When the private value conditional distribution is absolutely continuous with a probability density function (p.d.f) $f_k(\cdot|X)$ positive on its support $[V_k(0|X), V_k(1|X)] \subset \mathbb{R}_+$, as considered from now on, $V_k(\alpha|X)$ is the reciprocal function $F_k^{-1}(\alpha|X)$. For simplicity, we will suppress the conditionality on X and assume that $K = N$ so that we can write $k = i$.

Suppose that asymmetry among the bidders is characterized by a parameter $\lambda_i \in \mathbb{R}_+$ and that $F_i(\cdot)$ is related with the parent distribution of private values $F(\cdot)$ as

$$F_i(v) = [F(v)]^{\lambda_i}. \tag{2.2}$$

Note that for any $\lambda_i \in \mathbb{R}_+$, $F_i(\cdot)$ is well defined on the same support of $F(\cdot)$. From definition (2.1), we can write $F_i(V(\alpha)) = F(V(\alpha))^{\lambda_i}$ and since the αth-quantile of the private value distribution satisfies $F(V(\alpha)) = \alpha$,

$$\begin{aligned}
F_i(V(\alpha)) &= F(V(\alpha))^{\lambda_i} \\
V(\alpha) &= F_i^{-1}\left(F(V(\alpha))^{\lambda_i}\right) \\
V(\alpha) &= V_i(\alpha^{\lambda_i})
\end{aligned} \tag{2.3}$$

Equation (2.3) gives the relationship between the quantiles of the parent and bidder ith's private value distributions. In other words, it shows that the αth-quantile of the private value distribution is equal to the (α^{λ_i})th-quantile of bidder i's private value distribution.

In the ascending auction considered here, bidders raise continuously their prices and drop out of the auction as the price reaches their valuation. As a result, the winner is the bidder with the highest valuation and the winning bid is given by the second-highest valuation, if it is above the reservation price, as in the assumption below.

Assumption 1 *The transaction price in an ascending auction is the greater of the reservation price and the second-highest bidder's willingness to pay.*

Assumption 1 is an assumption about equilibrium play and has been also considered in Aradillas-Lopez et al. (2013) and Gimenes (2017).

Let $G_{(N-1:N)}(w,i)$ denote the distribution function of the winning bid at an auction won by a bidder whose identity/class is i. Bidder i wins at a price w when his valuation exceeds those of his opponents and all the valuations of his opponents are less than or equal to w. As studied in Bendstrup and Paarsch (2006),

$$\begin{aligned}G_{(N-1:N)}(w,i) &= \mathbb{P}\left[(v_i \geq v_j)\,\text{and}\,(v_j \leq w,\ j \neq i)\right] \\ &= (1-F_i(w))\,\Pi_{j\neq i}F_j(w) + \int_0^w \Pi_{j\neq i}F_j(u)\,dF_i(u) \\ &= \int_0^w (1-F_i(u))\,d\Pi_{j\neq i}F_j(u)\,,\end{aligned}$$

where the last equality is obtained by integration by parts. From (2.2), $G_{(N-1:N)}(w,i)$ can be written as

$$G_{(N-1:N)}(w,i) = F(w)^{\sum_{j\neq i}\lambda_j} - \left[\frac{\sum_{j\neq i}\lambda_j}{\sum_i \lambda_i}\right] F(w)^{\sum_i \lambda_i}\,. \tag{2.4}$$

In this paper, we propose a quantile approach to identify the unknown private value quantile function $V(\cdot|X)$ and parameters of asymmetry $\lambda_i, i=1,\ldots,N$, from the observed winning bids and knowledge of winner's identity. The identification strategy is based on the distribution of winning bids conditional on bidder i, whose type is λ_i, winning the auction. Let B_i be bidder i's bid, the distribution of winning bids conditional on bidder i, with type λ_i, winning is

$$G_{(N-1:N)}(w|\text{“}\lambda_i\text{ wins”}) = \mathbb{P}\left[B_i \leq w|\text{“}\lambda_i\text{ wins”}\right] = \frac{\mathbb{P}\left[B_i \leq w, \text{“}\lambda_i\text{ wins”}\right]}{\mathbb{P}\left[\text{“}\lambda_i\text{ wins”}\right]},$$

where $\mathbb{P}\left[B_i \leq w, \text{“}\lambda_i\text{ wins”}\right]$ is the joint distribution given in (2.4). Bidder "i", whose type is λ_i, wins the auction when $w\to\infty$, which substituting into (2.4) gives $\mathbb{P}\left[\text{“}\lambda_i\text{ wins”}\right] = 1 - \left(\frac{\sum_{j\neq i}\lambda_j}{\sum_i \lambda_i}\right)$.

Hence,

$$\mathbb{P}\left[B_i \leq w | \text{“}\lambda_i \text{ wins”}\right] = \frac{F\left(w\right)^{\sum_{j\neq i}\lambda_j} - \left[\frac{\sum_{j\neq i}\lambda_j}{\sum_i \lambda_i}\right] F\left(w\right)^{\sum_i \lambda_i}}{1-\left(\frac{\sum_{j\neq i}\lambda_j}{\sum_i \lambda_i}\right)}, \tag{2.5}$$

which indeed characterizes a c.d.f..

Consider a linear quantile regression specification for the private value quantile function. The quantile regression model writes for a $(d+1)\times 1$ vector $X = (1, x')'$ and a conformable vector of parameters $\gamma(\alpha) = (\gamma_0(\alpha), \gamma_1(\alpha)')'$,

$$V(\alpha|X) = X'\gamma\left(\alpha\right) = \gamma_0\left(\alpha\right) + x'\gamma_1\left(\alpha\right), \tag{2.6}$$

for all $\alpha \in [0,1]$, where $\gamma_1\left(\alpha\right)$ groups the slope quantile regression parameters. For a fixed α, the specification above is parametric and can be estimated with a parametric rate $n^{1/2}$, where n is the sample size. However, since $\gamma\left(\alpha\right)$ is a functional of the quantile level α, the model contains some nonparametric features, which can make it flexible enough to fit many datasets (see Gimenes (2017) and Gimenes and Guerre (2015)). The next two assumptions are required for identification.

Assumption 2 $V\left(\alpha|X\right)$ *is strictly increasing and continuous on its support* $\left[V\left(0|X\right), V\left(1|X\right)\right]$.

Assumption 3 *The vector of auction specific variables,* x*, has dimension* $d\times 1$*, with a compact support in* $\mathcal{X} \subset (0,+\infty)^d$ *and a nonempty interior.*

Assumption 2 concerns the quantiles of the bidders' private value distribution. Assumption 3 ensures that if $x\gamma_1 = x\gamma_2$, for all $x \in \mathcal{X}$, thus $\gamma_1 = \gamma_2$.

Let $\lambda = (\lambda_1, \ldots, \lambda_N)'$ and define

$$\Lambda_N = \sum_{i=1}^{N}\lambda_i, \quad \Lambda_{N|i} = \Lambda_N - \lambda_i$$

and

$$\Psi_i(\alpha;\lambda) = \frac{\alpha^{\Lambda_{N|i}} - \left[\frac{\Lambda_{N|i}}{\Lambda_N}\right]\alpha^{\Lambda_N}}{1-\left(\frac{\Lambda_{N|i}}{\Lambda_N}\right)} = \frac{\Lambda_N\alpha^{\Lambda_{N|i}} - \Lambda_{N|i}\alpha^{\Lambda_N}}{\lambda_i}. \tag{2.7}$$

Since $F(V(\alpha)) = \alpha$, $\Psi_i(\alpha;\lambda)$ is the distribution of winning bids conditional on a bidder of type λ_i winning written as a function of the quantile level α. Let $W(\alpha)$ be the αth-quantile of the winning bids distribution. As above, the αth-quantile of $\Psi_i(F(w);\lambda)$ satisfies $\Psi_i(F(W(\alpha));\lambda) = \alpha$, which together with (2.6) gives the following quantile and quantile regression identification result,

Lemma 1 *Under the Independent Private Value paradigm and an ascending auction in which Assumption 1 and 2 hold and the identity of the winner is known,*

1. *If parametrization (2.2) applies,*

$$W(\Psi_i(\alpha;\lambda)|X) = V(\alpha|X). \tag{2.8}$$

2. *Assuming a linear model for $V(\alpha|X)$, $\alpha \in [0,1]$,*

   (i) There exists, for each $\alpha \in [0,1]$, a vector of coefficients $\beta(\alpha)$ such that,

$$W(\alpha|X) = X'\beta(\alpha);$$

   (ii) $\beta(\alpha)$ is uniquely defined and satisfies, for each $i = 1, 2, \cdots, N$,

$$\gamma(\alpha) = \beta(\Psi_i(\alpha;\lambda)), \tag{2.9}$$

   where $\Psi_i(\cdot;\lambda)$ is as defined in (2.7).

Lemma 1 gives the two cornerstone results of the quantile and quantile regression identification approaches. Lemma 1-(1) shows that the entire private value quantile function can be nonparametrically identified by the winning bid quantile function and the identity of the winner. More specifically, the α level of the private value parent quantile function, is nonparametrically identified by the $\Psi_i(\alpha;\lambda)$ level of the observed winning bid quantile function. Lemma 1-(2) gives the stability property of linear specifications. Due to (2.8), assuming that $V(\cdot|X)$ belongs to a linear model generates a winning bid quantile function within the same linear model family. The proof of Lemma 1 is given in the appendix.

3 Optimal Reservation Price

This section studies the optimal reservation price in a quantile perspective when bidders are asymmetric. Consider a binding reservation price set by the seller, i.e. $R(X) \in [V(0|X), V(1|X)]$. The reservation price plays the role of a screening level because it prevents bidders with private value $V(\alpha|X) < R(X)$ from participating in the auction. Denote the screening level by $\alpha_R(X)$ such that $R(X) = V(\alpha_R|X)$.

Let the seller's payoff in an auction with reservation price r be defined as

$$\pi(r) = W\mathbb{I}(W \geq r) + V_0(1 - \mathbb{I}(W \geq r)), \tag{3.10}$$

where W is the winning bid, V_0 the seller's private value and $\mathbb{I}(A)$ an indicator function equal to 1 if the event A holds and 0 otherwise. The following proposition gives a quantile version for the seller's expected payoff, a candidate for the optimal screening level $\alpha_R^*(X, V_0) = \alpha_R^*(X, V_0(X))$ and for the corresponding optimal reservation price $R_*(X, V_0) = R_*(X, V_0(X))$. Let $\Pi(\alpha|X, V_0, \lambda, N)$ be the seller's expected payoff given (X, V_0, λ, N) when the screening level is α.

Proposition 2 *Suppose an ascending auction in which the Independent Private Value paradigm holds, bidders are asymmetric, the screening level is α and assumptions 1-2 hold.*

(i) The probability of selling the auctioned unit is $(1 - \alpha^{\Lambda_N})$, where $\Lambda_N = \sum_{i=1}^{N} \lambda_i$ and $\lambda_i, i = 1, 2, \ldots, N$, are bidder i's type;

(ii) The seller's expected payoff is

$$\Pi(\alpha|V_0, X, N, \Lambda_N) = V_0(X)\alpha^{\Lambda_N} + R(X)\sum_{i=1}^{N}\alpha^{\Lambda_{N|i}}(1-\alpha^{\lambda_i}) + \int_{\alpha}^{1} V(t|X)\left[(1-N)\Lambda_N t^{\Lambda_N - 1} + \sum_{i=1}^{N}\Lambda_{N|i}t^{\Lambda_{N|i}-1}\right]dt, \tag{3.11}$$

where, $V_0(X)$ is the seller's private value, $R(X)$ the reservation price and $\Lambda_{N|i} = \Lambda_N - \lambda_i$.

(iii) The optimal reservation price $R_*(X, V_0) = V(\alpha^*|X)$, when a bidder of type λ_i wins, satisfies

$$V_0(X) = R_*(X, V_0) - V^{(1)}(\alpha^*|X)\frac{\alpha^*(X, V_0)}{\Lambda_N}\sum_{i=1}^{N}\left(\alpha^*(X, V_0)^{-\lambda_i} - 1\right), \tag{3.12}$$

where $\alpha^*(X, V_0)$ is the optimal screening level.

The proof of Proposition 2 is given in the appendix.

4 Estimation

Consider L independent ascending auctions $\{w_\ell, x_\ell, Z_{i\ell}, i = 1, \cdots, N_\ell, \ell = 1, \cdots, L\}$, where w_ℓ is the winning bid at auction ℓ, x_ℓ a specific characteristic of the good auctioned at auction ℓ and $Z_{i\ell}$ a specific characteristic of type i at auction ℓ. In what follows, let $X_\ell = (1, x_\ell')' \in \mathcal{X}$ be a column vector of dimension $(d+1)$ and $Z_{i\ell}$ a column vector of dimension $K \leq N$ grouping bidder/type's specific characteristics. We will assume that the parameter of asymmetry is fully determined by each type own characteristic, that is

$$\lambda_i = \exp\left(Z_i'\eta\right),$$

where η is a conformable vector of coefficients.

In this paper, we propose a two-step procedure to estimate the parameters of asymmetry and the parameters characterizing the private value conditional quantile function. Since $\mathbb{P}[\text{“}\lambda_i \text{ wins”}] = \lambda_i / \sum_{i=1}^{N}\lambda_i$, a simple way to estimate λ_i is via the empirical probability of type λ_i winning. If the econometrician has information on $Z_{i\ell}$, then $\lambda_{i\ell}, i = 1, \ldots, N_\ell, \ell = 1, \cdots, L$ can be estimated via $\lambda_{i\ell} = \exp(Z'_{i\ell}\eta)$ where η maximizes the likelihood function

$$\mathcal{L}(\eta) = \sum_{\ell=1}^{L} \sum_{i=1}^{N_\ell} \mathbb{I}\left(\text{“i wins at } \ell\text{”}\right) \ln \frac{\exp\left(Z'_{i\ell}\eta\right)}{\sum_{i=1}^{N_\ell} \exp\left(Z'_{i\ell}\eta\right)}, \tag{4.13}$$

over a grid of $\left[\underline{\eta}_1, \overline{\eta}_1\right] \times \left[\underline{\eta}_2, \overline{\eta}_2\right] \times \ldots \left[\underline{\eta}_K, \overline{\eta}_K\right]$. The parameters $\widehat{\eta}$ can then be used in the second step to estimate the parameters of the private value conditional quantile function via quantile regression method,

$$\widehat{\gamma}(\alpha) = \arg\min_{\gamma \in \Gamma} \sum_{\ell=1}^{L} \sum_{i=1}^{N_\ell} \rho_{\Psi(\alpha;\widehat{\eta})}\left(w_\ell - X'_\ell\gamma\right), \tag{4.14}$$

where $\rho_\alpha(u) = u(\alpha - \mathbb{I}(u < 0))$, $\Psi_i(\cdot;\eta)$ is as defined in (2.7) and Γ is a compact subset of $\mathbb{R}^{d+1}$.

5 Monte Carlo Simulation

5.1 Performance of the Two-Step Estimator

In this section, we present the results of a Monte Carlo simulation designed to evaluate the performance of the two-step estimation procedure above.

Data Generating Process. We simulate $L = 200$ ascending auctions with $N = 5$ bidders assigned to $K = 2$ different classes: type 1 and type 2, with specific characteristics $(Z_{1\ell}, Z_{2\ell})$ randomly draw from $\mathcal{U}_{[0,1]}$ and $\mathcal{U}_{[2,3]}$, respectively, at each auction ℓ. The parent private value conditional quantile function is generated as

$$V(\alpha|X_\ell) = \gamma_0(\alpha) + \gamma_1(\alpha)x_{1\ell},$$

where the auctioned good characteristics $x_{1\ell} \sim \mathcal{U}_{[1,5]}$ and the true quantile regression coefficients are

$$\gamma_0(\alpha) = -\log\left(1-\left(1-\frac{1}{e}\right)\alpha\right)$$

and

$$\gamma_1(\alpha) = 1-\frac{1}{e^{\alpha}}.$$

By equation (2.3), type k's private value conditional quantile function is

$$V_k(\alpha|X_\ell) = V\left(\alpha^{1/\lambda_{k\ell}}|X_\ell\right), \quad k=1,2,$$

where $\lambda_{k\ell} = \exp(Z_{k\ell}\eta)$ and η a conformable vector with parameters $\eta_k, k = 1,2$. One of the parameters is restricted to zero to allow identification. At each auction ℓ, 3 private values are randomly draw from $V_1(\alpha|X_\ell)$ and 2 from $V_2(\alpha|X_\ell)$. The winning bid is given by the second-highest private value and the identity/type of the winner is stored.

Estimation. The estimation is conducted in two steps. In the first step, η is estimated via maximum likelihood over a grid of points $[0.01, 0.02, \ldots, 1]$ by maximizing (4.13). For each $k = 1,2$ and $\ell = 1,\ldots,L$, $\widehat{\lambda}_{k\ell} = \exp(Z_{k\ell}\widehat{\eta})$, which is then used into the second step to estimate $(\gamma_0(\alpha), \gamma_1(\alpha))'$ via (4.14).

The estimation results for a median auction are given in the following three figures. Figure (1), (2) and (3) show the estimation of the private value conditional quantile functions (parent and type-2)[1] for sample sizes $L = 20$, $L = 50$ and $L = 200$, respectively.

[1] Type-1 private value conditional quantile function is not shown because η_1 is set to zero.

Figure 1: Private Value Conditional Quantile Function
$L = 20$

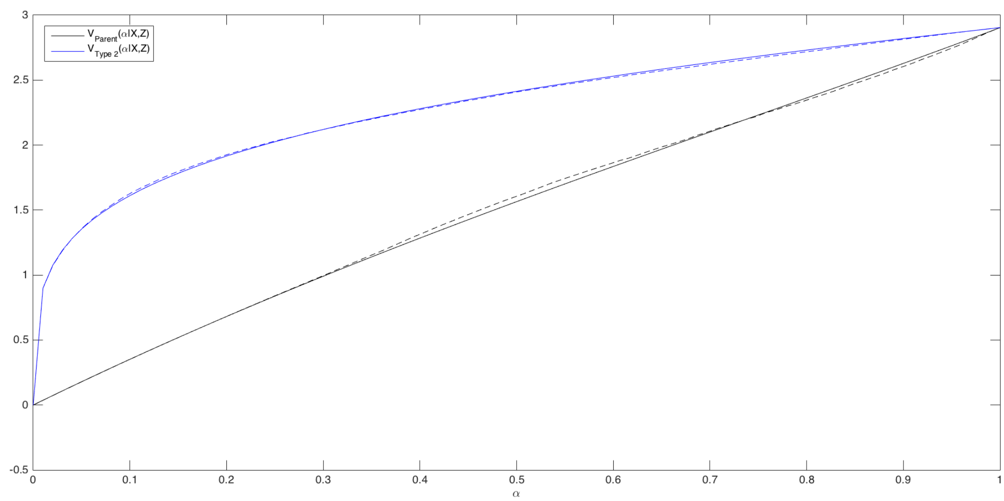


Figure 2: Private Value Conditional Quantile Function
$L = 50$

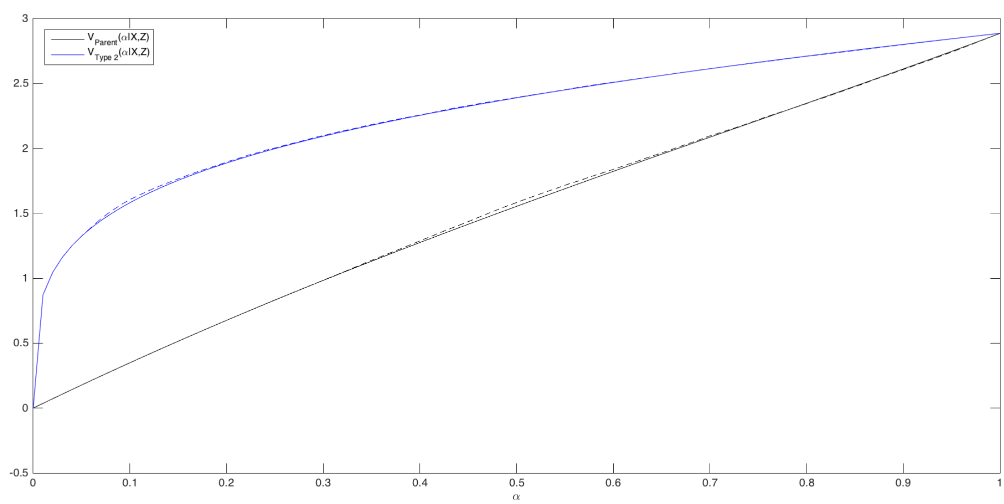
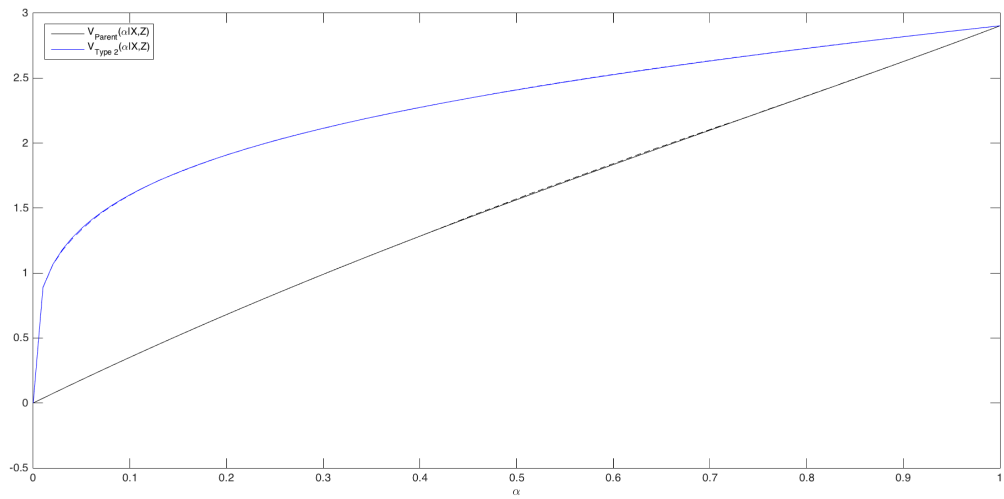


Figure 3: Private Value Conditional Quantile Function
$L = 200$



5.2 Effect of Asymmetry on Optimal Reservation Price: An Example

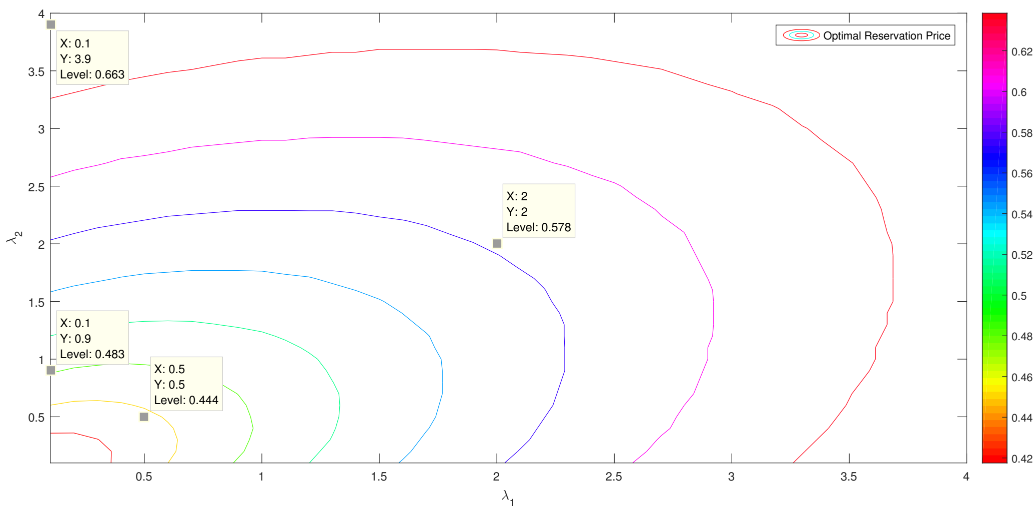
This section demonstrates how the presence of asymmetry among bidders in an auction affects its optimal reservation price. In the quantile setup, the optimal reservation price is $R_*(X) = V(\alpha_R^*|X)$, where α_R^* maximizes the expected revenue given in (3.11). For computational simplicity, consider an auction with two bidders and assume the parent distribution to be uniform. Two-buyer and uniform distribution are common assumptions used in the literature of asymmetric auctions to demonstrate complex concepts more easily (see, for example, Maskin and Riley (2000)). Under these simplifying assumptions for our model, the parent distribution quantile is $V(\alpha) = \alpha$, and the bidder i's private value distribution quantile is $V_i(\alpha) = \alpha^{1/\lambda_i}$. Further, the reservation price $R_* = V(\alpha_R^*) = \alpha_R^*$. Proposition 2-(iii), under the assumptions of this example, is reduced as follows,

$$\alpha_R^* - \frac{\alpha_R^*}{\Lambda_2}({\alpha_R^*}^{-\lambda_1} + {\alpha_R^*}^{-\lambda_2} - 2) = 0 \tag{5.15}$$

where, $\Lambda_2 = \lambda_1 + \lambda_2$.

We solve the above equation for the optimal reservation price, α_R^*, by varying the asymmetry parameters in the range $0 < \lambda_1, \lambda_2 \leq 4$. Figure 4 shows the contour plot of optimal reservation price.

Figure 4: Contour Plot for Optimal Reservation Price under Asymmetry

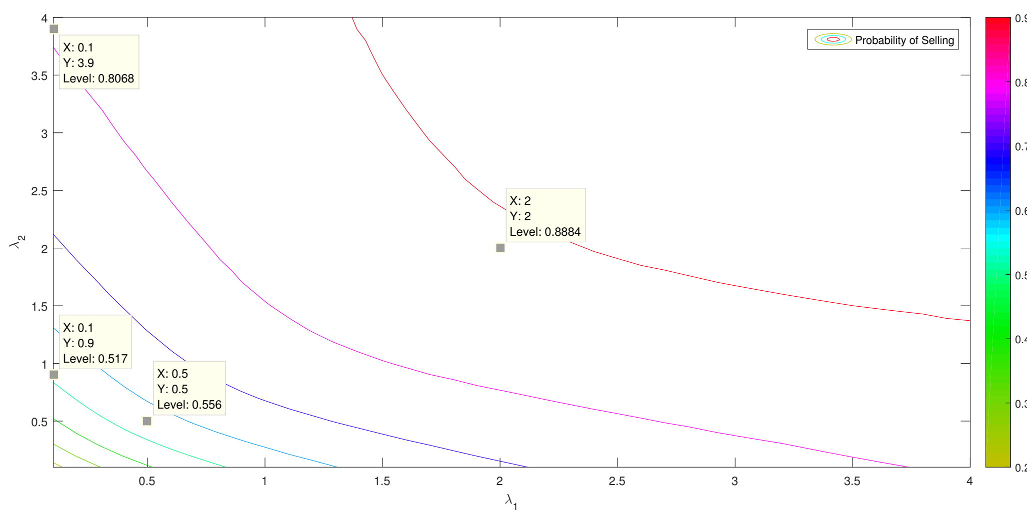


Following are some key observations from Figure 4, regarding how asymmetry affects the optimal reservation price. First, we see that for a given Λ_2, different pairs of λ_1 and λ_2 yield different values for the optimal reservation price. This has been shown in the figure by some data-points for two such different combinations, both for $\Lambda_2 = 1$ and $\Lambda_2 = 4$. Second, the highest reservation price that can be obtained for a particular Λ_2 increases with Λ_2. In the figure, the highest reservation price that is associated with all combinations of λ_1 and λ_2 such that $\Lambda_2 = 1$ is 0.4830 while $\Lambda_2 = 4$ has such combinations of λ_1 and λ_2 that produce a greater optimal reservation price of 0.6630. And lastly, for a given Λ_2, the higher the asymmetry, the higher is the optimal reservation price. For instance, in the figure, for $\Lambda_2 = 4$, a greater asymmetry as characterized by $\lambda_1 = 0.1$ and $\lambda_2 = 3.9$

produces an optimal reservation price of 0.6630, while upon reducing the asymmetry to the extreme case where asymmetry no longer exists such that $\lambda_1 = \lambda_2 = 2$, the optimal reservation price falls to 0.5780.

Note that the probability of selling the auctioned good, as shown in Proposition 2-(i) , is $(1 - {\alpha_R^*}^{\Lambda_2})$. Hence, in an auction with a given Λ_2, there exists an inverse relationship between the reservation price and the probability of selling. This can be seen in Figure 5: for a given Λ_2, the probability of selling falls with an increase in asymmetry, as expected. But important here is to note that, the optimal reservation price for a particular Λ_2 is chosen in a way that it maximizes the expected revenue for the seller, while the probability of selling is not reduced prohibitively. For instance, a high level of asymmetry characterized by $\lambda_1 = 0.1$ and $\lambda_2 = 3.9$, associated with a high optimal reservation price of 0.6630, still yields a high probability of selling of over 0.8.

Figure 5: Contour Plot for Probability of Selling under Asymmetry



5.2.1 Limitations of using symmetric estimation on an asymmetric model

Since a visual analysis of the above graphs reveal that for a given Λ_2, the individual asymmetry parameters, λ_i, $(i = 1, 2)$, play an important role in choosing the optimal reservation price that achieves the seller's objective of maximizing his expected revenue, estimating these asymmetry parameters separately seems important. In order to rigorously analyze and demonstrate the need for such an unconstrained estimation of λ_1 and λ_2, we study the effect of applying the constrained symmetric estimation procedure of Gimenes (2017) on the asymmetric example model considered here. The optimal reservation price under symmetric estimation, when the model is truly asymmetric, is computed using the limit private value distribution of the symmetric estimator, as detailed below.

Since the parent distribution is assumed to be uniform such that $F(v) = v$, the marginal cdf associated with the bidder i is $F_i(v) = v^{\lambda_i}$, $i = 1, 2$. Let the individual values, V_1 and V_2, be independent. Since the second highest private value is the winning bid, W, we have,

$$W = \min\left(V_1, V_2\right),$$

and the cdf of the winning bids is given by

$$\begin{aligned}
\mathbb{P}\left(W \le w\right) &= 1 - \mathbb{P}\left(W > w\right) = 1 - \mathbb{P}\left[\min\left(V_1, V_2\right) > w\right] = 1 - \mathbb{P}\left(V_1 > w\right)\mathbb{P}\left(V_2 > w\right) \\
&= 1 - \left(1 - \mathbb{P}\left(V_1 \le w\right)\right)\left(1 - \mathbb{P}\left(V_2 \le w\right)\right) = 1 - \left(1 - w^{\lambda_1}\right)\left(1 - w^{\lambda_2}\right) \\
&= w^{\lambda_1} + w^{\lambda_2} - w^{\lambda_1 + \lambda_2}
\end{aligned} \tag{5.16}$$

Let this cdf be denoted by $F_\lambda^W(w)$ and let $W_\lambda(\alpha)$ be the associated winning bid quantile function. Let $\Psi(t) = \Psi_2(t) = 2t - t^2$. As detailed in Gimenes (2017), $\Psi\left(F^V(\cdot)\right)$ represents the symmetric distribution of the winning bids. Hence, the limit private value distribution $F_\lambda^V(\cdot)$ obtained with the symmetric estimation procedure, when the distribution of (V_1, V_2) is as above, satisfies,

$$\Psi\left[F_\lambda^V(w)\right] = F_\lambda^W(w)$$

Hence,

$$F_\lambda^V(v) = \Psi^{-1}\left[F_\lambda^W(v)\right]$$
$$\Rightarrow F_\lambda^V(W_\lambda(\alpha)) = \Psi^{-1}(\alpha)$$
$$\Rightarrow W_\lambda(\alpha) = V_\lambda(\Psi^{-1}(\alpha))$$

which gives the limit private value quantile function as follows:

$$V_\lambda(\alpha) = W_\lambda(\Psi(\alpha)) \tag{5.17}$$

The corresponding optimal reservation price $R_{S,\lambda} = V_\lambda\left(\alpha_\lambda^R\right)$ computed under symmetry (see Gimenes (2017), equation 2.10 for details) is such that

$$V_\lambda\left(\alpha_\lambda^R\right) - V_\lambda^{(1)}\left(\alpha_\lambda^R\right)\left(1-\alpha_\lambda^R\right) = V_0 \tag{5.18}$$

where V_0 is the seller's private value for the good, assumed to be 0 in the present analysis. Hence, for different pairs of λ_1 and λ_2, the true asymmetric distribution of the winning bids, $F_\lambda^W(w)$, is computed according to equation 5.16 and the limit private value quantile function is then derived according to equation 5.17, and used in equation 5.18 to solve for the optimal reservation price under the symmetric estimation procedure.

Finally, the expected revenue for the seller, from both using the true asymmetric optimal reservation price and the symmetric optimal reservation price, denoted by $\Pi(R^*)$ and $\Pi(R^S)$ respectively, is calculated according to (3.11) in Proposition 2, as follows:

$$\begin{aligned}
\Pi(R^*) &= R^* \sum_{i=1}^{2} {\alpha_R^*}^{\Lambda_{2|i}}(1-{\alpha_R^*}^{\lambda_i}) + \int_{\alpha_R^*}^{1} V(t|X)\left[\sum_{i=1}^{2}\Lambda_{2|i}t^{\Lambda_{2|i}-1} - \Lambda_2 t^{\Lambda_2-1}\right]dt \\
\Pi(R^S) &= R_{S,\lambda} \sum_{i=1}^{2} {\alpha_R^S}^{\Lambda_{2|i}}(1-{\alpha_R^S}^{\lambda_i}) + \int_{\alpha_R^S}^{1} V(t|X)\left[\sum_{i=1}^{2}\Lambda_{2|i}t^{\Lambda_{2|i}-1} - \Lambda_2 t^{\Lambda_2-1}\right]dt
\end{aligned} \tag{5.19}$$

where, $V(\alpha)$ is the quantile function associated with the parent distribution $F(v)$, $R^* = V(\alpha_R^*)$ is the true optimal reservation price, $R_{S,\lambda} = V_\lambda\left(\alpha_\lambda^R\right)$ is the optimal reservation price under symmetry, $\alpha_R^S = F(R_{S,\lambda})$, and $\Lambda_{2|i} = \Lambda_2 - \lambda_i$.

Table 1 compares the optimal reservation price and the expected seller revenue obtained from the limit symmetric estimation procedure versus the analytically solved true values, for values of λ_1 and λ_2 that sum to $\Lambda_2 = 1$. All values have been rounded off to four decimal digits.

Table 1: Analytical vs limit symmetric estimator with $\Lambda_2 = 1$

		Optimal Reservation Price		Expected Seller Revenue		
λ_1	λ_2	Analytical	Symmetric	Analytical	Symmetric	Increase (%)
0.1	0.9	0.4830	0.4420	0.2550	0.2535	0.6158
0.2	0.8	0.4680	0.4433	0.2593	0.2590	0.1420
0.3	0.7	0.4550	0.4442	0.2627	0.2627	0.0029
0.4	0.6	0.4470	0.4440	0.2648	0.2648	0.0003
0.5	0.5	0.4440	0.4449	0.2655	0.2655	0.0000

From the above table we see that, whenever asymmetry exists the optimal reservation price obtained through the constrained symmetric estimation procedure is lesser than the true analytical values (rows 1 to 4). With higher asymmetry, the constrained estimation gives a poorer value for the optimal reservation price. Using the true optimal reservation price, that can be obtained through an unconstrained estimation of λ_1 and λ_2, as against the sub-optimal symmetric reservation price, can generate an increase in the seller's expected revenue. More the asymmetry, more is the potential for obtaining a higher expected revenue gain. When asymmetry reduces, however, the performance of the constrained estimation procedure, as expected, improves, getting close to the true values for the symmetric case of $\lambda_1 = \lambda_2 = 0.5$, such that no gain in expected revenue ensues.

As noted earlier, the highest value of optimal reservation price associated with a particular Λ_2 increases with Λ_2, and that this highest optimal reservation price corresponds to the highest level of asymmetry that exists within that Λ_2. Hence, a symmetric estimation procedure, wrongly applied to such a model, should then produce the highest potential expected revenue gain from using a correct asymmetric estimation instead. Table 2 demonstrates this, by considering some pairs of λ_1 and λ_2 that sum to $\Lambda_2 = 4$, and have high asymmetry. We see that compared to the previous table, the magnitude of gain in expected revenue that can now be achieved from using a correct unconstrained estimation of individual λ_i significantly improves.

Table 2: Analytical vs limit symmetric estimator with $\Lambda_2 = 4$

		Optimal Reservation Price		Expected Seller Revenue		
λ_1	λ_2	Analytical	Symmetric	Analytical	Symmetric	Increase (%)
0.1	3.9	0.6630	0.5451	0.5389	0.5059	6.5080
0.2	3.8	0.6580	0.5481	0.5427	0.5158	5.2151
0.3	3.7	0.6520	0.5520	0.5464	0.5250	4.0848
0.4	3.6	0.6460	0.5559	0.5501	0.5333	3.1421
0.5	3.5	0.6410	0.5583	0.5537	0.5406	2.4202

Not only does the performance of the symmetric estimation procedure deteriorate with increasing asymmetry, it also remains heavily reliant on the functional form of the parent distribution. To see this, we repeat the above exercise by considering a non-uniform parent distribution of the form $F(v) = 1 - (1-v)^v$, v in $[0,1]$. The rest of the assumptions are unchanged, that is, there are two bidders of type λ_1 and λ_2, and the seller's private value $V_0 = 0$. The individual cdf, then, is given by $F_i(v) = [1-(1-v)^v]^{\lambda_i}$, $(i = 1,2)$. We compute the true asymmetric reservation price, $R_* = V(\alpha_R^*)$, by solving equation (3.12) given in Proposition 2 for $0 < \lambda_1, \lambda_2 \leq 4$, as earlier.

For applying the limit symmetric estimator, we first derive the true asymmetric cdf of winning bids as:

$$\begin{aligned}\mathbb{P}(W \leq w) &= 1 - \mathbb{P}(W > w) = 1 - \mathbb{P}[\min(V_1, V_2) > w] = 1 - \mathbb{P}(V_1 > w)\,\mathbb{P}(V_2 > w) \\ &= 1 - (1 - \mathbb{P}(V_1 \leq w))\,(1 - \mathbb{P}(V_2 \leq w)) \\ &= 1 - \left[1 - \{1-(1-w)^w\}^{\lambda_1}\right]\left[1 - \{1-(1-w)^w\}^{\lambda_2}\right]\end{aligned}$$

The limit private value distribution is then computed as described previously, using equation

5.17, and it is then used to solve equation 5.18 to get the new symmetric optimal reservation price. Expected revenues, under both the true asymmetric optimal reservation price and the symmetric optimal reservation price, are calculated using equation 5.19, as earlier. Table 3 reports the analytically calculated values for optimal reservation price and expected seller revenue, and the corresponding results obtained from the symmetric estimation procedure. We observe that for the same Λ_2 and the same levels of asymmetry as reported in table 2, there is, now, a steeper fall in the optimal reservation price obtained from the symmetric estimation procedure, which results in a greater potential for increasing the seller's expected revenue. For instance, with $\lambda_1 = 0.1$ and $\lambda_2 = 3.9$, the seller can potentially increase his expected revenue through unconstrained estimation of individual λ_i $(i = 1, 2)$, by about 6.51% if the parent distribution were uniform, while a non-uniform distribution of the form considered here can almost double the gain to 13.18%.

Table 3: Analytical vs limit symmetric estimator with new parent distribution

		Optimal Reservation Price		Expected Seller Revenue		
λ_1	λ_2	Analytical	Symmetric	Analytical	Symmetric	Increase (%)
0.1	3.9	0.7062	0.5512	0.6316	0.5581	13.1838
0.2	3.8	0.7016	0.5608	0.6362	0.5788	9.9138
0.3	3.7	0.6964	0.5685	0.6407	0.5960	7.5128
0.4	3.6	0.6911	0.5768	0.6451	0.6110	5.5872
0.5	3.5	0.6858	0.5823	0.6495	0.6232	4.2100

The above analysis, carried out for both uniform and a non-uniform parent distribution with varying levels of asymmetry, makes it clear that a symmetric estimation procedure, when applied to a model that is truly asymmetric, produces inefficient and unreliable results. The inefficiency arises because the symmetric optimal reservation price is always less than the true one, whenever asymmetry is present, increasing with the level of asymmetry. The unreliability stems from the fact that such a constrained procedure chooses a $\lambda_1 = \lambda_2$ in such a way that the level by which the optimal reservation price thus obtained is lower from its true value depends on the functional form of the underlying parent distribution of private values, even for the same level of asymmetry.

A combined effect of the two results in a lowered expected revenue for the seller, whose magnitude can be high or low depending on the level of asymmetry and the functional form of parent cdf.

Since the constrained estimation procedure picks up a value of $\lambda_1 = \lambda_2$ which gives a reservation price that is truly sub-optimal, in the sense that the seller's expected revenue is not maximized at that reservation price, it is important that the individual asymmetry parameters λ_i $(i = 1, 2)$ is estimated in an unconstrained manner. By estimating the individual asymmetry parameters separately first, and then using them to estimate private value quantile function and expected revenue, the estimation procedure also remains robust to any underlying functional form of parent distribution.

Appendix - Proof Section

Proof. **Lemma 1:** Since (2.7) is strictly increasing[2] in α, $\alpha \in (0, 1)$, and the αth-quantile of $\Psi_i(F(w); \lambda)$ satisfies $\Psi_i(F(W(\alpha)); \lambda) = \alpha$,

$$\begin{aligned} \Psi_i\left(F\left(W\left(\alpha\right)\right); \lambda\right) &= \alpha \\ F\left(W\left(\alpha\right)\right) &= \Psi_i^{-1}\left(\alpha; \lambda\right) \\ W\left(\alpha\right) &= F^{-1}\left(\Psi_i^{-1}\left(\alpha; \lambda\right)\right) \\ W\left(\alpha\right) &= V\left(\Psi_i^{-1}\left(\alpha; \lambda\right)\right) \\ W\left(\Psi_i\left(\alpha; \lambda\right)\right) &= V\left(\alpha\right), \end{aligned}$$

which by (2.3), gives (2.8). ■

[2]Let $\Psi_i^{\alpha}(\alpha; \lambda) = \frac{\partial \Psi_i(\alpha;\lambda)}{\partial \alpha}$, where

$$\Psi_i^{\alpha}\left(\alpha; \lambda\right) = \frac{\Lambda_{N|i}\Lambda_N}{\lambda_i}\alpha^{\Lambda_{N|i}-1}\left(1 - \alpha^{\lambda_i}\right) > 0.$$

Proof. Proposition 2: In an auction with screening level α, the auctioned good will not be sold if all the players have valuation below $R(X)$. Therefore, the probability of trading is $1-\alpha^{\Lambda_N}$, which gives Proposition 2-(i). For simplicity of exposition, we will drop the conditionality on X. Consider the seller's payoff defined in (3.10). Under assumption 1, the seller possible payoffs are

$$\pi(r)=\begin{cases} V_0 & \text{if } V_{N:N}<r \\ r & \text{if } V_{N-1:N}<r\leq V_{N:N}, \\ V_{N-1:N} & \text{if } r\leq V_{N-1:N}, \end{cases}$$

where $V_{i:N}$ is the ith-lowest order statistics of private values, i.e. $V_{N:N}$ is the first-highest order statistic. Suppose that α is the screening level such that $r=V(\alpha)$. The next three statements evaluate each event above.

1. $\mathbb{P}(V_{N:N}<r) = \mathbb{P}(V_i<r, \forall i=1,\ldots,N) = \Pi_{i=1}^N F_i(r) = \Pi_{i=1}^N [F(V(\alpha))]^{\lambda_i} = \alpha^{\Lambda_N}$. The associated seller's payoff under this event is $\pi_1(r)=V_0\alpha^{\Lambda_N}$;

2. $\mathbb{P}(V_{N-1:N}<r\leq V_{N:N}) = \sum_{i=1}^N \Pi_{j\neq i}F_j(r)(1-F_i(r)) = \sum_{i=1}^N \alpha^{\Lambda_{N|i}}(1-\alpha^{\lambda_i})$, where i is the winner's type. The corresponding seller's payoff under this second event is $\pi_2(r) = V(\alpha)\sum_{i=1}^N \alpha^{\Lambda_{N|i}}(1-\alpha^{\lambda_i})$; and

3. Let $F_{N-1:N}(v)$ denote the c.d.f. of the second-highest order statistic $V_{N-1:N}$, which is given by[3]

$$F_{N-1:N}(v)=\prod_{i=1}^N F_i(v)+\sum_{i=1}^N\prod_{j\neq i}F_j(v)(1-F_i(v)).$$

[3]See, for example, David and Nagaraja (1970).

Under parametrization (2.2),

$$
\begin{aligned}
F_{N-1:N}(v) &= [F(v)]^{\Lambda_N} + \sum_{i=1}^{N} \left[(1-(F(v))^{\lambda_i}).(F(v))^{\Lambda_{N|i}}\right] \\
&= [F(v)]^{\Lambda_N} + \sum_{i=1}^{N}[(F(v))^{\Lambda_{N|i}} - (F(v))^{\Lambda_N}] \\
&= (1-N)(F(v))^{\Lambda_N} + \sum_{i=1}^{N}(F(v))^{\Lambda_{N|i}} \\
&= (F(v))^{\Lambda_N}\left[(1-N) + \sum_{i=1}^{N}(F(v))^{-\lambda_i}\right],
\end{aligned}
$$

and a change of variable $v = V(t) = F^{-1}(t)$, we can write

$$
dF_{N-1:N}(F^{-1}(t)) = \left[\Lambda_N t^{\Lambda_N - 1}(1-N) + \sum_{i=1}^{N}\Lambda_{N|i}t^{\Lambda_{N|i}-1}\right] dt. \tag{5.20}
$$

The seller's payoff for this third event is

$$
\begin{aligned}
\pi_3(r) &= \int_{\alpha}^{1} F^{-1}(t)\left[\Lambda_N t^{\Lambda_N - 1}(1-N) + \sum_{i=1}^{N}\Lambda_{N|i}t^{\Lambda_{N|i}-1}\right] dt \\
&= \int_{\alpha}^{1} V(t)\left[\Lambda_N t^{\Lambda_N - 1}(1-N) + \sum_{i=1}^{N}\Lambda_{N|i}t^{\Lambda_{N|i}-1}\right] dt
\end{aligned}
$$

From the three events above, we find the expected seller's revenue by $\pi_1(r) + \pi_2(r) + \pi_3(r)$, which gives (3.11). Hence, Proposition 2-(ii) is proved.

Now, if the seller's expected revenue $\Pi(\alpha|V_0, N, \Lambda_N)$ achieves a maximum for $\alpha^*(V_0) \in [0,1]$, it must satisfy a first order condition $\frac{\partial}{\partial\alpha}\Pi(\alpha|V_0, N, \Lambda_N) = 0$. The first derivative of $\Pi(\alpha|V_0, N, \Lambda_N)$

with respect to α is

$$\frac{\partial}{\partial\alpha}\Pi\left(\alpha|V_0,N,\Lambda_N\right)=V_0\Lambda_N\alpha^{\Lambda_N-1}+V^{(1)}(\alpha)\sum_{i=1}^{N}\left(\alpha^{\Lambda_{N|i}}-\alpha^{\Lambda_N}\right)$$
$$+V(\alpha)\sum_{i=1}^{N}(\Lambda_{N|i}\alpha^{\Lambda_{N|i}-1}-\Lambda_N\alpha^{\Lambda_N-1})$$
$$-V(\alpha)\left[\Lambda_N\alpha^{\Lambda_N-1}+\sum_{i=1}^{N}(\Lambda_{N|i}\alpha^{\Lambda_{N|i}-1}-\Lambda_N\alpha^{\Lambda_N-1})\right]$$

A candidate for the optimal screening level is

$$V_0\Lambda_N\alpha^*(V_0)^{\Lambda_N-1}+V^{(1)}(\alpha^*)\sum_{i=1}^{N}\left(\alpha^*(V_0)^{\Lambda_{N|i}}-\alpha^*(V_0)^{\Lambda_N}\right)-V(\alpha^*)\Lambda_N\alpha^*(V_0)^{\Lambda_N-1}=0$$
$$\Rightarrow V_0=V(\alpha^*)-V^{(1)}(\alpha^*)\sum_{i=1}^{N}\left(\frac{\alpha^*(V_0)^{1-\lambda_i}}{\Lambda_N}-\frac{\alpha^*(V_0)}{\Lambda_N}\right),$$

which proves Proposition 2-(iii). ■

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