Common factors in a panel with two cross-sectional dimensions

Håvard Hungnes
Statistics Norway
hhu@ssb.no

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Introduction

- The presence of some cross-sectional correlation of errors in panel data applications is likely to be the rule rather than the exception.
- Two approaches are used in the presence of common factors;
  - the principal component approach, PC
  - the common correlated effects, CCE
- Both PC and CCE are based on **one** cross-sectional dimension
- Here; CCE is extended to **two** cross-sectional dimensions

An heterogenous model with two cross-sectional dimensions:

$$Y_{ijt} = \alpha'_{ijt} D_t + \beta'_{ij} + E_{ijt}, \tag{1}$$

where the exogenous variables follow the process

$$X_{ijt} = A'_{ij} D_t + V_{ijt}. \tag{2}$$

Combining theses yields

$$Z_t \equiv \begin{pmatrix} Y_{ijt} \\ X_{ijt} \end{pmatrix} = B'_{ij} D_t + \varepsilon_{ijt} \tag{3}$$

The errors can have one of the following multi-factor structures:

$$\varepsilon_{ijt} = \begin{cases} U_{ijt} & \text{alternative 0} \\ C'_{ij} f_t + U_{ijt} & \text{alternative I} \\ C^{A\prime}_{ij} f^A_{it} + U_{ijt} & \text{alternative II} \\ C'_{ij} f_t + C^{A\prime}_{ij} f^A_{it} + U_{ijt} & \text{alternative III} \\ C'_{ij} f_t + C^{A\prime}_{ij} f^A_{it} + C^{B\prime}_{ij} f^B_{jt} + U_{ijt} & \text{alternative IV} \end{cases} \tag{4}$$

- The multi-factor structure in alternative I is similar to the one considered in Pesaran (2006). This formulation of multi-factor structure implies that we do not consider the two cross-sectional dimensions explicitly. Hence, we could have stacked the two cross-sectional dimensions into one cross-sectional dimension.
- The multi-factor structure in alternative II is also similar to the one considered in Pesaran (2006) when each of the N^A cross-sectional data sets is considered separately.
- The multi-factor structure in alternative III implies that we combine overall common factors (f_t) with common factors that differ across the A-dimension. This multi-factor structure is a combination of I and II.
- The multi-factor structure in alternative IV implies that common factors that are specific to both of the two cross-sectional dimensions are included. These are included in addition to some overall common factors.

Summary of results:

- With multi-factor structure I, the vector of observable variables $\left(D_t', \overline{Z}_t'\right)'$ with $\overline{Z}_t = \sum_{j=1}^{N^B} w_j^A \sum_{i=1}^{N^A} w_i^B Z_{ijt}$ can be used as a proxy for the common factors.
- With multi-factor structure II, the vector of observable variables $\left(D_t', \overline{Z}_{i.t}'\right)'$ with $\overline{Z}_{i.t} = \sum_{j=1}^{N^B} w_j^A Z_{ijt}$ can be used as a proxy for the common factors. This result follows from Pesaran (2006).
- With multi-factor structure III, the vector of observable variables $\left(D_t', \overline{Z}_t', \overline{Z}_{i.t}\right)'$, with $\overline{Z}_t$ and $\overline{Z}_{i.t}$ defined above, can be used as a proxy for the common factors.
- With multi-factor structure IV, the vector of observable variables $\left(D_t', \overline{Z}_t', \overline{Z}_{i.t}, \overline{Z}_{.jt}\right)'$, with $\overline{Z}_{.jt} = \sum_{i=1}^{N^A} w^B Z_{ijt}$ and with $\overline{Z}_t$ and $\overline{Z}_{i.t}$ defined above, can be used as a proxy for the common factors.

How the results are obtained:

Taking averages over eq. (3) with (4):

$$
\begin{aligned}
\overline{Z}_t &= \overline{B}'_w D_t + \sum_i \sum_j w_j^A w_i^B \left[C_{ij}^{A\prime} f_{it}^A + C_{ij}^{B\prime} f_{jt}^B + C'_{ij} f_t \right] + \overline{U}_t, \\
\overline{Z}_{i.t} &= \overline{B}'_{iw} D_t + \sum_j w_j^A \left[C_{ij}^{A\prime} f_{it}^A + C_{ij}^{B\prime} f_{jt}^B + C'_{ij} f_t \right] + \overline{U}_{it}^A, \\
\overline{Z}_{.jt} &= \overline{B}'_{wj} D_t + \sum_i w_i^B \left[C_{ij}^{A\prime} f_{it}^A + C_{ij}^{B\prime} f_{jt}^B + C'_{ij} f_t \right] + \overline{U}_{jt}^B,
\end{aligned}
$$

From this it follows that f_t, f_{it}^A, f_{jt}^B $(\forall i, j)$ are functions of $\overline{Z}_t, \overline{Z}_{i.t}, \overline{Z}_{.jt}$.

Note: In the special case of multi-factor structure IV with $f_t = 0$ (i.e., no overall common factor), the remaining common factors (f_{it}^A, f_{jt}^B) are still functions of $\overline{Z}_t, \overline{Z}_{i.t}, \overline{Z}_{.jt}$.

The exogenous variables

The vector X_{ijt} contains k variables that we assume differ in at least one of the cross-sectional dimensions. In this presentation I will assume that all observations in X_{ijt} are unique in both cross-sectional dimensions. However, it will be convenient to partition this vector as $X'_{ijt} = \left(x^{A\prime}_{it}, x^{B\prime}_{jt}, x'_{ijt}\right)$, where x^{A}_{it} is a vector of the k_1 variables that only differs in the first dimension, x^{B}_{jt} is a vector of k_2 variables that only differs in the second dimension, and x_{ijt} is a vector of the k_3 variables that differs in both dimensions; $k = k_1 + k_2 + k_3$. The coefficient vector β_{ij} is partitioned similarly; $\beta'_{ij} = \left(\beta^{A\prime}_{ij}, \beta^{B\prime}_{ij}, \beta^{C\prime}_{ij}\right)$. When including both industry-specific and input factor-specific common factors, β^{A}_{ij} and β^{B}_{ij} are not identifiable because the effect from the exogenous variables x^{A}_{it} and x^{B}_{jt} cannot be distinguished from the common factors. Hence, only β^{C}_{ij} can be identified.

Proposition

Under the multi-factor structure in alternative IV and Assumptions given in the paper, the following limiting result for the estimator is given by:

$$\begin{aligned}\widehat{b}_{ij} - \beta_{ij} &= \left(\frac{X_{ij}' M_{gij} X_{ij}}{T}\right)^{-1} \left(\frac{X_{ij}' M_{gij} \varepsilon_{ij}}{T}\right) \\ &\quad + O_p\left(\frac{1}{N^A}\right) + O_p\left(\frac{1}{N^B}\right) + O_p\left(\frac{1}{\sqrt{N^A T}}\right) + O_p\left(\frac{1}{\sqrt{N^B T}}\right),\end{aligned}$$

where $M_{gij} = I_T - G_{ij}\left(G_{ij}' G_{ij}\right)^{-} G_{ij}'$. Since ε_{ij} is distributed independently of X_{ij} and G_{ij}, then, for a fixed T and $N^A \to \infty$ and $N^A \to \infty$, we have $E\left(\widehat{b}_{ij} - \beta_{ij}\right) = 0$.

The proposition shows that the proposed estimator is asymptotically unbiased.

Monte Carlo

$$y_{ijt} = \beta_{ij} x_{ijt} + \gamma^{A}_{yij} f^{A}_{it} + \gamma^{B}_{yij} f^{B}_{jt} + \varepsilon_{ijt}, \quad (5)$$

$$x_{ijt} = \gamma^{A}_{xij} f^{A}_{it} + \gamma^{B}_{xij} f^{B}_{jt} + v_{ijt}, \quad (6)$$

with

$$f^{A}_{it} = \rho^{A}_{fi} f^{A}_{it-1} + \upsilon^{A}_{fit}, \text{ with } \upsilon^{A}_{fit} \sim NIID\,(0,1) \text{ for } t = 1,\ldots,T$$

$$\text{with } f^{A}_{i0} \sim NIID\left(0, \frac{1}{\left(1-\rho^{A}_{i0}\right)^{2}}\right) \text{ if } |\rho^{A}_{fi}| < 1 \text{ for } i = 1,\ldots,N^{A}$$

$$f^{B}_{jt} = \rho^{B}_{fj} f^{B}_{jt-1} + \upsilon^{B}_{fjt}, \text{ with } \upsilon^{B}_{fjt} \sim NIID\,(0,1) \text{ for } t = 1,\ldots,T$$

$$\text{with } f^{B}_{j0} \sim NIID\left(0, \frac{1}{\left(1-\rho^{B}_{j0}\right)^{2}}\right) \text{ if } |\rho^{B}_{fj}| < 1 \text{ for } j = 1,\ldots,N^{A}$$

$$v_{ijt} = \rho_{ij} v_{ijt-1} + \upsilon_{ijt}, \text{ with } \upsilon_{ijt} \sim NIID\,(0,1) \text{ for } t = 1,\ldots,T$$

$$\text{with } v_{ijt} \sim NIID(0, \frac{1}{(1-\rho_{ij})}) \text{ if } |\rho_{ij}| < 1 \text{ for } i = 1,\ldots,N^{A} \text{ and } j = 1,.$$

Results from simulations: $\hat{\beta}_{MG}$, non-stationary case

Table reports "Common Correlated Effects Mean Group Estimator" (CCEMG) $\hat{\beta}_{MG} = \left(N^A N^B\right)^{-1} \sum_{i=1}^{N^A} \sum_{j=1}^{N^B} \hat{\beta}_{ij}$.

	infeasible	naïve	overall c.f.	c.f. in the A-dim.	both	all	special
Proxies	$\begin{pmatrix} f_{it}^A \\ f_{jt}^B \end{pmatrix}$	0	$\begin{pmatrix} \bar{y}_t \\ \bar{x}_t \end{pmatrix}$	$\begin{pmatrix} \bar{y}_{it} \\ \bar{x}_{it} \end{pmatrix}$	$\begin{pmatrix} \bar{y}_t \\ \bar{x}_t \\ \bar{y}_{it} \\ \bar{x}_{it} \end{pmatrix}$	$\begin{pmatrix} \bar{y}_t \\ \bar{x}_t \\ \bar{y}_{it} \\ \bar{x}_{it} \\ \bar{y}_{jt} \\ \bar{x}_{jt} \end{pmatrix}$	$\begin{pmatrix} \bar{y}_{it} \\ \bar{x}_{it} \\ \bar{y}_{jt} \\ \bar{x}_{jt} \end{pmatrix}$
Asympt. bias	0	2/3	2/3	1/2	1/2	0	0

In the table, "c.f." is used for "common factors" and 'A-dim" is short for "A-dimension", "both" is the joint of "overalll c.f." and "c.f. in A-dim."

	infeasible	naïve	overall c.f.	c.f. in the A-dim.	both	all	special
mean	1.0000	1.6671	1.6648	1.5002	1.5002	1.0000	1.0195
min	0.9976	1.6161	1.6321	1.4546	1.4614	0.9970	1.0045
max	1.0024	1.6994	1.6945	1.5490	1.5402	1.0030	1.0595
st.d.*100	0.0760	1.1934	0.9910	1.4416	1.2710	0.0962	0.7768

In this simulation: 1000 replications, $N^A = N^B = 100$, $T = 100$,
$\rho_{fi}^A = \rho_{fj}^B = \rho_{ij} = \rho_{\varepsilon ij} = 1$, and $\rho_{\varepsilon ij} = 0.5, \forall i,j,\ \beta_{ij} = 1, \forall i,j,\ \gamma_{yij}^A = \gamma_{yij}^B = \gamma_{xij}^A = \gamma_{xij}^B = 1, \forall i,j.$

	infeasible	naïve	overall c.f.	c.f. in the A-dim.	both	all	special
mean	1.0019	1.6666	1.6445	1.4991	1.4971	1.0001	1.1482
min	0.8278	1.4313	1.4619	1.2347	1.2036	0.5601	0.8721
max	1.2190	1.8680	1.8667	1.7943	1.8376	1.5572	1.4509
st.d.*100	5.6764	5.6071	5.6310	7.3794	8.3871	12.165	8.4186

In this simulation: As above, except $N^A = N^B = 10$, $T = 10$.

Illustration: Substitution elasticity in factor demand

$$v_{ijt} = \sigma_i \ln \delta_{ijt} - \frac{1}{\kappa_i}\theta_{it} - \sigma_i \left(p_{ijt} - p_{iAt}\right) + \frac{1}{\kappa_i}x_{it}, \qquad (7)$$

where v_{ijt} is (log of) input factor j in industry i (at time t); $p_{ijt} - p_{iAt}$ is the relative input price; and x_{it} is log of production in industry i. Finally; σ_i is the elasticity of substitution in industry i.

The joint process of the factor-neutral technological level and the distribution parameters follows a deterministic trend and some common factors:

$$\sigma_i \ln \delta_{ijt} - \frac{1}{\kappa_i}\theta_{it} = \mu_{ij} + \gamma_{ij}t + \lambda'_{ij}f^*_{ijt}, \qquad (8)$$

Estimated elasticity of substitution

Proxies for f^*_{ijt}	no	overall	industry-specific	both	all types
	0	$\begin{pmatrix} x_{.t} \\ v_{..t} \end{pmatrix}$	$(v_{i.t})$	$\begin{pmatrix} x_{.t} \\ v_{..t} \\ v_{i.t} \end{pmatrix}$	$\begin{pmatrix} x_{.t} \\ p_{.jt} - p_{.At} \\ v_{i.t} \\ v_{..t} \\ v_{.jt} \end{pmatrix}$
Industry	Est. (std. err.)	Est. (std. err.)	Est. (std. err.)	Est. (std. err.)	Est. (std. err.)
01	0.522 (0.052)	0.478 (0.052)	0.451 (0.048)	0.399 (0.046)	0.216 (0.081)
02	**-0.382 (0.047)**	**-0.277 (0.045)**	**-0.338 (0.049)**	**-0.121 (0.048)**	0.195 (0.061)
03	0.241 (0.052)	0.227 (0.050)	0.232 (0.047)	0.218 (0.045)	0.952 (0.071)
04	0.566 (0.046)	0.531 (0.046)	0.553 (0.043)	0.484 (0.043)	0.458 (0.096)
05	0.328 (0.044)	0.292 (0.044)	0.340 (0.041)	0.284 (0.041)	0.271 (0.082)
06	0.662 (0.044)	0.663 (0.044)	0.689 (0.041)	0.613 (0.042)	1.054 (0.059)
07	**-0.141 (0.024)**	**-0.099 (0.023)**	**-0.104 (0.023)**	**-0.053 (0.022)**	0.211 (0.023)
08	0.823 (0.039)	0.691 (0.039)	0.813 (0.038)	0.626 (0.038)	0.939 (0.064)
09	0.356 (0.053)	0.336 (0.052)	0.332 (0.052)	0.264 (0.051)	0.521 (0.122)
10	0.439 (0.066)	0.473 (0.065)	0.676 (0.064)	0.444 (0.062)	0.836 (0.095)
11	0.597 (0.048)	0.551 (0.048)	0.271 (0.057)	0.236 (0.056)	0.402 (0.064)
12	**-0.134 (0.103)**	**-0.206 (0.103)**	0.011 (0.097)	0.028 (0.097)	0.552 (0.147)
13	0.137 (0.044)	0.119 (0.043)	0.129 (0.043)	0.120 (0.039)	0.403 (0.106)
14	0.554 (0.052)	0.540 (0.050)	0.641 (0.049)	0.524 (0.045)	0.357 (0.091)
15	0.306 (0.053)	0.390 (0.055)	0.383 (0.052)	0.438 (0.050)	0.850 (0.089)
16	0.790 (0.046)	0.851 (0.047)	0.809 (0.044)	0.860 (0.042)	0.428 (0.092)
17	0.678 (0.073)	0.615 (0.070)	0.558 (0.074)	0.631 (0.069)	0.837 (0.092)
log.lik.	-5525.6	-3487.21	-3596.77	-1082.02	4478.2
no. of par.	446	732	589	875	1161

Conclusions

- Presented how the CCE approach can be extended to a panel with two cross-sectional dimensions
- Showed that the estimator is unbiased
- By Monte Carlo simulations, shown that not taking the common factors properly into account can lead to large biases
- By Monte Carlo simulations, also shown that the approach works for non-stationary data
- Illustrated the approach by estimating the elasticity of substitution between (up to) 10 input factors in 17 industries in Norway