STRUCTURAL ESTIMATION OF DYNAMIC DIRECTIONAL GAMES WITH MULTIPLE EQUILIBRIA*

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Many important real life economic processes involve intertemporal strategic interactions between economic agents. Yet, there is relatively small number of empirical applications producing counterfactual predictions of the behavior in these complex interactions. Dynamic game models provide a theoretical framework to describe such economic processes but are numerically difficult to solve and estimate. The main reason is the existence of multiple equilibria, particularly in discrete games, which presents methodological challenges for the estimation of these models. Existing applied work typically rely on solution algorithms and estimation methods that do not fully explore, or even assumes away, this multiplicity, thereby inadvertently operating as an equilibrium selection mechanism. However, rather than haphazardly selecting an equilibrium depending on for example how the solution algorithm is initialized, a robust estimator *should compute all equilibria and select one, or several of them on a priori economic grounds or based on what we can learn from the data.*

In this paper, we develop a robust algorithm for computing the "full solution" maximum-likelihood estimator of a particular class of dynamic stochastic games with multiple equilibria,

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namely directional dynamic games. Our method allows for multiple equilibria having been played in data without making any assumptions on the equilbirium selection rule. It is based on Rust's (1987) Nested Fixed Point (NFXP) maximum likelihood estimator, but uses the Recursive Lexicographic Search (RLS) algorithm (Iskhakov, Rust and Schjerning 2016) within the "the inner loop" of NFXP to solve for *all* Markov perfect equilibria (MPE) of the game at each evaluation of the likelihood function. Until recently, NFXP approach have generally not been possible to implement for dynamic games with multiple equilibria since no algorithm was guaranteed to find all MPE. RLS provides this capability for directional dynamic games.

In models with numerous equilibria, the full solution method that combines NFXP and RLS (NRLS), even though theoretically ideal, may be impractical because of the computational burden of finding all equilibria. Utilizing the special structure of equilibria in directional dynamic games, we develop a specialized combinatorial optimization algorithm that can be fine tuned to balance the computational cost and the statistical efficiency of the estimator.

Dynamic directional games are a subclass of stochastic games with finite state space where under all feasible Markovian strategies the game transitions through the points of the state space in a directed fashion from a subset of initial states to a subset of absorbing states. The *state recursion algorithm* developed in (Iskhakov, Rust and Schjerning 2016) solves these games by backward induction on the state space, sequentially computing the solutions to the system of Bellman equations for particular points, and using the solutions already computed for the points downstream. In this manner, the large system of equations characterizing the equilibria of the whole game is decomposed into a number of smaller computational tasks (fixed point problems for the infinite horizon games or non-linear systems of equations otherwise).

In cases when the game has multiple equilibria, the state recursion algorithm itself is invoked by RLS sequentially and recursively for particular subsets of the state points. The RLS algorithm can be thought of as a tree traversal algorithm that simultaneously discovers the structure of the tree and computes the required statistics for each branch. The tree is composed of all MPE of the game: each branching corresponds to a point in the state space where the system of Bellman equations of the players for that point has multiple solutions. Each MPE in the whole game can be mapped into a path from the origin of the tree to each individual leaf.

Ideally, the discrete optimization problem of finding the equilibrium with the maximum likelihood for a given candidate value of the structural parameters has to be solved exactly. To solve this problem we employ the *branch and bound* optimization algorithm (Land and Doig 1960) which is an ideal computational tool for the task due to the tree structure of the equilibria and certain properties of the likelihood function. We show that just the use of this algorithm in its original form significantly reduces the computational burden compared to full enumeration on average.

The efficiency of the branch and bound algorithm in computing the maximum likelihood over the discrete set of equilibria, however, depends on how informative the data is in every points of the state space, and especially in points close to the trunk of the tree that correspond to the later stages of the game. To enhance the overall computational performance of the estimator in small samples, we develop a refinement of the branch and bound algorithm for our problem which sharpens the bounds using a statistical criterion, and does not explore the branches that appear particularly unlikely. The refined version of the algorithm computes an approximation to the true maximum of equilibrium specific likelihoods, and therefore trades off statistical efficiency for computational feasibility. However, as the sample size increases, the exact solution by the original branch and bound algorithm proves to be computationally feasible, resulting in fully efficient MLE estimator.

There has been considerable progress in the development of algorithms for computing Markov perfect equilibria, starting with the pioneering work by Pakes and McGuire (1994) and recent progress on homotopy methods for finding multiple equilibria of both static and dynamic games (Borkovsky, Doraszelski and Kryukov 2010, Besanko, Doraszelski, Kryukov and Satterthwaite 2010). Still, it is an extremely challenging problem to find even a single MPE of a dynamic game, much less all of them. The computational complexity has continuously led researchers to propose estimation methods that do not require to repeatedly compute the solution of the dynamic game during the course of estimation. Various two-step estimators have been proposed in the literature (see references in Egesdal, Lai and Su (2015)), but these are known to suffer from potential large biases in finite samples. In an effort to reduce the finite-sample biases associated with the two stage pseudo maximum likelihood (2S-PML) estimator, Aguirregabiria and Mira (2007) propose an algorithm to "swap the nesting" in the NFXP algorithm, and suggested *nested pseudo likelihood* (NPL) algorithm that should be robust to multiple equilibria as long as only *one* is played in the data. However, Pesendorfer and Schmidt-Dengler (2010) show that NPL can fail to produce consistent estimates such that we cannot expect to uncover structural parameters using NPL in even very large samples. Kasahara and Shimotsu (2012) propose a two-step modified version of NPL (NPL-Λ) that perform better. However, Egesdal, Lai and Su (2015) demonstrated that the lack of convergence of NPL and $NPL-\Lambda$ is not a trivial issue in practice.

To circumvent both the small-sample bias and the repeated solution of all equilibria, Su and Judd (2012) advocated for a constrained optimization approach to structural estimation which they called *mathematical programming with equilibrium constraints*. Egesdal, Lai and Su (2015) implement MPEC for the entry/exit dynamic game of Aguirregabiria and Mira (2007) and argue MPEC overcomes two computational challenges: it does not need to solve for all Markov perfect equilibria and even with multiple equilibria the constrained optimization problem is smooth. They conduct Monte Carlo experiments to investigate the numerical performance and finite-sample properties of their constrained optimization approach and argue that the MPEC approach is a favorable method for estimating dynamic games.

We show that NRLS is computationally efficient and asymptotically equivalent to MLE and present Monte Carlo evidence to compare it's performance to a variety of existing estimators including MPEC. Our Monte Carlo evidence is based on simulations from equilibria in the dynamic Bertrand investment game in Iskhakov, Rust and Schjerning (2013). The models is a simple dynamic discrete choice extension of the classic static model of Bertrand price competition where competing duopolists are allowed to undertake cost-reducing investments in an attempt to "leapfrog" their rival to attain temporary low-cost leadership. A characteristic aspect of the findings in this paper is a possibility of plethora of equilibria: even a simple finite state, dynamic extension of the standard static textbook model of Bertrand price competition may result in *hundreds of millions of Markov Perfect Equilibria* with a big variety of different investment dynamics.

RLS gives us the ability to fully solve the Bertrand investment game and simulate data from any equilibrium in this model, and therefore provide a perfect testbed for the performance of estimators in case of multiple equilibria. We study the behavior of existing estimation methods in situations that have generally not been possible previously. Specifically, we simulate data from the Bertrand investment game where we vary the number of equilibria by choosing the appropriate model parameters. By mixing data from several of these equilibria we investigate the performance of existing methods when multiple equilibria are played in the data. We also analyze the consequences of having a vast multiplicity of equilibria in the theoretical model when only a single one is played in the data.

We find NRLS to be remarkably robust, computational fast and able to both obtain efficient MLE of the structural parameters and at the same time identify the equilibrium selection played in the data (out of millions of potential MPEs). Moreover, NRLS allows us to relax assumptions for the equilibrium selection rules to allow for different equilibria to be played at different markets without much additional computational cost.

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