Partial Identification of Nonseparable Models with Binary Instruments

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January 24, 2017

Abstract

We consider partial identification of nonseparable models with continuous endogenous variables and binary instrumental variables. We show that the structural function is partially identified when the structural function is monotone or concave in a explanatory variable. D'Haultfœuille and Février (2015) and Torgovitsky (2015) prove point identification of the structural function under two key assumptions; (a) the conditional distribution functions of an endogenous variable given instruments have intersections, and (b) the structural function is strictly increasing in a scalar unobservable variable. We show that, even if these two assumptions do not hold, monotonicity or concavity provides identifying power. Point identification is achieved when the structural function is flat or linear in explanatory variables in a particular interval.

Keywords: Nonseparable models, partial identification, endogeneity, monotonicity, concavity, discrete outcomes, unobserved heterogeneity, instrumental variables.

1 Introduction

In this paper, we study the identification of a system of structural equations that takes the form:

$$\begin{aligned} Y &= g(X, \epsilon) \\ X &= h(Z, \eta), \end{aligned} \tag{1}$$

where $Y \in \mathbb{R}$ is a scalar response variable, $X \in \mathbb{R}$ is a continuous endogenous variable, $Z \in \{0, 1\}$ is a binary instrument, and ϵ and η are unobservable scalar variables. For simplicity, we suppose that X is a scalar variable. It is straightforward to extend the results to the case in which X is a vector. This specification is nonseparable in the unobservable variable ϵ and captures the unobserved heterogeneity in the effect of X on Y. Models of this type are also considered by, for example, D'Haultfœuille and Février (2015) and Torgovitsky (2015).

D'Haultfœuille and Février (2015) and Torgovitsky (2015) show that g is point identified when $g(x, e)$ and $h(z, v)$ are strictly increasing in e and v, respectively, and Z is

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independent of (ϵ, η). Their results are important for empirical work because many instruments employed in empirical work are binary or discrete, such as the intent-to-treat in a radomized controlled experiment and quarter of birth used in Angrist and Keueger (1991). For nonseparable models such as (1) with continuously distributed X, other nonparametric point identification results require Z to be continuously distributed. See, for example, Newey and Powell (2003), Chernozhukov, Imbens, and Newey (2007), and Imbens and Newey (2009).

D'Haultfœuille and Février (2015) and Torgovitsky (2015) use two key assumptions when they establish point identification of g. First, $F_{X|Z}(x|0)$ and $F_{X|Z}(x|1)$ have intersections. Second, $g(x, e)$ is strictly increasing in e. There are, however, many empirically important models that do not satisfy these assumptions. For example, $F_{X|Z}(x|0)$ and $F_{X|Z}(x|1)$ have no intersection when Z has strictly monotonic effect on X such as linear models $X = \beta_0 + \beta_1 Z + \eta$. Further, in many applications, instrumental variables have strictly monotonic effect on endogenous variables, such as LATE proposed by Imbens and Angrist (1994). For example, as seen in Baird, McIntosh, and Özler (2011), cash transfer programs were conducted in many countries. Then, if one uses a treatment indicator Z as the instrumental variable of income X, Z has strictly monotonic effect on X, which violates the intersection assumption. When Y is discrete or censored, $g(x, e)$ is not strictly increasing in e. Many problem in economics involve dependent variables that are discrete or censored. For example, development economists may wish to analyze the effect of income changes on child education. If school attendance is used as the dependent variable, then Y is discrete. As another example, suppose one want to analyze the effect of income changes on education expenditure. Then, education expenditure is censored at 0 when children do not attend school.

This paper shows that, when $g(x, e)$ is monotone or concave in x, we can partially identify g even if $F_{X|Z}(x|0)$ and $F_{X|Z}(x|1)$ have no intersections and $g(x, e)$ is not strictly increasing in e. $g(x, e)$ is monotone or concave in x in many economic models. For example, the Engel function of a normal good is monotonically increasing in income, the demand function is ordinarily decreasing in price, and economic analyses of production often suppose that the production function is monotone and concave in inputs. Manski (1997) assumes these assumptions and shows that the average treatment response is partially identified. The partial identification approach using the concavity assumption introduced in this paper is slightly similar to that considered by D'Haultfoeuille, Hoderlein, and Sasaki (2013).

There is a rich literature on the identification of nonseparable models with endogeneity. For example, Chernozhukov and Hansen (2005), Chesher (2007), Chernozhukov et al. (2007), and Imbens and Newey (2009) consider the identification of nonseparable models with endogeneity. Especially, Imbens and Newey (2009) consider very similar models to (1) and prove the point identification of g. Their paper allows ϵ to be multivariate and shows that the quantile function of $g(x, \epsilon)$ is point identified, while ϵ is imposed to be scalar in this paper. Their result needs continuous instruments, whereas D'Haultfœuille and Février (2015), Torgovitsky (2015) and this paper do not require continuous instruments.

This paper uses the control function approach, which is also used by, for example, Imbens and Newey (2009), D'Haultfœuille and Février (2015), or Torgovitsky (2015). This approach needs the strict monotonicity of $h(z, v)$ in v and the strong exogeneity: Z is independent of (ϵ, η). The estimating equation approach followed by, for example, Newey and Powell (2003), Chernozhukov and Hansen (2005) or Chernozhukov et al. (2007) does not need these assumptions. Chernozhukov and Hansen (2005) allow η to be multivariate and correlated with Z. However, they assume independent variables and instrumental variables are discrete.

In this paper, we suppose that the instrumental variable Z is binary. D'Haultfœuille and Février (2015) consider the case where the instrumental variable takes more than two values. In this case, they show point identification can be achieved by using group and dynamical systems theories even when $F_{X|Z}(x|z)$ and $F_{X|Z}(x|z')$ have no intersections.

In Section 2, we introduce the assumptions employed in this paper. In Section 3 and 4, we show partial identification of g under the monotonicity and concavity assumption when the conditional distributions have no intersections. In Section 5, we extend the result of Section 3 to the general case, where we allow Y to be discrete or censored. Section 6 concludes.

2 The Model

The following two assumptions are the same as those of D'Haultfœuille and Février (2015) and Torgovitsky (2015).

Assumption 1. *The instrument is independent of the unobservable variables:* $Z \perp\!\!\!\perp (\epsilon, \eta)$.

Assumption 2. *(i) The function g is everywhere continuous and $g(x, e)$ is strictly increasing in e for all $x \in \mathcal{X}$. (ii) For all $z \in \{0, 1\}$, $h(z, v)$ is continuous and strictly increasing in v.*

Assumption 1 is the strong exogeneity assumption and this assumption is typically employed when using the control function approach. See, for example, Imbens and Newey (2009), D'Haultfœuille and Février (2015), and Torgovitsky (2015). Imbens and Newey (2009) also need Assumption 2 (ii) but they allow ϵ to be multivariate. Chernozhukov and Hansen (2005) and Chernozhukov et al. (2007) employ Assumption 2 (i) but they do not need Assumption 2 (ii). We relax a part of Assumption 2 (i) in Section 5. In Section 5, we assume that $g(x, e)$ is nondecreasing in e.

Next assumption on the conditional distributions of X conditional on Z is different from the assumption imposed by D'Haultfœuille and Février (2015) and Torgovitsky (2015). For any random variables U and W, let $F_{U|W}$ denote the conditional distribution function of U conditional on W. Let $\mathcal{X}_z$ be an interior of the support of $X|Z = z$ and $\mathcal{Y}_{x,z}$ be an interior of the support of $Y|X = x, Z = z$.

Assumption 3. *(i) $F_{X|Z}(x|z)$ is continuous in x for all $z \in \{0, 1\}$ and $F_{X|Z}(x|0) < F_{X|Z}(x|1)$ for all $x \in \mathcal{X}_0 \cap \mathcal{X}_1$. (ii) $\mathcal{X}_0 = (\underline{x}_0, \overline{x}_0)$, $\mathcal{X}_1 = (\underline{x}_1, \overline{x}_1)$ and $-\infty < \underline{x}_1 < \underline{x}_0 < \overline{x}_1 < \overline{x}_0 < \infty$.*

Conditions (i) and (ii) imply that $F_{X|Z}(x|z)$ is strictly increasing and continuous in x on $\mathcal{X}_z$. This assumption implies that $F_{X|Z}(x|0)$ and $F_{X|Z}(x|1)$ do not have any intersections on the support of X and $X|Z = 0$ stochastically dominates $X|Z = 1$. Thus, Z has strictly monotonic effect on X. This case is ruled out by D'Haultfœuille and Février (2015) and Torgovitsky (2015) because they assume $F_{X|Z}(x|0)$ and $F_{X|Z}(x|1)$ have intersections.

Torgovitsky (2015) shows that the point identification of g holds when $\underline{x}_1 = \underline{x}_0$, that is, $F_{X|Z}(x|0)$ and $F_{X|Z}(x|1)$ have an intersection at $x = \underline{x}_1 = \underline{x}_0$.

Next, we impose restrictions to the conditional distributions of Y conditional on X and Z.

Assumption 4. *(i) For all $(z, x, y) \in \{0,1\} \times \mathcal{X}_z \times \mathcal{Y}_{x,z}$, $F_{Y|X,Z}(y|x, z)$ is continuous in x and y. (ii) For all $(z, x) \in \{0,1\} \times \mathcal{X}_z$, we have $\mathcal{Y}_{x,z} = \mathcal{Y} = (\underline{y}, \overline{y})$ where $-\infty \leq \underline{y} < \overline{y} \leq \infty$.*

D'Haultfœuille and Février (2015) and Torgovitsky (2015) also assume the condition (i) but do not assume the condition (ii). The conditions (i) and (ii) imply that $F_{Y|X,Z}(y|x, z)$ is strictly increasing and continuous in y on $\mathcal{Y}$. Hence, the conditional quantile function of Y conditional on X and Z is the inverse of $F_{Y|X,Z}(y|x, z)$. The condition (ii) is not necessary for this paper's result but the result is cumbersome without this assumption. We relax the conditions (i) in Section 5 and allow Y to be discrete or censored.

Finally, we impose the normalization assumption on unobservable variables and the support condition of $\epsilon | X = x, Z = z$.

Assumption 5. *(i) $\epsilon \sim U(0,1)$ and $\eta \sim U(0,1)$. (ii) For all $(z, x) \in \{0,1\} \times \mathcal{X}_z$, the interior of the support of $\epsilon | X = x, Z = z$ is $(0,1)$.*

The condition (i) is the usual normalization in nonseparable model (see, Matzkin (2003)). Torgovitsky (2015) does not use this normalization but D'Haultfœuille and Février (2015) normalize ϵ to be distributed uniformly. D'Haultfœuille and Février (2015) do not assume $\eta \sim U(0,1)$. The theorems in this paper hold without this assumption, but the proof becomes more complicated. The condition (ii) implies that $g(x, e) \in (\underline{y}, \overline{y}) = \mathcal{Y}$ for all $(x, e) \in \mathcal{X} \times (0,1)$.

3 Partial Identification via Monotonicity

Let $\overline{\mathcal{Y}}$ be the closure of $\mathcal{Y}$. We establish the partial identification of g by showing that we can identify functions $T^U_{x',x}(y) : \overline{\mathcal{Y}} \to \overline{\mathcal{Y}}$ and $T^L_{x',x}(y) : \overline{\mathcal{Y}} \to \overline{\mathcal{Y}}$ that (i) are strictly increasing in y, (ii) are surjective, i.e. $T^U_{x',x}\left([\underline{y}, \overline{y}]\right) = T^L_{x',x}\left([\underline{y}, \overline{y}]\right) = [\underline{y}, \overline{y}]$, and (iii) satisfy the following inequalities:

$$g(x', e) \leq T^U_{x',x}\left(g(x, e)\right), \tag{2}$$

$$g(x', e) \geq T^L_{x',x}\left(g(x, e)\right). \tag{3}$$

$T^U_{x',x}(y)$ and $T^L_{x',x}(y)$ correspond to $Q_{x'x}$ in D'Haultfœuille and Février (2015). If $T^U_{x',x}(y)$ is identified for all $x, x' \in \mathcal{X} \equiv \mathcal{X}_0 \cup \mathcal{X}_1$, we can obtain a lower bound of the structural function $g(x, e)$ by the following way. Here, define $G^L_x(u) \equiv \int F_{Y|X=x'}\left(T^U_{x',x}(u)\right) dF_X(x')$. If $T^U_{x',x}(y)$ satisfying (2) is obtained for all $x, x' \in \mathcal{X}$, then we have

$$\begin{aligned}
G^L_x\left(g(x, e)\right) &= \int F_{Y|X=x'}\left(T^U_{x',x}\left(g(x, e)\right)\right) dF_X(x') \\
&\geq \int F_{Y|X=x'}\left(g(x', e)\right) dF_X(x') \\
&= \int P(g(x', \epsilon) \leq g(x', e) | X = x') dF_X(x') \\
&= \int P(\epsilon \leq e | X = x') dF_X(x') = e,
\end{aligned} \tag{4}$$

where the first inequality follows from (2) and the monotonicity of $F_{Y|X}(y|x')$ in y and the third equality follows from the strict monotonicity of $g(x,e)$ in e. Furthermore, $G_x^L(u)$ is invertible because $T_{x',x}^U(y)$ is strictly increasing in y. Because $T_{x',x}^U(y)$ is surjective, we have $G_x^L\left([\underline{y},\overline{y}]\right)=[0,1]$. Hence, for all $e\in(0,1)$, we have

$$g(x,e)\geq\left(G_x^L\right)^{-1}(e). \tag{5}$$

Similarly, define $G_x^U(u)\equiv\int F_{Y|X=x'}\left(T_{x',x}^L(u)\right)dF_X(x')$, then we have

$$g(x,e)\leq\left(G_x^U\right)^{-1}(e).$$

This idea is similar to that of D'Haultfœuille and Février (2015). They use a function $T_{x',x}(y)$ that is strictly increasing in y and satisfies $g(x',e)=T_{x',x}\left(g(x,e)\right)$. Define $G_x(u)\equiv\int F_{Y|X=x'}\left(T_{x',x}(u)\right)dF_X(x')$. Then, similarly to above inequality, we have $G_x\left(g(x,e)\right)=e$. Hence, if $T_{x',x}(y)$ is strictly increasing in y, then $G_x(u)$ is invertible and we have $g(x,e)=\left(G_x\right)^{-1}(e)$.

We propose how to construct the functions $T_{x',x}^U(y)$ and $T_{x',x}^L(y)$ that satisfy (2) and (3). As in Torgovitsky (2015), define $\pi(x):\mathcal{X}_0\to\mathcal{X}_1$ and $\pi^{-1}(x):\mathcal{X}_1\to\mathcal{X}_0$[1] as

$$\begin{aligned}\pi(x) &\equiv Q_{X|Z}\left(F_{X|Z}(x|0)|1\right)\\ \pi^{-1}(x) &\equiv Q_{X|Z}\left(F_{X|Z}(x|1)|0\right).\end{aligned} \tag{6}$$

Here, for any random variables U and W, let $Q_{U|W}(\tau|w)$ denote the conditional τ-th quantile of U conditional on $W=w$, i.e. $Q_{U|W}(\tau|w)\equiv\inf\{u:F_{U|W}(u|w)\geq\tau\}$. The following result is essentially proven by D'Haultfœuille and Février (2015) (Theorem 1). We state this result as a proposition because this result plays a central role in the following and our assumptions are slightly different from cited2015identification.

Proposition 1.

$$\begin{aligned}\tilde{T}_x^{(1)}(y) &\equiv Q_{Y|X,Z}\left(F_{Y|X,Z}\left(y|x,0\right)|\pi(x),1\right),\\ \tilde{T}_x^{(-1)}(y) &\equiv Q_{Y|X,Z}\left(F_{Y|X,Z}\left(y|x,1\right)|\pi^{-1}(x),0\right).\end{aligned}$$

Then, under Assumptions 1-5, we have

$$\begin{aligned}g\left(\pi(x),e\right) &= \tilde{T}_x^{(1)}\left(g(x,e)\right),\\ g\left(\pi^{-1}(x),e\right) &= \tilde{T}_x^{(-1)}\left(g(x,e)\right).\end{aligned}$$

Proof. Step.1 We show that, for all $e\in(0,1)$,

$$P(\epsilon\leq e|X=x,Z=0)=P(\epsilon\leq e|X=\pi(x),Z=1). \tag{7}$$

First, we examine the variable $V\equiv F_{X|Z}(X|Z)$. This is called "control variable" in Imbens and Newey (2009). Let $h^{-1}(z,x)$ be the inverse function of $h(z,v)$ with respect to v. We have, for all $(z,x)\in\{0,1\}\times\mathcal{X}_z$,

$$\begin{aligned}F_{X|Z}(x|z) &= P\left(h(z,\eta)\leq x|Z=z\right)\\ &= P\left(\eta\leq h^{-1}(z,x)|Z=z\right)\\ &= P\left(\eta\leq h^{-1}(z,x)\right)=h^{-1}(z,x),\end{aligned}$$

[1]Those functions correspond to s_{ij} in D'Haultfœuille and Février (2015).

where the second equality follows from the strict monotonicity of $h(x,v)$ in v and the third equality follows from $Z \perp\!\!\!\perp (\epsilon,\eta)$. Therefore, we obtain

$$V = h^{-1}(Z,X) = \eta.$$

Next, we show that the conditional distribution of ϵ conditional on $(X,Z)=(x,z)$ is the same as the conditional distribution of ϵ conditional on $V=F_{X|Z}(x|z)$. Because $(x,z)\to(F_{X|Z}(x|z),z)$ is one-to-one, $F_{X|Z}(x|z)$ is continuous in x, $Z \perp\!\!\!\perp (\epsilon,\eta)$, and $V=\eta$, we have

$$\begin{aligned} P\left(\epsilon \le e|X=x,Z=z\right) &= P\left(\epsilon \le e|V=F_{X|Z}(x|z),Z=z\right) \\ &= P\left(\epsilon \le e|V=F_{X|Z}(x|z)\right). \end{aligned} \tag{8}$$

Hence, the conditional distribution of ϵ conditional on X and Z depends on only $V = F_{X|Z}(X|Z)$.

By definition, the functions $\pi(x)$ and $\pi^{-1}(x)$ satisfy that

$$\begin{aligned} F_{X|Z}(\pi(x)|1) &= F_{X|Z}(x|0) \quad \text{and} \\ F_{X|Z}(\pi^{-1}(x)|0) &= F_{X|Z}(x|1). \end{aligned} \tag{9}$$

Hence, the events $\{X=x,Z=0\}$ and $\{X=\pi(x),Z=1\}$ have the same $V=F_{X|Z}(X|Z)$, and (7) follows from (8).

<u>Step.2</u> We show that (7) implies $g\left(\pi(x),e\right)=\tilde{T}_x^{(1)}\left(g(x,e)\right)$. For all $(x,e)\in\mathcal{X}_0\times(0,1)$, we have

$$\begin{aligned} \tilde{T}_x^{(1)}\left(g(x,e)\right) &= Q_{Y|X=\pi(x),Z=1}\left(F_{Y|X=x,Z=0}\left(g(x,e)\right)\right) \\ &= Q_{Y|X=\pi(x),Z=1}\left(P(\epsilon\le e|X=x,Z=0)\right) \\ &= Q_{Y|X=\pi(x),Z=1}\left(P(\epsilon\le e|X=\pi(x),Z=1)\right) \\ &= Q_{Y|X=\pi(x),Z=1}\left(F_{Y|X=\pi(x),Z=1}\left(g(\pi(x),e)\right)\right) \\ &= g(\pi(x),e), \end{aligned}$$

where the third equality follows from (7).

Similarly, we can prove that $g\left(\pi^{-1}(x),e\right)=\tilde{T}_x^{(-1)}\left(g(x,e)\right)$. □

By definition, $\tilde{T}_x^{(1)}(y)$ and $\tilde{T}_x^{(-1)}(y)$ are strictly increasing, $\tilde{T}_x^{(1)}([\underline{y},\overline{y}])=[\underline{y},\overline{y}]$, and $\tilde{T}_x^{(-1)}([\underline{y},\overline{y}])=[\underline{y},\overline{y}]$. Define $\pi^n(x)\equiv\pi\circ\cdots\circ\pi(x)$ for $n\in\mathbb{N}$ if $\pi^n(x)$ exists. Then, we obtain $g(\pi^n(x),e)=\tilde{T}_{\pi^{n-1}(x)}^{(1)}\circ\cdots\circ\tilde{T}_x^{(1)}\left(g(x,e)\right)$. Define $\tilde{T}_x^{(n)}(y)\equiv\tilde{T}_{\pi^{n-1}(x)}^{(1)}\circ\cdots\circ\tilde{T}_x^{(1)}(y)$ for $n\in\mathbb{N}$, then we have

$$g(\pi^n(x),e)=\tilde{T}_x^{(n)}\left(g(x,e)\right).$$

In addition, $\tilde{T}_x^{(n)}(y)$ is strictly increasing in y and $\tilde{T}_x^{(n)}([\underline{y},\overline{y}])=[\underline{y},\overline{y}]$. Similarly, define $\pi^{-n}(x)\equiv\pi^{-1}\circ\cdots\circ\pi^{-1}(x)$ and $\tilde{T}_x^{(-n)}(y)\equiv\tilde{T}_{\pi^{-(n-1)}(x)}^{(-1)}\circ\cdots\circ\tilde{T}_x^{(-1)}(y)$ for $n\in\mathbb{N}$, then we have

$$g(\pi^{-n}(x),e)=\tilde{T}_x^{(-n)}\left(g(x,e)\right).$$

$\tilde{T}_x^{(-n)}(y)$ is strictly increasing in y, and $\tilde{T}_x^{(-n)}([\underline{y},\overline{y}])=[\underline{y},\overline{y}]$. For all $x\in\mathcal{X}$, define $\tilde{T}_x^{(0)}(y)\equiv y$.

Here, we examine the properties of $\pi(x)$ and $\pi^{-1}(x)$. Because $F_{X|Z}(x|0) < F_{X|Z}(x|1)$ on $\mathcal{X}$, we have

$$\begin{aligned} \pi(x) = & \quad Q_{X|Z}\left(F_{X|Z}(x|0)|1\right) \\ & < Q_{X|Z}\left(F_{X|Z}(x|1)|1\right) = x, \\ \pi^{-1}(x) = & \quad Q_{X|Z}\left(F_{X|Z}(x|1)|0\right) \\ & > Q_{X|Z}\left(F_{X|Z}(x|0)|0\right) = x. \end{aligned} \tag{10}$$

Figure 1 illustrates this intuitively. Because $X|Z = 0$ stochastically dominates $X|Z = 1$ and the functions $\pi(x)$ and $\pi^{-1}(x)$ satisfy (9), these inequalities hold.

In order to facilitate the illustration of our identification result, we first review the identification approach of D'Haultfœuille and Février (2015) and Torgovitsky (2015) when $\underline{x}_0 = \underline{x}_1 = \xi$, even though Assumption 3 rules out the case $\underline{x}_0 = \underline{x}_1 = \xi$. Pick an initial point $x_0 \in \mathcal{X}$ (i.e. $x_0 > \xi$) and form a recursive sequence $x_{n+1} = \pi(x_n)$ for $n > 0$. Because $\underline{x}_0 = \underline{x}_1 = \xi$ implies $\mathcal{X}_1 \subset \mathcal{X}_0$, we have $\pi(x) \in \mathcal{X}_0$ for all $x \in \mathcal{X}$ and there exists a sequence $\{\pi^n(x)\}_{n=1}^{\infty}$. The sequence $\{x_n\}$ is decreasing by (10) and $x_n > \xi$ for all $n \geq 0$ by the definition of $\pi(x)$. Hence, the sequence $\{x_n\}$ converges to a limiting point. Because (9) implies

$$F_{X|Z}(x_{n+1}|1) = F_{X|Z}(x_n|0)$$

and $F_{X|Z}(x|z)$ is continuous in x, we have $F_{X|Z}(\lim_{n\to\infty} x_n|1) = F_{X|Z}(\lim_{n\to\infty} x_n|0)$. Because $F_{X|Z}(x|0) < F_{X|Z}(x|1)$ on $(\xi, \overline{x}_0)$ and $F_{X|Z}(\xi|0) = F_{X|Z}(\xi|1) = 0$, the sequence $\{x_n\}$ converges to ξ for any initial point $x_0 \in \mathcal{X}$. Figure 2 illustrates this intuitively. Define $\tilde{T}_x^{(\infty)}(y) \equiv \lim_{n\to\infty} \tilde{T}_x^{(n)}(y)$, which is strictly increasing and invertible in y. By the continuity of g, we obtain, for all $x \in \mathcal{X}$,

$$\tilde{T}_x^{(\infty)}(g(x,e)) = \lim_{n\to\infty} g(\pi^n(x), e) = g(\xi, e).$$

A similar argument gives $\tilde{T}_{x'}^{(\infty)}(g(x',e)) = g(\xi,e)$. Therefore, $\tilde{T}_x^{(\infty)}(g(x,e)) = \tilde{T}_{x'}^{(\infty)}(g(x',e))$ holds for any x, x', and

$$g(x',e) = \left(\tilde{T}_{x'}^{(\infty)}\right)^{-1}\left(\tilde{T}_x^{(\infty)}\left(g(x,e)\right)\right).$$

Define $T_{x',x}(y) \equiv \left(\tilde{T}_{x'}^{(\infty)}\right)^{-1}\left(\tilde{T}_x^{(\infty)}(y)\right)$, then $T_{x',x}(y)$ is strictly increasing and satisfies $g(x',e) = T_{x',x}\left(g(x,e)\right)$. Hence, as discussed above, g is point identified.

Under our Assumption 3, this approach is not available because a convergent sequence $\{\pi^n(x)\}_{n=1}^{\infty}$ does not exist. When $F_{X|Z}(x|0)$ and $F_{X|Z}(x|1)$ have no intersections, $\pi^n(x)$ lies in $\mathcal{X}_1 \cap \mathcal{X}_0^c = (\underline{x}_1, \underline{x}_0]$ when n is sufficiently large. If $\pi^n(x)$ is in $\mathcal{X}_1 \cap \mathcal{X}_0^c$ then $\pi^{n+1}(x)$ does not exist. As seen in Lemma 1, for all $x \in \mathcal{X}$, the set $\{n : \pi^n(x) \text{ exists.}\}$ is a finite set under Assumption 3. For example, in Figure 1, $\pi(x)$, $\pi^{-1}(x)$, and $\pi^{-2}(x)$ exist but $\pi^2(x)$ and $\pi^{-3}(x)$ do not exist.

If we do not impose any additional restrictions, then we cannot construct the function $T_{x',x}^U(y)$ and $T_{x',x}^L(y)$ that satisfy (2) and (3) respectively. First, we show that a set $\Pi_{x',x}^M$ defined below is nonempty and finite even when $F_{X|Z}(x|0)$ and $F_{X|Z}(x|1)$ have no intersections. Next, we show that we can partially identify $g(x,e)$ by using the set $\Pi_{x',x}^M$ when $g(x,e)$ is nondecreasing in x.

For $(x,x') \in \mathcal{X} \times \mathcal{X}$, define the set $\Pi^M_{x',x}$ as

$$\begin{aligned}\Pi^M_{x',x} \quad \equiv \quad & \{(n,m) : \pi^n(x') \leq \pi^m(x), \\ & n,m \in \mathbb{Z},\ \pi^n(x') \text{ and } \pi^m(x) \text{ exist.}\}, \end{aligned} \tag{11}$$

where $\pi^0(x) \equiv x$. In Figure 1, $\Pi^M_{x',x} = \{(-1,-2),(0,-2),(0,-1),(1,-2),(1,-1),(1,0)\}$. The following lemma shows that the set $\Pi^M_{x',x}$ is nonempty and finite even when $F_{X|Z}(x|0)$ and $F_{X|Z}(x|1)$ have no intersections.

Lemma 1. *Under Assumptions 1-5, the set $\Pi^M_{x',x}$ defined by (11) is nonempty and finite for all $(x,x') \in \mathcal{X} \times \mathcal{X}$.*

Proof. Observe that, if $\pi^n(x)$ exists and $\pi^n(x) \in \mathcal{X}_0$, then $\pi^{n+1}(x)$ also exists from (6). Suppose that there does not exist $n \in \mathbb{N} \cup \{0\}$ such that $\pi^n(x) \in \mathcal{X}_1 \cap \mathcal{X}_0^c = (\underline{x}_1, \underline{x}_0]$. Then, there exists a sequence $\{x_n\}_{n=0}^\infty$ such that $x_n = \pi^n(x)$. By (10), $\{x_n\}_{n=0}^\infty$ is a decreasing sequence. Because $x_n > \underline{x}_0$, $\{x_n\}_{n=0}^\infty$ converges to $x_\infty \in [\underline{x}_0, \overline{x}^0)$. Because

$$F_{X|Z}(x_{n+1}|1) = F_{X|Z}(x_n|0)$$

holds by (9), $F_{X|Z}(x_\infty|1) = F_{X|Z}(x_\infty|0)$ by the continuity of $F_{X|Z}$. However, this equation violates the Assumption 3. Hence, for all $x \in \mathcal{X}$, there exists $n \in \mathbb{N} \cup \{0\}$ such that $\pi^n(x) \in \mathcal{X}_1 \cap \mathcal{X}_0^c$. Consequently, $\pi^{n'}(x)$ does not exist for $n' > n$. Similarly, for all $x \in \mathcal{X}$, there exists $n \in \mathbb{N} \cup \{0\}$ such that $\pi^{-n}(x) \in \mathcal{X}_0 \cap \mathcal{X}_1^c$. Then, $\pi^{-n'}(x)$ does not exist for $n' > n$. Therefore, $\Pi^M_{x',x}$ is finite for all $(x,x') \in \mathcal{X} \times \mathcal{X}$ because the set $\{(n,m) \in \mathbb{Z} \times \mathbb{Z} : \pi^n(x') \text{ and } \pi^m(x) \text{ exist.}\}$ is finite.

We proceed to show the nonemptyness of $\Pi^M_{x',x}$. For all $x, x' \in \mathcal{X}$, there exist $(n,m) \in \mathbb{Z} \times \mathbb{Z}$ such that $\pi^n(x') \in \mathcal{X}_1 \cap \mathcal{X}_0^c = (\underline{x}_1, \underline{x}_0]$ and $\pi^m(x) \in \mathcal{X}_0 \cap \mathcal{X}_1^c = [\overline{x}_1, \overline{x}_0)$. It follows from Assumption 3 (ii) that $\pi^n(x') < \pi^m(x)$. □

Under Assumptions 1-5, the set $\{n \in \mathbb{Z} : \pi^n(x) \text{ exists.}\}$ is finite from Lemma 1. Hence, g can not be point identified by using the method proposed by D'Haultfœuille and Février (2015) and Torgovitsky (2015)).

We impose the following assumption:

Assumption M (Monotonicity)**.** *For all $e \in (0,1)$, $g(x,e)$ is nondecreasing in x.*

Monotonicity assumption holds in many economic models. For example, Engel function of a normal good is monotonically increasing in income, the demand function is ordinarily monotonically decreasing in price, and economic analyses of production often suppose that the production function is monotonically increasing in input. Monotonicity assumptions of this type is employed in many papers. Manski (1997) imposes a monotonicity assumption on a response function and shows that the average treatment response is partially identified.

If $(n,m) \in \Pi^M_{x',x}$, then Assumption M implies that

$$\tilde{T}^{(n)}_{x'}\left(g(x',e)\right) = g(\pi^n(x),e) \leq g(\pi^m(x),e) = \tilde{T}^{(m)}_{x}\left(g(x,e)\right).$$

Because $\tilde{T}^{(n)}_{x'}(y)$ is strictly increasing in y and $\tilde{T}^{(n)}_{x'}\left([\underline{y},\overline{y}]\right) = [\underline{y},\overline{y}]$, we have $g(x',e) \leq \left(\tilde{T}^{(n)}_{x'}\right)^{-1}\left(\tilde{T}^{(m)}_{x}\left(g(x,e)\right)\right)$. Hence, we have

$$g(x',e) \leq \min_{(n,m)\in\Pi^M_{x',x}} \left(\tilde{T}^{(n)}_{x'}\right)^{-1}\left(\tilde{T}^{(m)}_{x}\left(g(x,e)\right)\right).$$

Define

$$\begin{aligned} T^{MU}_{x',x}(y) &\equiv \min_{(n,m)\in\Pi^M_{x',x}} \left(\tilde{T}^{(n)}_{x'}\right)^{-1}\left(\tilde{T}^{(m)}_{x}(y)\right), \\ T^{ML}_{x',x}(y) &\equiv \max_{(n,m)\in\Pi^M_{x,x'}} \left(\tilde{T}^{(m)}_{x'}\right)^{-1}\left(\tilde{T}^{(n)}_{x}(y)\right). \end{aligned} \tag{12}$$

Then, $T^{MU}_{x',x}(y)$ is strictly increasing and satisfies

$$g(x',e) \le T^{MU}_{x',x}(g(x,e)). \tag{13}$$

$T^{MU}_{x',x}(y)$ achieves the tightest lower bound of g because the function $\left(\tilde{T}^{(n)}_{x'}\right)^{-1}\left(\tilde{T}^{(m)}_{x}(y)\right)$ satisfies (2) for any $(n,m)\in\Pi^M_{x',x}$. Similarly, $T^{ML}_{x',x}(y)$ is strictly increasing and satisfies

$$g(x',e) \ge T^{MU}_{x',x}(g(x,e)). \tag{14}$$

Define

$$\begin{aligned} G^{ML}_x(u) &\equiv \int F_{Y|X=x'}\left(T^{MU}_{x',x}(u)\right) dF_X(x'), \\ G^{MU}_x(u) &\equiv \int F_{Y|X=x'}\left(T^{ML}_{x',x}(u)\right) dF_X(x'), \\ B^{ML}(x,e) &\equiv \sup_{y:y\le x}\left\{\left(G^{ML}_y\right)^{-1}(e)\right\}, \\ B^{MU}(x,e) &\equiv \inf_{y:y\ge x}\left\{\left(G^{MU}_y\right)^{-1}(e)\right\}. \end{aligned}$$

$G^{ML}_x(u)$ and $G^{MU}_x(u)$ provide lower and upper bounds on $g(x,e)$ via the argument (4) and (5). $B^{ML}(x,e)$ and $B^{MU}(x,e)$ strengthen these bounds.

Theorem 1. *Under Assumptions 1-5 and M, for all $(x,e)\in\mathcal{X}\times(0,1)$, we have*

$$B^{ML}(x,e) \le g(x,e) \le B^{MU}(x,e).$$

Proof. Step.1 First, we show that

$$\left(G^{ML}_x\right)^{-1}(e) \le g(x,e) \le \left(G^{MU}_x\right)^{-1}(e). \tag{15}$$

The function $T^{ML}_{x',x}(y)$ defined in (12) is strictly increasing in y because $\tilde{T}^{(n)}_x(y)$ is strictly increasing in y for all $n\in\mathbb{Z}$ and $x\in\mathcal{X}$, and $\Pi^M_{x',x}$ is finite by Lemma 1. Similarly, $T^{MU}_{x',x}(y)$ is strictly increasing in y. Because $\tilde{T}^{(n)}_x([\underline{y},\overline{y}])=[\underline{y},\overline{y}]$ for all $n\in\mathbb{Z}$ and $x\in\mathcal{X}$, we have $\left(\tilde{T}^{(m)}_{x'}\right)^{-1}\left(\tilde{T}^{(n)}_x([\underline{y},\overline{y}])\right)=[\underline{y},\overline{y}]$. Because $\left(\tilde{T}^{(m)}_{x'}\right)^{-1}\left(\tilde{T}^{(n)}_x([\underline{y},\overline{y}])\right)=[\underline{y},\overline{y}]$ and $\Pi^M_{x',x}$ is finite, it follows that $G^{ML}_x\left([\underline{y},\overline{y}]\right)=[0,1]$. Similarly, $G^{MU}_x\left([\underline{y},\overline{y}]\right)=[0,1]$. Therefore, the inequalities (15) hold from (4) and (5).
Step.2 Next, we show that

$$B^{ML}(x,e) \le g(x,e) \le B^{MU}(x,e). \tag{16}$$

Because $g(x,e)$ is nondecreasing in x and $g(x,e) \geq \left(G_x^{ML}\right)^{-1}(e)$, we have $g(x,e) \geq g(y,e) \geq \left(G_y^{ML}\right)^{-1}(e)$ for all $y \leq x$. Hence, it follows that

$$g(x,e) \geq \sup_{y: y \leq x} \left\{ \left(G_y^{ML}\right)^{-1}(e) \right\}.$$

Similarly, $g(x,e) \leq g(y,e) \leq \left(G_y^{MU}\right)^{-1}(e)$ for all $y \geq x$ and it follows that

$$g(x,e) \leq \inf_{y: y \geq x} \left\{ \left(G_y^{MU}\right)^{-1}(e) \right\}.$$

Therefore, the inequalities (16) hold. □

In the first step, we show $\left(G_x^{ML}\right)^{-1}(e) \leq g(x,e) \leq \left(G_x^{MU}\right)^{-1}(e)$. In the second step, we strengthen these bounds to $B^{ML}(x,e) \leq g(x,e) \leq B^{MU}(x,e)$. Figure 3 illustrates this proof intuitively. This idea is similar to Manski (1997). He consider the case where the response function $y(t)$ is increasing, where $y(t)$ is a latent outcome with treatment t. He use the monotonicity of $y(t)$ to partially identify the average response function $E[y(t)]$ when the support of outcome is bounded. Our approach does not require the information of the infimum and supremum of the support of outcomes.

To illustrate Theorem 1, we consider the following example:

$$\begin{aligned} Y &= X^{1/3} exp\left(\alpha + \beta \Phi^{-1}(\epsilon)\right) \\ X &= d + c(1 - Z) + \eta, \end{aligned}$$

where $\Phi(\cdot)$ is the standard normal distribution function and Z is a random Bernoulli variable with $p = 0.5$. Suppose that

$$\begin{aligned} \epsilon &= \Phi(U) \\ \eta &= \Phi(V) \\ (U,V) &\sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right). \end{aligned}$$

Thus, $\epsilon \sim U(0,1)$ and $\eta \sim U(0,1)$. In this example, $F_{X|Z}(x|1) = x - d$ for $x \in \mathcal{X}_1 = (d, 1+d)$ and $F_{X|Z}(x|0) = x - c - d$ for $x \in \mathcal{X}_0 = (c+d, 1+c+d)$. Hence, the conditional distribution functions $F_{X|Z}(x|0)$ and $F_{X|Z}(x|1)$ do not have any intersections. This example satisfies the assumptions of Theorem 1. We calculate the bounds of $g(x, 0.5)$ by using Theorem 1 when $\alpha = 0.5, \beta = 0.5, c = 0.5, d = 0.05$ and $\rho = 0.3$. These bounds are depicted in Figure 4.

These bounds become tighter as the difference between $g(x', e)$ and $T^U_{x',x}(g(x,e))$ (or $T^L_{x',x}(g(x,e))$) is smaller. The following theorem shows that, if $g(x,e)$ is flat in x in a particular interval, the inequalities (2) and (3) become equalities and the structural function g is point identified.

Theorem 2. *Under Assumptions 1-5 and M, if there exists $\tilde{x} \in \mathcal{X}$ such that $g(x,e)$ is constant in x on $[\pi(\tilde{x}), \pi^{-1}(\tilde{x})]$, then $g(x,e)$ is point identified for all $(x,e) \in \mathcal{X} \times (0,1)$. This result holds even when an interval $[\pi(\tilde{x}), \pi^{-1}(\tilde{x})]$ is unknown.*

Proof. Step.1 First, we show that for all $x \in \mathcal{X}$, there exists $n \in \mathbb{Z}$ such that $\pi^n(x), \pi^{n+1}(x) \in [\pi(\tilde{x}), \pi^{-1}(\tilde{x})]$. Because $\pi^n(x)$ is strictly increasing in x, for all $n \in \mathbb{Z}$, we have

$$x \leq y \quad \Rightarrow \quad \pi^n(x) \leq \pi^n(y). \tag{17}$$

We consider the following four cases: (i) $\pi(\tilde{x}) \leq x \leq \tilde{x}$, (ii) $\tilde{x} \leq x \leq \pi^{-1}(\tilde{x})$, (iii) $x < \pi(\tilde{x})$, and (iv) $x > \pi^{-1}(\tilde{x})$. In the case (i), it follows from (17) that $\pi(\tilde{x}) \leq x \leq \tilde{x} \leq \pi^{-1}(x) \leq \pi^{-1}(\tilde{x})$. In the case (ii), it follows from (17) that $\pi(\tilde{x}) \leq \pi(x) \leq \tilde{x} \leq x \leq \pi^{-1}(\tilde{x})$. In the case (iii), by the definition of π, we have $\pi(\tilde{x}) \in \mathcal{X}_1$. It follows from the proof of Lemma 1 that there exists $n' \in \mathbb{Z}$ such that $\pi(\tilde{x}) \leq \pi^{-n'}(x) \in \mathcal{X}_0 \cap \mathcal{X}_1^c$. Hence, there exists $n \in \mathbb{Z}$ such that $\pi^{n+2}(x) \leq \pi(\tilde{x}) \leq \pi^{n+1}(x)$, and we can obtain $\pi(\tilde{x}) \leq \pi^{n+1}(x) \leq \tilde{x} \leq \pi^n(x) \leq \pi^{-1}(\tilde{x})$ from (17). In the case (iv), similarly to the case (iii), there exist $n \in \mathbb{Z}$ such that $\pi^n(x), \pi^{n+1}(x) \in [\pi(\tilde{x}), \pi^{-1}(\tilde{x})]$.
Step.2 Next, we show that g is point identified. From Step 1, for all $x, x' \in \mathcal{X}$, there exists $n, m \in \mathbb{Z}$ such that $\pi^n(x'), \pi^{n+1}(x'), \pi^m(x), \pi^{m+1}(x) \in [\pi(\tilde{x}), \pi^{-1}(\tilde{x})]$. Then, from (17), we have either $\pi^{n+1}(x') \leq \pi^{m+1}(x) \leq \pi^n(x') \leq \pi^m(x)$ or $\pi^{m+1}(x) \leq \pi^{n+1}(x') \leq \pi^m(x) \leq \pi^n(x')$. If $\pi^{n+1}(x') \leq \pi^{m+1}(x) \leq \pi^n(x') \leq \pi^m(x)$, then we have $(n+1, m+1), (n, m) \in \Pi^M_{x',x}$ and $(m+1, n) \in \Pi^M_{x,x'}$. If $\pi^{m+1}(x) \leq \pi^{n+1}(x') \leq \pi^m(x) \leq \pi^n(x')$, then we have $(n+1, m) \in \Pi^M_{x',x}$ and $(m+1, n+1), (m, n) \in \Pi^M_{x,x'}$. Because $g(x,e)$ is constant in x on $[\pi(\tilde{x}), \pi^{-1}(\tilde{x})]$, $\pi^n(x'), \pi^m(x) \in [\pi(\tilde{x}), \pi^{-1}(\tilde{x})]$ implies that

$$\tilde{T}^{(n)}_{x'}\left(g(x', e)\right) = \tilde{T}^{(m)}_{x}\left(g(x, e)\right).$$

Therefore, $g(x', e) = T^{MU}_{x',x}\left(g(x,e)\right)$ and $g(x', e) = T^{ML}_{x',x}\left(g(x,e)\right)$. Hence, $g(x,e)$ is identified because the inequalities (13) and (14) become equalities. □

In the first step, we show that for all $x \in \mathcal{X}$, there exists $n \in \mathbb{Z}$ such that $\pi^n(x), \pi^{n+1}(x) \in [\pi(\tilde{x}), \pi^{-1}(\tilde{x})]$. In the second step, we show that g is point identified. Because $g(x,e)$ is constant in x on $[\pi(\tilde{x}), \pi^{-1}(\tilde{x})]$, we have $g(x', e) = T^{MU}_{x',x}\left(g(x,e)\right)$ and $g(x', e) = T^{ML}_{x',x}\left(g(x,e)\right)$ for all $x, x' \in \mathcal{X}$ and $e \in (0,1)$. Hence, $g(x,e)$ is point identified because the inequalities (13) and (14) become equalities.

Example 1 (Engel curve). *Consider the relationship between household's expenditure on food and household's income. Let Y denote household's expenditure on food and X denote a household's income. Then, ϵ represents household's preference. In this case, the Engel function $g(X, \epsilon)$ is usually nondecreasing in X. Each household has the basic consumption $\underline{g}(\epsilon)$ which is invariant with respect to income. If a household has a sufficiently low income, then $Y = g(X, \epsilon) = \underline{g}(\epsilon)$. Hence, in this case, the structural function $g(x, e)$ is constant in x when x is small. If the interval in which $g(x, e) = \underline{g}(e)$ is sufficiently long, then Theorem 2 can be applied.*

4 Partial Identification via Concavity

In this section, we propose how to construct lower and upper bounds of $g(x,e)$ when $g(x,e)$ is concave in x.

First, we show that a set $\Pi^C_{x',x}$ defined below is nonempty and finite. Next, we show that we can partially identify g by using the set $\Pi^C_{x',x}$, when $g(x,e)$ is concave in x.

For $(x,x') \in \mathcal{X} \times \mathcal{X}$, define the set $\Pi^C_{x',x}$ as

$$\begin{aligned}\Pi^C_{x',x} \equiv & \left\{(n,m) : \pi^n(x') \le \pi^m(x) \le \pi^{n-1}(x'),\right. \\ & \left. n,m \in \mathbb{Z},\ \pi^n(x'),\ \pi^{n-1}(x) \text{ and } \pi^m(x) \text{ exist.}\right\}. \end{aligned} \tag{18}$$

In Figure 1, $\Pi^C_{x',x} = \{(0,-1),(1,0)\}$. The following lemma shows that the set $\Pi^C_{x',x}$ is nonempty and finite similarly to Lemma 1.

Lemma 2. *Under Assumptions 1-5, the set $\Pi^C_{x',x}$ defined by (18) is nonempty and finite for all $(x,x') \in \mathcal{X} \times \mathcal{X}$.*

Proof. From the proof of Lemma 1, the set $\Pi^C_{x',x}$ is finite. Hence, we show the nonemptyness of $\Pi^C_{x',x}$. From the proof of Lemma 1, for all $x \in \mathcal{X}$, there exist $n,m \in \mathbb{Z}$ such that $\pi^m(x), \pi^n(x') \in \mathcal{X}_1 \cap \mathcal{X}_0^c$. Without loss of generality, we assume $\pi^n(x') \le \pi^m(x)$. Then, $\pi^{m-1}(x)$ and $\pi^{n-1}(x')$ exist because $\pi^m(x) \in \mathcal{X}_1$ and $\pi^n(x') \in \mathcal{X}_1$. It follows from (17) that $\pi^n(x') \le \pi^m(x) \le \pi^{n-1}(x') \le \pi^{m-1}(x)$ and $(n,m) \in \Pi^C_{x',x}$. Therefore, the set $\Pi^C_{x',x}$ is nonempty. □

Similarly to Section 3, we impose the following assumption:

Assumption C (Concavity)**.** *For all $e \in (0,1)$, $g(x,e)$ is concave in x.*

Concavity assumption holds in many economic models. For example, economic analyses of production often suppose that the production function is concave in inputs. Manski (1997) assumes a concavity assumption and shows that the average treatment response is partially identified. D'Haultfoeuille et al. (2013) achieves the partial identification of the average treatment on treated effect by using a locally concavity assumption.

As seen in Section 3, if we identify the functions $T^U_{x',x}(y)$ and $T^L_{x',x}(y)$ that are strictly increasing in y and surjective, and satisfy (2) and (3), we can obtain lower and upper bounds of $g(x,e)$. Hence, we consider how to construct functions $T^U_{x',x}(y)$ and $T^L_{x',x}(y)$ that are strictly increasing in y and surjective, and satisfy (2) and (3).

If $(n,m) \in \Pi^C_{x',x}$, by Assumption C, we have

$$\left[t_{x',x}(n,m)\tilde{T}^{(n)}_{x'} + (1 - t_{x',x}(n,m))\tilde{T}^{(n-1)}_{x'}\right](g(x',e)) \le \tilde{T}^{(m)}_x(g(x,e)),$$

where $\left[t_{x',x}(n,m)\tilde{T}^{(n)}_{x'} + (1 - t_{x',x}(n,m))\tilde{T}^{(n-1)}_{x'}\right](y) = t_{x',x}(n,m)\tilde{T}^{(n)}_{x'}(y) + (1 - t_{x',x}(n,m))$ $\tilde{T}^{(n-1)}_{x'}(y)$ and $t_{x',x}(n,m) = (\pi^{n-1}(x') - \pi^m(x)) / (\pi^{n-1}(x') - \pi^n(x'))$. Because $\tilde{T}^{(n)}_{x'}(y)$ and $\tilde{T}^{(n-1)}_{x'}(y)$ are strictly increasing in y, $\tilde{T}^{(n)}_{x'}([\underline{y},\overline{y}]) = [\underline{y},\overline{y}]$, and $\tilde{T}^{(n-1)}_{x'}([\underline{y},\overline{y}]) = [\underline{y},\overline{y}]$, we have

$$g(x',e) \le \min_{(n,m)\in\Pi^C_{x',x}} \left[t_{x',x}(n,m)\tilde{T}^{(n)}_{x'} + (1 - t_{x',x}(n,m))\tilde{T}^{(n-1)}_{x'}\right]^{-1}\left(\tilde{T}^{(m)}_x(g(x,e))\right).$$

Define

$$\begin{aligned} T^{CU}_{x',x}(y) &\equiv \min_{(n,m)\in\Pi^C_{x',x}} \left[t_{x',x}(n,m)\tilde{T}^{(n)}_{x'} + (1 - t_{x',x}(n,m))\tilde{T}^{(n-1)}_{x'}\right]^{-1}\left(\tilde{T}^{(m)}_x(y)\right), \\ T^{CL}_{x',x}(y) &\equiv \max_{(n,m)\in\Pi^C_{x,x'}} \left(\tilde{T}^{(m)}_{x'}\right)^{-1}\left(\left[t_{x,x'}(n,m)\tilde{T}^{(n)}_x + (1 - t_{x,x'}(n,m))\tilde{T}^{(n-1)}_x\right](y)\right). \end{aligned} \tag{19}$$

Then, $T^{CU}_{x',x}(y)$ defined in (19) is strictly increasing and satisfies

$$g(x',e) \leq T^{CU}_{x',x}\left(g(x,e)\right). \tag{20}$$

Similarly, $T^{CL}_{x',x}(y)$ defined in (19) is strictly increasing and satisfies

$$g(x',e) \geq T^{CL}_{x',x}\left(g(x,e)\right). \tag{21}$$

Define

$$\begin{aligned} G^{CL}_x(u) &\equiv \int F_{Y|X=x'}\left(T^{CU}_{x',x}(u)\right)dF_X(x'), \\ G^{CU}_x(u) &\equiv \int F_{Y|X=x'}\left(T^{CL}_{x',x}(u)\right)dF_X(x'), \end{aligned}$$

$$B^{CL}(x,e) \equiv \sup_{y,y':y<x<y'}\left\{\left(\frac{x-y}{y'-y}\right)\left(G^{CL}_{y'}\right)^{-1}(e) + \left(\frac{y'-x}{y'-y}\right)\left(G^{CL}_{y}\right)^{-1}(e)\right\},$$

$$\begin{aligned} B^{CU}(x,e) \equiv \ \min\Big[&\inf_{y,y':x<y<y'}\left\{\left(\tfrac{x-y}{y'-y}\right)B^{CL}(y',e) + \left(\tfrac{y'-x}{y'-y}\right)\left(G^{CU}_{y}\right)^{-1}(e)\right\}, \\ &\inf_{y,y':y'<y<x}\left\{\left(\tfrac{y-x}{y-y'}\right)B^{CL}(y',e) + \left(\tfrac{x-y'}{y-y'}\right)\left(G^{CU}_{y}\right)^{-1}(e)\right\}\Big]. \end{aligned}$$

$G^{CL}_x(u)$ and $G^{CU}_x(u)$ provide lower and upper bounds on $g(x,e)$ via the argument (4) and (5). $B^{CL}(x,e)$ and $B^{CU}(x,e)$ strengthen these bounds.

Theorem 3. *Under Assumptions 1-5 and C, for all $(x,e) \in \mathcal{X} \times (0,1)$, we have*

$$B^{CL}(x,e) \leq g(x,e) \leq B^{CU}(x,e).$$

Proof. Similarly to the proof of Theorem 1, we can obtain

$$\left(G^{CL}_x\right)^{-1}(e) \leq g(x,e) \leq \left(G^{CU}_x\right)^{-1}(e).$$

Because $g(x,e)$ is concave in x, if $x = ty' + (1-t)y$ and $t \in (0,1)$, then we have $g(x,e) \geq tg(y',e) + (1-t)g(y,e) \geq t\left(G^{CL}_{y'}\right)^{-1}(e) + (1-t)\left(G^{CL}_{y}\right)^{-1}(e)$. Hence, we have

$$g(x,e) \geq \sup_{y,y':y<x<y'}\left\{\left(\frac{x-y}{y'-y}\right)\left(G^{CL}_{y'}\right)^{-1}(e) + \left(\frac{y'-x}{y'-y}\right)\left(G^{CL}_{y}\right)^{-1}(e)\right\}.$$

Because $g(x,e)$ is concave in x, if $x = ty' + (1-t)y$ and $t < 0$, then we have $g(x,e) \leq tg(y',e) + (1-t)g(y,e)$. Because $B^{CL}(x,e) \leq g(x,e) \leq \left(G^{CU}_x\right)^{-1}(e)$, $t < 0$, and $1-t > 0$, we have $g(x,e) \leq tB^{CL}(y',e) + (1-t)\left(G^{CU}_y\right)^{-1}(e)$. Similarly, if $x = ty + (1-t)y'$ and $t > 1$, then we have $g(x,e) \leq tg(y,e) + (1-t)g(y',e) \leq t\left(G^{CU}_y\right)^{-1}(e) + (1-t)B^{CL}(y',e)$. Hence, we have

$$\begin{aligned} g(x,e) \leq \ \min\Big[&\inf_{y,y':x<y<y'}\left\{\left(\tfrac{x-y}{y'-y}\right)B^{CL}(y',e) + \left(\tfrac{y'-x}{y'-y}\right)\left(G^{CU}_{y}\right)^{-1}(e)\right\}, \\ &\inf_{y,y':y'<y<x}\left\{\left(\tfrac{y-x}{y-y'}\right)B^{CL}(y',e) + \left(\tfrac{x-y'}{y-y'}\right)\left(G^{CU}_{y}\right)^{-1}(e)\right\}\Big]. \end{aligned}$$

□

Similarly to Theorem 1, we can show that $\left(G_x^{CL}\right)^{-1}(e) \leq g(x,e) \leq \left(G_x^{CU}\right)^{-1}(e)$. We strengthen the bounds to $B^{CL}(x,e) \leq g(x,e) \leq B^{CU}(x,e)$ by using the concavity of $g(x,e)$ in x. Figure 5 illustrates this proof intuitively. The similar approach is used by Manski (1997). Manski (1997) uses the concavity of the response function to partially identify the average response function when the support of outcome is bounded. Our approach does not require the information of the infimum and supremum of the support of outcomes.

This identification approach is slightly similar to that considered by D'Haultfoeuille et al. (2013). They show that under the assumption that $A_t|V_t \equiv F_{X_t}(X_t) = v \sim A_s|V_s \equiv F_{X_s}(X_s) = v$, the average treatment on treated effect $\Delta^{ATT}(x,x') \equiv E[g_T(x,A_T) - g_T(x',A_T)|X_T = x]$ is identified when $F_{X_T}(x) = F_{X_t}(x')$. Under their assumption, $\Delta^{ATT}(x,x')$ is not identified if $F_{X_T}(x) \neq F_{X_t}(x')$ for all $t \in \{1,\cdots,T-1\}$. However, they show that $\Delta^{ATT}(x,x')$ is partially identified if g is locally concave in x.

Similarly to Theorem 2 in Section 3, the following theorem shows that if $g(x,e)$ is linear in x in a particular interval, then the inequalities (20) and (21) become equalities and the structural function g is point identified.

Theorem 4. *Under Assumptions 1-5 and C, if there exists $\tilde{x} \in \mathcal{X}$ such that $g(x,e)$ is linear in x on $[\pi(\tilde{x}), \pi^{-1}(\tilde{x})]$, then $g(x,e)$ is point identified. This result holds even if an interval $[\pi(\tilde{x}), \pi^{-1}(\tilde{x})]$ is unknown.*

Proof. Similarly to Theorem 2, for all $x, x' \in \mathcal{X}$, there exist $n, m \in \mathbb{Z}$ such that $\pi^n(x')$, $\pi^{n-1}(x')$, and $\pi^m(x)$ are in $[\pi(\tilde{x}), \pi^{-1}(\tilde{x})]$ and $\pi^n(x') \leq \pi^m(x) \leq \pi^{n-1}(x')$. Because $g(x,e)$ is linear in x, we have

$$\left[t_{x',x}(n,m)\tilde{T}_{x'}^{(n)} + (1 - t_{x',x}(n,m))\tilde{T}_{x'}^{(n-1)}\right](g(x',e)) = \tilde{T}_x^{(m)}(g(x,e)).$$

Similarly, for all $x, x' \in \mathcal{X}$, there exist $n, m \in \mathbb{Z}$ such that

$$\tilde{T}_{x'}^{(m)}(g(x',e)) = \left[t_{x,x'}(n,m)\tilde{T}_x^{(n)} + (1 - t_{x,x'}(n,m))\tilde{T}_x^{(n-1)}\right](g(x,e)).$$

Hence, as described above, $g(x,e)$ is point identified because the inequalities (20) and (21) become equalities. □

Example 2 (quantile regression)**.** *Suppose that $g(X,\epsilon) = \theta_0(\epsilon) + \theta_1(\epsilon)X$, where X is positive and $\theta_0(e)$ and $\theta_1(e)$ are strictly increasing in e. This model is a quantile regression model with endogeneity. The τ-th quantile function of $g(x,\epsilon)$ is $\theta_0(\tau) + \theta_1(\tau)x$. In this case, the structural function $g(x,e) = \theta_0(e) + \theta_1(e)x$ is linear in x. Hence, Theorem 4 shows that $\theta_0(e)$ and $\theta_1(e)$ are identified if binary instruments are available.*

5 Extension: General Models

In this section, we extend the result of Section 3 to more general models and allow Y to be discrete or censored. If outcomes are discrete or censored, then Assumptions 2 and 4 are not satisfied. Hence, we replace these assumptions for the following assumptions:

Assumption 2'. *(i) The function $g(x,e)$ is nondecreasing in e for all $x \in \mathcal{X}$. (ii) For all $z \in \{0,1\}$, $h(z,v)$ is continuous and strictly increasing in v.*

Assumption 4'. *For all* $(z, x) \in \{0,1\} \times \mathcal{X}_z$*, we have* $\overline{\mathcal{Y}}_{x,z} = \overline{\mathcal{Y}}$*. Here, define* $\overline{y} \equiv \sup\{y : y \in \overline{\mathcal{Y}}\}$ *and* $\underline{y} \equiv \inf\{y : y \in \overline{\mathcal{Y}}\}$*.*

Assumption 2' (i) is different from Assumption 2 (i). Assumption 2 (i) imposes the strict monotonicity of $g(x, e)$ in e, while Assumption 2' (i) requires only the weak monotonicity of $g(x, e)$ in e. For example, if we consider the model

$$g(x, e) = \mathbf{1}\{e > (1 + \exp(\beta_0 + \beta_1 x))^{-1}\},$$

then $g(x, e)$ is not strictly increasing in e. Chesher (2010) also uses a weak monotonicity condition. Assumption 4 implies that Y is continuously distributed, but Assumptions 2' and 4' allow that outcomes are discrete or censored.

D'Haultfœuille and Février (2015) and Torgovitsky (2015) do not consider the case where outcomes are discrete or censored because they assume that $g(x, e)$ is strictly increasing in e. Chesher (2010) considers instrumental variable models for discrete outcome. He shows that the structural function is partially identified by using instruments.

In this section, we show that $g(x, e)$ is partially identified under Assumptions 1, 2', 3, 4', 5 and M.

Define

$$\begin{aligned}
F^{+}_{Y|X,Z}(y|x,z) &\equiv P(Y \leq y|X = x, Z = z), \\
F^{-}_{Y|X,Z}(y|x,z) &\equiv P(Y < y|X = x, Z = z), \\
Q^{+}_{Y|X,Z}(\tau|x,z) &\equiv \sup\{y : F^{-}_{Y|X,Z}(y|x,z) \leq \tau\} \wedge \overline{y}, \\
Q^{-}_{Y|X,Z}(\tau|x,z) &\equiv \inf\{y : F^{+}_{Y|X,Z}(y|x,z) \geq \tau\} \vee \underline{y}.
\end{aligned}$$

$F^{+}_{Y|X,Z}(y|x,z)$ and $Q^{+}_{Y|X,Z}(\tau|x,z)$ are right continuous in y and τ respectively. $F^{-}_{Y|X,Z}(y|x,z)$ and $Q^{-}_{Y|X,Z}(\tau|x,z)$ are left continuous in y and τ respectively. Under Assumptions 2' and 4', Proposition 1 does not hold. Instead, we show the following proposition.

Proposition 2. *Define*

$$\begin{aligned}
\hat{T}^{(1)}_x(y) &\equiv Q^{-}_{Y|X,Z}\left(F^{-}_{Y|X,Z}(y|x,0)\Big|\pi(x),1\right), \\
\check{T}^{(1)}_x(y) &\equiv Q^{+}_{Y|X,Z}\left(F^{+}_{Y|X,Z}(y|x,0)\Big|\pi(x),1\right), \\
\hat{T}^{(-1)}_x(y) &\equiv Q^{-}_{Y|X,Z}\left(F^{-}_{Y|X,Z}(y|x,1)\Big|\pi^{-1}(x),0\right), \\
\check{T}^{(-1)}_x(y) &\equiv Q^{+}_{Y|X,Z}\left(F^{+}_{Y|X,Z}(y|x,1)\Big|\pi^{-1}(x),0\right).
\end{aligned}$$

Then, under Assumption 1, 2', 3, 4', and 5, we have

$$\begin{aligned}
g\left(\pi(x), e\right) &\geq \hat{T}^{(1)}_x\left(g(x,e)\right), \\
g\left(\pi(x), e\right) &\leq \check{T}^{(1)}_x\left(g(x,e)\right), \\
g\left(\pi^{-1}(x), e\right) &\geq \hat{T}^{(-1)}_x\left(g(x,e)\right), \\
g\left(\pi^{-1}(x), e\right) &\leq \check{T}^{(-1)}_x\left(g(x,e)\right).
\end{aligned}$$

Proof. Because $g(x,e)$ is nondecreasing in e, we have

$$\begin{aligned}
F^{-}_{Y|X,Z}\left(g(x,e)|x,0\right) &= P\left(g(x,\epsilon) < g(x,e)|X=x,Z=0\right) \\
&\leq P\left(\epsilon < e|X=x,Z=0\right) \\
&= P\left(\epsilon \leq e|X=\pi(x),Z=1\right) \\
&\leq P\left(g\left(\pi(x),\epsilon\right) \leq g\left(\pi(x),e\right)|X=\pi(x),Z=1\right) \\
&= F^{+}_{Y|X,Z}\left(g(\pi(x),e)|\pi(x),1\right), \qquad (22)
\end{aligned}$$

where the first inequality follows from $\{\epsilon : g(x,\epsilon) < g(x,e)\} \subset \{\epsilon : \epsilon < e\}$ and the second inequality follows from $\{\epsilon : \epsilon \leq e\} \subset \{\epsilon : g(x,\epsilon) \leq g(x,e)\}$. From the definition of $Q^{-}_{Y|X,Z}(\tau|x,z)$, it follows that $Q^{-}_{Y|X,Z}\left(F^{+}_{Y|X,Z}(y|x,z)\Big|x,z\right) = \inf\{y' : F^{+}_{Y|X,Z}(y'|x,z) \geq F^{+}_{Y|X,Z}(y|x,z)\} \vee \underline{y} \leq y$ for all $y \in \overline{\mathcal{Y}}$. Hence, the inequality (22) implies that

$$\begin{aligned}
\hat{T}^{(1)}_{x}\left(g(x,e)\right) &= Q^{-}_{Y|X,Z}\left(F^{-}_{Y|X,Z}(g(x,e)|x,0)\Big|\pi(x),1\right) \\
&\leq Q^{-}_{Y|X,Z}\left(F^{+}_{Y|X,Z}\left(g(\pi(x),e)|\pi(x),1\right)\Big|\pi(x),1\right) \\
&\leq g(\pi(x),e).
\end{aligned}$$

Similarly, because $g(x,e)$ is nondecreasing in e, we have

$$F^{+}_{Y|X,Z}\left(g(x,e)|x,0\right) \geq F^{-}_{Y|X,Z}\left(g(\pi(x),e)|\pi(x),1\right).$$

Because $Q^{+}_{Y|X,Z}\left(F^{-}_{Y|X,Z}(y|x,z)\Big|x,z\right) = \sup\{y' : F^{-}_{Y|X,Z}(y'|x,z) \leq F^{-}_{Y|X,Z}(y|x,z)\} \wedge \overline{y} \geq y$ for all $y \in \overline{\mathcal{Y}}$, we have

$$g(\pi(x),e) \leq \check{T}^{(1)}_{x}\left(g(x,e)\right).$$

Similarly, we have two inequalities $g(\pi^{-1}(x),e) \geq \hat{T}^{(-1)}_{x}\left(g(x,e)\right)$ and $g(\pi^{-1}(x),e) \leq \check{T}^{(-1)}_{x}\left(g(x,e)\right)$. □

This approach is similar to the identification approaches of Athey and Imbens (2006) and Chesher (2010). Athey and Imbens (2006) show that the counterfactual distribution is partially identified by using right continuous quantile function and left continuous quantile function when outcomes are discrete. Chesher (2010) uses the result that the weak monotonicity of $h(x,u)$ in u implies $\{u : h(x,u) \leq h(x,\tau)\} \supset \{u : u \leq \tau\}$ and $\{u : h(x,u) < h(x,\tau)\} \subset \{u : u < \tau\}$, and shows that the structural function h is partially identified.

Define $\hat{T}^{(n)}_{x}(y) \equiv \hat{T}^{(1)}_{\pi^{n-1}(x)} \circ \cdots \circ \hat{T}^{(1)}_{x}(y)$ and $\check{T}^{(n)}_{x}(y) \equiv \check{T}^{(1)}_{\pi^{n-1}(x)} \circ \cdots \circ \check{T}^{(1)}_{x}(y)$ for $n \in \mathbb{N}$, then we have

$$\begin{aligned}
g(\pi^{n}(x),e) &\geq \hat{T}^{(n)}_{x}\left(g(x,e)\right), \\
g(\pi^{n}(x),e) &\leq \check{T}^{(n)}_{x}\left(g(x,e)\right).
\end{aligned}$$

Similarly, define $\hat{T}^{(-n)}_{x}(y) \equiv \hat{T}^{(-1)}_{\pi^{-(n-1)}(x)} \circ \cdots \circ \hat{T}^{(-1)}_{x}(y)$ and $\check{T}^{(-n)}_{x}(y) \equiv \check{T}^{(-1)}_{\pi^{-(n-1)}(x)} \circ \cdots \circ \check{T}^{(-1)}_{x}(y)$ for $n \in \mathbb{N}$, then we have

$$\begin{aligned}
g(\pi^{-n}(x),e) &\geq \hat{T}^{(-n)}_{x}\left(g(x,e)\right), \\
g(\pi^{-n}(x),e) &\leq \check{T}^{(-n)}_{x}\left(g(x,e)\right).
\end{aligned}$$

Define $\hat{T}_x^{(0)}(y) = y$ and $\check{T}_x^{(0)}(y) = y$.

If $(n,m) \in \Pi_{x',x}^M$, then Assumption M implies that

$$\hat{T}_{x'}^{(n)}\left(g(x',e)\right) \leq g(\pi^n(x'),e) \leq g(\pi^m(x),e) \leq \check{T}_x^{(m)}\left(g(x,e)\right).$$

Define

$$\left(\hat{T}_{x'}^{(n)}\right)^{\rightarrow}(u) \equiv \sup\{y : \hat{T}_{x'}^{(n)}(y) \leq u\} \wedge \overline{y},$$

then we have $\left(\hat{T}_{x'}^{(n)}\right)^{\rightarrow}\left(\hat{T}_{x'}^{(n)}(y)\right) = \sup\{y' : \hat{T}_{x'}^{(n)}(y') \leq \hat{T}_{x'}^{(n)}(y)\} \wedge \overline{y} \geq y$ for all $y \in \overline{\mathcal{Y}}$. Hence, we obtain

$$g(x',e) \leq \min_{(n,m)\in\Pi_{x',x}^M} \left(\hat{T}_{x'}^{(n)}\right)^{\rightarrow}\left(\check{T}_x^{(m)}(g(x,e))\right).$$

Define $T_{x',x}^{GU}(y) \equiv \min_{(n,m)\in\Pi_{x',x}^M} \left(\hat{T}_{x'}^{(n)}\right)^{\rightarrow}\left(\check{T}_x^{(m)}(y)\right)$, then $T_{x',x}^{GU}(y)$ satisfies that

$$g(x',e) \leq T_{x',x}^{GU}\left(g(x,e)\right). \tag{23}$$

Similarly, define $T_{x',x}^{GL}(y) \equiv \max_{(n,m)\in\Pi_{x,x'}^M} \left(\check{T}_{x'}^{(m)}\right)^{\leftarrow}\left(\hat{T}_x^{(n)}(y)\right)$ and $\left(\check{T}_{x'}^{(m)}\right)^{\leftarrow}(u) \equiv \inf\{y : \check{T}_{x'}^{(m)}(y) \geq u\} \vee \underline{y}$, then $T_{x',x}^{GL}(y)$ satisfies that

$$g(x',e) \geq T_{x',x}^{GL}\left(g(x,e)\right). \tag{24}$$

Define

$$\begin{aligned}
G_x^{GL}(u) &\equiv \int F_{Y|X}^{+}\left(T_{x',x}^{GU}(u)|x'\right)dF(x'),\\
G_x^{GU}(u) &\equiv \int F_{Y|X}^{-}\left(T_{x',x}^{GL}(u)|x'\right)dF(x'),\\
B^{GL}(x,e) &\equiv \sup_{y:y\leq x}\left\{\inf\{u : G_y^{GL}(u) \geq e\}\right\} \vee \underline{y},\\
B^{GU}(x,e) &\equiv \inf_{y:y\geq x}\left\{\sup\{u : G_y^{GU}(u) \leq e\}\right\} \wedge \overline{y},
\end{aligned}$$

where $F_{Y|X}^{+}(y|x) \equiv P(Y \leq y|X = x)$ and $F_{Y|X}^{-}(y|x) \equiv P(Y < y|X = x)$. $G_x^{GL}(u)$ and $G_x^{GU}(u)$ provide lower and upper bounds on $g(x,e)$ via the argument similar to (4) and (5). $B^{GL}(x,e)$ and $B^{GU}(x,e)$ strengthen these bounds.

Theorem 5. *Under Asssumptions 1, 2', 3, 4', 5, and M, for all $(x,e) \in \mathcal{X} \times (0,1)$, we have*

$$B^{GL}(x,e) \leq g(x,e) \leq B^{GU}(x,e).$$

Proof. First, we show that

$$\inf\{u : G_x^{GL}(u) \geq e\} \vee \underline{y} \leq g(x,e) \leq \sup\{u : G_x^{GU}(u) \leq e\} \wedge \overline{y}. \tag{25}$$

Because $T^{GU}_{x',x}(y)$ satisfies (23), we have

$$\begin{aligned} G^{GL}_x\left(g(x,e)\right) &= \int F^{+}_{Y|X=x'}\left(T^{GU}_{x',x}(g(x,e))\right)dF(x') \\ &\leq \int F^{+}_{Y|X=x'}\left(g(x',e)\right)dF(x') \\ &= \int P\left(g(x',\epsilon)\leq g(x',e)|X=x'\right)dF(x') \\ &\leq \int P\left(\epsilon\leq e|X=x'\right)dF(x')=e, \end{aligned}$$

where the second inequality follows from $\{\epsilon:\epsilon\leq e\}\subset\{\epsilon:g(x',\epsilon)\leq g(x',e)\}$. Because $g(x,e)\geq \underline{y}$, we can obtain $g(x,e)\geq \inf\{u:G^{GL}_x(u)\geq e\}\vee\underline{y}$. Similarly, because $T^{GL}_{x',x}(y)$ satisfies (24), we have

$$\begin{aligned} G^{GU}_x\left(g(x,e)\right) &\leq \int F^{-}_{Y|X=x'}\left(g(x',e)\right)dF(x') \\ &= \int P\left(g(x',\epsilon)< g(x',e)|X=x'\right)dF(x') \\ &\leq \int P\left(\epsilon< e|X=x'\right)dF(x')=e, \end{aligned}$$

where the second inequality follows from $\{\epsilon:g(x',\epsilon)<g(x',e)\}\subset\{\epsilon:\epsilon<e\}$. Hence, we can obtain $g(x,e)\geq \sup\{u:G^{GU}_x(u)\leq e\}\wedge\overline{y}$.

Because $g(x,e)$ is nondecreasing in x and (25) holds, similarly to Theorem 1, we have $B^{GL}(x,e)\leq g(x,e)\leq B^{GU}(x,e)$. □

6 Conclusion

We consider partial identification of nonseparable models with discrete instruments. We show that partially identification can be achieved when $g(x,e)$ is monotone or concave in x, even if X is continuous and Z is binary. D'Haultfœuille and Février (2015) and Torgovitsky (2015) show that g is point identified without monotonicity and concavity. They use two key assumptions to establish point identification of g. First, $F_{X|Z}(x|0)$ and $F_{X|Z}(x|1)$ have intersections. Second, $g(x,e)$ is strictly increasing in a scalar unobservable. There are, however, many empirically important models that do not satisfy these assumption. We provides bounds of the structural function without these assumptions.

Appendix:Figures

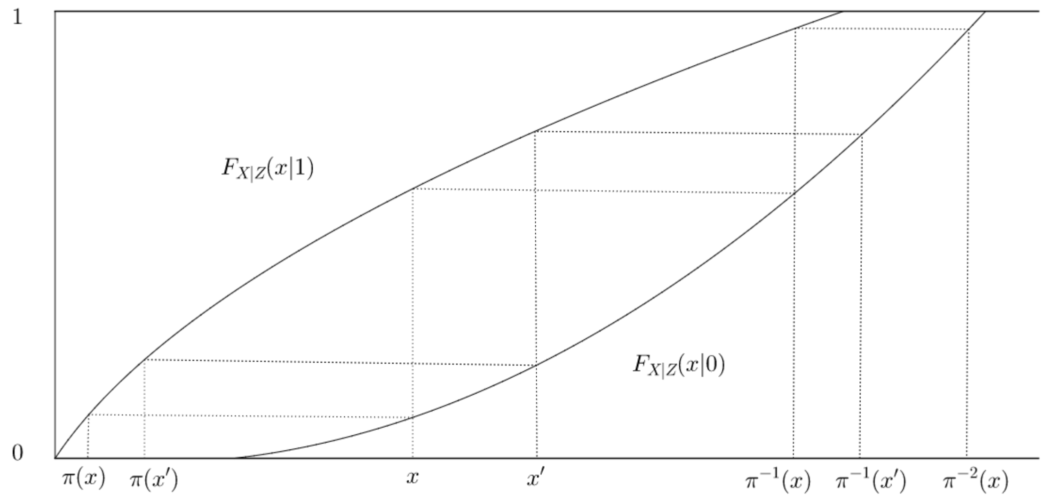

Figure 1: The left function is $F_{X|Z}(x|1)$ and the right function is $F_{X|Z}(x|0)$.

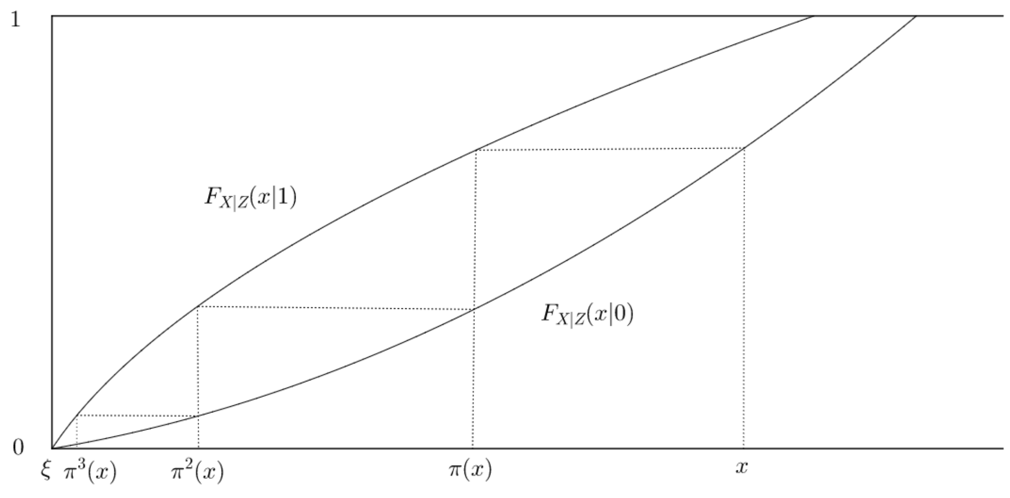

Figure 2: The left function is $F_{X|Z}(x|1)$ and the right function is $F_{X|Z}(x|0)$.

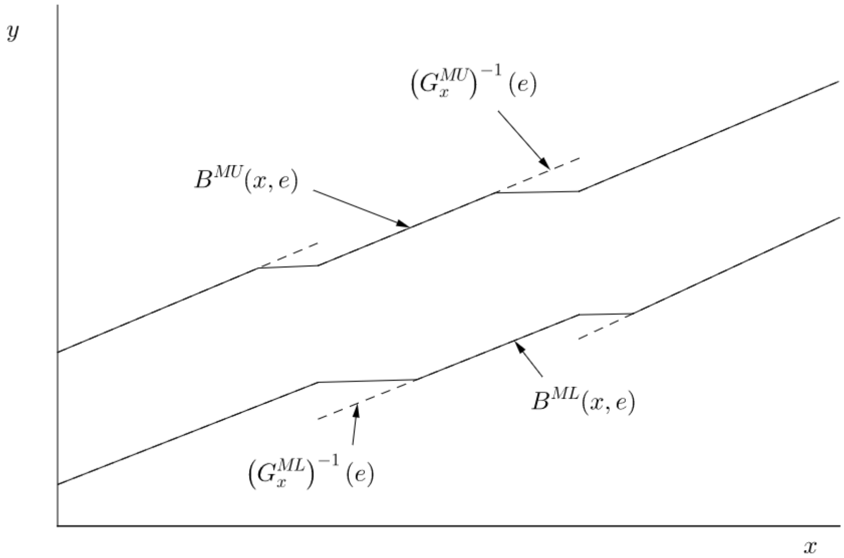

Figure 3: The dashed lines denote $\left(G_x^{ML}\right)^{-1}(e)$ and $\left(G_x^{MU}\right)^{-1}(e)$. The solid lines denote $B^{ML}(x, e)$ and $B^{MU}(x, e)$.

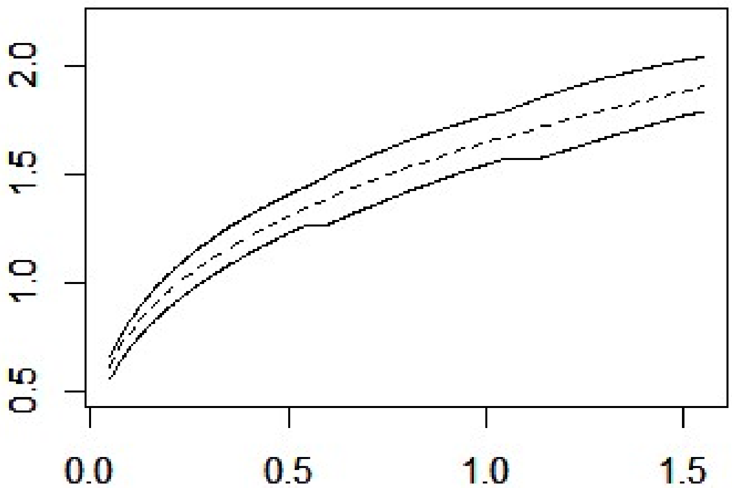

Figure 4: The dashed line denotes $g(x, 0.5)$. The solid lines are the bounds of $g(x, 0.5)$, $B^{ML}(x, 0.5)$ and $B^{MU}(x, 0.5)$. $\alpha = 0.5, \beta = 0.5, c = 0.5, d = 0.05, \rho = 0.3$.

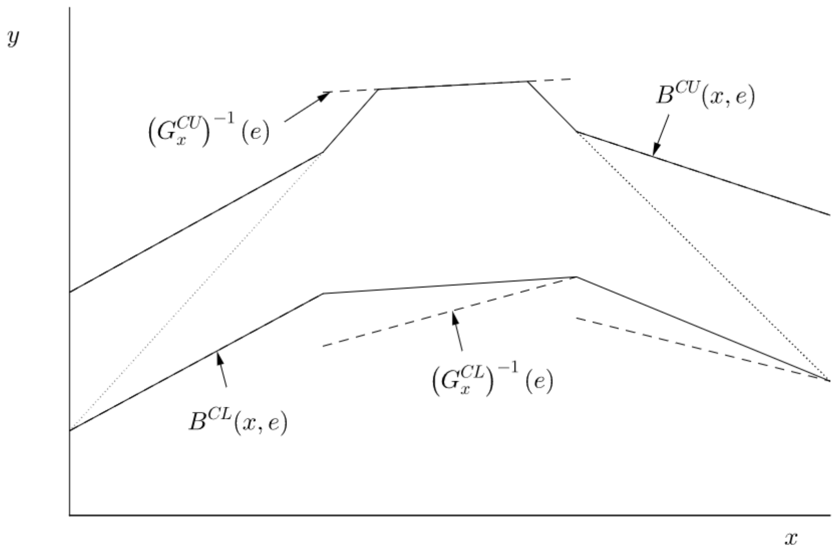

Figure 5: The dashed lines denote $\left(G_x^{CL}\right)^{-1}(e)$ and $\left(G_x^{CU}\right)^{-1}(e)$. The solid lines denote $B^{CL}(x,e)$ and $B^{CU}(x,e)$.

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