Forecasting Value-at-Risk by Estimating the Quantiles of the Intra-Day Low and High Series

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Abstract

Although value at risk (VaR) is an intuitive and popular measure of risk, it is inherently challenging to estimate, as it requires the forecasting of extreme quantiles of the time series of daily returns. This paper estimates the VaR for series of daily returns using the intra-day low and high time series, which are, respectively, the time series of lowest and highest log prices occurring on each day. In contrast to intra-day observations, the intra-day low and high are widely available for many financial assets. We discover that extreme quantiles of the daily returns can be well approximated by less extreme quantiles of the intra-day low and high series. Several conditional autoregressive value at risk (CAViaR) time series models are used to estimate the quantiles of the intra-day low and high series. In addition, two novel multivariate multi-quantile (MV-MQ-CAViaR) models are proposed to estimate the quantiles of the intra-day low and high series simultaneously. We provide empirical support for the new proposals using daily stock index data.

Keywords: Value at Risk; CAViaR; Quantile Regression; Intra-day Low and High.

JEL: C22, C53, G10

1 Introduction

Value at risk (VaR) is currently the most prevalent risk measure and a standard tool for risk management in financial and insurance institutions (Berkowitz et al. 2011). It involves measuring the amount a certain portfolio can lose given a probability level. Mathematically, VaR is defined to be the absolute value of the quantile of the distribution of a financial return series, with the probability level usually chosen as 1% or 5%. The accurate forecasting of VaR is fundamental for internal risk control and financial regulation. Despite the simplicity of the concept of VaR, its measurement is a challenging problem and has received considerable attention in recent years.

The CAViaR models, which estimate the quantiles directly, have been shown to perform competitively in comparison with other VaR models (Manganelli and Engle 2004; Chen et al. 2012). Our aim in this paper is to improve the CAViaR models to generate more accurate VaR forecasts. First, we describe an interesting relationship between the intra-day low and high series and the returns. In short, we find that the quantiles of the intra-day low and high series are good approximations of the quantiles of the returns. Secondly, we consider several CAViaR models to estimate the quantiles of the intra-day low and high series, including the CAViaR models based on the returns and the CAViaR models based on the intra-day data. In particular, we incorporate the intra-day range, which is the difference between the intra-day high and the intra-day low, and the overnight return, which is the difference between the log opening price on one day and the log closing price on the previous day in the CAViaR models. Thirdly, we propose two MV-MQ-CAViaR models based on the work of White et al. (2015) to estimate the upper quantiles and the lower quantiles simultaneously to capture the co-movements between the upper tails and the lower tails. We find that some of the proposed methods are able to outperform the benchmark models in the empirical study.

The remainder of the paper is organized as follows. Section 2 gives a brief review of the established VaR methods. Section 3 introduces our proposed VaR approaches. Section 4 uses five series of stock returns to evaluate the performance of the proposed methods, and to compare their VaR estimation accuracy with the established models. Section 5 provides a summary and some remarks.

2 Review of VaR Methods

2.1 Standard VaR Models

2.1.1 VaR Models Based on Daily Returns

We divide the standard VaR literature into three categories using the classification of Manganelli and Engle (2004): parametric, nonparametric and semiparametric.

Parametric models involve a parameterization of the conditional volatility and a specification of the shape of the distribution of a financial return. The VaR associated with any probability level can then be obtained from the estimated distribution of the return. Typical examples of the parameterization of the conditional volatility are the GARCH models. As for the conditional distribution, a Gaussian distribution is computationally convenient, but the fact that the distribution of the financial return is often fat-tailed has prompted the use of the Student t distribution. Although parametric models provide an estimate of the complete conditional distribution of the return series, they usually suffer from some degree of misspecification, either from the model structure or from the distributional assumption.

In contrast to the parametric models, nonparametric methods propose no parameterization for the conditional volatility and no assumption for the distribution of the return. Historical simulation and kernel density estimation are two examples. With historical simulation, the VaR estimates are generated as the quantiles of the sample distribution of the return over some window length (Butler and Schachter 1997; Pritsker 2006). With kernel density estimation, the quantiles are forecasted from the estimated distribution, which is constructed by assigning some kernel function (typically a Gaussian density function) to each observation within some window length (Butler and Schachter 1997). Nonparametric methods are model free and easy to implement, which is appealing in practice (Manganelli and Engle 2004). However, the historical simulation and the kernel density estimation methods require implicitly that the distribution of the returns remain at least roughly the same within the specified window length, which may be inappropriate (Manganelli and Engle 2004). Moreover, the choice of the window length is a difficult task: a small number might result in sizeable sampling errors while a large number may cause the model to adapt slowly to the changes in the dynamics of the return.

Finally, semiparametric methods involve the use of extreme value theory or quantile regression, such as the CAViaR models proposed by Engle and Manganelli (2004). The CAViaR models involve an explicit modeling of the dynamics of the conditional quantile but do not make any distributional assumption about the shape of the return. A generic CAViaR model has the following expression:

$$q_t(\boldsymbol{\beta}) = f(y_{t-1}, \boldsymbol{\Psi}_{t-1} \ldots y_1, \boldsymbol{\Psi}_1; \boldsymbol{\beta}) \tag{1}$$

where $\boldsymbol{\beta}$ is a vector containing the parameters, y_t is the return and $\boldsymbol{\Psi}_t$ is a vector of explanatory variables. In particular, Engle and Manganelli (2004) consider the following four specifications of the generic CAViaR model:

CAViaR Adaptive (CAViaR-Adaptive):

$$q_t(\boldsymbol{\beta}) = q_{t-1}(\boldsymbol{\beta}) + \beta_1 \Big[\theta - I(y_{t-1} < q_{t-1}(\boldsymbol{\beta}))\Big] \tag{2}$$

CAViaR Symmetric Absolute Value (CAViaR-SAV):

$$q_t(\boldsymbol{\beta}) = \beta_1 + \beta_2 q_{t-1}(\boldsymbol{\beta}) + \beta_3 |y_{t-1}| \tag{3}$$

CAViaR Asymmetric Slope (CAViaR-AS):

$$q_t(\boldsymbol{\beta}) = \beta_1 + \beta_2 q_{t-1}(\boldsymbol{\beta}) + \beta_3 (y_{t-1})^+ + \beta_4 (y_{t-1})^- \tag{4}$$

CAViaR Indirect GARCH (CAViaR-IndG):

$$q_t(\boldsymbol{\beta}) = sgn(\theta - 0.5)(\beta_1 + \beta_2 q_{t-1}(\boldsymbol{\beta})^2 + \beta_3 y_{t-1}^2)^{\frac{1}{2}} \tag{5}$$

where $(x)^+ = \max(x, 0)$, $(x)^- = -\min(x, 0)$, and θ is the probability level of interest. The parameter vector $\boldsymbol{\beta}$ is chosen to be any vector that minimizes the following quantile regression check loss function:

$$\sum_{t=1}^{T} \Big[y_t - q_t(\boldsymbol{\beta})\Big] \Big[\theta - I(y_t < q_t(\boldsymbol{\beta}))\Big] \tag{6}$$

where T is the number of the observations in the in-sample period.

The CAViaR models have many advantages. The fact that they make no distributional assumption avoids any potential misspecification of the shape of the distribution of the return. They allow different quantiles to have distinct dynamics because the models associated with different probability levels can have different parameters. Moreover, the CAViaR models allow the conditional distribution to be time-varying (Engle and

Manganelli 2004).

2.1.2 VaR Models Based on Intra-day Range and Overnight Return

The unobservable nature of daily volatility is a challenge for volatility estimation. Intra-day data provides information regarding the distribution of the price process within a day, and this can be used to help provide insight into the understanding of volatility. However, intra-day data, recorded at a relatively high frequency, such as minute-by-minute, is generally expensive and not available for a long period. In contrast, the daily opening, closing, intra-day high and intra-day low prices are readily available for most tradable assets for the last thirty years. In this paper, we consider the use of the intra-day range in quantile estimation, which is not only easily obtainable but is also a highly efficient volatility estimator compared to the absolute return (Andersen and Bollerslev 1998; Brandt and Jones 2006; Kumar and Maheswaran 2015). Although an abundance of literature can be found in volatility estimation based on the intra-day range (see, for example, Alizadeh et al. 2002; Brandt and Jones 2006; Corsi 2009), surprisingly little attention has been devoted to improving quantile estimation models using the intra-day range. Another variable we consider in this paper, which somehow has not been widely studied in quantile regression models, is the overnight return, despite its usefulness in volatility estimation. Hua and Manzan (2013) point out that the positions of the investors, who hold their portfolios more than a few days, can be affected by the market overnight variability. Hansen and Lunde (2006) find that ignoring the overnight return in an intra-day volatility measure might lead to a improper proxy for the true daily volatility. To the author's knowledge, the only literature in quantile forecasting, that consider the intra-day range, are the studies of Chen et al. (2012) and Meng and Taylor (2015), and the only literature considering the overnight return in quantile regression models is Meng and Taylor (2015). Chen et al. (2012) propose the following CAViaR models with range information, some of which are shown to outperform the CAViaR models without range information. Meng and Taylor (2015) add the overnight return to one of the models based on intra-day range proposed by Chen et al. (2012), and they show that the inclusion of the overnight return improve the VaR forecasting accuracy:

CAViaR Range Value (CAViaR-Range)

$$q_t(\boldsymbol{\beta}) = \beta_1 + \beta_2 q_{t-1}(\boldsymbol{\beta}) + \beta_3 Range_{t-1} \tag{7}$$

CAViaR Threshold Range Value (CAViaR-Range-T):

$$q_t(\boldsymbol{\beta}) = \begin{cases} \beta_1 + \beta_2 q_{t-1}(\beta) + \beta_3 Range_{t-1} & if\ q_{t-1}(\boldsymbol{\beta}) \leq \gamma \\ \beta_4 + \beta_5 q_{t-1}(\beta) + \beta_6 Range_{t-1} & if\ q_{t-1}(\boldsymbol{\beta}) > \gamma \end{cases} \tag{8}$$

CAViaR Threshold Range Indirect GARCH(1, 1) (CAViaR-Range-TIndG):

$$q_t(\boldsymbol{\beta}) = \begin{cases} sgn(\theta - 0.5)\left[\beta_1 + \beta_2 q_{t-1}(\boldsymbol{\beta})^2 + \beta_3 Range_{t-1}^2\right]^{\frac{1}{2}} & if\ y_{t-1} \leq \gamma \\ sgn(\theta - 0.5)\left[\beta_4 + \beta_5 q_{t-1}(\boldsymbol{\beta})^2 + \beta_6 Range_{t-1}^2\right]^{\frac{1}{2}} & if\ y_{t-1} > \gamma \end{cases} \tag{9}$$

where γ is s a chosen constant value representing the threshold. The threshold value γ comes from the idea that the quantiles might respond differently depending on the sign of $y_t - \gamma$.

The CAViaR model proposed by Meng and Taylor (2015) has the following form: CAViaR model with the intra-day range and the overnight return (CAViaR-Range-N):

$$q_t(\boldsymbol{\beta}) = \beta_1 + \beta_2 q_{t-1}(\boldsymbol{\beta}) + \beta_3 Range_{t-1} + \beta_4 |y_{N,t-1}| \tag{10}$$

where N denotes the overnight return, and $y_{N,t-1}$ denote the overnight return, which is defined as the log ratio between the opening price of day $t-1$ and the closing price of day $t-2$.

2.2 Multivariate Multi-Quantile CAViaR Models

In many situations, it is desirable to estimate the quantiles corresponding to more than one probability level. For example, quantiles of different probability levels can represent some basic features of a distribution (Koenker 2005). While the parametric models and the nonparametric methods produce the complete distribution of the returns, from which multiple quantiles can be obtained easily, CAViaR models focus on the quantile corresponding to just one probability level. To address this, White et al. (2010) propose the MQ-CAViaR model. Essentially, a MQ-CAViaR model is a vector version of a CAViaR model, which, for a single return series, simultaneously estimates quantiles corresponding to several different probability levels. Let $\theta_1, \theta_2, \ldots \theta_p$ denote the probability levels of interest, let $q_{j,t}(\boldsymbol{\beta})$ be the estimated $\theta_j th$ quantile of y_t at time t, and let $\boldsymbol{q}_t = (q_{1,t}, q_{2,t}, \ldots, q_{p,t})'$ be the column vector of all quantiles of interest. A generic linear MQ-CAViaR model can be expressed as follows[1],

$$q_{j,t}(\boldsymbol{\beta}) = \boldsymbol{\Psi}_t' \boldsymbol{\alpha}_j + \sum_{l=1}^{n} \boldsymbol{q}_{t-l}'(\boldsymbol{\beta}) \boldsymbol{\gamma}_{j,l} \tag{11}$$

where n is the number of lags, $\boldsymbol{\Psi}_t$ is a vector of explanatory variables, $\boldsymbol{\alpha}_j$ and $\boldsymbol{\gamma}_{j,l}$ are row vectors of parameters,and $\boldsymbol{\beta}$ is the column vector $(\boldsymbol{\alpha}_1', \boldsymbol{\gamma}_1', \boldsymbol{\alpha}_2', \boldsymbol{\gamma}_2', \ldots \boldsymbol{\alpha}_p', \boldsymbol{\gamma}_p')'$. In particular, White et al. (2010) consider a MQ-CAViaR model, where each model is assumed to follow a CAViaR-SAV model. This model has the following expression:

[1] Expression (11) has a simple linear form, but the model is not linear since it is autoregressive.

MQ-CAViaR symmetric absolute value (MQ-CAViaR-SAV):

$$q_{j,t}(\boldsymbol{\beta}) = \alpha_{j,0} + \alpha_{j,1}|y_{t-1}| + \boldsymbol{q}_{t-1}^{'}(\boldsymbol{\beta})\boldsymbol{\gamma}_j \tag{12}$$

Since the intra-day range and the overnight return have been shown to be useful for VaR forecasting (Meng and Taylor 2015), we also consider the following MQ-CAViaR model:

MQ-CAViaR model with intra-day range and the overnight return (MQ-CAViaR-Range-N):

$$q_{j,t}(\boldsymbol{\beta}) = \alpha_{j,0} + \alpha_{j,1}Range_{t-1} + \alpha_{j,2}|Overnight_{t-1}| + \boldsymbol{q}_{t-1}^{'}(\boldsymbol{\beta})\boldsymbol{\gamma}_j \tag{13}$$

where the parameters are estimated by minimizing the quantile regression check loss function in expression (14).

The parameter vector $\boldsymbol{\beta}$ can be chosen as any solution that minimizes the following quantile regression check loss function:

$$\sum_{j=1}^{p}\left\{\sum_{t=1}^{T}\Big[y_t - q_{j,t}(\boldsymbol{\beta})\Big]\Big[\theta_j - I(y_t < q_{j,t}(\boldsymbol{\beta}))\Big]\right\} \tag{14}$$

In other words, expression (14) is just the sum of expression (6) for each quantile.

White et al. (2015) generalize the MQ-CAViaR models to the MV-MQ-CAViaR models to estimate different quantiles of different return series. Let $y_{1,t}, y_{2,t}, \ldots y_{m,t}$ denote the return series, let $\theta_{i,1}, \theta_{i,2}, \ldots \theta_{i,p}$ denote the probability levels of interest of $y_{i,t}$, let $q_{i,j,t}$ be the $\theta_{i,j}th$ quantile of $y_{i,t}$ at time t, let $\boldsymbol{q}_t = (q_{1,1,t}, q_{1,2,t}, \ldots, q_{m,p,t})'$ be the column vector of all quantiles. A generic linear[2] multivariate MQ-CAViaR model can

[2]Expression (15) has a simple linear form, but the model is not linear since it is autoregressive.

be expressed as follows,

$$q_{i,j,t}(\cdot,\boldsymbol{\beta}) = \boldsymbol{\Psi}_t^{'}\boldsymbol{\alpha}_{i,j} + \sum_{l=1}^{n} \boldsymbol{q}_{t-l}^{'}(\cdot,\boldsymbol{\beta})\boldsymbol{\gamma}_{i,j,l} \tag{15}$$

where n is the number of lags, $\boldsymbol{\Psi}_t$ is a vector of explanatory variables, $\boldsymbol{\alpha}_{i,j}$ and $\boldsymbol{\gamma}_{i,j,l}$ are row vector of parameters, and $\boldsymbol{\beta}$ is the column vector $(\boldsymbol{\alpha}_{1,1}^{'}, \boldsymbol{\gamma}_{1,1}^{'}, \boldsymbol{\alpha}_{1,2}^{'}, \boldsymbol{\gamma}_{1,2}^{1}, \ldots \boldsymbol{\alpha}_{m,p}^{'}, \boldsymbol{\gamma}_{m,p}^{1})'$. Then $\boldsymbol{\beta}$ can be chosen as any solution that minimizes the following quantile regression check loss function:

$$\sum_{i=1}^{m}\sum_{j=1}^{P}\left\{\sum_{t=1}^{T}[y_{i,t} - q_{i,j,t}(\boldsymbol{\beta})][\theta_{i,j} - I(y_{i,t} < q_{i,j,t}(\boldsymbol{\beta}))]\right\} \tag{16}$$

3 Proposed Methods for Value at Risk

3.1 Data

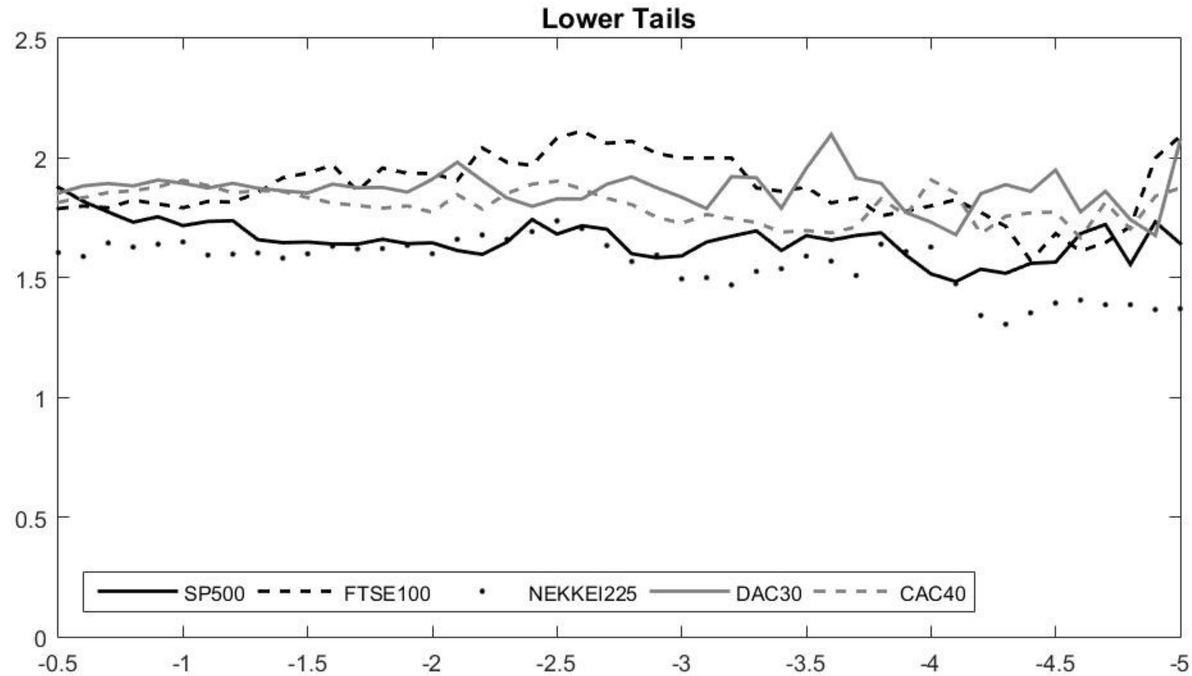
We first describe the data we used to illustrate our proposed methods. We used the daily opening, daily closing, intra-day high and intra-day low prices of the following five stocks: S&P500, FTSE100, Nikkei225, DAX30, CAC40. The data were obtained from the Oxford-Man Institute's realized library Version 0.2 (Heber et al. 2009). The sample period used in our study consisted of 15 years of data, from 2 Jan 2000 to 20 May 2014. Note that different markets can have different numbers of observations within a certain period because the public holidays of different markets can be different. We chose our period so that the observations of the stocks shared the same end date. The numbers of observations considered for each of the five stocks are 3513, 3513, 3476, 3526, and 3534, respectively. We subtract the log closing price of the previous day from the intra-day low and the intra-day high, so that the intra-day low and high series and the return can be compared directly. Following Engle and Manganelli (2004), we multiply the return, the overnight return, the intra-day low, intra-day high, and the intra-day range by 100.

3.2 Motivation

The accuracy of VaR forecasting is influenced by the number of observations in the tail of the distribution of the returns. When the data is scarce in the tail, the performance of the VaR estimation methods, and even the estimates of the unconditional empirical quantiles can be unreliable (Manganelli and Engle 2004). The unreliability could result from the parameter estimation procedure rather than model misspecification. The scarcity of observations in the tail of the daily returns distribution motivates us to consider the series of intra-day high and low values. In the remainder of this section, our discussion focuses on the intra-day low, but the ideas that we discuss are similar for the intra-day high.

The main inspiration of this paper comes from some properties of a standard Brownian motion. Brownian motion has been widely used to model the log price in the context of stochastic volatility and option price modeling. It is a well-known result that the distribution of the infimum L_t of a standard Brownian motion y_s over a closed interval $[0, t]$ and the distribution of y_t are closely related via the expression, $P(L_t < x) = 2P(y_t < x)$ for $x < 0$. This property can be easily derived using the reflection principle of Brownian motion (see, for example, Mörters and Peres 2010). The expression implies that the θth quantiles of the returns are equal to the $(2\theta)th$ quantiles of the intra-day low, which means that we will have twice as many observations beyond the VaR if we use the intra-day low series than if we use the daily returns series. This is very appealing in terms of estimation accuracy. Moreover, suppose we use some CAViaR model to estimate the θth quantiles of the returns and use the same model to estimate the $(2\theta)th$ quantiles of the intra-day low, then it can be shown that it is more efficient to estimate

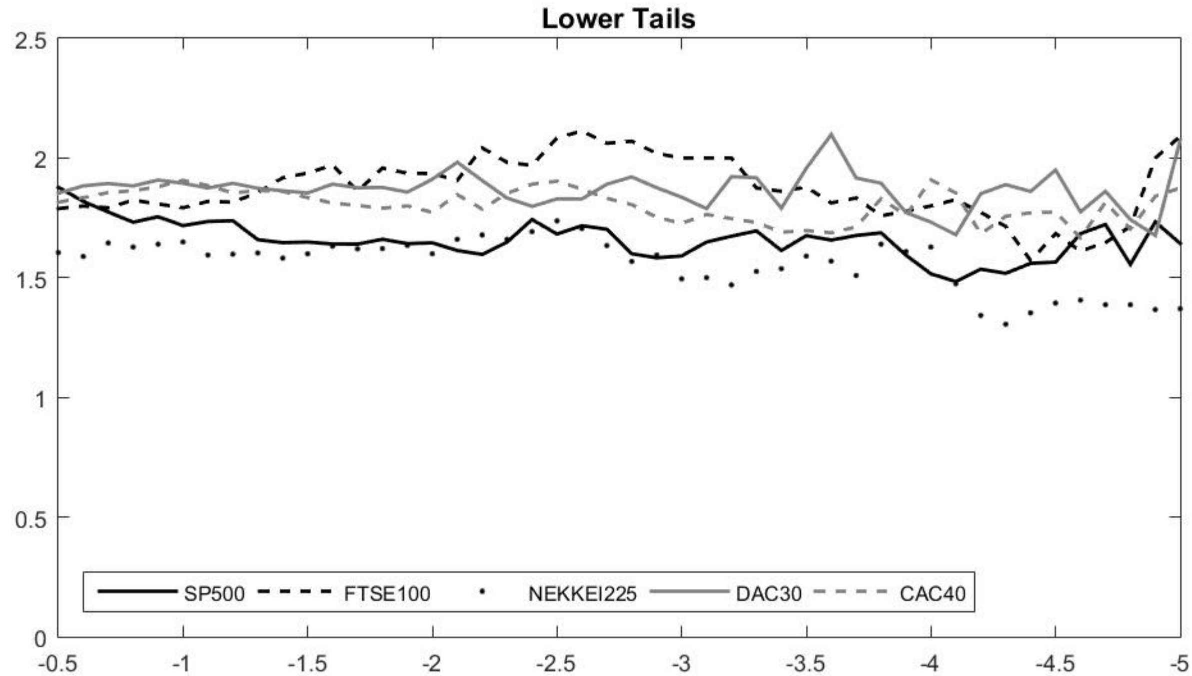
Figure 1: The plot of $\frac{\#\{L_t<x\}}{\#\{y_t<x\}}$ for the lower tails.



the parameters for the intra-day low. Empirical evidence is that daily financial asset returns do not precisely follow Brownian motion. In fact, provided the ratio $\frac{P(L_t<x)}{P(y_t<x)}$ is any constant, greater than 1, which is always true in practice, improved efficiency will be achieved by estimating the quantile for the intra-day low series. We include a proof in Appendix A.

The findings in Brownian motion prompt us to look for similar behavior in financial data. To do this, we consider the ratio $\frac{\#\{L_t<x\}}{\#\{y_t<x\}}$ for each of the five data sets. We calculate the values of the ratio from -0.5 to -5 (from 0.5 to 5 for the intra-day high), at an interval of length 0.01. The resultant values are shown in Figure 1 and Figure 2. It can be seen that the ratios are close to constants in the interval $[-0.5, -2.5]$. Beyond -2.5, the ratios become unstable, which could result from the insufficient number of observations.

Figure 2: The plot of $\frac{\#\{H_t > x\}}{\#\{y_t > x\}}$ for the upper tails.



3.3 Proposed CAViaR Methods

Following the argument in Section 3.2, we propose to use the quantiles of the intra-day low and high series as the approximations of the quantiles of the returns. The probability levels corresponding to the quantiles of the intra-day low and high series and the quantiles of the returns are different. In other words, we need to calculate the ratio $\frac{P(L_t < x)}{P(y_t < x)}$. We approach this as follows. Suppose the θth ($\theta < 0.5$) quantiles of the returns are of interest, we determine the ratio by the empirical distribution in the in-sample period, which is $\lambda = \frac{\#\{L_t < x\}}{\#\{y_t < x\}}$, where x is the empirical θth quantile of the returns in the in-sample period. Once the ratio λ is determined, we estimate the $(\lambda\theta)th$ quantiles of the intra-day low. Similarly, for $\theta > 0.5$, it can be shown that the corresponding probability level for the intra-day high is $1 - \frac{\#\{H_t > x\}}{\#\{y_t > x\}}$. For simplicity, we use $\tilde{\theta}$ to denote the probability levels for the intra-day low and high series.

Next, we need to estimate the quantiles of the intra-day low and high series. The obvious candidates are the CAViaR models. For example, the CAViaR-SAV model for the intra-day low is an option, which has the following form: $\tilde{q}_t(\boldsymbol{\beta}) = \beta_1 + \beta_2\tilde{q}_{t-1}(\boldsymbol{\beta}) + \beta_3|L_{t-1}|$, where $\tilde{}$ is used to distinguish the quantiles of the intra-day low and high series from the quantiles of the returns. But it is a poor model for the quantile of the intra-day low and high series. Because the main purpose of the structure of the CAViaR models is to capture the clustering effect of the daily volatility and the absolute return or the intra-day range serves as the proxy for the daily volatility. On the other hand, the intra-day low and high series are very poor proxies for the daily volatility. For example, the intra-day low is zero if the market price increases monotonically in one day, in which case the zero value of the intra-day low does not give the CAViaR model any innovation. Therefore, we use the CAViaR models in Section 2, without any modification, to model the quantiles of the intra-day low and high series. To stress the fact that we are estimating the quantiles of the intra-day low and high series, we add the word 'LH' at the end of the name of each model, where 'LH' stands for intra-day low and intra-day high. In theory, any CAViaR model has its corresponding model for the intra-day low and high, but in particular, we consider the following five models in this paper:

CAViaR Symmetric Absolute Value for the intra-day low and high (CAViaR-SAV-LH):

$$\tilde{q}_t(\boldsymbol{\beta}) = \beta_1 + \beta_2\tilde{q}_{t-1}(\boldsymbol{\beta}) + \beta_3|y_{t-1}| \tag{17}$$

CAViaR Asymmetric Slope for the intra-day low and high (CAViaR-AS-LH):

$$\tilde{q}_t(\boldsymbol{\beta}) = \beta_1 + \beta_2\tilde{q}_{t-1}(\boldsymbol{\beta}) + \beta_3(y_{t-1})^+ + \beta_4(y_{t-1})^- \tag{18}$$

CAViaR Indirect GARCH for for the intra-day low and high (CAViaR-IndG-LH):

$$\tilde{q}_t(\boldsymbol{\beta}) = sgn(\theta - 0.5)(\beta_1 + \beta_2 \tilde{q}_{t-1}(\boldsymbol{\beta})^2 + \beta_3 y_{t-1}^2)^{\frac{1}{2}} \tag{19}$$

CAViaR model with the intra-day range for the intra-day low and high (CAViaR-Range-LH):

$$\tilde{q}_t(\boldsymbol{\beta}) = \beta_1 + \beta_2 \tilde{q}_{t-1}(\boldsymbol{\beta}) + \beta_3 Range_{t-1} \tag{20}$$

CAViaR model with the intra-day range and the overnight return for the intra-day low and high (CAViaR-Range-N-LH):

$$\tilde{q}_t(\boldsymbol{\beta}) = \beta_1 + \beta_2 \tilde{q}_{t-1}(\boldsymbol{\beta}) + \beta_3 Range_{t-1} + \beta_4 |y_{N,t-1}| \tag{21}$$

In addition to the above proposed new CAViaR models for the intra-day low and high, we take the view that there is valuable and different information provided by the intra-day data and the simultaneous modeling of the quantiles corresponding to different probability levels, which leads us to consider the simultaneous estimation of the quantiles of the intra-day low and the quantiles of the intra-day high based on the intraday range and the overnight return. We consider a model, which takes the structure of the MV-MQ-CAViaR model in expression (15), as the following form:

MV-MQ-CAViaR model with intra-day range and the overnight return for the intra-day low and the intra-day high (MV-MQ-CAViaR-Range-N-LH):

$$\tilde{q}_{i,j,t}(\boldsymbol{\beta}) = \alpha_{i,j,0} + \alpha_{i,j,1} Range_{t-1} + \alpha_{i,j,2} |y_{N,t-1}| + \tilde{\boldsymbol{q}}_{t-1}^{'}(\boldsymbol{\beta}) \boldsymbol{\gamma}_j \tag{22}$$

where the notations are as in expression (15). We also consider a MV-MQ-CAViaR model, similar to the MQ-CAViaR-SAV model in expression (12), for the intra-day low and the intra-day high based on the returns, in which each quantile is assumed to follow a CAViaR-SAV type model. The model has the following form:

MV-MQ-CAViaR symmetric absolute value model for the intra-day low and the intra-day high (MV-MQ-CAViaR-Range-N-LH):

$$\tilde{q}_{i,j,t}(\boldsymbol{\beta}) = \alpha_{i,j,0} + \alpha_{i,j,1}|y_{i,t}| + \tilde{\boldsymbol{q}}_{t-1}^{'}(\boldsymbol{\beta})\boldsymbol{\gamma}_j \tag{23}$$

In expression (22) and expression (23), the parameters are estimated by minimizing the quantile regression check loss function in expression (16).

4 Empirical Study of VaR Estimation

To compare the accuracy of the quantile forecasts from our proposed methods with benchmark methods, we used the data described in Section 3.1. We considered the 0.5%, 1%, 5%, 95%, 99% and 99.5% probability levels. A rolling window of 1800 days was used for estimation, and a post-sample period of 1500 days was used to evaluate day-ahead quantile estimation. The post-sample period covered the very volatile financial crisis period and a recent relatively tranquil period.

4.1 Methods Used for Estimating VaR

4.1.1 Benchmark Methods

In this section, we describe the benchmark methods that are implemented in this study. Note that the benchmark models can utilize the daily return as well as intra-day information, however, they are estimated using the daily return only. The nonparametric benchmark methods performed very poorly, thus we omit the results from this paper. As benchmark methods, we implemented the GARCH(1,1) model, the GJR-GARCH(1,1) model, and the FIGARCH(1,d,1) model. The error term of each model was assumed to follow a Student t distribution. As semiparametric benchmarks, we implemented three standard CAViaR models based on the daily return, one CAViaR model based on intra-day range, and one CAViaR model based on the intra-day range and the overnight return. The three standard CAViaR models are the CAViaR-SAV model in expression (3), the CAViaR-AS model in expression (4) and the CAViaR-IndG model in expression (5), the CAViaR model based on the intra-day range returns is the CAViaR-Range model in expression (7), and the CAViaR model based on the intra-day range and the overnight return is the CAViaR-Range-N model in expression (10). We left out the CAViaR-Adaptive model expression (2) because this model is not good at capturing the underlying dynamics of the quantiles, and has been shown to perform not as competitively as the other CAViaR models (Manganelli and Engle 2004). The five benchmark CAViaR models were estimated by minimizing the quantile regression check loss function in expression (6), as used by Engle and Manganelli (2004). As two additional benchmarks, we implemented two MQ-CAViaR models, with estimation performed by following the approach of White et al. (2010), which involves minimizing the quantile regression check loss function of expression (14). The first model is the MQ-CAViaR-SAV model based on the returns in expression (12), and the second model is the MQ-CAViaR-Range-N model based on the intra-day range and the overnight return in expression (13). Note that in a MQ-CAViaR model, the number of parameters is $5 * (k + 3)$, where k is the number of different quantiles involved. This number is very large for $k \geq 2$, which makes the parameter estimation procedure numerically challenging. Therefore, in this paper, we only implemented a MQ-CAViaR model with two probability levels, θ and $1 - \theta$, where $\theta = 0.5\%$, 1 or 5%.

4.1.2 Proposed Methods

Turning to our proposed new approaches, we implemented all the five CAViaR models and two MV-MQ-CAViaR models to estimate the quantiles of the intra-day low and high series. The corresponding probability levels $\tilde{\theta}$ were calculated as described in Section 3.3, and they were also re-estimated for each new moving window. The five CAViaR models for the intra-day low and high series, which essentially have the same structure as the five benchmark CAViaR models for the returns, are as follows: the CAViaR-SAV-LH in expression (17), the CAViaR-AS-LH model in expression (18), the CAViaR-IndG-LH model in expression (19), the CAViaR-Range-LH model in expression (20), and the CAViaR-Range-N-LH model in expression (21). These models were estimated by minimizing the quantile regression check loss function in expression (6), with the returns replaced by intra-day low/high. Two MV-MQ-CAViaR models are the MV-MQ-CAViaR-SAV-LH model in expression (23) and the MV-MQ-CAViaR-Range-N-LH model in expression (22), with estimation performed by following the approach of White et al. (2015), which involves minimizing the quantile regression check loss function of expression (16). For the same consideration as the case of the MQ-CAViaR models, we only implemented the MV-MQ-CAViaR models with two probability levels, $\tilde{\theta}$ and $1 \tilde{-} \theta$.

All the CAViaR models were estimated using a procedure similar to that described by Engle and Manganelli (2004). For each model, let d be the number of the parameters involved in the model, then a total number of 10^{d+1} initial trial vectors of parameters were uniformly randomly generated between 0 and 1. We calculated the value of the corresponding quantile regression check loss function for each initial trial vector. The 24 vectors producing the lowest values of the quantile regression check loss function were passed on to the Nelder-Mead algorithm as the starting vectors. The resulting vector that produced the lowest quantile regression check loss function value was selected as the optimal parameter vector. For the MQ-CAViaR models and the MV-MQ-CAViaR models, we adopted an approach similar to that described by White et al. (2015). We took the parameter estimates of the corresponding CAViaR models and set the remaining parameters to zero, and used the parameter values as the primal starting vector in the Nelder-Mead optimization algorithm. To increase the accuracy of the estimation procedure, we considered 42 different starting vectors near the primal starting vector in the optimization routine, 21 of which were obtained by adding random vectors drawn from a normal distribution with mean 0 and standard deviation $\frac{1}{20}$, and the rest of which were obtained by adding random vectors drawn from a normal distribution with mean 0 and standard deviation $\frac{1}{50}$. The numbers $\frac{1}{20}$ and $\frac{1}{50}$ were the choice of White et al. (2015) in their Matlab code. We increased the number of the different starting vectors, which is 20 in the Matlab code of White et al. (2015), to 42 in order to improve the accuracy of the numerical estimation procedure. The number 42 is chosen so as to fully utilize the parallel computing power on our computer with 6 core processors. Our Matlab code is available on request.

4.2 Evaluation Methods

We evaluated the post-sample performance of the models using two back-testing methods: a test of unconditional coverage and the dynamic quantile test of Engle and Manganelli (2004). The unconditional coverage test is based on the hit percentage, which is the proportion of the observations of the returns that fell below the estimated quantiles. For the quantile with probability level θ, the ideal value of the proportion is θ, and this is the null hypothesis in a standard test for a sample proportion. The dynamic quantile test, which jointly tests both the unconditional coverage and the independence of violations jointly, evaluates whether the random variable $Hit_t = I(y_t < q_t) - \theta$ has an i.i.d. Bernoulli distribution with parameter θ, and is independent of the lags of the Hit_t variable, and the lagged conditional quantile forecast. Following Engle and Manganelli (2004), we set the number of lagged Hit_t variables involved in the dynamic quantile test to be four.

4.3 Post-Sample Results

In this section, we present our empirical results. Table 1 presents the parameters of the MQ-CAViaR-Range-N model of expression (13) and the MV-MQ-CAViaR-Range-N-LH model of expression (22). To save space, we only present the parameter estimates for the FTSE100 series. Since the parameters are re-estimated for each rolling window, there are 1500 different estimated parameter vectors, so we present the estimated parameters derived using the first moving window of 1800 days. Table 2 presents the number of rejections for the joint null hypothesis that the coefficients β_5 and β_9 equal zero at the 5% level. These coefficients represent the co-movement between the quantiles. It can be seen that the estimated parameters obtained using the two different estimation methods are different. In each case, the joint null hypothesis that the coefficients β_5 and β_9, equal zero is rejected for the majority of cases, indicating that the MQ-CAViaR and the MV-MQ-CAViaR modeling does enable a richer modeling of quantile dynamics than the individual CAViaR modeling.

We now consider the post-sample evaluation results. We first report the evaluation results for the FTSE100. Table 3 presents the hit percentage results, where the significance of the unconditional coverage test at 5% and 1% levels are indicated by * and **, respectively. The p-values of the dynamic quantile test are presented in Table 4 for

Table 1: Parameter estimates of the MQ-CAViaR-Range-N and the MV-MQ-CAViaR-Range-N-LH for FTSE100, derived using the first moving window of 1800 days.

	MQ-CAViaR-Range-N			MV-MQ-CAViaR-Range-N-LH		
$100 \times \theta$	0.5 & 99.5	1 & 99	5 & 95	0.5 & 99.5	1 & 99	5 & 95
β_1	0.160	0.020	0.057	0.138	0.039	-0.002
β_2	0.643	0.451	0.246	0.444	0.366	0.200
β_3	0.246	0.319	0.073	0.258	0.390	0.241
β_4	0.547	0.777	0.911	0.820	0.768	0.956
β_5	0.110	-0.051	-0.155	-0.138	-0.009	-0.166
β_6	-0.007	0.034	0.036	0.079	0.028	-0.008
β_7	0.211	0.266	0.179	0.274	0.253	0.183
β_8	0.244	0.106	0.077	0.151	0.121	0.241
β_9	0.051	0.004	-0.057	-0.088	-0.017	-0.029
β_{10}	0.784	0.805	0.877	0.890	0.831	0.835

Table 2: Number of rejections for the joint null hypothesis that all off-diagonal coefficients β_5 and β_9 are equal to zero at the 5% level for the MQ-CAViaR-Range-N model and the MV-MQ-CAViaR-Range-N-LH model.

	MQ-CAViaR-Range-N			MV-MQ-CAViaR-Range-N-LH		
$\theta \times 100$	0.5 & 99.5	1 & 99	5 & 95	0.5 & 99.5	1 & 99	5 & 95
SP500	1487	1461	1498	1500	1498	1494
FTSE100	1490	1500	1492	1500	1500	1492
NIKKEI225	1101	851	1500	1263	1500	1484
DAX30	1490	1497	1500	1500	1500	1499
CAC40	1403	1500	1500	1489	1500	1499

the FTSE 100. In Table 3 and Table 4, the numbers of rejections from the tests at 5% significance level are reported in the final column. Larger p-values and smaller values for the number of test rejections are preferred. Table 5 summarizes the unconditional coverage and the dynamic quantile test results for the five stock indices. It is impressive see that the performance is encouraging for the models that employ the intra-day low and high series. Each CAViaR model, based on the intra-day low and high data, outperforms the same model based on the daily returns. In other words, every model with the notation 'LH' outperforms its counterpart without 'LH'. For example, the CAViaR-SAV-LH model outperforms the CAViaR-SAV model. The CAViaR-GARCH-LH model performs the best among the CAViaR models. Moving to the MQ-CAViaR models and the MV-MQ-CAViaR models, the MV-MQ-CAViaR-Range-N-LH model outperforms the MQ-CAViaR-Range-N model, and the MV-MQ-CAViaR-SAV-LH and the MQ-CAViaR-SAV model perform similarly. Overall, the MV-MQ-CAViaR-Range-N-LH model and the CAViaRRange-LH model perform the best among all the VaR models implemented.

To summarize, the results of the proposed methods are promising. It is encouraging to find consistently good results from the CAViaR models based on the intra-day low and high data. In fact, all of these CAViaR models and the MV-MQCAViaR-Range-N-LH model outperform the CAViaR and MQCAViaR-Range models for the returns, despite the fact that the quantiles of the intra-day low and high are only approximations of the quantiles of the returns. The only exception is the MV-MQCAViaR-SAV-LH model, but it is reassuring to see that the MV-MQ-CAViaRSAV-LH model is not outperformed by the MQ-CAViaR-SAV model.

Table 3: Hit percentage, the unconditional coverage test results, and the number of test rejections at 5% significance for the FTSE100.

$\theta \times 100$	0.5	1	5	95	99	99.5	Number of rejections
Benchmarks							
GARCH-t	0.5	1.5	6.7**	95.3	99.3	99.7	1
GJR-t	0.8	1.5	6.1	95.5	99.5*	99.7	1
CAViaR-SAV	0.5	0.9	5.9	94.9	98.7	99.3	0
CAViaR-AS	0.9	1.1	5.2	94.3	98.3*	98.8**	2
CAViaR-IndG	0.4	0.7	5.7	95.1	98.8	99.3	0
CAViaR-Range	0.7	0.9	5.7	95.0	98.4*	99.0*	2
CAViaR-Range-N	0.7	1.1	5.1	95.3	98.5	98.9**	1
MQ-CAViaR-SAV	0.5	1.0	5.9	94.8	98.5	99.4	0
MQ-CAViaR-Range-N	0.6	1.3	5.4	95.1	98.6	99.1*	1
Proposed Models for Intra-day Low/High							
CAViaR-SAV-LH	0.3	0.8	5.4	94.9	98.9	99.6	0
CAViaR-AS-LH	0.9	1.1	5.2	94.4	98.9	99.4	0
CAViaR-IndG-LH	0.3	0.7	5.4	94.8	99.1	99.4	0
CAViaR-Range-LH	0.4	0.7	6.1	94.9	98.9	99.5	0
CAViaR-Range-N-LH	0.5	0.8	5.8	95.1	98.7	99.5	0
MV-MQ-CAViaR-SAV-LH	0.4	0.8	5.5	94.7	98.9	99.3	0
MV-MQ-CAViaR-Range-N-LH	0.5	0.9	5.6	94.5	98.7	99.5	0

Table 4: Dynamic quantile test p-values and the number of test rejections at 5% significance for the FTSE100.

$\theta \times 100$	0.5	1	5	95	99	99.5	Number of rejections
Benchmarks							
GARCH-t	0.000	0.003	0.029	0.760	0.915	0.811	3
GJR-t	0.750	0.200	0.266	0.363	0.580	0.949	0
CAViaR-SAV	0.001	0.066	0.078	0.732	0.746	0.895	1
CAViaR-AS	0.004	0.495	0.591	0.470	0.052	0.000	2
CAViaR-IndG	0.000	0.155	0.398	0.322	0.916	0.902	1
CAViaR-Range	0.849	0.422	0.625	0.322	0.220	0.181	0
CAViaR-Range-N	0.941	0.538	0.997	0.386	0.237	0.003	1
MQ-CAViaR-SAV	0.001	0.069	0.192	0.574	0.239	0.994	1
MQ-CAViaR-Range-N	0.975	0.444	0.903	0.458	0.404	0.325	0
Proposed Models for Intra-day Low/High							
CAViaR-SAV-LH	0.985	0.020	0.642	0.351	0.956	0.993	1
CAViaR-AS-LH	0.323	0.957	0.624	0.288	0.976	0.771	0
CAViaR-IndG-LH	0.981	0.146	0.612	0.602	0.996	0.997	0
CAViaR-Range-LH	0.919	0.167	0.244	0.178	0.953	0.952	0
CAViaR-Range-N-LH	0.999	0.987	0.680	0.170	0.860	0.961	0
MV-MQ-CAViaR-SAV-LH	0.996	0.020	0.428	0.623	0.975	0.902	1
MV-MQ-CAViaR-Range-N-LH	0.962	0.295	0.884	0.524	0.775	0.972	0

Note: Larger values of the dynamic quantile test p-value are better.

Table 5: Summary of the unconditional coverage and the dynamic quantile test results for all five indices: Number of test rejections at 5% significance.

	Unconditional Coverage							Dynamic Quantile						
$\theta \times 100$	0.5	1	5	95	99	99.5	Total	0.5	1	5	95	99	99.5	Total
Benchmarks														
GARCH-t	0	1	3	0	2	1	7	3	2	2	0	0	0	7
GJR-t	0	1	2	0	2	0	5	2	1	0	0	1	0	4
CAViaR-SAV	0	1	0	0	0	1	2	4	1	1	0	1	0	7
CAViaR-AS	0	1	0	0	2	3	6	3	1	0	0	0	3	7
CAViaR-IndG	0	0	0	0	0	1	1	3	2	0	0	0	0	5
CAViaR-Range	0	0	1	1	2	2	6	2	2	1	1	1	0	7
CAViaR-Range-N	0	1	0	1	1	2	5	2	0	0	1	1	2	6
MQ-CAViaR-SAV	0	1	0	0	0	1	2	5	1	1	0	0	0	7
MQ-CAViaR-Range-N	1	1	0	0	1	2	5	3	1	0	0	1	1	6
Proposed Models for Intra-day Low/High														
CAViaR-SAV-LH	0	0	0	1	0	1	2	2	3	0	1	0	1	7
CAViaR-AS-LH	1	1	0	0	0	1	3	2	1	1	0	1	1	6
CAViaR-IndG-LH	0	0	0	0	0	0	0	2	1	0	1	0	0	4
CAViaR-Range-LH	0	0	1	1	0	1	3	2	1	1	2	0	0	6
CAViaR-Range-N-LH	0	0	0	1	0	1	2	2	0	0	1	0	1	4
MV-MQ-CAViaR-SAV-LH	1	0	0	0	2	1	4	2	2	0	2	0	0	6
MV-MQ-CAViaR-Range-N-LH	0	0	0	0	0	1	1	2	0	0	1	0	0	3

Note: Smaller values of the rejection numbers are better.

5 Conclusion

In this paper, we have introduced a new approach to estimating conditional VaR using quantile regression. The main contribution of this paper is that we show that we have found an interesting relationship between the intra-day low and high series and the daily returns, that is, the quantiles of the intra-day low and high are good approximations of the quantiles of the returns. Several CAViaR models and MV-MQ-CAViaR models are proposed to estimate the quantiles of the intra-day low and high series. A further contribution of the paper is that we have found that the intra-day range and the overnight return are two useful explanatory variables for VaR estimation. Our empirical study suggests that the proposed methods improve the VaR forecasting accuracy.

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Appendix A: Proofs

In this appendix, we show that if the conditions $P(L_t < x) = \lambda P(y_t < x)$ and $P(H_t > x) = \lambda P(y_t > x)$ hold, then the parameter estimation of any (MV-MQ-) CAViaR model for the intra-day low and high series is more efficient than the parameter estimation of the same (MQ-) CAViaR model for the returns. We start with the CAViaR models, in which case we only establish the result for the lower quantiles without loss of generality.

Theorem 1. Let $q_t(\boldsymbol{\beta}) = f(y_{t-1}, \boldsymbol{\Psi}_{t-1} \ldots y_1, \boldsymbol{\Psi}_1; \boldsymbol{\beta})$ be any CAViaR model for the θth quantiles of the returns, and let $\tilde{q}_t(\tilde{\boldsymbol{\beta}}) = f(y_{t-1}, \boldsymbol{\Psi}_{t-1} \ldots y_1, \boldsymbol{\Psi}_1; \tilde{\boldsymbol{\beta}})$ be the same model for the $(\lambda\theta)th$ quantiles of the intra-day low. In other words, $q_t(\boldsymbol{\beta})$ and $\tilde{q}_t(\tilde{\boldsymbol{\beta}})$ have the same structure, and only differ by the parameters. By Theorem 2 in Engle and Manganelli (2004), we can obtain the asymptotic covariance matrices of $\boldsymbol{\beta}$ and $\tilde{\boldsymbol{\beta}}$, which are denotes as $\boldsymbol{D}_T^{-1}\boldsymbol{A}_T\boldsymbol{D}_T^{-1}$ and $\tilde{\boldsymbol{D}}_T^{-1}\tilde{\boldsymbol{A}}_T\tilde{\boldsymbol{D}}_T^{-1}$. Then it can be shown that $\boldsymbol{D}_T^{-1}\boldsymbol{A}_T\boldsymbol{D}_T^{-1} - \tilde{\boldsymbol{D}}_T^{-1}\boldsymbol{A}_T\tilde{\boldsymbol{D}}_T^{-1}$ is positive definite.

Proof of Theorem 1

Proof. Since $P(L_t < x) = \lambda P(y_t < x)$, we have that the $\tilde{\theta}th$ quantiles of L_t equal to the θth quantiles of y_t. Since $q_t(\boldsymbol{\beta})$ and $\tilde{q}_t(\tilde{\boldsymbol{\beta}})$ have the same model structure, so $\tilde{\boldsymbol{\beta}}^0 = \boldsymbol{\beta}^0$ by the unique identification condition Assumption E7 in Engle and Manganelli (2004), where $\tilde{\boldsymbol{\beta}}^0$ and $\boldsymbol{\beta}^0$ are the true underlying parameter vectors of $\tilde{q}_t(\tilde{\boldsymbol{\beta}})$ and $q_t(\boldsymbol{\beta})$, respectively. Consequently, $\nabla q_t(\tilde{\boldsymbol{\beta}}^0) = \nabla q_t(\boldsymbol{\beta}^0)$.

Since $\boldsymbol{A}_T = E\left[\theta(1-\theta)\sum_{t=1}^{T}\nabla q_t(\boldsymbol{\beta}^0)\nabla' q_t(\boldsymbol{\beta}^0)\right]$ and $\tilde{\boldsymbol{A}}_T = E\left[\lambda\theta(1-\lambda\theta)\sum_{t=1}^{T}\nabla q_t(\boldsymbol{\beta}^1) \nabla' q_t(\boldsymbol{\beta}^1)\right]$, it can be shown that $\tilde{\boldsymbol{A}}_T^{-1} = \frac{(1-\theta\lambda)\lambda}{(1-\theta)}(\boldsymbol{A}_T)^{-1}$. Similarly, we can get $\tilde{\boldsymbol{D}}_T^{-1} = \frac{1}{\lambda}(\boldsymbol{D}_T)^{-1}$. Therefore, $\tilde{\boldsymbol{D}}_T^{-1}\tilde{\boldsymbol{A}}_T\tilde{\boldsymbol{D}}_T^{-1} = \frac{(1-\theta\lambda)\lambda}{(1-\theta)}\boldsymbol{D}_T^{-1}\boldsymbol{A}_T\boldsymbol{D}_T^{-1}$. Since λ is always larger than

1 by its definition, $\frac{(1-\theta\lambda)\lambda}{(1-\theta)}$ is always positive:

$$
\begin{aligned}
& \frac{(1-\theta\lambda)\lambda}{(1-\theta)} > 1 \\
\iff \quad & (1-\theta\lambda)\lambda > 1-\theta \\
\iff \quad & \theta\lambda - \lambda + (1-\theta) < 0 \\
\iff \quad & (\lambda-1)[\theta\lambda - (1-\theta)] < 0 \\
\iff \quad & \lambda \in (1, \frac{1-\theta}{\theta})
\end{aligned}
$$

So, $\tilde{\boldsymbol{D}}_T^{-1}\tilde{\boldsymbol{A}}_T\tilde{\boldsymbol{D}}_T^{-1} - \boldsymbol{D}_T^{-1}\boldsymbol{A}_T\boldsymbol{D}_T^{-1} = \textit{positive constant} \times \boldsymbol{D}_T^{-1}\boldsymbol{A}_T\boldsymbol{D}_T^{-1}$, which is positive definite by Assumption AN3 in Engle and Manganelli (2004).

□

Next, we establish the relative efficiency for the MV-MQ-CAViaR models against the MQ-CAViaR model. The comparison is more difficult than the CAViaR models, so we only establish the results for the bivariate case considered in the empirical study, where the θth and the $(1-\theta)th$ quantiles for the returns, where $\theta < 0.5$:

Theorem 2. Let $q_{i,t}(\boldsymbol{\beta}) = f(y_{t-1}, \boldsymbol{\Psi}_{t-1} \ldots y_1, \boldsymbol{\Psi}_1; \boldsymbol{\beta})$ be any CAViaR model for the θth quantiles and the $(1-\theta)th$ quantiles of the returns, and let $\tilde{q}_{i,j,t}(\tilde{\boldsymbol{\beta}}) = f(y_{t-1}, \boldsymbol{\Psi}_{t-1} \ldots y_1, \boldsymbol{\Psi}_1; \tilde{\boldsymbol{\beta}})$ be the same model for the $(\lambda\theta)th$ quantiles of the intra-day low and the $(1-\lambda\theta)th$ of the intra-day high. That is, $q_{i,t}(\boldsymbol{\beta})$ and $\tilde{q}_{i,t}(\tilde{\boldsymbol{\beta}})$ have the same structure, and only differ by the parameters. Note that j is 1 since there is only one probability level corresponding to the intra-day low or the intra-day high, so we omit the indicator j for simplicity, in which case $\tilde{q}_{i,j,t}(\tilde{\boldsymbol{\beta}})$ is denoted by $\tilde{q}_{i,,t}(\tilde{\boldsymbol{\beta}})$. By Theorem 2 in White et al. (2015), we can obtain the asymptotic covariance matrices of $\boldsymbol{\beta}$ and $\tilde{\boldsymbol{\beta}}$, which are denotes as $\boldsymbol{D}_T^{-1}\boldsymbol{A}_T\boldsymbol{D}_T^{-1}$ and $\tilde{\boldsymbol{D}}_T^{-1}\tilde{\boldsymbol{A}}_T\tilde{\boldsymbol{D}}_T^{-1}$. Then it can be shown that $\boldsymbol{D}_T^{-1}\boldsymbol{A}_T\boldsymbol{D}_T^{-1} - \tilde{\boldsymbol{D}}_T^{-1}\boldsymbol{A}_T\tilde{\boldsymbol{D}}_T^{-1}$

is positive definite.[3]

Before we proceed to prove Theorem 2, an extra condition, which seems always to be satisfied in practice, needs to be imposed. We also state two lemmas that are used in the proof. In the MV-MQ-CAViaR model, we use $\tilde{y}_{1,t}$ and $\tilde{y}_{2,t}$ to denote L_t and H_t respectively, and we use $\theta_1$and θ_2 to denote θ and $1-\theta$ respectively.

Assumption 1. For any return series, let $q^*_{1,t}$ and $q^*_{2,t}$ be the true θth and $(1-\theta)th$ quantiles of the return series, the quantity $1 - \frac{\mathrm{P}(\{L_t<q^*_{1,t}\}\cap\{H_t>q^*_{2,t}\})}{\lambda(\lambda-1)\theta}$ is positive.

Assumption 1 seems to be always met in practice when θ is sufficiently small. For the all five series considered, we found $\frac{\mathrm{P}(\{L_t<q^*_{1,t}\}\cap\{H_t>q^*_{2,t}\})}{\theta} < 0.15$ for $\theta \leq 2.5\%$ (see Figure 3), and $\lambda > 1.154$ (see Figure 1), in which case $\frac{\mathrm{P}(\{L_t<q^*_{1,t}\}\cap\{H_t>q^*_{2,t}\})}{\lambda(\lambda-1)\theta} \leq \frac{0.15}{(1.154-1)1.153} < 0.85 < 1$.

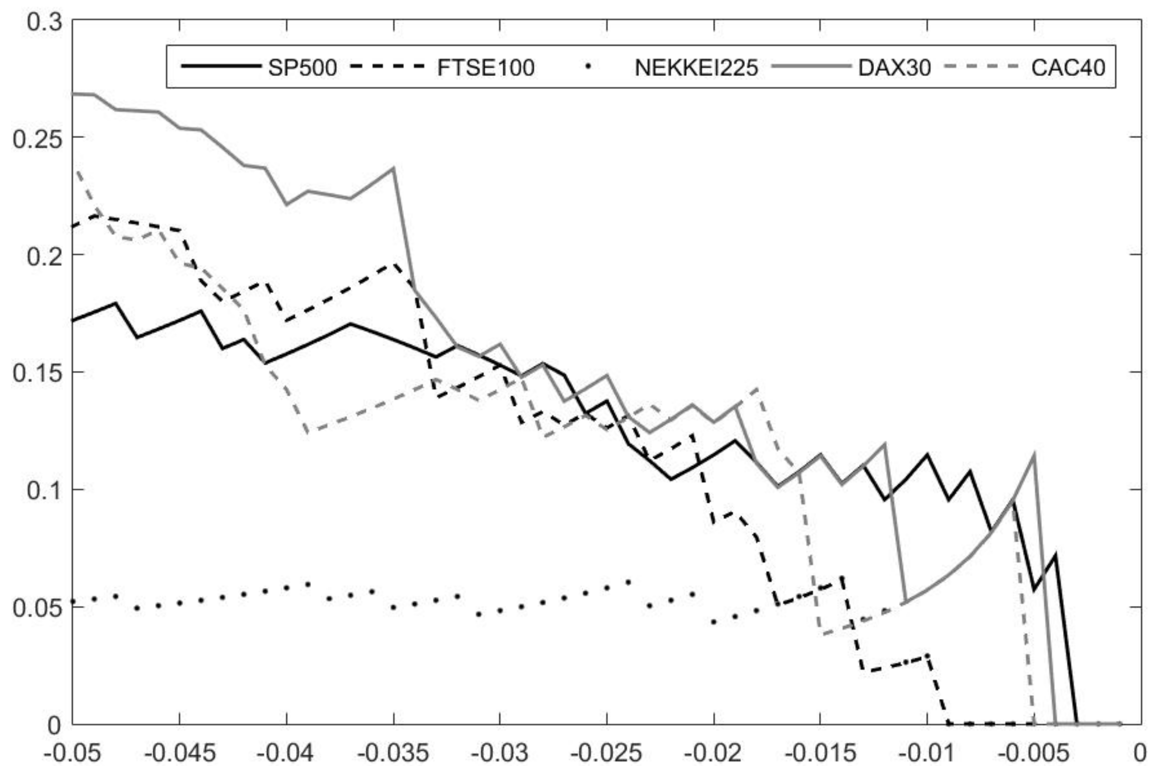
Next, we state a few lemmas that are used in our proof:

Lemma 1. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are two symmetric matrices of the same size, if $\boldsymbol{A}$ is nonsingular and $\boldsymbol{B}$ is semi-positive definite, then $\boldsymbol{ABA}$ is semi-positive definite. In particular, if $\boldsymbol{B}$ is positive definite, then $\boldsymbol{ABA}$ is also positive definite.

Proof. If $\boldsymbol{B}$ is semi-positive definite, we show that $\boldsymbol{x}^t\boldsymbol{ABAx}$ is non-negative for any column vector $\boldsymbol{x}$. Since $\boldsymbol{A}$ is symmetric, $(\boldsymbol{Ax})^t = \boldsymbol{x}^t\boldsymbol{A}$. Let $\boldsymbol{y} = \boldsymbol{Ax}$, we obtain $\boldsymbol{x}^t\boldsymbol{ABAx} = \boldsymbol{y}^t\boldsymbol{By}$. As $\boldsymbol{B}$ is semi-positive definite, $\boldsymbol{x}^t\boldsymbol{ABAx} = \boldsymbol{y}^t\boldsymbol{By}$ is always non-negative. If $\boldsymbol{B}$ is positive definite, let $\boldsymbol{x}$ be a nonzero vector, then $\boldsymbol{y} = \boldsymbol{Ax}$ is nonzero since $\boldsymbol{A}$ is nonsingular. Therefore, $\boldsymbol{y}^t\boldsymbol{By}$ is positive by the positive definiteness of $\boldsymbol{B}$, which completes the proof.

[3]Note that $\boldsymbol{A}_T(\tilde{\boldsymbol{A}}_T)$ and $\boldsymbol{D}_T(\tilde{\boldsymbol{D}}_T)$ are the matrices $\boldsymbol{V}^*$ and $\boldsymbol{Q}^*$ in Theorem 2 in White et al. (2015). We choose these notations so as to be consistent with Theorem 1 in this paper.

Figure 3: The ratio $\frac{\mathrm{P}(\{L_t<q_{1,t}\}\cap\{H_t>q_{2,t}\})}{\theta}$



□

Lemma 2. Suppose the model considered is a bivariate MQ-CAViaR model, then $\boldsymbol{A}_T = \sum_{i=1}^{2}\sum_{j=1}^{2} E\Big\{\Big\{-\theta_j\theta_i + \min(\theta_j,\theta_i)\Big\}\nabla q_{j,t}(\boldsymbol{\beta}^0)\nabla' q_{i,t}(\boldsymbol{\beta}^0)\Big\}$ and $\tilde{\boldsymbol{A}}_T = \sum_{i=1}^{2}\sum_{j=1}^{2} E\Big\{\Big\{-\tilde{\theta}_j\tilde{\theta}_i + \mathrm{P}(\{\tilde{y}_{i,t} < \tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0)\}\cap\{\tilde{y}_{j,t} < \tilde{q}_{i,t}(\tilde{\boldsymbol{\beta}}^0)\}\Big\}\nabla \tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0)\nabla' \tilde{q}_{i,t}(\tilde{\boldsymbol{\beta}}^0)\Big\}$, where $\boldsymbol{A}_T$ and $\tilde{\boldsymbol{A}}_T$ are as in Theorem 2.

Proof. We first compute $\tilde{\boldsymbol{A}}_T = E(\boldsymbol{\eta}_t\boldsymbol{\eta}_t')$, where $\boldsymbol{\eta}_t = \sum_{j=1}^{2}\nabla\tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0)\psi_{\theta_j}(\tilde{y}_{j,t} - \tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0))$ as in Theorem 2 in White et al. (2015):

$$
\begin{aligned}
\tilde{\boldsymbol{A}}_T &= \sum_{i=1}^{2}\sum_{j=1}^{2} E\Big\{\nabla\tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0)\psi_{\tilde{\theta}_j}(\tilde{y}_{j,t} - \tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0))\psi_{\tilde{\theta}_i}(\tilde{y}_{i,t} - \tilde{q}_{i,t}(\tilde{\boldsymbol{\beta}}^0))\nabla'\tilde{q}_{i,t}(\tilde{\boldsymbol{\beta}}^0)\Big\}\\
&= \sum_{i=1}^{2}\sum_{j=1}^{2} E\Big\{\psi_{\tilde{\theta}_j}(\tilde{y}_{j,t} - \tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0))\psi_{\tilde{\theta}_i}(\tilde{y}_{i,t} - \tilde{q}_{i,t}(\tilde{\boldsymbol{\beta}}^0))\nabla\tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0)\nabla'\tilde{q}_{i,t}(\tilde{\boldsymbol{\beta}}^0)\Big\}\\
&= \sum_{i=1}^{2}\sum_{j=1}^{2} E\Big\{\Big\{\psi_{\tilde{\theta}_j}(\tilde{y}_{j,t} - \tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0))\psi_{\tilde{\theta}_i}(\tilde{y}_{i,t} - \tilde{q}_{i,t}(\tilde{\boldsymbol{\beta}}^0))\Big\}\Big\{\nabla\tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0)\nabla'\tilde{q}_{i,t}(\tilde{\boldsymbol{\beta}}^0)\Big\}\Big\}\\
&= \sum_{i=1}^{2}\sum_{j=1}^{2} E\Big\{\psi_{\tilde{\theta}_j}(\tilde{y}_{j,t} - \tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0))\psi_{\tilde{\theta}_i}(\tilde{y}_{i,t} - \tilde{q}_{i,t}(\tilde{\boldsymbol{\beta}}^0))\Big\}E\Big\{\nabla\tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0)\nabla'\tilde{q}_{i,t}(\tilde{\boldsymbol{\beta}}^0)\Big\}\\
&= \sum_{i=1}^{2}\sum_{j=1}^{2} E\Big\{-\tilde{\theta}_j\tilde{\theta}_i + \tilde{\theta}_i I(\tilde{y}_{j,t} < \tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0)) + \tilde{\theta}_j I(\tilde{y}_{i,t} < \tilde{q}_{i,t}(\tilde{\boldsymbol{\beta}}^0))\\
&\quad + I(\tilde{y}_{i,t} < \tilde{q}_{i,t}(\tilde{\boldsymbol{\beta}}^0))I(\tilde{y}_{j,t} < \tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0))\Big\}E\Big\{\nabla\tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0)\nabla'\tilde{q}_{i,t}(\tilde{\boldsymbol{\beta}}^0)\Big\}\Big\}\\
&= \sum_{i=1}^{2}\sum_{j=1}^{2}\Big\{-\tilde{\theta}_j\tilde{\theta}_i + \mathrm{P}(\{\tilde{y}_{i,t} < \tilde{q}_{i,t}(\tilde{\boldsymbol{\beta}}^0)\}\cap\{\tilde{y}_{j,t} < \tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0)\})\Big\}\\
&\quad E\Big\{\nabla\tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0)\nabla'\tilde{q}_{i,t}(\tilde{\boldsymbol{\beta}}^0)\Big\}
\end{aligned}
$$

The fourth line uses the fact that the random variable $\psi_{\tilde{\theta}_j}(\tilde{y}_{j,t} - \tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0))$ is independent of the past information and that $\nabla\tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0)\nabla'\tilde{q}_{i,t}(\tilde{\boldsymbol{\beta}}^0)$ is determined by the past information, and the last line uses the fact that $E\Big\{I(\tilde{y}_{j,t} < \tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0))\Big\} = \tilde{\theta}_j$ and

$$E\Big\{I(\tilde{y}_{i,t} < \tilde{q}_{i,t}(\tilde{\boldsymbol{\beta}}^0))I(\tilde{y}_{j,t} < \tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0))\Big\} = \mathrm{P}(\{\tilde{y}_{i,t} < \tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0)\}\cap\{\tilde{y}_{j,t} < \tilde{q}_{i,t}(\tilde{\boldsymbol{\beta}}^0)\}).$$

The derivation for $\boldsymbol{A}_T$ is similar. We replace $\tilde{\theta}_j$ by θ, $\tilde{y}_{j,t}$ by y_t, and $\nabla\tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0)$ by $q_{j,t}(\boldsymbol{\beta}^0)$ in the above equations, and also use the face that $\mathrm{P}(\{y_t < q_{i,t}(\boldsymbol{\beta}^0)\}\cap\{y_t < q_{j,t}(\boldsymbol{\beta}^0)\}) = \min(\theta_i, \theta_j)$.

□

Proof of Theorem 2

Proof. For simplicity, let $\boldsymbol{C}_{i,j}$ denote $\nabla q_{i,t}(\boldsymbol{\beta}^0)\nabla' q_{j,t}(\boldsymbol{\beta}^0)$ and let c denote the probability $\mathrm{P}(\{L_t < q_{1,t}(\tilde{\boldsymbol{\beta}}^0)\}\cap\{H_t > q_{2,t}(\tilde{\boldsymbol{\beta}}^0)\})$. By the assumption that $P(L_t < x) = \lambda P(y_t < x)$ and $P(H_t > x) = \lambda P(y_t > x)$, we have that the $(\tilde{\theta}_j)th$ quantiles of $y_{j,t}$ equal to the $(\theta_j)th$ quantiles of y_t. So $\tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0) = q_{j,t}(\boldsymbol{\beta}^0)$, and by the unique identification condition Assumption 4 in White et al. (2015) we have $\tilde{\boldsymbol{\beta}}^0 = \boldsymbol{\beta}^0$. So $\nabla\tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0) = \nabla q_{j,t}(\boldsymbol{\beta}^0)$, and hence $\tilde{\boldsymbol{D}}_T = \lambda\boldsymbol{D}_T$. So it is equivalent to show that $\boldsymbol{D}_T^{-1}\boldsymbol{A}_T\boldsymbol{D}_T^{-1} - \frac{1}{\lambda^2}\boldsymbol{D}_T^{-1}\tilde{\boldsymbol{A}}_T\boldsymbol{D}_T^{-1}$ is semi-positive definite. Note that $\boldsymbol{D}_T$ is symmetric by its definition, and it is positive definite hence nonsingular by Assumption 6 in White et al. (2015). By Lemma 1, it suffices to show that $\boldsymbol{A}_T - \frac{1}{\lambda^2}\tilde{\boldsymbol{A}}_T$ is semi-positive definite. Suppose $0 < \theta_1 < 0.5 < \theta_2 < 1$, we calculate $\boldsymbol{A}_T - \frac{1}{\lambda^2}\tilde{\boldsymbol{A}}_T$ using Lemma 2:

$$\begin{aligned}
\boldsymbol{A}_T - \frac{1}{\lambda^2}\tilde{\boldsymbol{A}}_T &= \sum_{i=1}^{2}\sum_{j=1}^{2}\Big\{\theta_j\theta_i - \min(\theta_j,\theta_i)\Big\}E\Big\{\boldsymbol{C}_{i,j}\Big\} - \\
&\quad \frac{1}{\lambda^2}\sum_{i=1}^{2}\sum_{j=1}^{2}\Big\{\tilde{\theta}_j\tilde{\theta}_i - \mathrm{P}(\{\tilde{y}_{i,t} < \tilde{q}_{i,t}(\tilde{\boldsymbol{\beta}}^0)\}\cap\{\tilde{y}_{j,t} < \tilde{q}_{j,t}(\tilde{\boldsymbol{\beta}}^0)\})\Big\}E\Big\{\boldsymbol{C}_{i,j}\Big\} \\
&= E\Big\{\theta(1-\theta)\boldsymbol{C}_{11} + \theta(1-\theta)\boldsymbol{C}_{22} + \theta^2(\boldsymbol{C}_{12} + \boldsymbol{C}_{21}) \\
&\quad -\frac{1}{\lambda^2}\Big\{\lambda\theta(1-\lambda\theta)\boldsymbol{C}_{11} + \lambda\theta(1-\lambda\theta)\boldsymbol{C}_{22} \\
&\quad +\Big\{-\tilde{\theta}_1\tilde{\theta}_2 + \mathrm{P}(\{L_t < \tilde{q}_{1,t}(\tilde{\boldsymbol{\beta}}^0)\}\cap\{H_t < \tilde{q}_{2,t}(\tilde{\boldsymbol{\beta}}^0)\})(\boldsymbol{C}_{12} + \boldsymbol{C}_{21})\Big\}\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= E\Bigg\{\Big\{\theta(1-\theta)-\frac{\lambda\theta(1-\lambda\theta)}{\lambda^2}\Big\}\boldsymbol{C}_{11}+\Big\{\theta(1-\theta)-\frac{\lambda\theta(1-\lambda\theta)}{\lambda^2}\Big\}\boldsymbol{C}_{22}+ \\
&\quad \Big\{\theta^2-\frac{-\tilde{\theta}_1\tilde{\theta}_2+\mathrm{P}(\{L_t<\tilde{q}_{1,t}(\tilde{\boldsymbol{\beta}}^0)\}\cap\{H_t<\tilde{q}_{2,t}(\tilde{\boldsymbol{\beta}}^0)\})}{\lambda^2}\Big\}(\boldsymbol{C}_{12}+\boldsymbol{C}_{21})\Bigg\} \\
&= E\Bigg\{\Big\{\theta(1-\theta)-\frac{\lambda\theta(1-\lambda\theta)}{\lambda^2}\Big\}\boldsymbol{C}_{11}+\Big\{\theta(1-\theta)-\frac{\lambda\theta(1-\lambda\theta)}{\lambda^2}\Big\}\boldsymbol{C}_{22}+ \\
&\quad \Big\{\theta^2-\frac{-\tilde{\theta}_1\tilde{\theta}_2+\mathrm{P}(L_t<\tilde{q}_{1,t}(\tilde{\boldsymbol{\beta}}^0))}{\lambda^2} \\
&\quad +\frac{-\mathrm{P}(\{L_t<\tilde{q}_{1,t}(\tilde{\boldsymbol{\beta}}^0)\}\cap\{H_t>\tilde{q}_{2,t}(\tilde{\boldsymbol{\beta}}^0)\})}{\lambda^2}(\boldsymbol{C}_{12}+\boldsymbol{C}_{21})\Bigg\} \\
&= E\Bigg\{\Big\{\theta(1-\theta)-\frac{\theta(1-\lambda\theta)}{\lambda}\Big\}\boldsymbol{C}_{11}+\Big\{\theta(1-\theta)-\frac{\theta(1-\lambda\theta)}{\lambda}\Big\}\boldsymbol{C}_{22}+ \\
&\quad \Big\{\theta^2-\frac{-\tilde{\theta}_1\tilde{\theta}_2+\tilde{\theta}_1-c}{\lambda^2}\Big\}(\boldsymbol{C}_{12}+\boldsymbol{C}_{21})\Bigg\} \\
&= E\Bigg\{\Big\{\theta(1-\theta)-\frac{\theta(1-\lambda\theta)}{\lambda}\Big\}\boldsymbol{C}_{11}+\Big\{\theta(1-\theta)-\frac{\theta(1-\lambda\theta)}{\lambda}\Big\}\boldsymbol{C}_{22}+ \\
&\quad \Big\{\theta^2-\frac{\tilde{\theta}_1(1-\tilde{\theta}_2)-c}{\lambda^2}\Big\}(\boldsymbol{C}_{12}+\boldsymbol{C}_{21})\Bigg\} \\
&= E\Bigg\{\Big\{\theta(1-\theta)-\frac{\theta(1-\lambda\theta)}{\lambda}\Big\}\boldsymbol{C}_{11}+\Big\{\theta(1-\theta)-\frac{\theta(1-\lambda\theta)}{\lambda}\Big\}\boldsymbol{C}_{22}+ \\
&\quad \Big\{(1-\frac{\lambda^2}{\lambda^2})\theta^2+\frac{c}{\lambda^2}\Big\}(\boldsymbol{C}_{12}+\boldsymbol{C}_{21})\Bigg\} \\
&= E\Bigg\{(\frac{\lambda-1}{\lambda}\theta)\boldsymbol{C}_{11}+(\frac{\lambda-1}{\lambda}\theta)\boldsymbol{C}_{22}+\frac{c}{\lambda^2}(\boldsymbol{C}_{12}+\boldsymbol{C}_{21})\Bigg\} \\
&= (\frac{\lambda-1}{\lambda}\theta)E\Bigg\{\boldsymbol{C}_{11}+\boldsymbol{C}_{22}+\frac{c}{\lambda(\lambda-1)\theta}(\boldsymbol{C}_{12}+\boldsymbol{C}_{21})\Bigg\}
\end{aligned}
$$

Note that $\frac{\lambda-1}{\lambda}\theta>0$ since $\lambda>1$. We show that $E\Big\{\boldsymbol{C}_{11}+\boldsymbol{C}_{22}+\frac{c}{\lambda(\lambda-1)\theta}(\boldsymbol{C}_{12}+\boldsymbol{C}_{21})\Big\}$

is semi-positive definite:

$$
\begin{aligned}
& E\left\{\boldsymbol{C}_{11}+\boldsymbol{C}_{22}+\frac{c}{\lambda(\lambda-1)\theta}(\boldsymbol{C}_{12}+\boldsymbol{C}_{21})\right\} \\
=& \frac{c}{\lambda(\lambda-1)\theta}E\left\{\boldsymbol{C}_{11}+\boldsymbol{C}_{22}+(\boldsymbol{C}_{12}+\boldsymbol{C}_{21})\right\}+\left\{1-\frac{c}{\lambda(\lambda-1)\theta}\right\}E\left\{\boldsymbol{C}_{11}+\boldsymbol{C}_{22}\right\} \\
=& \frac{c}{\lambda(\lambda-1)\theta}E\left\{\left\{\frac{\sqrt{c}}{\lambda}\nabla q_{1,t}(\boldsymbol{\beta}^0)+\frac{\sqrt{c}}{\lambda}\nabla q_{2,t}(\boldsymbol{\beta}^0)\right\}\left\{\frac{\sqrt{c}}{\lambda}\nabla' q_{1,t}(\boldsymbol{\beta}^0)+\frac{\sqrt{c}}{\lambda}\nabla' q_{2,t}(\boldsymbol{\beta}^0)\right\}\right\} \\
& +\left\{1-\frac{c}{\lambda(\lambda-1)\theta}\right\}E\left\{\boldsymbol{C}_{11}+\boldsymbol{C}_{22}\right\}
\end{aligned}
$$

In the last equation, both terms are semi-positive definite, since any matrix of the form AA' is semi-positive definite and hence so is its expectation, and $1-\frac{c}{h(h-1)\theta}$ is positive by Assumption 1. Therefore, $\boldsymbol{D}_T^{-1}\boldsymbol{A}_T\boldsymbol{D}_T^{-1}-(\tilde{\boldsymbol{D}}_T)^{-1}\tilde{\boldsymbol{A}}_T(\tilde{\boldsymbol{D}}_T)^{-1}$ is semi-positive definite.

□