

# Posterior Inference for Portfolio Weights\*

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## Abstract

We investigate estimation uncertainty in portfolio weights through their posterior distributions in a Bayesian regression framework. While we derive analytical posterior results for shrinkage variants of the global minimum variance portfolio (GMVP), the main advantage of our novel approach is the direct specification of the prior on the optimal portfolio weights. We show how to incorporate economic views about the asset returns in our framework as a shrinkage target and how to account for the investor's uncertainty about these views through a hierarchical set-up. In a series of empirical experiments we explore the effect of estimation errors on the performance of the optimal portfolio and propose various practical trading strategies derived from the posterior distribution, which are highly beneficial to the investor.

**Keywords:** Bayesian inference, estimation risk, global minimum variance portfolio, shrinkage

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# 1 Introduction

Stabilizing portfolio weights to tackle estimation errors is a challenging task of empirical portfolio allocation. Although the framework of [Markowitz \(1952\)](#) delivers maximum expected utility portfolio weights against the risky nature of future returns, there is empirical evidence that neglecting the uncertainty in parameter estimates of the asset return distribution results in portfolios with poor out-of-sample performance ([Jobson and Korkie, 1980](#); [DeMiguel et al., 2009](#)).

So far, Bayesian methods in portfolio optimization have mainly been used to update beliefs and to diminish the effect of estimation risk on the economic decision through the predictive return distributions that marginalize out unknown distributional parameters ([Klein and Bawa, 1976](#)). However, the Bayesian toolbox can also be applied to place a certain structure on the portfolio weights directly and to quantify the impact of estimation uncertainty around point estimates.

This study offers a new perspective on handling estimation error and proposes a new method of regularization of portfolio weights by means of Bayesian regressions. The key idea rests on representing the empirical GMVP weights in terms of the linear regression model derived by [Kempf and Memmel \(2006\)](#). Adopting the Bayesian framework of [Frey and Pohlmeier \(2015\)](#), this allows to (i) estimate the portfolio weights directly, to (ii) obtain a joint posterior distribution for the portfolio weights vector, to (iii) shrink that towards a predefined target (such as the naive  $1/N$  portfolio) and to (iv) impose restrictions on the weights by an appropriate choice of the prior distribution.<sup>1</sup>

While the  $1/N$  portfolio seems to be a natural shrinkage target, most of the focus in the literature has been placed on finding the optimal shrinkage intensity ([Frahm and Memmel, 2010](#)) whereas the question concerning the optimal shrinkage target has merely been neglected. In practice, assigning priors to portfolio weights is equivalent to incorporating prior beliefs into the 'optimal' portfolio allocation through the prior mean (shrinkage

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<sup>1</sup>We are not the first to study distributions of estimated portfolio weights. While [Okhrin and Schmid \(2006\)](#) derive the distribution for the sample-based estimated GMVP weights under Gaussian returns, [Frahm \(2010\)](#) proposes an hypothesis test for the portfolio weights under normality. [Kan and Smith \(2008\)](#) derive the finite sample distribution and moments of the sample minimum-variance frontier when returns are independent multivariate normally distributed. [Bodnar et al. \(2015\)](#) derive exact posterior distribution for the weights of the GMVP. However, they first propose Bayesian priors for the mean and variance of the asset returns and then derive the posterior distribution for the weights through a different parameterization.

target). This can be done for example by reflecting non-sample information such as stock characteristics in the fashion of Brandt et al. (2009), a previous portfolio allocation in order to reduce transaction costs or an estimated portfolio vector that corresponds to an alternative strategy. While the uncertainty of the investors' beliefs in this 'target portfolio' can be expressed through the prior variance, the uncertainty inherent in the target vector that comes from estimating it has to be modeled separately. Our regression framework allows to place a hyper-prior distribution on the target vector and to marginalize out the inherent estimation uncertainty through a hierarchical set-up by 'averaging' across different target portfolios.

In what follows, we first derive the posterior distribution for GMVP portfolio weights and explain its Bayesian interpretation in section 2. Then, section 3 offers an overview about various empirical application and generalizations of the proposed regression framework: The range of applications reaches from investigating the well-documented *home bias*, stating that investors do not diversify enough into international assets, to proposing trading strategies that maximize the out-of-sample utility of an investor *ex ante* or restrict trading to those days when significant changes in distribution of weights occur. In order to incorporate non-sample information, we also implement a hierarchical Bayesian regression framework centered on Brandt et al. (2009) optimal factor weights. The empirical exercises in this section show that using the information content of joint posterior weight distribution is beneficial for the investor. We compare the proposed Bayesian shrinkage strategies to popular frequentist approaches and find that the former show better out-of-sample performance based on various criteria. Finally, section 4 concludes and gives an outlook on possible paths of future research.

## 2 Posterior distributions of portfolios weights

Consider an investor with  $N$  risky financial assets who chooses his portfolio weights such that the risk in terms of portfolio variance is minimized. Let  $r_t = (r_{t,1}, \dots, r_{t,N})' \in \mathbb{R}^N$  be the return vector with mean vector  $\mu$  and variance-covariance matrix  $\Sigma$ . The vector of

GMVP weights is defined as the solution to the minimization problem

$$\omega_{mvp} = \arg \min_{\omega \in \mathbb{R}^N, \omega' \iota_N = 1} \omega' \Sigma \omega = \frac{\Sigma^{-1} \iota_N}{\iota_N' \Sigma^{-1} \iota_N}. \quad (1)$$

If  $r_t$  follows an *iid* process, an unbiased estimate of  $\Sigma$  is obtained by the sample covariance matrix  $\hat{\Sigma} = \frac{1}{T-1} \sum_{t=1}^T (r_t - \hat{r})(r_t - \hat{r})'$  with  $\hat{r} = \left( \frac{1}{T} \sum_{s=1}^T r_{s,1}, \dots, \frac{1}{T} \sum_{s=1}^T r_{s,N} \right)'$ . Replacing  $\Sigma$  in (1) by its sample counterpart yields the plug-in estimator for the GMVP weights

$$\hat{\omega}_{mvp} = \frac{\hat{\Sigma}^{-1} \iota_N}{\iota_N' \hat{\Sigma}^{-1} \iota_N}. \quad (2)$$

Kempf and Memmel (2006) show that the weights of the plug-in estimator can be obtained by least squares estimation of

$$r_{t,N} = \mu_{mvp} + \sum_{i=1}^{N-1} \omega_i (r_{t,N} - r_{t,i}) + \varepsilon_t, \quad (3)$$

in which the adding-up constraint to invest all available wealth of the investor is imposed through the weight for the  $N$ -th asset  $\omega_N = 1 - \sum_{i=1}^{N-1} \omega_i$ .<sup>2</sup> It holds that demeaning all assets and omitting the intercept in equation (3) yields the same OLS estimates for  $\omega = (\omega_1, \dots, \omega_N)'$ . Denote by  $r_i^d = (r_{1,i} - \hat{r}_i, \dots, r_{T,i} - \hat{r}_i)'$  the demeaned return vector for asset  $i$ . Then in matrix notation the regression equation reads

$$Y = X\omega^* + \varepsilon \quad (4)$$

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<sup>2</sup>Note that due to the adding-up constraint, the ordering of the assets is arbitrary such that the left-hand side variable can be any of the  $N$  assets.

where

$$Y = r_N^d \quad (5)$$

$$X = \begin{pmatrix} r_{1,N}^d - r_{1,1}^d & \cdots & r_{1,N}^d - r_{1,N-1}^d \\ \vdots & \ddots & \vdots \\ r_{T,N}^d - r_{T,1}^d & \cdots & r_{T,N}^d - r_{T,N-1}^d \end{pmatrix} \quad (6)$$

$$\omega^* = (\omega_1, \dots, \omega_{N-1})'. \quad (7)$$

Here, the superscript  $*$  denotes the truncation of a vector by neglecting the last element.

Under a Gaussian likelihood function with homoskedastic error terms, the conditional distribution of the dependent variable is normal:  $Y|\omega^*, h \sim N(X\omega^*, h^{-1}I_T)$ . A conventional choice of a prior for  $\omega$  in the normal linear regression model has been the conditional conjugate  $g$ -prior proposed by Zellner (1986):

$$\omega^*|h \sim N(\beta^*, gh^{-1}(X'X)^{-1}) \text{ where } g > 0 \quad (8)$$

$$h \sim \text{Gamma}(s^{-1}, v). \quad (9)$$

Appealingly, the  $g$ -prior reduces the choice of the  $(N-1) \times (N-1)$  prior variance-covariance matrix for  $\omega$  to a single scalar hyperparameter  $g$ , governing the degree of shrinkage towards the prior mean. While this prior has been widely applied for Bayesian model averaging, the purpose here will be to link the  $g$  parameter to optimizing the investor's objective function. The marginal posterior distribution of  $\omega^*$  is given by

$$\omega^*|\{Y, X\} \sim t_{(T)}(\bar{\omega}^*, \bar{s}^2\bar{V}), \quad (10)$$

where

$$\bar{V} = \frac{g}{1+g}(X'X)^{-1} \quad (11)$$

$$\bar{\omega}^* = \frac{1}{1+g}\beta^* + \frac{g}{1+g}\hat{\omega}_B^* \quad (12)$$

$$\bar{s}^2 = \frac{1}{T} \left( (Y - X\hat{\omega}_B^*)'(Y - X\hat{\omega}_B^*) + \frac{1}{1+g}(\hat{\omega}_B^* - \beta^*)'X'X(\hat{\omega}_B^* - \beta^*) \right). \quad (13)$$

Due to the adding-up constraint, it follows that the unconditional posterior distribution of the portfolio weight for the  $N$ -th asset  $\omega_N$  is also a multivariate t distribution with  $T$  degrees of freedom:

$$\omega_N = 1 - \omega^{*\prime} \iota_{N-1} \sim t_{(T)}(1 - \bar{\omega}^{*\prime} \iota_{N-1}, \iota'_{N-1} \bar{s}^2 \bar{V} \iota_{N-1}). \quad (14)$$

As  $\omega_N = 1 - \omega^{*\prime} \iota_{N-1}$  is determined by the  $N - 1$  regression coefficients,  $(\omega^*) \rightarrow (\omega^*, \omega_N) =: \omega$  is an affine transformation of the multivariate t distribution with  $T$  degrees of freedom and first moment

$$E(\omega) = (\bar{\omega}^{*\prime}, 1 - \bar{\omega}^{*\prime} \iota_{N-1})' \in \mathbb{R}^N.$$

Further, the scale matrix of  $\omega$  is a stacked  $N \times N$  matrix in which  $\bar{V}$  is the  $(N - 1) \times (N - 1)$  dimensional sub-matrix in the upper left hand corner, the  $(N, j)$ -th and  $(j, N)$ -th entries for  $j = 1, \dots, N - 1$  in the bottom row and right column have the value  $\text{cov}(\omega_N, \omega_j) = -\left(\sum_{i=1}^{N-1} \text{cov}(\omega_i, \omega_j)\right) = -\sum_{i=1}^{N-1} C_{i,j}$  and  $\iota'_{N-1} \bar{V} \iota_{N-1}$  appears in the lower right hand corner of the matrix.

The posterior distribution of the portfolio weights depends on (i) the choice of the reference portfolio  $\beta$  and (ii) on the choice of hyperparameter  $g$  in equation (8). The choice of  $g$  determines certainty of the investor in the optimality of the reference portfolio: While for  $g \rightarrow \infty$  the prior becomes completely uninformative on  $\omega^*$  and yields the plug-in estimator  $\omega_{mvp}$ , a completely informative (deterministic) prior for  $g \rightarrow 0$  leads to the reference strategy. It follows that the posterior distribution of  $\omega_{mvp}$  under the non-informative prior by letting  $g \rightarrow \infty$  is

$$\omega_{mvp}^* | \{X, Y\} \sim t_{(T)}\left(\omega_{mvp}^*, \frac{1}{T}(Y - X\omega_{mvp}^*)'(Y - X\omega_{mvp}^*)(X'X)^{-1}\right). \quad (15)$$

Furthermore, the first two posterior moments of  $\omega_{mvp}$  are

$$E(\omega_{mvp}|D) = \hat{\omega}_{mvp} \quad \text{and} \quad (16)$$

$$\begin{aligned} \Sigma_{mvp} := & \frac{1}{T-1} (Y - X\hat{\omega}_{mvp}^*)'(Y - X\hat{\omega}_{mvp}^*) \\ & \times \begin{pmatrix} (X'X)^{-1} & (X'X)^{-1}\iota_{N-1} \\ \iota'_{N-1}(X'X)^{-1} & \iota'_{N-1}(X'X)^{-1}\iota_{N-1} \end{pmatrix}. \end{aligned} \quad (17)$$

### 3 Empirical applications

Equation (10) provides the base to account for the uncertainty incorporated in the parameter estimates for the portfolio weights and offers a number of applications which can be used to improve the out-of-sample portfolio performance. In the following, we will elaborate on various portfolio allocation applications incorporating the posterior information of the portfolio weights.

#### 3.1 Zellners's $g$ -prior and adjusted risk aversion

The posterior mean of the portfolio weights in equation (12) is a convex combination of the GMVP and the reference portfolio, in which the hyperparameter  $g$  determines the degree of shrinkage between the two. Choosing the *estimated efficient portfolio*

$$\hat{\omega}_e(\gamma) := \operatorname{argmax}_{\omega \in \mathbb{R}^N, \iota'_N \omega = 1} \left\{ \hat{\mu}'\omega - \frac{\gamma}{2} \omega' \hat{\Sigma} \omega \right\} = \frac{\hat{\Sigma}^{-1} \iota_N}{\iota'_N \hat{\Sigma}^{-1} \iota_N} + \frac{1}{\gamma} \left( \hat{\Sigma}^{-1} - \frac{\hat{\Sigma}^{-1} \iota_N \iota'_N \hat{\Sigma}^{-1}}{\iota'_N \hat{\Sigma}^{-1} \iota_N} \right) \hat{\mu}, \quad (18)$$

as the reference portfolio, allows us to link the risk aversion parameter of the investor  $\gamma$  to the parameter  $g$ . The idea is that an investor with risk aversion factor  $\gamma$  does not obtain satisfactory results out-of-sample by investing only into the estimated efficient portfolio  $\hat{\omega}_e(\gamma)$  if the uncertainty regarding the portfolio weights is large. We address this issue via the choice of the hyperparameter  $g$  to construct an informed prior and illustrate the impact of choosing a  $g$ -prior by shrinking  $\hat{\omega}_e(\gamma)$  along the efficient frontier towards  $\hat{\omega}_{mvp}$  within our regression framework. This approach allows to compute an adjusted risk aversion factor  $\tilde{\gamma}$  accounting for the inherent estimation risk.

### Lemma 3.1

Define the shrinkage target  $\beta := \hat{\omega}_e(\gamma)$ . Under the  $g$ -prior the Bayesian decision rule is equivalent to the efficient portfolio  $\hat{\omega}_e(\tilde{\gamma})$  with an adjusted relative risk aversion factor  $\tilde{\gamma} := (1 + g)\gamma$ .

**Proof:** The proof can be found in the Appendix A.1.  $\square$

As  $0 < g$  it holds that  $\tilde{\gamma} > \gamma$ . Denoting a completely uninformed prior by letting  $g \rightarrow \infty$  corresponds to an implied adjusted risk aversion factor  $\tilde{\gamma} \rightarrow \infty$ . Letting  $g \rightarrow 0$  corresponds to a prior without any uncertainty in the parameters. Therefore, the investor chooses the efficient portfolio  $\hat{\omega}_e(\tilde{\gamma}) = \hat{\omega}_e(\gamma)$ .

Figure 1 visualizes the risk adjustment procedure and gives the intuition behind the impact of applying Zellners'  $g$ -prior: The blue line corresponds to the empirically efficient frontier in the  $(\sqrt{\sigma^2_P}, \mu_P)$ -plane based on  $\hat{\mu}$  and  $\hat{\Sigma}$ . The rightmost black dot corresponds to the location of the efficient portfolio for an investor with  $\gamma$ . The second black dot corresponds to the expected parameters of the efficient portfolio with adjusted risk aversion  $\tilde{\gamma}$  and the red dot is assigned to  $\hat{\omega}_{mvp}$ . Clearly, with decreasing risk aversion factor  $\gamma$  the decision of the investor moves along the efficiency frontier towards regions with higher expected return and associated risk. To reinforce the cautions regarding the huge impact of the estimation risk on the portfolio allocation decision, credible regions are included into the figure.<sup>3</sup> It can be deciphered from Figure 1 that the stochastic nature of the portfolio weights can harm the investment significantly, as, especially with a lower risk aversion factor, the range of the credible set in the  $(\sqrt{\sigma^2_{PF}}, \mu_{PF})$ -plane is large.<sup>4</sup> Therefore, an investor with risk aversion factor  $\gamma$  takes into account that although choosing a  $g$ -prior results in a Bayesian decision rule that deviates from the theoretically efficient portfolio, the impact of estimation risk can be reduced.

### 3.2 Out-of-sample utility maximization through $g$

Building on the considerations of the previous section, we propose to chose the shrinkage parameter  $g$  in order to maximize the out-of-sample certainty equivalent (CE) or any other objective function of an investor *ex ante*. By choosing a coherent objective function,

<sup>3</sup>This Bayesian methodology follows the frequentist approach of Okhrin and Schmid (2007).

<sup>4</sup>The ellipsoids in the figure denote the  $\alpha = 0.05$  significance level.



the choice of  $g$  already takes into account the uncertainty incorporated in the portfolio allocation. For the following, we assume that the reference portfolio is the naive  $1/N$  portfolio, which is supposed to have good out-of-sample characteristics (e.g. DeMiguel et al., 2009). Assume further an investor with risk aversion factor  $\gamma = \infty$  chooses to shrink his portfolio  $\hat{\omega}_{mvp}$  towards the naive portfolio. Maximizing the posterior out-of-sample CE is then equivalent to minimizing the out-of-sample portfolio variance. This corresponds to

$$\arg \min_{g \in \mathbb{R}_+} \text{Var}(\bar{\omega}' r_{t+1} | D). \quad (19)$$

with  $\bar{\omega} = (\bar{\omega}^*, 1 - \bar{\omega}^* \iota_{N-1})$ .

**Lemma 3.2**

Let  $c = \frac{g}{1+g}$ . Then, the portfolio  $\bar{\omega}_{opt} := c_{opt} \hat{\omega}_{mvp} + (1 - c_{opt}) \frac{1}{N} \iota_N$  with

$$c_{opt} := \frac{(\frac{1}{N} \iota_N - \hat{\omega}_{mvp})' \hat{\Sigma} \frac{1}{N} \iota_N}{\hat{\mu}' \Sigma_{mvp} \hat{\mu} + \text{tr}(\Sigma_{mvp} \hat{\Sigma}) + (\hat{\omega}_{mvp} - \frac{1}{N} \iota_N)' \hat{\Sigma} (\hat{\omega}_{mvp} - \frac{1}{N} \iota_N)} \quad (20)$$

minimizes the objective function  $\min_{c \in \mathbb{R}} \text{Var}(\bar{\omega}' r_{t+1} | D) = \text{tr}(\Sigma_{\bar{\omega}} \hat{\Sigma}) + \bar{\omega}' \hat{\Sigma} \bar{\omega} + \hat{\mu}' \Sigma_{\bar{\omega}} \hat{\mu}$ .

**Proof:** The proof can be found in the Appendix A.2. □

The mixing factor  $c_{opt}$  incorporates the uncertainty regarding the estimation risk of  $\hat{\omega}_{mvp}$  into the decision framework and provides a weighting scheme which takes lower values (and subsequently invests more into the naive portfolio) if the estimation risk is higher. The derived proportion  $c_{opt}$  increases proportional to the difference between the naive portfolio and  $\hat{\omega}_{mvp}$ . The measure for parameter uncertainty -  $\Sigma_{mvp}$  - enters only through the denominator and reduces  $c_{opt}$ . In a setting ignoring the parameter uncertainty, the denominator of Equation (20) would only contain the last term  $(\hat{\omega}_{mvp} - \frac{1}{N} \iota_N)' \hat{\Sigma} (\hat{\omega}_{mvp} - \frac{1}{N} \iota_N)$ . Therefore, our approach delivers a more conservative way of asset allocation.

We investigate a simple horse race based on monthly returns of six national MSCI stock market indexes (CAN, GER, JP, UK, USA, SUI) from 1969 to 2015. Hereby we shrink between the naive portfolio and  $\hat{\omega}_{mvp}$  with shrinkage intensity  $c_{opt}$ . As benchmark portfolios serve the naive portfolio  $1/N \iota_N$ ,  $\hat{\omega}_{mvp}$  and the approach proposed by (DeMiguel et al., 2009, eq. (11)), a combination of the two which was shown to perform competitive across

a range of benchmark portfolios. The latter strategy ignores the uncertainty regarding the true portfolio weights ( $\omega_{mvp}$ ) but chooses the mixing factor  $c_{dm}$  in order to minimize the in-sample variance of the portfolio:  $\min_c \omega' \hat{\Sigma} \omega$ . Table 1 illustrates annualized mean results after taking transaction costs of 50 basis points into account.<sup>5</sup> TO denotes the average turnover,  $\mu$  is the annualized return,  $\sigma$  is the standard deviation of the portfolio return,  $SR := \mu/\sigma$  is the Sharpe ratio, CE reports the certainty equivalent for a risk aversion  $\gamma = 1$  and RL means return loss against the  $1/N$  portfolio. The main message is that our proposed approach  $\omega_{opt}$  outperforms the benchmark portfolios in terms of all evaluation criteria but the turnover rate. In Figure 2 the time series of fractions invested into the GMVP are plotted for the two dynamic portfolio strategies. At all points in time our approach  $\omega_{opt}$  invests a higher fraction of wealth into the naive portfolio. The horse-race suggests that incorporating estimation risk is beneficial to the investor: It is not clear why one should invest into  $\hat{\omega}_{mvp}$  in the presence of estimation risk at all. However, the results indicate that the Bayesian mixing strategy balances the gains from using historical data and relying on the naive portfolio superior compared to  $\omega_{dm}$ .

### 3.3 Sparse rebalancing

Determining when to rebalance a portfolio is a complex problem in the presence of transaction costs and structural breaks. Commonly, models rely on calendar-based or other ad hoc methods that do not reflect the associated costs and market conditions. We propose a dynamic rebalancing strategy, based on a test whether the portfolio weight at time  $t$  differs significantly from the asset weight just before reallocation  $\omega_{t-1,+} := \frac{\omega_{t-1} \circ R_t}{\|\omega_{t-1} \circ R_t\|}$  by performing Bayesian significance tests. The procedure is straightforward: At time  $t$ , we compute the posterior distribution of  $\omega_t$  based on Equation (10). Next, we check if  $\omega_{t-1,+}$  falls into the elliptical credible set of  $\omega_{t,i}$ . If this is the case, rebalancing the portfolio is not justified due to the high uncertainty. Therefore, conditional on  $\omega_{t-1,+}$ , the proposed

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<sup>5</sup>Transaction costs are assumed to be proportional to absolute weight changes:  $c_t \propto \sum_{i=1}^N \left| \omega_{i,t+1} - \frac{\omega_{i,t}(1+r_{i,t})}{1'(\omega_t \circ (1+r_t))} \right|$  where  $\circ$  denotes element-wise multiplication.

dynamic decision rule is:

$$\omega_{dyn}^t = \begin{cases} \omega_t & \text{if } \omega_{t-1,+} \notin [\omega_{t,i} \pm \text{Var}(\omega^{t,i}|D)^{\frac{q_t(\xi)}{2}}] \text{ for } i = 1, \dots, N \\ \omega_{t-1,+} & \text{else} \end{cases}$$

where  $q_t(\xi)$  is the  $\xi$  quantile of the posterior distribution. This is an intuitive procedure as an investor taking into account the uncertainty of the true model parameters should weigh the costs associated with rebalancing and the amount of new information that leads to an updated allocation. If it cannot be ruled out that the reallocation is not truly performance increasing by not reflecting significant changes of the portfolio weights, rebalancing should be rejected to avoid transaction costs. We implement an horse race to present the appealing features of incorporating a dynamic rebalancing rule. Based on the six national stock market indexes used in the previous section we compare the performance of the portfolio strategies  $\omega_{mvp}^1$ ,  $\omega_{mvp}^6$ ,  $\omega_{mvp}^{12}$ ,  $\omega_{mvp}^{24}$  and  $\omega_{mvp}^{dyn}$ , defined as the GMVP weights, readjusted monthly, semi-annual, annual, every second year and with the dynamic procedure.<sup>6</sup> In the spirit of DeMiguel et al. (2009) we assume that transaction costs  $c := 50$  bp are proportional to monthly turnover  $\iota'(|\omega_{i,t} - \omega_{i,t-1,+}|)$ . Annualized out-of-sample mean results of the horse race are presented in Table 2. Clearly the turnover of the dynamic strategy is only slightly more than  $\text{TO}(\omega_{mvp}^{24})$ . Compared to the other strategies, the restriction regarding rebalancing reduces turnover dramatically. The return parameter raises evidence that our proposed strategy is indeed able to capture the relevant dates for trading, whereas it avoids transaction costs when there is high uncertainty if the readjustment represents a structural break in the portfolio weights. Indeed, the out-of-sample Sharpe ratio after transaction costs is highest for the dynamic strategy.

To visualize the properties of the sparse reallocation strategy, Figure 3 shows the cumulative returns after accounting for transaction costs. The vertical black lines visualize the dates of readjustment for the dynamic portfolio strategy. Obviously the dates are not randomly distributed over the whole time span of interest but are clustered around specific time periods, especially during the financial crisis.

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<sup>6</sup> $\omega_{mvp}^{dyn}$  is computed using  $\xi = 0.01$ .

### 3.4 The *home bias* puzzle

Allocating wealth into a portfolio of international assets promises diversification of individual country risks among several states in the world. Nonetheless, French and Poterba (1991) observe that investors prefer to allocate wealth into assets of their own countries. This *home bias puzzle* is subject to an ongoing discussion in the field of behavioral finance.<sup>7</sup> Instead of following the attempts to explain this puzzle as irrational behavior, Britten-Jones (1999) uses derived distributional properties of the tangency portfolio weights  $\omega_{tan}$  to investigate whether the estimation risk in the underlying parameters can rationally justify the home bias and finds that for none of the 11 assets investigated the hypothesis  $H_0 : \omega_{tan}^i = 0$  can be rejected. Furthermore, from the perspective of an US investor, he shows that the hypothesis  $H_0 : \omega_{tan}^i = 0, i \neq US$  cannot be rejected for the two periods between 1977 to 1986 and 1987 to 1996. Intuitively speaking, this suggest that the high estimation risk inherent in the data of past returns does not justify the assumption that no diversification is irrational but that the home bias could rather reflect the uncertainty of the investor.

We apply the Bayesian concept of hypothesis testing to the field of international asset diversification to investigate whether  $\omega_{mvp}^{t,i} = 0$  for  $t \in \{1, \dots, T\}$  and  $i \in \{1, \dots, N\}$ . Again, based on the dataset of stock market indexes from six developed countries - Canada, Germany, Japan, Switzerland, United Kingdom and the United States of America - a rolling window approach with an estimation window size of  $h = 60$  months is applied.<sup>8</sup> For each month we compute the posterior distribution for  $\omega_{mvp}$  and perform a test in the spirit of Lindley (1965) for the hypothesis  $\omega_{mvp}^i = 0, i \in 1, \dots, 6$ . For each asset we compute a  $1 - \alpha$  credible region and check if 0 falls outside this interval. The advantage of the Bayesian testing approach is that we are able to evaluate the posterior distribution of the weights for each point in time. The resulting credible region allows to quantify the probability that a certain element of  $\hat{\omega}_{mvp}$  is 0.

The results of this test are visualized in Figure 4. The bold line represents the Bayesian decision rule for each national stock market index separately (mean posterior estimate), the dashed lines form the 5 and 95 percent quantiles corresponding to a 90 percent credible interval and the dotted lines correspond to the 99 percent credible region.

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<sup>7</sup>See Barberis and Thaler (2003) for a survey.

<sup>8</sup>The dataset of monthly MSCI returns reaches from December 1969 until August 2015.

Our results are in line with Britten-Jones (1999) in the context of the home bias puzzle: Ambiguity aversion of investors takes estimation risk in the portfolio allocation context into account. As the credible regions are broad, a lack of diversification can be explained by the belief that the true portfolio weights do not differ from 0, except for the US asset, for which we find a significant positive weight at almost all times. For all other country indexes, the credible intervals of the portfolio weights are too wide and include zero for almost the whole sample only until the recent financial crisis, where for example the optimal portfolio requires a significant short position in the German asset.

We summarize our analysis of the credible regions by computing Bayesian F-tests to check, whether the home bias can be justified for any of the investors located in the 6 countries analyzed. We construct high posterior density regions to test whether  $\omega_{mvp}^i = 0 \forall i \neq j$  and  $j \in \{\text{CAN, SUI, JP, GER, USA, UK}\}$ . As the family of t distributions is closed under linear transformations,  $\delta_j := R_j \hat{\omega}$  with  $R_j = (e_1, \dots, e_{j-1}, 0, e_{j+1}, \dots)$  is t distributed as well.<sup>9</sup> Consequently, we can compute analytically the posterior of  $\zeta_j := \frac{T}{N-1} (R_j \hat{\omega})' (R \Sigma_{mvp} R')^{-1} (R_j \hat{\omega}) \sim F(N-1, T)$ . Our procedure allows to compute  $\zeta_j$  and to compare this value with the  $1 - \alpha$  quantile of the  $F(N-1, T)$  distribution. It can be seen in Figure 5 that the data gives confidence into our hypothesis at each point in time and for every country investigated. These results do not aim in explaining the home bias but they show that the uncertainty in the optimal portfolio allocation is indeed so high that international diversification is not necessarily justified with the mean-variance approach.

Additionally, the computation of the posterior distribution of  $\omega_{mvp}$  allows to investigate the global market uncertainty the investor faces. In Figure 4 it becomes obvious that the range of the credible regions widened during the financial crisis, especially for the United Kingdom, Germany and Japan. Interpreting the posterior variance of the portfolio weights as an measure for uncertainty incorporated into the decision of the investor gives the possibility to draw conclusions from the time series of inter-quantile ranges. Figure 6 shows the mean of the range between the 0.95 and 0.05 credible region for all 6 assets investigated as a time series. We associate an increasing value with higher uncertainty of the allocation conditional on the prior information. It is easy to see how the uncertainty

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<sup>9</sup> $e_j \in \mathbb{R}^N$  denotes a vector with 1 in the  $j$ -th element and 0 elsewhere.

incorporated into the data varies over time and is especially affected by events that can be expected to influence the perfect allocation. For example, the Black Monday in 1987 and the Lehman crash in 2008 increased the inter-quantile range substantially. However, we also note that the sudden decreases in the mean interquantile range can not be associated with any particular economic event.

### 3.5 Hierarchical model

In this section we depart from using the naive  $1/N$  portfolio as the shrinkage target. Instead, we allow the target portfolio to be stochastic itself, which might be the case when an investor wants to shrink the GMVP towards another estimated portfolio strategy, for example the tangency portfolio. We will now augment our regression framework by placing a hyper-prior distribution on the target vector and marginalize out the estimation uncertainty in it through a hierarchical set-up by ‘averaging’ across different target portfolios.

Another motivation for an hierarchical model could also be that the investor wants to take into account external information to form prior beliefs. This idea connects the spirit of [Brandt et al. \(2009\)](#) and [Black and Litterman \(1992\)](#) as it allows to form priors based on external information and to ‘shrink’ the resulting allocation towards the GMVP portfolio conditional on the historical stock market returns. [Brandt et al. \(2009\)](#) model the portfolio weight  $\omega_t$  in each asset directly as a function of several stock characteristics  $C_t \in \mathbb{R}^{N \times k}$  via a simple linear specification:

$$\omega_b^t = \frac{1}{N} (\iota + C_t \Theta). \quad (21)$$

In this setting,  $C_t \Theta$  is standardized such that the deviations from the naive reference portfolio sum up to 0. The coefficients  $\Theta$  are found by maximizing the in-sample expected utility:

$$\Theta_0 = \arg \max_{\Theta} E[u(r_{p,t+1})] = \arg \max_{\Theta} \frac{1}{h} \sum_{t=0}^h u \left( \frac{1}{N} (\iota + C_t \Theta)' r_{t+1} \right). \quad (22)$$

The estimated portfolio allocation  $\omega_b$  is a natural candidate for the reference portfolio: It reflects what can be learned from the explanatory variables taken into account. However, as

$\Theta_0$  is chosen to optimize in-sample utility, ambiguity aversion should reflect the uncertainty in the predictive power of the model.

Our model provides an intuitive approach to combine the predictive power of the parametric model and to reflect the inherent uncertainty by accounting for variations in past observed returns. In order to account for specification (21) we propose a linear hierarchical Bayesian regression model:

$$\text{Likelihood: } Y = X\omega_{hb}^* + \nu \text{ where } \nu \sim N(0, h^{-1}I) \quad (23)$$

$$\text{Prior: } \omega_{hb}^* | \Theta, C^*, h \sim N\left(\frac{1}{N}(\iota + C^*\Theta), gh^{-1}(X'X)^{-1}\right) \quad (24)$$

$$\Theta \sim N(\Theta_0, \Sigma_\Theta) \quad (25)$$

$$h \sim \text{Gamma}(s^{-1}, v). \quad (26)$$

Hereby,  $Y$  and  $X$  are identical to equation (4). The prior mean of the regression coefficient  $\omega_{hb}$  is now determined by  $\omega_b^*$ . This approach closes a gap between [Brandt et al. \(2009\)](#) and [Black and Litterman \(1992\)](#) by incorporating the investment views through the asset characteristics to form priors of the portfolio weights. Comparable to their model, our regression framework uses the historical returns to capture uncertainty and to 'weight' the informed input with the evidence given by the data.

We implement the hierarchical framework described in equation (23) - (26) to evaluate the performance in another horse race. We expect our model to generate significant out-of-sample improvements when incorporating external stock information into the decision framework. Accounting for estimation risk should lead to more stable results compared to the standard approach of [Brandt et al. \(2009\)](#). Sampling from the posterior is possible with a MCMC scheme. Hereby we tried to be rather uninformative in  $h$  by setting  $s$  and  $v$  to 0.001.  $\Sigma_\Theta$  is chosen as  $\tau I$  with  $\tau := 1000$ . The shrinkage intensity  $g$  is arbitrarily set to 1, optimizing  $g$  is left for future research.

The underlying data for our horse race consists of the assets listed in the Dow Jones index. We use weekly data for the time horizon from May 1999 until October 2015. As explanatory variables we use market value measured as the price per share times the number of shares outstanding, the price-earnings ratio and the dividend yield. Based on

those three characteristics we compute weekly estimates of  $\omega_b$  as defined in equation (21). The time horizon is fixed as a rolling window of size  $h = 52$  weeks (1 Year). Based on the estimates for  $\Theta_0$  we set up a Gibbs scheme generating draws from the posterior distribution of  $\omega_{hb}$  defined as in model (23). The posterior mean is used as the portfolio allocation vector  $\omega_{hb}$ . In order to compare the performance of those parametric portfolio policies, Table 3 lists the annualized out-of-sample results after accounting for transaction costs also for the standard GMVP portfolio  $\hat{\omega}_{mvp}$ , the naive portfolio  $\frac{1}{N}\iota_N$  and the shrinkage portfolio with optimal re-balancing derived in Lemma 3.2. As illustrated in Table 3 the hierarchical model outperforms the competing methods in terms of average return and Sharpe-ratio by far. As expected, taking into account estimation risk reduces turnover compared to  $\omega_b$ . However, in terms of out-of-sample utility the Bayesian approach does not outperform the naive portfolio.

## 4 Concluding remarks

Our Bayesian framework allows to investigate the posterior uncertainty in point estimates of asset allocation vectors. We propose to assign a prior on the portfolio weights directly in order to avoid the usual two-step procedure of first estimating the moments of the asset return distribution and second calculating optimal portfolio weights. We further outline several applications and find that the information content in the posterior distribution is sustainable and can be used for reducing welfare losses from turnover costs.

The proposed framework in this paper offers several obvious ways for extensions: An hierarchical prior set-up can also include a hyper prior distribution for the shrinkage parameter  $g$ . This would allow for varying shrinkage intensities for different assets in the portfolio. The intuition here comes from the results in the home bias example: While an investor can almost always be sure to include the US asset in a long position, shrinking other portfolio weights to zero in order to exclude these assets from the portfolio might be reasonable given the varying uncertainty in the estimated portfolio weights. Eventually, more sophisticated portfolio strategies can be incorporated through different Bayesian testing procedures.



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## A Appendix

### Proofs

#### Proof A.1 (Proof of Lemma 3.1)

Under quadratic loss the Bayesian decision rule corresponds to the posterior mean  $E(\hat{\omega}|X, Y)$ . The posterior mean is  $E(\hat{\omega}|X, Y) = \frac{1}{1+g}\hat{\omega}_e(\gamma) + \frac{g}{1+g}\hat{\omega}_{mvp}$ . The closed form solution for the mean-variance optimization problem in terms of CE maximization can be derived as

$$\begin{aligned}\hat{\omega}_e(\gamma) &= \hat{\omega}_{mvp} + \frac{1}{\gamma} \left( \hat{\Sigma}^{-1} - \frac{\hat{\Sigma}^{-1} \iota_N \iota_N' \hat{\Sigma}^{-1}}{\iota_N' \hat{\Sigma}^{-1} \iota_N} \right) \hat{\mu} = \hat{\omega}_{mvp} + \frac{1}{\gamma} A \hat{\mu} \\ \implies E(\bar{\omega}|X, Y) &= \frac{1}{1+g} \hat{\omega}_e(\gamma) + \frac{g}{1+g} \hat{\omega}_{mvp} = \hat{\omega}_{mvp} + \underbrace{\left( \frac{1}{\gamma(1+g)} \right)}_{:= \frac{1}{\tilde{\gamma}}} A \hat{\mu} = \hat{\omega}_e(\tilde{\gamma}).\end{aligned}$$

□

#### Proof A.2 (Proof of Lemma 3.2)

It holds that  $\Sigma_{\bar{\omega}} = c^2 \Sigma_{mvp}$ , where  $\Sigma_{mvp}$  takes the form derived in Equation (17).<sup>10</sup> Plug-in into the objective function  $Z(c) := \text{tr}(\Sigma_{\bar{\omega}} \hat{\Sigma}) + \bar{\omega}' \hat{\Sigma} \bar{\omega} + \hat{\mu}' \Sigma_{\bar{\omega}} \hat{\mu}$  leads to

$$\begin{aligned}Z &= c^2 \text{tr}(\Sigma_{mvp} \hat{\Sigma}) + \left( c \omega_{mvp} + (1-c) \frac{1}{N} \iota_N \right)' \hat{\Sigma} \left( c \omega_{mvp} + (1-c) \frac{1}{N} \iota_N \right) \\ &\quad + c^2 \hat{\mu}' \Sigma_{mvp} \hat{\mu} \\ &= c^2 \left( \hat{\mu}' \Sigma_{mvp} \hat{\mu} + \text{tr}(\Sigma_{mvp} \hat{\Sigma}) + (\omega_{mvp} - \frac{1}{N} \iota_N)' \hat{\Sigma} (\omega_{mvp} - \frac{1}{N} \iota_N) \right) \\ &\quad + \frac{2c}{N} \left( \omega_{mvp} - \frac{1}{N} \iota_N \right)' \hat{\Sigma} \iota_N + \frac{1}{N^2} \iota_N' \hat{\Sigma} \iota_N \\ \implies \frac{dZ}{dc} &= 2c \left( \hat{\mu}' \Sigma_{mvp} \hat{\mu} + \text{tr}(\Sigma_{mvp} \hat{\Sigma}) + (\omega_{mvp} - \frac{1}{N} \iota_N)' \hat{\Sigma} (\omega_{mvp} - \frac{1}{N} \iota_N) \right) + \\ &\quad \frac{2}{N} \left( \omega_{mvp} - \frac{1}{N} \iota_N \right)' \hat{\Sigma} \iota_N \stackrel{!}{=} 0 \\ \Leftrightarrow c^* &= -\frac{1}{N} \frac{(\omega_{mvp} - \frac{1}{N} \iota_N)' \hat{\Sigma} \iota_N}{\hat{\mu}' \Sigma_{mvp} \hat{\mu} + \text{tr}(\Sigma_{mvp} \hat{\Sigma}) + (\omega_{mvp} - \frac{1}{N} \iota_N)' \hat{\Sigma} (\omega_{mvp} - \frac{1}{N} \iota_N)}\end{aligned}$$

This procedure is equivalent to a Bayesian setting with  $g$ -prior where  $\frac{g}{g+1} = c^*$  and reference

<sup>10</sup>Due to the absence of estimation risk for deterministic portfolio weights,  $\Sigma_{\frac{1}{N}} = 0$ .

portfolio  $\beta = \frac{1}{N}\iota_N$ .

□

## Figures

### In sample efficient frontier

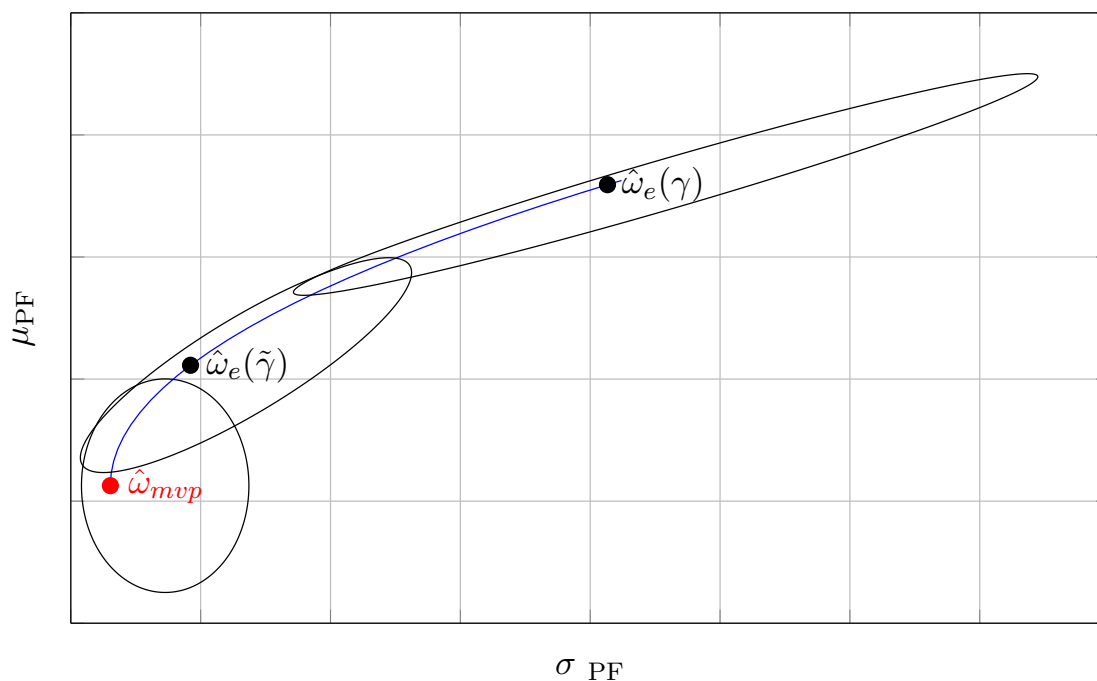


Figure 1: The blue line corresponds to the efficient frontier in the  $(\sqrt{\sigma^2}, \mu)$  plane. The red dot depicts the location of  $\hat{w}_{mvp}$ , the two black dots show the efficient portfolios derived for investors with  $\gamma$  and  $\tilde{\gamma}$ . The ellipsoids around the dots visualize the impact of the estimation risk on the location of the portfolios in the  $(\sqrt{\sigma^2}, \mu)$  plane, computed as credible sets for significance level  $\alpha = 0.05$ .

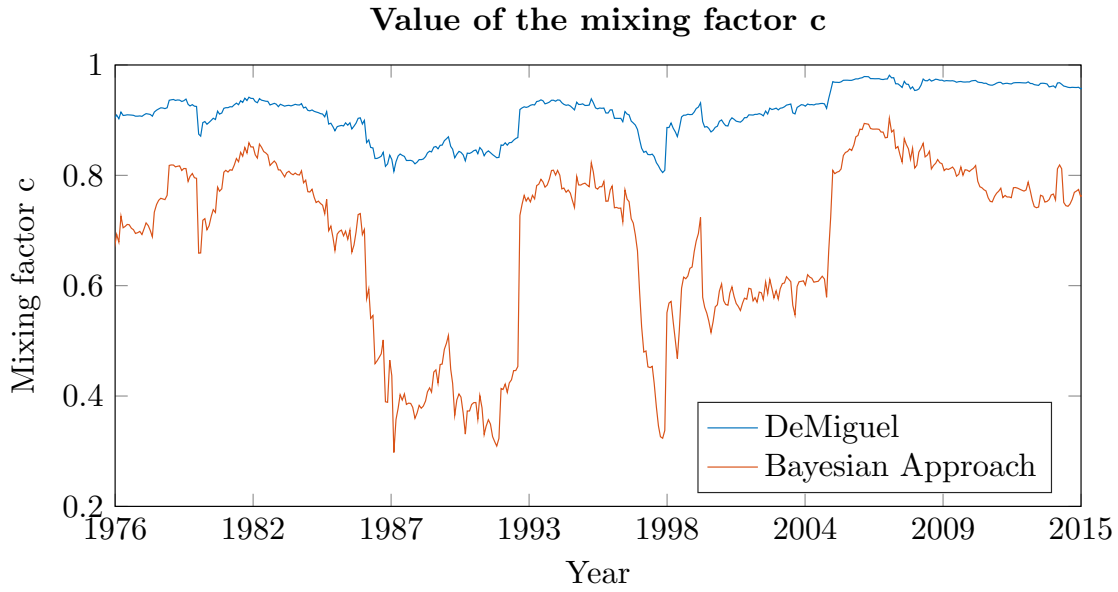


Figure 2: Time series of the mixing factor  $c$ , the fraction of wealth invested into the GMVP portfolio, based on the strategy of DeMiguel et al. (2009) and our bayesian approach.

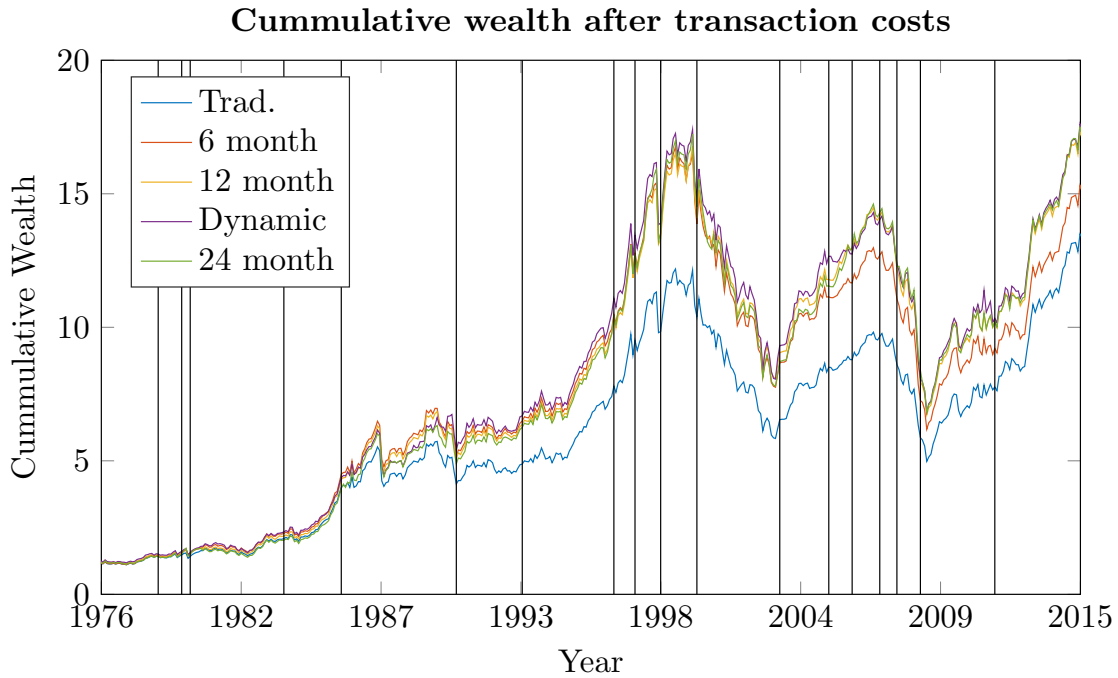


Figure 3: Cummulative wealth after transaction costs of the portfolio strategies based on monthly returns of 6 national MSCI stock market indexes (CAN,GER,JP,UK,USA,SUI) from 1969 to 2015. The portfolios applied differ in the timing of readjustment:  $\omega_{mvp}^1$ ,  $\omega_{mvp}^6$ ,  $\omega_{mvp}^{12}$ ,  $\omega_{mvp}^{24}$  and  $\omega_{mvp}^{dyn}$  are defined as the GMVP portfolio weights, readjusted monthly, semi-annual, annual, every second year and with the dynamic procedure (whereas  $\xi$  is set to 0.01). Transaction costs are computed as  $c = 50$  bp, proportional to monthly turnover  $\iota(|\omega_{i,t} - \omega_{i,t-1,+}|)$ . The vertical black lines visualize the dates of readjustment for the dynamic portfolio strategy.

## Credible Regions

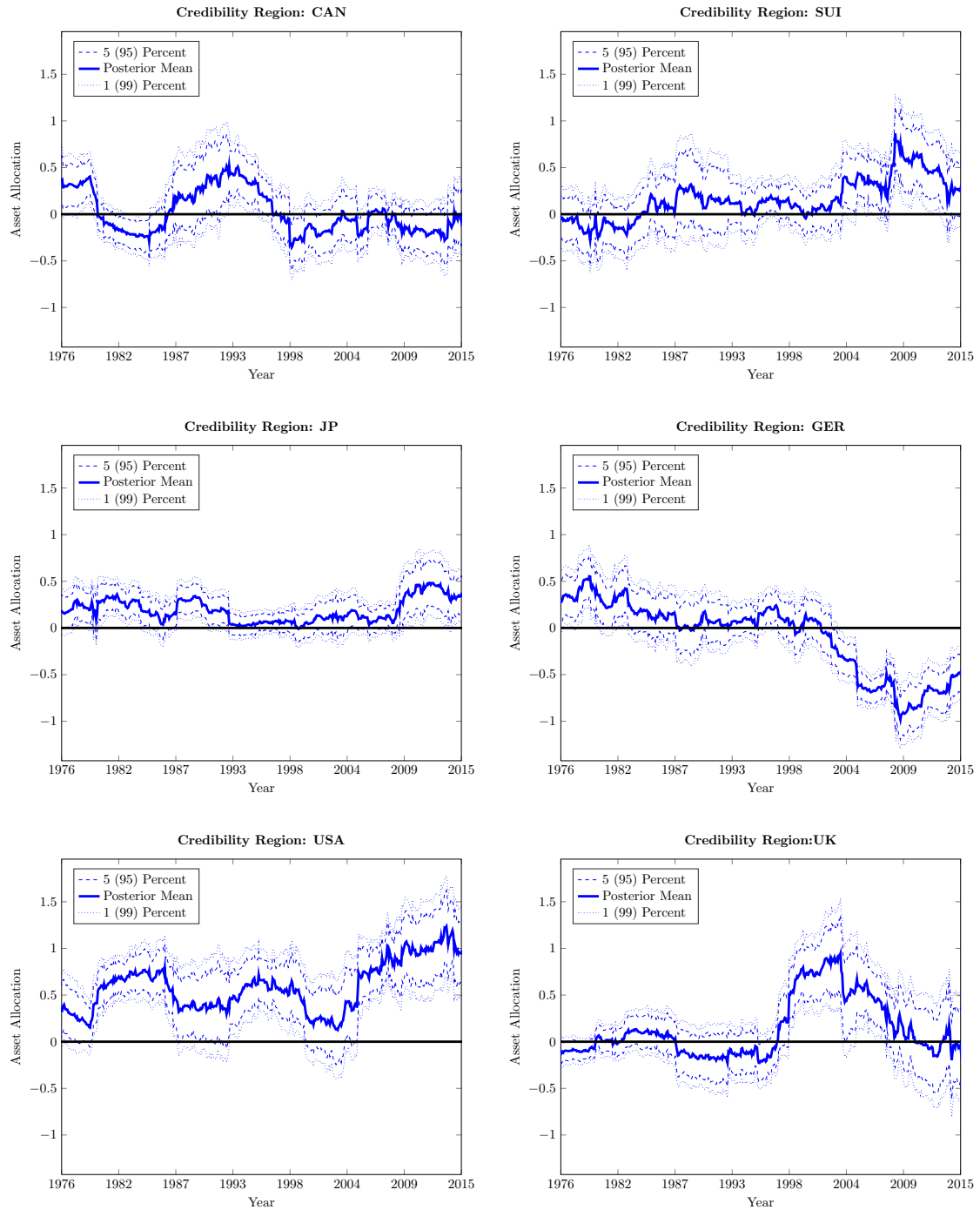


Figure 4: Credible regions for  $\omega_{mvp}^i$  for six market indexes (Canada, Germany, Japan, Switzerland, UK and USA) based on an estimated window of length 60 months. The time series of monthly returns, provided by MSCI, reaches from 1969 until 2015. The bold line corresponds to the posterior mean, the dashed line visualizes the 0.95 credible regions and the dotted line represents the 0.99 credible interval. The black line corresponds to 0 and indicates whether the posterior probability of the hypothesis  $\omega_{t,i}^{mvp} \neq 0$  is high.

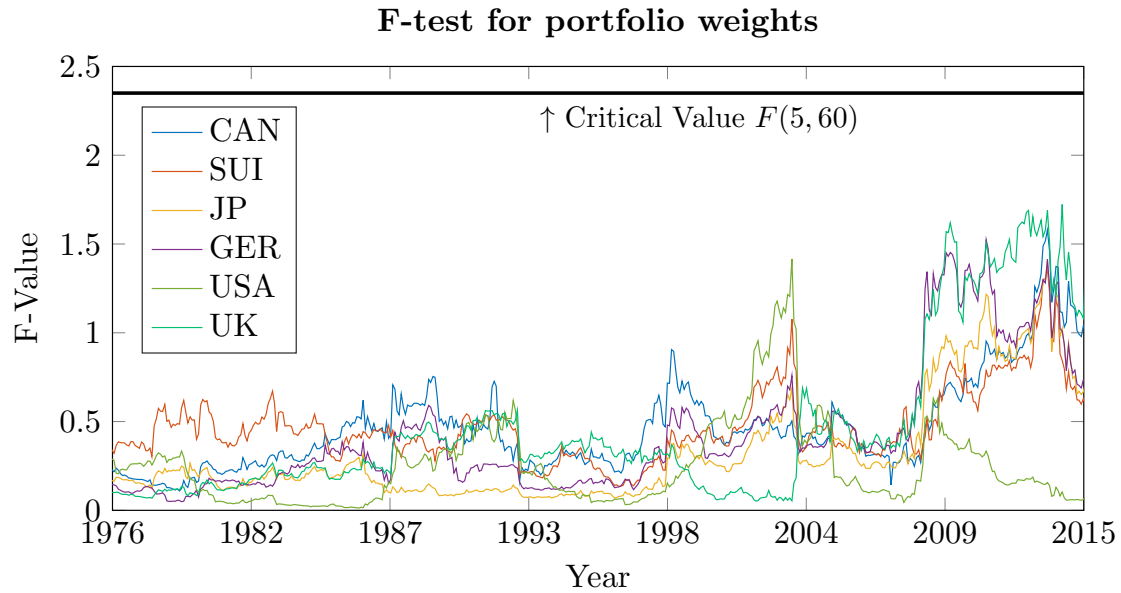


Figure 5: Value of the test statistic  $\zeta_j$  for an bayesian F-test, investigating the hypothesis  $\omega_{i,t} = 0 \forall i \neq j$  and  $j \in \{\text{CAN, SUI, JP, GER, USA, UK}\}$ . The posterior of  $\zeta_j$  follows a  $F(N, T)$  distribution.

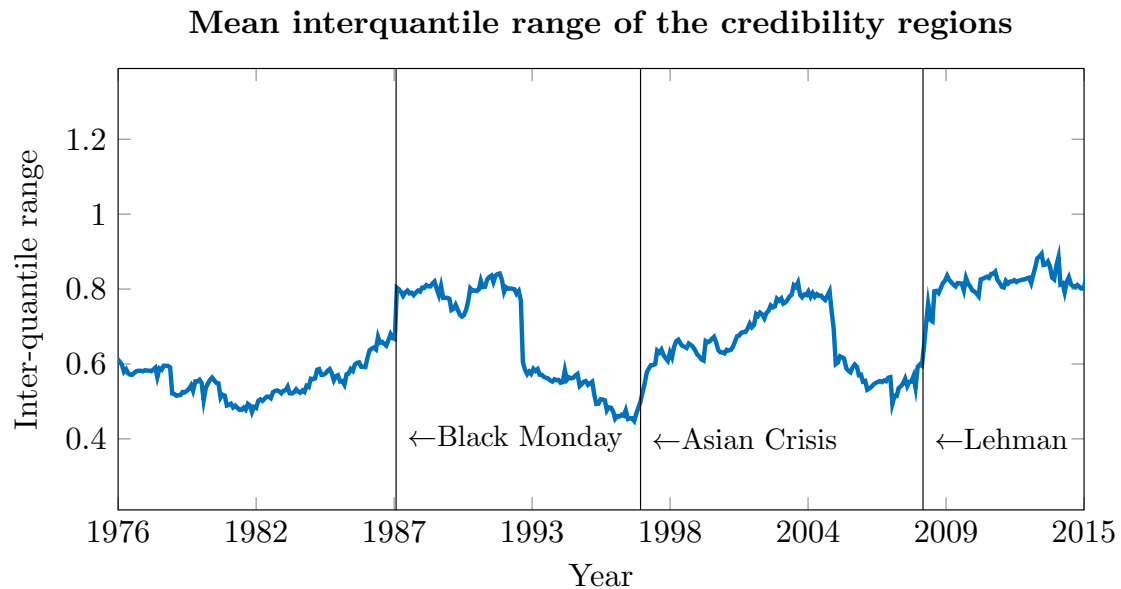


Figure 6: Mean interquantile-range for the 6 assets investigated over the time horizon between 1969 and 2015. The vertical lines indicate important dates to illustrate possible channels affecting the changes.



## Tables

Table 1: Out-of-sample results I:  $N = 6$  stock market indexes (CAN, GER, JP, UK, USA, SUI) from 1969 to 2015

	TO	$\mu$	$\sigma$	SR	CE	CE-Loss
$\omega_{mvp}$	1.7221	0.0728	0.1335	0.5457	0.0550	-0.0091
$\omega_{dm}$	1.2106	0.0723	0.1353	0.5348	0.0540	-0.0078
$\omega_{opt}$	1.3194	<b>0.0732</b>	<b>0.1335</b>	<b>0.5483</b>	<b>0.0554</b>	<b>-0.0095</b>
Naive	<b>0.3093</b>	0.0724	0.1517	0.4773	0.0494	-

*Note:* The table reports annualized mean results for out-of-sample portfolio returns for the rolling window based on monthly returns of 6 national MSCI stock market indexes (CAN, GER, JP, UK, USA, SUI) from 1969 to 2015. The portfolios applied are the GMVP ( $\hat{\omega}_{mvp}$ ), the shrinkage portfolio of (DeMiguel et al., 2009, eq. (11))  $\omega_{dm}$ , the optimal shrinkage portfolio  $\omega_{opt}$  derived in lemma 3.2 and the naive portfolio. All the return related values are after transaction costs, which are computed as  $c = 50$  bp, proportional to monthly turnover  $\iota'(|\omega_{i,t} - \omega_{i,t-1,+}|)$ . TO denotes the average turnover,  $\mu$  is the annualized return,  $\sigma$  is the standard deviation of the portfolio return and  $SR := \frac{\mu}{\sigma}$  is the Sharpe ratio, CE reports the certainty equivalents for a risk aversion  $\gamma = 1$  and RL means return loss against the  $1/N$  portfolio.

Table 2: Out-of-sample results II:  $N = 6$  stock market indexes (CAN, GER, JP, UK, USA, SUI) from 1969 to 2015

	TO	$\mu$	$\sigma$	SR	CE	RL
$\omega_{mvp}^1$	1.7221	0.0728	<b>0.1335</b>	0.5457	0.0550	-0.0091
$\omega_{mvp}^6$	0.7676	0.0763	0.1358	0.5616	0.0578	-0.0114
$\omega_{mvp}^{12}$	0.5497	0.0794	0.1363	0.5823	0.0608	-0.0143
$\omega_{mvp}^{24}$	0.3721	0.0801	0.1397	0.5734	0.0606	-0.0134
$\omega_{mvp}^{dyn}$	0.4049	<b>0.0801</b>	0.1369	<b>0.5852</b>	<b>0.0614</b>	<b>-0.0148</b>
Naive	<b>0.3093</b>	0.0724	0.1517	0.4773	0.0494	-

*Note:* The table reports annualized mean results for out-of-sample portfolio returns for the rolling window based on monthly returns of 6 national MSCI stock market indexes (CAN, GER, JP, UK, USA, SUI) from 1969 to 2015. The portfolios applied differ in the timing of readjustment:  $\omega_{mvp}^1$ ,  $\omega_{mvp}^6$ ,  $\omega_{mvp}^{12}$ ,  $\omega_{mvp}^{24}$  and  $\omega_{mvp}^{dyn}$  are defined as the GMVP portfolio weights, readjusted monthly, semi-annual, annual, every second year and with the dynamic procedure (whereas  $\alpha$  is set to 0.01). All the return related values are after transaction costs, which are computed as  $c = 50$  bp, proportional to monthly turnover  $\iota'(|\omega_{i,t} - \omega_{i,t-1,+}|)$ . TO denotes turnover,  $\mu$  is the average return,  $\sigma$  is the standard deviation of the portfolio return and  $SR := \frac{\mu}{\sigma}$  is the Sharpe ratio, CE reports the certainty equivalents for a risk aversion  $\gamma = 1$  and RL means return loss against the  $1/N$  portfolio.

Table 3: Out-of-sample results III:  $N = 28$  Dow Jones stocks

	TO	$\mu$	$\sigma$	SR	CE	CE-Loss
$\omega_{mvp}$	31.9128	0.0133	0.1628	0.0818	-0.0132	0.0305
$\omega_{hb}$	17.4199	<b>0.1019</b>	0.3033	<b>0.3360</b>	0.0099	-0.0203
$\omega_{opt}$	26.3459	0.0194	0.1528	0.1266	-0.0040	0.0218
$\omega_b$	105.5332	0.0477	<b>0.6251</b>	0.0763	-0.3431	0.1206
Naive	<b>1.0830</b>	0.0466	0.1732	0.2691	<b>0.0166</b>	-

*Note:* The table reports annualized mean results for out-of-sample portfolio returns for the rolling window based on weekly returns of 28 stocks listed in Dow Jones from May 1999 until October 2015. The portfolios applied are the GMVP ( $\hat{\omega}_{mvp}$ ), the Hierarchical Bayes weights  $\omega_{hb}$  derived in Equation (23), the naive portfolio  $\frac{1}{N} \iota_N$ , the optimal shrinkage portfolio  $\omega_{opt}$  derived in lemma 3.2, and the parametric portfolio strategy  $\omega_b$  as described in Brandt et al. (2009). The parametric portfolio policies  $\omega_b$  and  $\omega_{hb}$  are computed based on Market Value, Price-Earnings-Ratio and Dividend-yield. All the return related values are after transaction costs, which are computed as  $c = 50$  bp, proportional to weekly turnover  $\iota'(|\omega_{i,t} - \omega_{i,t-1,+}|)$ . TO denotes the average turnover,  $\mu$  is the average annual return,  $\sigma$  is the standard deviation of the portfolio return and  $SR := \frac{\mu}{\sigma}$  is the Sharpe ratio, CE reports the certainty equivalents for a risk aversion  $\gamma = 1$  and RL means return loss against the  $1/N$  portfolio.