Ambiguous Information, Permanent Income, and Consumption Fluctuations

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Abstract

This paper studies asymmetric responses in consumption where the asymmetries are endogenously generated by agents' preferences and incomplete knowledge about information quality. Agents form expectations about the future based on incomplete information which is assumed to be ambiguous and these future expectations, distorted by ambiguity, affect spending asymmetrically. With a noisy signal of uncertain quality, consumption features asymmetric responses: the absolute size of the responses depends on whether the signal delivers good or bad news. I estimate the model on U.S. data by maximum likelihood and the estimates suggest that ambiguity plays a non-negligible role in consumption fluctuations.

Keywords: ambiguity, consumption, permanent income hypothesis, news and noise.

JEL Classification Codes: D8, E20, E32.

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There is evidence that consumption growth is left-skewed. For example, consumption expenditure growth per capita[1] of the U.S. from 1948 to 2015 exhibits left-skewness with a negative estimate of -0.5654.[2] Van Nieuwerburgh and Veldkamp (2006) suggest that negative skewness in the distribution of changes represents the large number of small positive changes (or gradual increases) and the small number of large negative changes (or sudden decreases).[3] One potential reason for a left-skewed consumption growth is that positive and negative shocks have asymmetric effects on consumption.[4] Or it can also be explained by agents reacting symmetrically to asymmetric shocks. While two explanations (or a mixture of them) can easily justify a left-skewed unconditional distribution of consumption growth, it is not trivial to explain why agents react non-linearly to symmetric shocks.

Table 1: Left-skewed Consumption Growth

Sample	Country	Skewness
1870-2009	Canada	-0.7829
1820-2009	France	-0.8155
1851-2009	Germany	-7.8171
1861-2009	Italy	-1.3680
1870-2009	Japan	-2.2255
1830-2009	the United Kingdom	-0.7890

Notes: Data is yearly data. Per capital consumption data is from Barro and Ursua (2010) and can be accessed at http://scholar.harvard.edu/barro/publications/barro-ursua-macroeconomic-data

In this paper, by focusing on agents' preferences and information structures I attempt to suggest a possible explanation for the asymmetric effects of (symmetric) exogenous shocks in a simple forward looking consumption model where agents' beliefs are the key ingredients to explain consumption dynamics.[5] This follows a view on business cycles emphasizing the role of anticipating the future. Agents form expectations about the future based on incomplete information which is assumed to be ambiguous and these future expectations, distorted by ambiguity, affect spending asymmetrically. An interesting feature of the model

[1]Yang (2011) also documents that durable consumption growth in the U.S. is left-skewed.

[2]Table 1 reports skewness of consumption expenditure growth in the rest of the Group of Seven countries. Consumption growth is obtained by taking the first difference of log per-capita consumption.

[3]Van Nieuwerburgh and Veldkamp (2006) explain growth asymmetry in macro aggregates with learning about the aggregate technology level.

[4]The literature suggests that there is ample evidence of the asymmetric effects of shocks on key macro variables. Among others, Cover (1992) using the quarterly U.S. data suggests that while positive money supply shocks do not have an effect on output, negative ones do; Kandil (2002), using aggregate data of real output, price, and wage for the United States, provides evidence of the asymmetric effects of aggregate demand shocks; Hussain and Malik (2014) show that the effects of tax increase and decrease are asymmetric.

[5]Cao and Nie (2015) provides an explanation of asymmetric responses of the economy to symmetric exogenous productivity shocks with market incompleteness.

is that the possibility of agents responding *symmetrically* is not entirely ruled out such that it is possible to test which story (asymmetric or symmetric responses to symmetric shocks) fits data better statistically in a simple unified framework and provides numerical characterization of the conditional dynamics of consumption.

A common practice of modeling agents' expectations about future outcomes in macroeconomic analysis has been the use of rational expectations, often called model consistent expectations, where it requires, roughly speaking, that agents' beliefs about future variables coincide with expectations predicted by the model. While it has been the main ingredient of most dynamic general equilibrium models studied these days, the assumption imposes strong restrictions on agents' behaviors. For example, it is unlikely that consumers are fully aware of the underlying mechanisms governing firms' price-setting practices, technological progress, or other types of uncertainty regarding fundamentals of the economy.[6]

This paper relaxes restrictions imposed on agents' knowledge about the stochastic processes governing the economy: A stochastic signal about a permanent component of productivity is assumed to be not only noisy but also ambiguous in its information quality. As agents' beliefs about the state of the economy critically affect macroeconomic dynamics, how those expectations are formed under ambiguity turns out to be very important. In other words, agents face an additional challenge to perceive information of uncertain quality given their preferences. This, in turn, requires to model preferences under ambiguity, and I adopt the *maxmin* expected utility decision with multiple priors, where behavior derived from the decision rule is consistent with experimental evidence such as the Ellsberg Paradox. By assuming that agents exhibit aversion to uncertainty, processing a signal of uncertain quality - or updating beliefs - becomes estimating fundamentals consistent with a worst-case evaluation of ambiguous information, and conditional responses of the agents exhibit asymmetries: The absolute size of the responses depends on whether noisy sources of information deliver good or bad news.

Specifically, the theory is based on a model of business cycles driven by shocks to agents' expectations regarding productivity, where agents form anticipations about the future by observing noisy signals about productivity as in Blanchard, L'Huillier, and Lorenzoni (2013) and Cao and L'Huillier (2015). These signals sometimes turn out to be news and sometimes just noise, and consumers need to solve a signal extraction problem to decide their current spending. Later on, if information turns out to be news, agents adjust their expectations upward and the economy gradually adjusts to a new level of activity; if ex-post information

[6] A number of studies aim to relax such restrictions and to document subsequent macroeconomic outcomes. For instance, Bianchi and Melosi (2013) develop methods to study general equilibrium models where forward looking agents learn about the stochastic properties of realized events following waves of pessimism, optimism, and uncertainty and Adam and Marcet (2011) relax the rationality assumption to capture the notion that agents do not fully understand some underlying statistical properties.

turns out to be just noise, the economy returns to its original state of activity. In my version of the model, I modify this information structure such that agents are uncertain about the quality of noisy signals they receive and the uncertainty is captured by *the range of precisions*:

$$1/\sigma_\nu^2 \in [1/\bar{\sigma}_\nu^2, 1/\underline{\sigma}_\nu^2]$$

where $1/\sigma_\nu^2$ denotes the signal precision. In such a case, if agents are assumed to exhibit aversion toward ambiguity, they follow the *maxmin* optimization by which they make decisions that maximize their expected utility under a worst-case belief. The latter depends on the types of signals they receive. For a signal delivering bad news, a worst case is that the signal is very informative. Conversely, for a signal delivering good news, a worst case is such that the signal is very noisy.[7] This leads the agents to react more to bad news than to good news such that the size of the response is larger in an absolute value when bad news is delivered. In addition, when information quality becomes more ambiguous, the responses exhibit a larger degree of asymmetries.

This paper follows the tradition of a business cycle model where expectations play a significant role; the original thesis laid out in Pigou (1927) and a recent work by Beaudry and Portier (2004) are frequently cited works that started this strand of literature. Distinguishing shocks from the fundamental and transitory components of productivity are essential ingredients of this model and it is closely related to a business cycle model driven by shocks to agents' expectations on productivity as in Lorenzoni (2009), Blanchard et al. (2013), and Rousakis (2013). While sharing similar information structures and agent's information processing, I extend the setup to allow for uncertain quality of information and agents' aversion toward uncertainty.

To model ambiguity I follow the setup axiomatized by Gilboa and Schmeidler (1989) and recently adopted by Epstein and Schneider (2008), Ilut (2012), Ilut, Kehrig, and Schneider (2014), and Baqaee (2015).[8] Epstein and Schneider (2008) discuss asset markets in which ambiguity averse investors process news of uncertain quality with a worst-case assessment of information quality and the model generates more stronger reaction to bad news than good news which results in asymmetric responses in asset market; Ilut (2012) builds a model of exchange rate determination where an ambiguity averse agent solve a signal extraction problem with uncertain signal precision and take departures from uncovered interest rate parity; in Ilut, Kehrig, and Schneider (2014) firms' hiring decisions is

[7]Differentiating a signal delivering good news from the one delivering bad news is discussed in Section 1.

[8]In these models, agents possess multiple priors about the information quality of their signals and act upon their worst case prior to make decisions under ambiguity. In my model, in addition to an ambiguous signal, agents receive an additional signal which is assumed to be unambiguous.

modeled under ambiguous information. Firms receive ambiguous information about productivity of the economy and maximize multiple priors utility, which reflects firms' aversion to ambiguity; Baqaee (2015) attempts to incorporate ambiguous information and the signal extraction problem in order to explain downward wage rigidities where an equilibrium wage is more sensitive to inflation than to disinflation.[9]

Lastly, this paper is related to the random walk behavior of consumption in which the behavior of consumption is fully determined by the long-run level of productivity.[10] Combined with imperfect information structure, it requires that agents should forecast or estimate the long-run level of productivity to choose consumption such that, since agents' forecasts are distorted by the presence of ambiguity as we will see, ambiguity plays an important role in explaining consumption fluctuations.

The rest of the paper is organized as follow. Section 1 illustrates belief updating under ambiguity. Section 2 describes the model whereas Section 3 details the solution to the model. Section 4 discusses estimating the model and Section 5 concludes.

1 Belief Updating with Ambiguity

A crucial ingredient of this model is to determine how agents update beliefs under ambiguity or, in other words, how agents process new information which is assumed to be ambiguous and update beliefs about fundamentals. Specifically, since consumption depends solely on agents' expectation about productivity in the long run, it is essential for agents to estimate an unobserved permanent state (productivity) of the economy. However, with information quality being ambiguous, it is not trivial how agents update beliefs.

1.1 The One Signal Example

To begin with, consider the following case in which agents receive a signal each period to update beliefs about fundamentals.[11] Agents observe a noisy signal s_t about an unobservable fundamental x_t:

$$s_t = x_t + \nu_t$$

[9]Similarly, Ilut and Schneider (2013) build a state-of-the-art ambiguous business cycle model where shocks to confidence, which is modeled as changes in ambiguity, play important role in explaining fluctuations. Whereas Ilut and Schneider (2013) introduce ambiguity in a productivity shock with first-order effects, Masolo and Monti (2015) study the implication of introducing ambiguity in a monetary policy shock.

[10]This random walk behavior of consumption is discussed in detail in Blanchard et al. (2013) for a baseline New Keynesian model and in Cao and L'Huillier (2015) and in Cao, L'Huillier, and Yoo (2015) for a small open economy RBC model.

[11]The next section considers the example in which agents receive two signals each period.

where ν_t is an i.i.d. normal shock with variance σ_ν^2. The information quality of the signal is assumed to be ambiguous such that $\sigma_\nu^2 \in [\underline{\sigma}_\nu^2, \bar{\sigma}_\nu^2]$.[12] The fundamental x_t follows a random walk process:

$$x_t = x_{t-1} + \epsilon_t$$

where ϵ_t is an i.i.d. normal shock with variance σ_ϵ^2. The shocks ν_t and ϵ_t are assumed to be mutually independent. Observing the noisy signal, agents update beliefs about the fundamental:

$$\begin{aligned} x_{t|t} &= x_{t|t-1} + \omega_s (s_t - x_{t|t-1}) \\ \omega_s &= \sigma_x^2 / \left(\sigma_x^2 + \sigma_\nu^2\right) \end{aligned} \tag{1}$$

where $x_{t|t}$ is the estimate of the fundamental x_t with all available information at time t, and ω_s is the gain from revising the previous period estimate based on the surprise associated with the realization of the shocks.[13] Therefore, the revised belief is just a weighted average of the previous period's estimate of the fundamental and the observed signal where the relevant weights depend on a signal-to-noise ratio. The more precise the signal is, agents attach relatively more weights to the signal and vise versa. Since the quality of information is represented by *the range of precisions*, $[1/\underline{\sigma}_\nu^2, 1/\bar{\sigma}_\nu^2]$, it is unclear how agents choose relevant weights to update beliefs. The gain of observing the signal can take any value between $\underline{\omega}_s$ and $\bar{\omega}_s$ where

$$\underline{\omega}_s = \frac{\sigma_x^2}{\sigma_x^2 + \bar{\sigma}_\nu^2}, \; \bar{\omega}_s = \frac{\sigma_x^2}{\sigma_x^2 + \underline{\sigma}_\nu^2}$$

Assume that, for a given utility function $u(x)$, it is strictly increasing in x and that agents are ambiguity averse in the sense that they maximize expected utility under a

[12] For the rest of this paper, I use both the variance σ_ν^2 and the precision $1/\sigma_\nu^2$ interchangeably to describe information quality.

[13] In other words, it represents the relative importance of the error (or the surprise) with respect to the prior estimate and is a function of the precision $1/\sigma_\nu^2$. More precisely, $\frac{\partial \omega_s}{\partial \sigma_\nu^2}$ can be computed as follows:

$$\begin{aligned} \frac{\partial \omega_s}{\partial \sigma_\nu^2} &= \frac{\partial \left[\sigma_x^2 / \left(\sigma_x^2 + \sigma_\nu^2\right)\right]}{\partial \sigma_\nu^2} \\ &= \frac{\partial \sigma_x^2}{\partial \sigma_\nu^2} \left(\sigma_x^2 + \sigma_\nu^2\right)^{-1} - \left(\sigma_x^2 + \sigma_\nu^2\right)^{-1} \sigma_x^2 \left(\frac{\partial \sigma_x^2}{\partial \sigma_\nu^2} + 1\right) \\ &= \frac{\partial \sigma_x^2}{\partial \sigma_\nu^2} \left(\frac{\sigma_\nu^2}{\sigma_x^2 + \sigma_\nu^2}\right) - \frac{\sigma_x^2}{\sigma_x^2 + \sigma_\nu^2} \\ &< 0 \end{aligned}$$

The last inequality comes from the fact that $\partial \sigma_x^2 / \partial \sigma_\nu^2 < 0$.

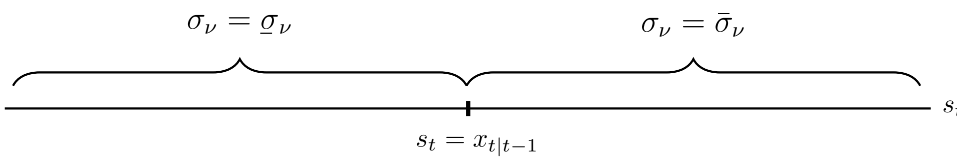

Figure 1: The Cutoff Rule for Belief Updating

Notes: $1/\sigma_\nu^2$ is the precision of information quality which is assumed to be ambiguous such that $1/\sigma_\nu^2 \in [1/\underline{\sigma}_\nu^2, 1/\bar{\sigma}_\nu^2]$.

worst case belief chosen from the family of priors.[14] This leads to agents updating beliefs according to the cutoff rule shown in Figure 1. From (1), it is easy to see that whenever $s_t > x_{t|t-1}$, the largest ω_s minimizes $x_{t|t}$. Similarly, when $s_t < x_{t|t-1}$, minimizing $x_{t|t}$ requires that ω_s takes the smallest value. As ω_s is strictly increasing in $1/\sigma_\nu^2$, only the maximum $(1/\underline{\sigma}_\nu^2)$ and the minimum $(1/\bar{\sigma}_\nu^2)$ from the range of precisions become relevant to update beliefs for ambiguity averse agents, which simplifies solving the model. Intuitively, ambiguity averse agents consider a signal very noisy when they receive good signals. On the contrary, they interpret a signal as very informative when receiving bad signals.[15] For the limiting case in which a signal is related to a single likelihood $(\underline{\sigma}_\nu^2 = \bar{\sigma}_\nu^2 = \sigma_\nu^2)$, the gain of observing noisy signals is pinned down by $\omega_s = \sigma_x^2/(\sigma_x^2 + \sigma_\nu^2)$ regardless of whether the signal delivers good $(s_t > x_{t|t-1})$ or bad news $(s_t < x_{t|t-1})$.

1.2 The Multiple Signal Case

Assume now that agents receive multiple signals about the fundamental where one of the signals is ambiguous. Here, let the number of signals be two.[16] Specifically, in addition to the noisy signal s_t described in the previous section, agents receive an additional signal a_t whose quality is measured by the signal precision $1/\sigma_\eta^2$:

$$a_t = x_t + \eta_t$$

where η_t is an i.i.d. normal shock with variance σ_η^2. Agents use the two signals to update beliefs about the fundamental x_t. Let $\mathbb{E}_t[x_t] = \mathbb{E}[x_t|\mathcal{I}_t] = x_{t|t}$ and $\mathbb{E}_{t-1}[x_t] = \mathbb{E}[x_t|\mathcal{I}_{t-1}] = x_{t|t-1}$ respectively represent the estimates of x_t with the current information set $(\mathcal{I}_t)$ and

[14]Here, the family of priors refers to the range of precisions.

[15]To clarify definition, note that whenever a signal is greater than agents' ex-ante expectations, that signal delivers a good news and vice versa.

[16]Extending the discussion to the case where the number of signals is N and $N-1$ of them are unambiguous is trivial.

the lagged information set ($\mathcal{I}_{t-1}$):

$$\mathbb{E}[x_t|\mathcal{I}_t] = x_{t|t-1} + K(S_t - S_{t|t-1}) \tag{2}$$

where $S_t = (a_t,\ s_t)'$ is a vector of signals and $K = (k_a\ k_s)$ is a row vector representing the gains of observing the signals. Specifically, k_i denotes the gain of observing the signal i.

From (2) the updated estimate on x_t with two signals can be summarized by a weighted average of the previous period estimate of the fundamental $x_{t|t-1}$ and of revisions based on the surprises associated with the realization of each shock:

$$\begin{aligned} x_{t|t} &= \left(\frac{\sigma_\nu^2\sigma_\eta^2}{\sigma_\nu^2\sigma_\eta^2 + \sigma_\nu^2\sigma_x^2 + \sigma_\eta^2\sigma_x^2}\right) x_{t|t-1} + \left(\frac{\sigma_x^2\sigma_\eta^2}{\sigma_\nu^2\sigma_\eta^2 + \sigma_\nu^2\sigma_x^2 + \sigma_\eta^2\sigma_x^2}\right) s_t + \left(\frac{\sigma_x^2\sigma_\nu^2}{\sigma_\nu^2\sigma_\eta^2 + \sigma_\nu^2\sigma_x^2 + \sigma_\eta^2\sigma_x^2}\right) a_t \\ &= x_{t|t-1} + \omega_s(s_t - x_{t|t-1}) + \omega_a(a_t - x_{t|t-1}) \end{aligned} \tag{3}$$

where ω_s and ω_a represent the relative importance of the errors (the surprises) with respect to the prior estimate. From (3), it can be shown that ambiguity averse agents update their beliefs according to the following decision criteria:

$$\sigma_\nu^2 = \begin{cases} \underline{\sigma}_\nu^2, & \text{if } a_t > x_{t|t-1} \text{ and } s_t < x_{t|t-1} \\ \bar{\sigma}_\nu^2, & \text{if } a_t < x_{t|t-1} \text{ and } s_t > x_{t|t-1} \end{cases}$$

As ω_s (ω_a) is increasing (decreasing) in the signal precision ($1/\sigma_\nu^2$), whenever revisions to the previous period estimate of the fundamental following the signals, $(a_t - x_{t|t-1})$ and $(s_t - x_{t|t-1})$, have different signs, it is easy to pin down the signal precision to minimize $x_{t|t}$. For instance, when $a_t > x_{t|t-1}$ and $s_t < x_{t|t-1}$, ω_s should take the largest values and ω_a smallest value. Thus, $1/\underline{\sigma}_\nu^2$ minimizes the estimate of the fundamental $x_{t|t}$. Similarly, when $a_t < x_{t|t-1}$ and $s_t > x_{t|t-1}$, minimizing $x_{t|t}$ requires ω_s to take the smallest value and ω_a to take the largest possible value such that $1/\bar{\sigma}_\nu^2$ is chosen to update beliefs. Intuitively, as a particular signal becomes less precise, the gain from observing that particular signal is relatively smaller; at the same time, you gain relatively more from observing the other signal. Left panel in Figure 2 depicts this cutoff rule of ambiguity averse agents. When both signals are greater than (the upper-right quadrant) or smaller than (the lower-left quadrant) the previous period estimate of the fundamental, (3) does not seem to produce simple decision criteria.

Now, let the agents update beliefs sequentially such that they first update beliefs with the unambiguous signal a_t and then with the ambiguous signal s_t. Then, the updating of

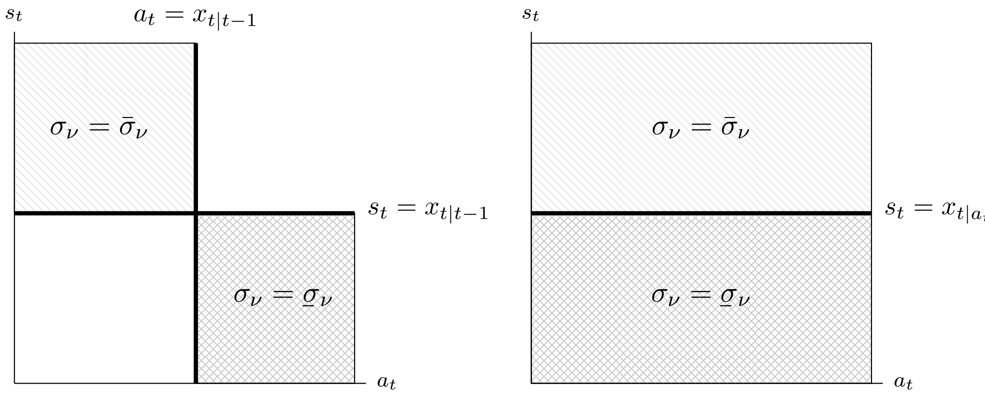

Figure 2: The Cutoff Rules: from (3) (left) and (4) (right)

Notes: Left panel represents the decision rule of simultaneous belief updating, whereas right panel depicts the decision rule of sequential belief updating. In this particular case, it is assumed that $a_t = x_{t|t-1}$.

beliefs can be given by

$$\begin{aligned} \mathbb{E}[x_t|\mathcal{I}_{t-1}, a_t] &= x_{t|a_t} = \left(\frac{\sigma_\eta^2}{\sigma_\eta^2 + \sigma_x^2}\right) x_{t|t-1} + \left(\frac{\sigma_x^2}{\sigma_\eta^2 + \sigma_x^2}\right) a_t \\ \mathbb{E}[x_t|\mathcal{I}_{t-1}, S_t] &= x_{t|t} = x_{t|a_t} + \left(\frac{\sigma_{xa}^2}{\sigma_{xa}^2 + \sigma_\nu^2}\right)\left(s - x_{t|a_t}\right) \end{aligned} \tag{4}$$

where $\sigma_{xa}^2 = \sigma_x^2 \sigma_\eta^2 / \left(\sigma_x^2 + \sigma_\eta^2\right)$ and ambiguity averse agents update beliefs according to the following decision rule:

$$\sigma_\nu^2 = \begin{cases} \bar{\sigma}_\nu^2, & \text{if } s_t > x_{t|a_t} \\ \underline{\sigma}_\nu^2, & \text{if } s_t < x_{t|a_t} \end{cases}$$

From the second term in (4), whenever $s_t > x_{t|a_t}$, $1/\bar{\sigma}_\nu^2$ (low precision) is chosen to update beliefs since the attached weight to the revision based on the surprise associated with the noise shock s_t is decreasing in the signal precision. Similarly, whenever $s_t < x_{t|a_t}$, $1/\underline{\sigma}_\nu$ (high precision) is chosen to update beliefs. Right panel in Figure 2 depicts this cutoff rule for the ambiguity averse agents. This is consistent with the way agents update beliefs such that they would use all information including any unambiguous information contemporaneously available in order to make decisions under ambiguity.

While two panels in Figure 2 do not look the same, two belief updating schemes are in fact identical: We can easily obtain (4) from (3), and vice versa. In other words, updating beliefs sequentially as in (4) is just a different way to describe updating beliefs simultaneously as in (3).

2 Model

2.1 Information Structure

Consider a "news and noise" information structure where productivity (in logs) is composed of two components - a permanent component x_t and a transitory component z_t:

$$a_t = x_t + z_t \tag{5}$$

where agents do not observe the two components separately. Instead, they observe productivity as a whole. The permanent component follows a trend that changes randomly due to permanent productivity shocks and it follows the stochastic process:

$$\Delta x_t = \rho_x \Delta x_{t-1} + \epsilon_t \tag{6}$$

whereas the transitory component follows the stationary stochastic process where it dies out after transitory productivity shocks:

$$z_t = \rho_z z_{t-1} + \eta_t \tag{7}$$

The persistent coefficient ρ_x and ρ_z are assumed to be in $[0, 1)$ and $\epsilon_t \sim N(0, \sigma_\epsilon^2)$ and $\eta_t \sim N(0, \sigma_\eta^2)$. Agents are assumed to know the precisions of the productivity shocks and both technologies have an identical persistence such that $\rho_x = \rho_z \equiv \rho$. As in Blanchard et al. (2013), I assume that the following condition holds:

$$\rho\sigma_\epsilon^2 = (1-\rho)^2\sigma_\eta^2$$

which implies that the univariate process for a_t is a random walk:

$$\mathbb{E}[a_{t+1}|a_t, a_{t-1}, \ldots] = a_t$$

In addition to productivity, agents observe a noisy signal concerning the permanent component of productivity:

$$s_t = x_t + \nu_t \tag{8}$$

where ν_t is an i.i.d. normal shock with mean zero and variance σ_ν^2. The processes $\{\epsilon_t\}_{t=0}^{\infty}$, $\{\eta_t\}_{t=0}^{\infty}$, and $\{\nu_t\}_{t=0}^{\infty}$ are assumed to be independent of the process $\{x_t\}_{t=0}^{\infty}$ and of each other. Due to the presence of the noise shock in the signal, agents are unable to perfectly identify the permanent shock to productivity. Following Epstein and Schneider (2008) there is incomplete knowledge about signal quality and the agents treat signals as ambiguous by updating beliefs as if they have multiple likelihoods. Specifically, the noisy signal s_t is related to the process x_t by a family of likelihoods through the signal precision:

$$1/\sigma_\nu^2 \in [1/\underline{\sigma}_\nu^2, 1/\bar{\sigma}_\nu^2]$$

Therefore, agents depart from Bayesian updating[17] and do not know the exact signal quality. Instead, the quality of information is captured by the range of precisions $[1/\bar{\sigma}_\nu^2, 1/\underline{\sigma}_\nu^2]$. In addition, agents are assumed not to be able to attach subjective probabilities to the priors; if they are allowed to do so, agents can simply form a subjective expectation to update beliefs.[18]

A signal is said to be more ambiguous if, given the lower $(1/\underline{\sigma}_\nu^2)$ or the upper bound $(1/\bar{\sigma}_\nu^2)$ of the signal precisions, the difference between the two is greater.[19] At the limit $(1/\underline{\sigma}_\nu = 1/\underline{\sigma}_\nu = 1/\sigma_\nu)$ agents update beliefs by a Bayesian process in which they use the standard Kalman filter with the signal precision given by $1/\sigma_\nu^2$ to estimate the fundamental. The range of precisions is assumed to remain constant over time and does not depend on other parameters. This is a strong assumption but it makes the analysis tractable.

Table 2: Left-skewed Consumption Growth Conditional on Productivity

Sample	Skewness	Number of observations
High productivity	-0.8470	128
Low productivity	-0.9572	141

Notes: The first sample (high productivity) contains observation for those with productivity growth higher than the average of the whole sample where the average productivity growth in the whole sample is 0.0043. Similarly, the second sample (low productivity) includes those observations with productivity growth lower than the average.

One of the reason to focus on the noisy signal being ambiguous rather than productivity comes from the observation that conditional on productivity process in the economy, we still observe negative-skewed consumption growth in the data. Table 2 documents skewness of consumption expenditure growth per capita of the U.S. from 1948:I to 2015:II conditional on productivity process. I split the sample into two sub-samples - the one with higher than the average productivity growth and the one with lower than the average productivity growth. Computing the skewness of consumption expenditure growth of the two samples, I find that skewness of consumption expenditure growth in both samples are still left-skewed

[17]If $1/\underline{\sigma}_\nu^2 = 1/\bar{\sigma}_\nu^2$, we are back to Bayesian updating.

[18]For example, if agents have a subjective belief such that $p(1/\bar{\sigma}_\nu^2) = 1/3$ and $p(1/\underline{\sigma}_\nu^2) = 2/3$, then they can construct a subjective expectation on $1/\sigma_\nu^2$: $\mathbb{E}[1/\sigma_\nu^2] = 1/3\left(1/\bar{\sigma}_\nu^2\right) + 2/3\left(1/\underline{\sigma}_\nu^2\right)$.

[19]Similarly, conditional on given $1/\sigma_\nu^2$ and ambiguity given as $\left[-\kappa\left(1/\sigma_\nu^2\right), \kappa\left(1/\sigma_\nu^2\right)\right]$, higher κ corresponds to the signal being more ambiguous.

even after conditioning the sample for productivity growth, which suggest that agents' responses to the noisy information be the likely source of the asymmetry.[20]

2.2 Consumption

Incorporating ambiguous information to belief updating requires an additional assumption on how consumers resolve ambiguity in decision-making. Specifically, assume that consumers maximize the multiple prior utility:

$$\max_{C_t} \min_{\Omega} \mathbb{E}\left[U(C_t)|\mathcal{I}_t\right]$$

where the prior is on information quality of the signal s_t: $\Omega = [1/\bar{\sigma}_\nu^2, 1/\underline{\sigma}_\nu^2]$ and $\mathcal{I}_t$ is the consumers' information available at time t. $\mathcal{I}_t$ includes observations up to time t such that $\mathcal{I}_t = \{s_j, a_j\}_{j=0}^t$ and the consumers' utility maximization adheres to the *maxmin* criterion. More precisely, consumers maximize expected utility under a worst-case evaluation of information quality chosen from Ω. With the *min* operator consumers evaluate different scenarios according to their priors and choose the worst case scenario available conditional on their decisions on the choice variables. With the *max* operator consumers maximize the worst case expected utility by choosing over the choice variables. For the rest of the paper, with an abuse of notation, I use $\widehat{\mathbb{E}}$ to denote the consumers' expectation based on a worst-case belief and reformulate the consumers' problem as follows:

$$\max_{C_t} \widehat{\mathbb{E}}\left[U(C_t)|\mathcal{I}_t\right]$$

subject to

$$C_t + B_{t-1} = Y_t + Q_t B_t$$

where R^* denotes the world interest rate, B_t is the external debt and Q_t is the price of this debt. Output Y_t is assumed to be produced using only labor through the linear production function:

$$Y_t = A_t N$$

where A_t is the productivity level and $A_t = e^{a_t}$. Abstracting from fluctuations on employment, I assume that consumers supply labor inelastically. The resource constraint is given by

$$C_t + NX_t = Y_t$$

[20] This is a primitive reasoning, I must admit, since I could as well divide the sample differently. But given that the noisy signal is unobservable for the econometrician, this is a decent alternative to examine the data.

where NX_t is the net exports and following Aguiar and Gopinath (2007) among others the price of debt is assumed to be sensitive to the level of outstanding debt:

$$\frac{1}{Q_t} = R_t = R^* + \psi \left\{ e^{\frac{B_{t+1}}{X_t} - b} - 1 \right\}$$

where ψ is elasticity of the interest rate, b represents the steady state level of debt, and $X_t = e^{x_t}$.

Let consumption c_t (in logs) satisfy the following Euler equation:[21]

$$c_t = \widehat{\mathbb{E}} \left[c_{t+1} | \mathcal{I}_t \right] = \widehat{\mathbb{E}}_t \left[c_{t+1} \right]$$

Thus, consumption at time t is equal to expected consumption under a worst case assessment of information quality at time $t+1$. As $c_t = \widehat{\mathbb{E}}_t \left[c_{t+1} \right], c_{t+1} = \widehat{\mathbb{E}}_{t+1} \left[c_{t+2} \right], \ldots$, by the law of iterated expectation

$$c_t = \lim_{j \to \infty} \widehat{\mathbb{E}}_t \left[c_{t+j} \right]$$

Having a long-run restriction such that

$$\lim_{j \to \infty} \widehat{\mathbb{E}}_t \left[c_{t+j} - a_{t+j} \right] = 0$$

where the lower case denotes a log transformation of a given variable, I get

$$c_t = \lim_{j \to \infty} \widehat{\mathbb{E}}_t \left[a_{t+j} \right]$$

Thus, consumption only depends on the consumers' expectations of productivity in the long run under a worst case assessment of information quality. Solving the model, consumption becomes a function of consumers' expectations under a worst-case belief:

$$c_t = \frac{1}{1-\rho} \left(\widehat{\mathbb{E}}_t \left[x_t \right] - \rho \widehat{\mathbb{E}}_t \left[x_{t-1} \right] \right) \tag{9}$$

where $\widehat{\mathbb{E}}_t \left[x_t \right]$ and $\widehat{\mathbb{E}}_t \left[x_{t-1} \right]$ represent the consumers' expectations on the current and the lagged permanent components of productivity under a worst-case belief.

It is such that consumers repeatedly estimates the long-run productivity by observing new signals and whenever the long-run productivity is estimated to be high, consumers increase their spending by importing more from abroad and vice versa. The important ingredient in this mechanism is that the long-run productivity estimate is consistent with

[21] The result is obtained by assuming that the real interest rate is (almost) constant. Online Appendix C contains detailed discussion on deriving the model.

a worst-case expectation.

Blanchard et al. (2013) theoretically show (Online Appendix Section 6.4.2) that a baseline New Keynesian (NK) model without capital converges to a simple permanent income model with a fixed real interest rate in which consumption is equal to the expectations of the long-run level of labor productivity and Cao and L'Huillier (2015) (theoretically) and Cao et al. (2014) (numerically) similarly show that small open economy without capital and constant labor converges to a simple permanent income model. Allowing for ambiguity on information quality, in Online Appendix C, the simple forward looking model of consumption discussed in this section is derived from the Small Open Economy RBC model where output is produced using only labor and the labor input is assumed to be constant.

3 Solving the model

Solving the model requires solving for consumption as a function of *beliefs about the long-run productivity* (BLR) under a worst case belief.[22] Consumers derive the expectations on the state vector

$$\mathbf{x}_t = (x_t,\ x_{t-1},\ z_t)'$$

using the Kalman filter. Let $x_{t|t} = \widehat{\mathbb{E}}\left[x_t|\mathcal{I}_t\right]$, $x_{t-1|t} = \widehat{\mathbb{E}}\left[x_{t-1}|\mathcal{I}_t\right]$ and $z_{t|t} = \widehat{\mathbb{E}}\left[z_t|\mathcal{I}_t\right]$ be the worst case current and lagged beliefs on the permanent component of productivity and the worst case current belief on the transitory component of productivity. Given new observations, the previous estimate of the permanent component is updated by applying the steady state Kalman filter:

$$\mathbf{x}_{t|t} = \left[I - KC\right] A\mathbf{x}_{t-1|t-1} + KS_t$$

where $\mathbf{x}_{t|t} = (x_{t|t},\ x_{t-1|t},\ z_{t|t})'$ and $\mathbf{x}_{t-1|t-1} = (x_{t-1|t-1},\ x_{t-2|t-1},\ z_{t-1|t-1})'$ are the worst case beliefs on $\mathbf{x}_t$ at time t and on $\mathbf{x}_{t-1}$ at time $t-1$ and $S_t = (a_t,\ s_t)'$ is a vector of observables. A and C are functions of underlying parameters of the model, K is the state

[22]Beliefs about the long-run productivity under a worst-case belief refer to

$$\lim_{j\to\infty} \widehat{\mathbb{E}}_t\left[a_{t+j}\right]$$

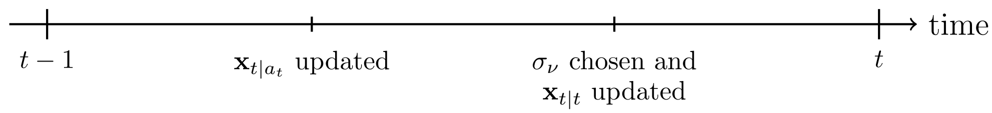

Figure 3: The Timing of Belief Updating

dependent gains of the observables, and I is the 3×3 identity matrix:

$$A = \begin{bmatrix} 1+\rho & -\rho & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

While K does not depend on time, as the noisy signal s_t is of uncertain quality, it depends on the signal precision.

Assumption 1 *Consumers sequentially update beliefs by first updating with productivity and then with a noisy signal.*

Under Assumption 1, the solution to the model can be tracked down by a simple cutoff rule on the ambiguity parameter σ_ν^2. As discussed in Online Appendix F, updated beliefs are the same whether consumers update beliefs *sequentially* as in Assumption 1 or consumers update beliefs *simultaneously*. In the simultaneous belief updating, agents would use all available information including current productivity to make decisions under ambiguity, which by definition is exactly the same as updating beliefs sequentially.[23] Figure 3 describes the timing of belief updating.

Proposition 1 (The sequential updating of beliefs) *The sequential updating of consumers' beliefs can be given by*

$$\mathbf{x}_{t|t} = \mathbf{A}\mathbf{x}_{t-1|t-1} + \mathbf{B}a_t + Gs_t \tag{10}$$

where $\mathbf{A} = [I - GC_2][I - HC_1]A$, $\mathbf{B} = [I - GC_2]H$, H *is the Kalman gain of observing productivity* a_t, G *is the Kalman gain of observing a noisy signal* s_t, *and* A, C_1, *and* C_2 *are the matrices of underlying parameters:*

$$A = \begin{bmatrix} 1+\rho & -\rho & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix}, \ C_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \ C_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

[23]This does not imply that there do not exist order effects. See Online Appendix F for details.

Proof. See Appendix A.1. ■

The types of news that the noisy signal delivers play a crucial role in updating beliefs in terms of choosing the appropriate signal precision. Formalizing the notion of the types of news, let good and bad news be defined as follows:

Definition 1 (Types of news) *A noisy signal delivers* ***good news*** *when it is greater than an ex-ante belief,* $x_{t|a_t}$*, where the ex-ante belief is an expectation about the permanent component of productivity updated with all "unambiguous information" available contemporaneously, which is* $\widehat{\mathbb{E}}[x_t|a_t, \mathcal{I}_{t-1}]$*. Similarly, it delivers* ***bad news*** *when it is smaller than* $x_{t|a_t}$*.*

A large noisy signal does not necessarily mean that it delivers good news. Instead, the types of news are related to the surprises carried by the signal relative to an ex-ante belief. The three shocks in the model are not identical in terms of the types of news that they deliver. Specifically, while a positive permanent productivity shock and a positive noise shock deliver good news to consumers, a positive transitory shock generates bad news. The intuition for these results is straightforward. A positive permanent shock to productivity increases a noisy signal one-to-one but due to the presence of the transitory component, agents underestimate the productivity increase such that $s_t > x_{t|a_t}$. Consequently, it delivers good news to consumers. Also, as a (positive) noise shock does not affect agents' beliefs, it also delivers good news. Finally, for a positive transitory shock, an increase in η_t positively affects productivity but it has no effect on a noisy signal. As $s_t < x_{t|a_t}$, it delivers bad news to consumers. Table 3 summarizes the relationship between the shocks and the types of news delivered. In addition, as discussed in Section 1, the types of news that signals deliver are directly associated with perceived information quality. For example, signals delivered by permanent productivity shocks and noise shocks are perceived (by the consumers) as low quality whereas transitory shocks to productivity generate signals of high perceived quality.

Table 3: Shocks, News, and Information Quality

Shocks (+)	Types of news	Information quality	Δ c	Δ a
Permanent tech shock (ϵ)	Good	Low	+	+
Transitory tech shock (η)	Bad	High	+	+
Noise shock (ν)	Good	Low	+	no change

Notes: Assume that the shocks are positive ones.

Proposition 2 (The cutoff rule with good news) *Let $\mathbf{x}_{t|t}$ be the beliefs updated with both productivity and a noisy signal and $\mathbf{x}_{t|a_t}$ be the beliefs updated with productivity:*

$$\mathbf{x}_{t|t} = \widehat{\mathbb{E}}\left[x_t|\{a_s\}_{s=0}^{t}, \{s_s\}_{s=0}^{t}\right]$$

$$\mathbf{x}_{t|a_t} = \widehat{\mathbb{E}}\left[x_t|\{a_s\}_{s=0}^{t}, \{s_s\}_{s=0}^{t-1}\right]$$

Then, if $s_t > s_{t|a_t}$ (delivering good news), the following conditions are satisfied:

(i) $x_{t|t} - x_{t|a_t} > 0$, $x_{t-1|t} - x_{t-1|a_t} > 0$, $z_{t|t} - z_{t|a_t} = 0$

(ii) $c_{t|t} - c_{t|a_t} > 0$

(iii) for ambiguity averse consumers, $\sigma_\nu^2 = \bar{\sigma}_\nu^2$

Proof. See Appendix A.2. ■

Proposition 3 (The cutoff rule with bad news) *Similarly, if $s_t < s_{t|a_t}$ (delivering bad news), the following conditions are satisfied:*

(i) $x_{t|t} - x_{t|a_t} < 0$, $x_{t-1|t} - x_{t-1|a_t} < 0$, $z_{t|t} - z_{t|a_t} = 0$

(ii) $c_{t|t} - c_{t|a_t} < 0$

(iii) for ambiguity averse consumers, $\sigma_\nu^2 = \underline{\sigma}_\nu^2$

Proof. See Appendix A.2. ■

Proposition 2 and 3 suggest that for ambiguity averse agents updating beliefs is consistent with choosing an extremum of the range of precisions. Only the boundaries of the range of precisions ($1/\bar{\sigma}_\nu^2$ and $1/\underline{\sigma}_\nu^2$) need to be evaluated to solve the model and the relevant gains of observing the noisy signal can be either $G(1/\underline{\sigma}_\nu^2)$ or $G(1/\bar{\sigma}_\nu^2)$, where $G(\cdot)$ represents the Kalman gain of observing the noise signal with the given precision of noise. Specifically, noisy signals delivering good news are treated as it they are uninformative and the ones delivering bad news are assumed to be very precise.

Definition 2 (The limit case) *A limit case refers to the specification in which the range of precisions degenerates to $1/\sigma_\nu^2$ such that $1/\sigma_\nu^2 = 1/\bar{\sigma}_\nu^2 = 1/\underline{\sigma}_\nu^2$.*

Solving the model, then, requires consumers to determine *beliefs about the long-run productivity* under a worst case belief, which is the right-hand side of (9), to satisfy consumers' aversion to ambiguity.[24] Figure 4 reports the responses of consumption to the shocks in

[24]Specifically, consumers' filtering in (10) is combined with (9) to determine consumption.

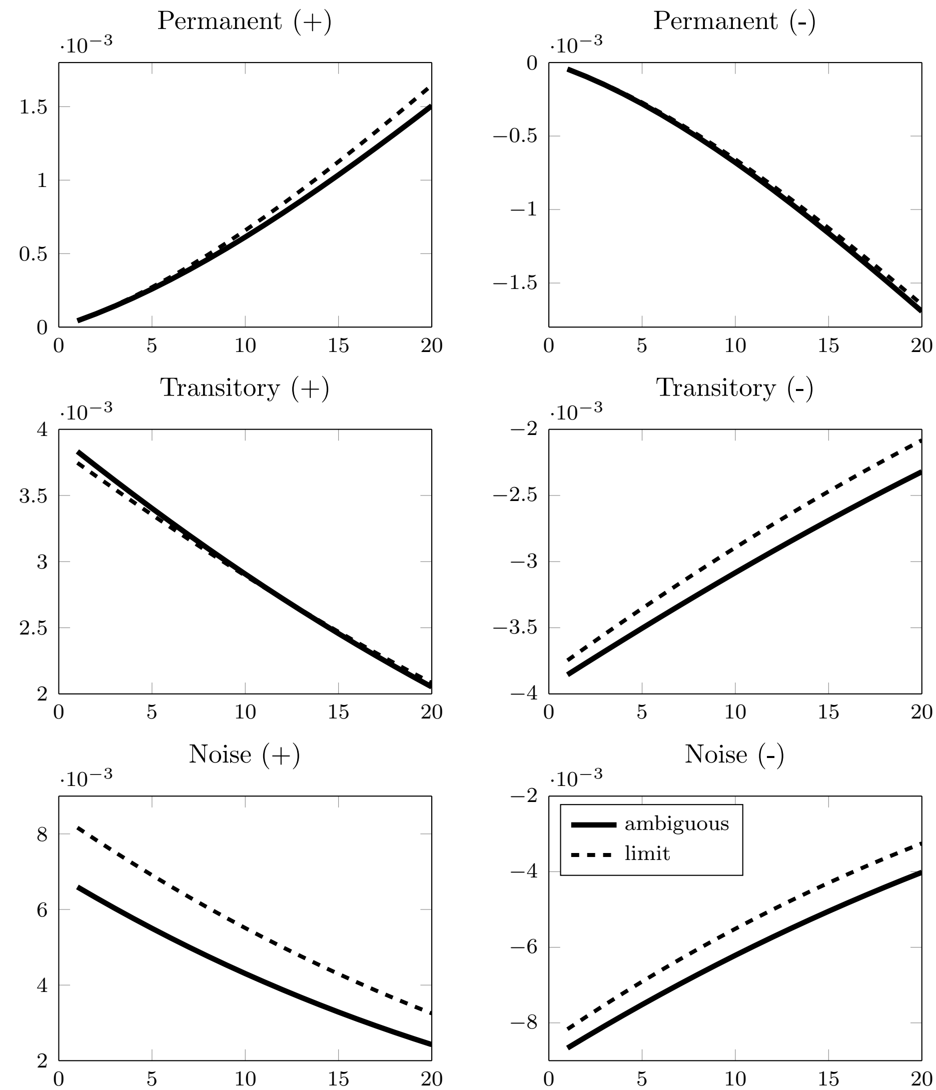

Figure 4: Impulse Responses: Consumption

Notes: Plots in the left column correspond to the IRFs of positive shock of one standard deviation and those in the right column correspond to the IRFs of negative shock of one standard deviation. The solid line corresponds to the case in which the noisy signal is ambiguous whereas the dotted line corresponds to the limit case in which $\underline{\sigma}_\nu^2 = \bar{\sigma}_\nu^2$.

two different set-ups; when the noisy signal is unambiguous (the limit case) and ambiguous (the benchmark). The time unit is one quarter and the impulse responses are one standard deviation positive and negative shocks. Responses of the positive shocks are depicted in the first column and those of the negative shocks are depicted in the second column. I use the estimated parameters in Section 4.2 as parameters. More precisely, the persistence parameter for productivity ρ is set to 0.99 and σ_u is set to 0.80% which implies that the standard deviations of the technology shocks are given by $\sigma_\epsilon = 0.008\%$ and $\sigma_\eta = 0.796\%$ and that of the noise shock, σ_ν, is set to 8.75% and *the range of precisions* is given by

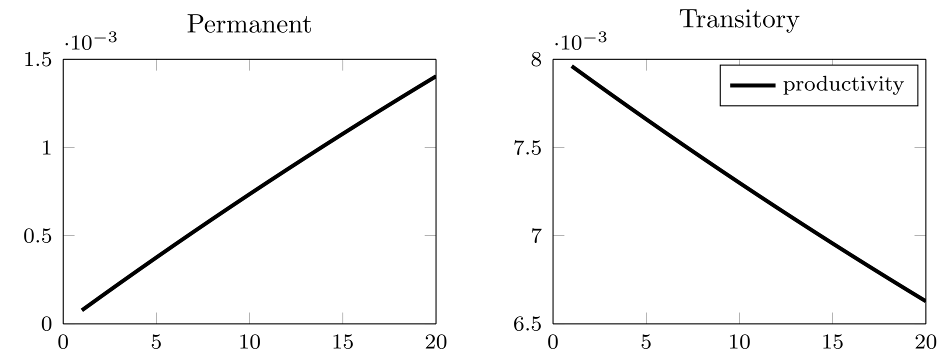

Figure 5: Impulse Responses: Productivity

Notes: The IRF of productivity is identical in both the limit case and ambiguous specification. Productivity does not respond to a noise shock.

$[8.71\%, 10.03\%]$. Since ambiguity has no effect on the dynamics of productivity, the responses are completely symmetric as depicted in Figure 5. Obviously, as productivity does not depend on the consumers' expectations, a noise shock does not move productivity at all.

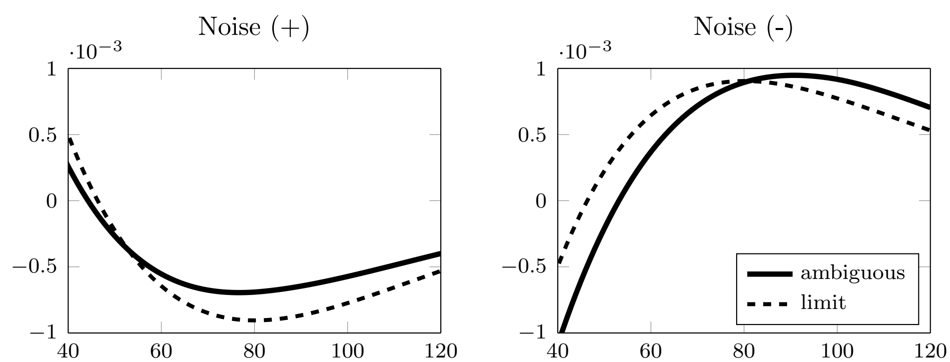

Figure 6: Consumption Responses to Noise Shocks (long horizons)

Notes: The figure displays the effects of a noise shock on consumption from 40 to 120 quarters. The solid line corresponds to the case in which a noisy signal is ambiguous whereas the dotted line corresponds to the limit case.

In response to a permanent technology shock ϵ_t, consumption increases slowly, which implies that the volatilities of other shocks, which cloud consumers' ability to recognize and adjust consumption, are large. In response to a transitory technology shock η_t, consumption initially increases but then declines. As productivity initially increases and then slowly declines, consumers partly believe that this increase in productivity is due to a permanent increase in productivity. However, consumers do learn over time that the increase in productivity is due to the transitory shock and consumption returns to the original level. For a noise shock ν_t, consumption increases and then returns to normal over time. It is such that the consumption responses are symmetric in the limit case. However, with ambiguity, consumption, in most cases, tends not to move as much as in the limit case following *positive shocks* but respond more with *negative shocks*, most notably with noise shocks.

For the long-run impacts, Figure 6 shows that consumption responses in the model with ambiguity are *not* always smaller than the limit counterpart. While it seems puzzling at first glance, intuition can be explained by that consumers project future growth, and subsequently choose their consumption, based on their expectations of both x_t and x_{t-1}. When the initial impulse hits, pessimistic beliefs are formed and such beliefs have prolonged effects since consumption depends not only on the expectation of the current permanent productivity but also on the expectation of the lagged permanent productivity as shown in (9). Specifically, beliefs on current permanent productivity ($x_{t|t}$) and lagged permanent productivity ($x_{t-1|t}$) have opposite effects on consumption decisions. Thus, conditional on pessimistic past beliefs, it might be that present consumption does not drop as much as consumers' current beliefs would indicate, which is illustrated in Figure 6.[25]

The effects of ambiguity can be observed from the asymmetric responses of consumption to the sign of the shocks. Under ambiguity, consumers are hesitant to respond to good news but are more willing to react to bad news. The intuition is that, with ambiguity, consumers become pessimistic about the future (which is not only uncertain but also ambiguous) and that such pessimism directly translates into consumption responses in this setup.

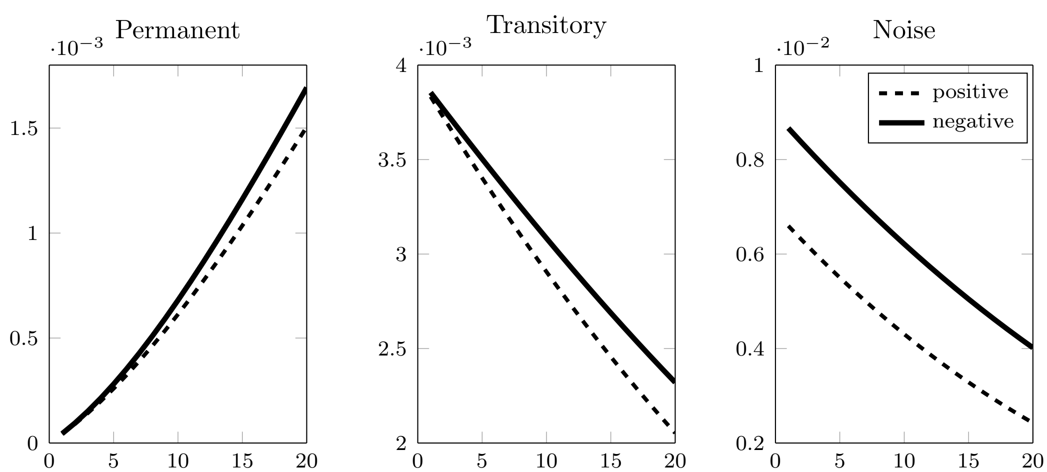

Figure 7: Size of Impulse Responses: Consumption

Notes: In order to compare the magnitude of the responses, I multiply -1 to the responses of consumption with negative shock and plot it with the responses to the positive shock. The solid line corresponds to the case in which the shocks are positive and the dotted line corresponds to consumption responses the following negative shocks.

[25]The figure only displays the impulse response to a noise shock, but the same logic applies to other two productivity shocks.

Figure 7 depicts the asymmetric responses of consumption more closely by comparing the size of consumption responses to the shocks with different signs. In the limit case, the magnitudes of effects are completely symmetric. However, when the noisy signal is ambiguous, the magnitudes of responses are greater with the negative shocks (solid line) than with the positive shocks (dotted line). To examine how the downward bias of ambiguity averse consumers translate into consumption dynamics, I conduct the following simulation exercise: I keep the underlying (unambiguous) parameters the same as in the previous exercise and use different values for the range of precisions of the ambiguity parameter σ_ν to evaluate the effects of ambiguity on consumption dynamics. Specifically, six different cases are considered where (1) $\sigma_\nu = 8.75\%$, (2) $\sigma_\nu = [7.87\%, 9.63\%]$, (3) $\sigma_\nu = [6.56\%, 10.94\%]$, (4) $\sigma_\nu = [4.37\%, 13.12\%]$, (5) $\sigma_\nu = [2.19\%, 15.31\%]$, and (6) $\sigma_\nu = [0.87\%, 16.62\%]$.[26]

Table 4: Consumption Moments Simulation

	$\sigma_\nu \in$	skewness	variance	mean
1.	$[0.0875, 0.0875]$	0.0000	0.0086	0.0000
2.	$[0.0787, 0.0963]$	-0.3465	0.0087	0.0000
3.	$[0.0656, 0.1094]$	-0.9196	0.0094	0.0000
4.	$[0.0437, 0.1312]$	-2.2541	0.0126	0.0000
5.	$[0.0219, 0.1531]$	-4.4812	0.0212	0.0000
6.	$[0.0087, 0.1662]$	-6.1124	0.0270	0.0000

Notes: The true signal precision is given by $1/0.0876^2$. For this exercise I assume that degrees of ambiguity is symmetric such that it is captured by the parameter α where $\sigma_\nu \in [(1-\alpha)\sigma, (1+\alpha)\sigma]$ and $\sigma = 0.0875$. More precisely, the six cases depicted here correspond to α being 0, 0.1, 0.25, 0.5, 0.75, and 0.9.

Table 4 reports the simulated consumption moments. The first row represents the limit case and the rest considers the case in which the signal is ambiguous. The degree of ambiguity is in increasing order from the second to the last row and the higher degree of ambiguity is associated with the more negatively skewed consumption growth and higher volatility.

Finally, Figure 8 depicts simulated consumption growth with varying degrees of ambiguity. First, by assuming that noisy signals are unambiguous, I run the Kalman smoother on U.S. data to extract the sequence of structural shocks and construct productivity series (a_t) and noisy signals (s_t) - the two signals that consumers are assumed to observe. Then, I feed these signals into my benchmark model with ambiguity and reconstruct consumption series. I use four different sets of values on the range of precisions: (1) $\sigma_\nu \in [9.80\%, 9.80\%]$, (2) $[8.82\%, 10.78\%]$, (3) $[7.35\%, 12.25\%]$, and (4) $[4.90\%, 14.70\%]$. It shows that agents consume less with more ambiguous the signals (that they receive) are. This may provide

[26] I fix the length of series to 1000 periods and the number of replication is set to 10000.

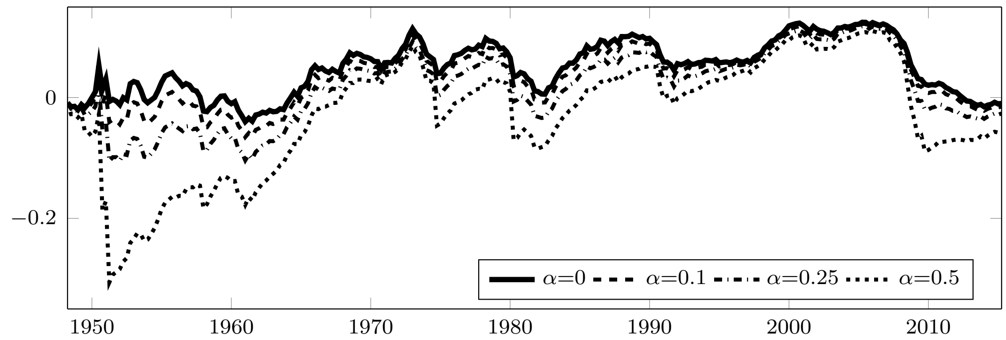

Figure 8: Reconstructed Consumption with Ambiguity

Notes: The figure shows the detrended consumption. The smoothed estimates of productivity and noisy signals are obtained from the U.S data with the limiting assumption ($\underline{\sigma}_\nu = \bar{\sigma}_\nu$). The solid line corresponds to the path of consumption without ambiguity. The dashed lines correspond to the counterfactual sample paths obtained with different degrees of ambiguity. The size of ambiguity is captured by the parameter α such that $\sigma_\nu \in [(1-\alpha)\,\sigma, (1+\alpha)\,\sigma]$ where $\sigma = 0.098$.

some interesting welfare consequences. Less (more) willingness to react to good (bad) news translates into underestimating the economy's long run potential, which in turn decreases contemporaneous consumption and is welfare worsening.

4 Estimation

4.1 Econometrician's Filtering

While the econometrician does not observe noisy signals, she observes consumption and the econometrician's set of observables include productivity and consumption series. The consumers' filtering suggests that only a maximum and minimum of the range of precisions needs to be evaluated for the ambiguity averse consumers' belief updating. The econometrician use this decision rule of consumers for his own filtering and the econometrician's filter becomes state dependent: the one with *the low precision* $(1/\bar{\sigma}_\nu^2)$ and the one with *the high precision* $(1/\underline{\sigma}_\nu^2)$. The underlying mechanism for constructing the econometrician's filter is based on the fact that even though the econometrician does not observe a noisy signal, she can fully recover the state (or the types of news that consumers received) each period. The econometrician is able to determine whether beliefs have been updated with the high or low precision using information available contemporaneously. In fact, similar to the consumers' cutoff rule to update beliefs, a cutoff rule to determine how consumers have updated beliefs at each period can be applied to the econometrician's filter. A likelihood function then can be constructed accordingly and the model is estimated through the maximum likelihood estimation.

Proposition 4 (The econometrician's cutoff rule) *Let* $\mathbf{x}_{t|t}$ *be the beliefs updated with both productivity and a noisy signal and* $\mathbf{x}_{t|a_t}$ *be the beliefs updated with productivity:*

$$\mathbf{x}_{t|t} = \widehat{\mathbb{E}}\left[x_t|\{a_s\}_{s=0}^{t}, \{s_s\}_{s=0}^{t}\right]$$

$$\mathbf{x}_{t|a_t} = \widehat{\mathbb{E}}\left[x_t|\{a_s\}_{s=0}^{t}, \{s_s\}_{s=0}^{t-1}\right]$$

Then,

1. $s_t > x_{t|a_t} \Longleftrightarrow c_t > c_{t|a_t}$

2. $s_t < x_{t|a_t} \Longleftrightarrow c_t > c_{t|a_t}$

where $c_{t|a_t}$ *is the consumption that consumers would have consumed had not observed the noisy signal* s_t*.*

Proof. See Appendix A.3. ■

By observing productivity, the econometrician is able to determine consumption that the consumers would have chosen without observing a noisy signal, which is denoted by $c_{t|a_t}$. Comparing this with the observed consumption in the data and applying the cutoff rule in Proposition 4, the econometrician can recover the types of news that consumers received. Essentially, much like the belief updated with productivity $x_{t|a_t}$ is used for the cutoff rule of the consumers, $c_{t|a_t}$ is similarly used for the cutoff rule of the econometrician.

For instance, consider the case in which the observed consumption c_t is greater than $c_{t|a_t}$. From Proposition 4 this implies that consumers have updated beliefs with the signal precision $1/\bar{\sigma}_\nu^2$. Intuitively, as the observed consumption is greater than the consumption consumers would have consumed without having observed a (contemporaneous) noisy signal, consumers must have received good news from the noisy signal and decided to consume more. This corresponds to believing that the signal is not so reliable such that the gain of observing the signal is small ($1/\bar{\sigma}_\nu^2$).

The econometrician's filtering can be obtained with the consumer's filter described in the previous section and the econometrician's cutoff rule in Proposition 4. Let the consumers' belief updating be given by equation (10) and the econometrician's state vector be $\mathbf{x}_t^E = (x_t,\ x_{t-1},\ z_t,\ x_{t|t},\ x_{t-1|t},\ z_{t|t})'$. Furthermore, let $c_{t|a_t}$ be the consumption after observing productivity a_t:

$$c_{t|a_t} = \frac{1}{1-\rho}\left(\widehat{\mathbb{E}}\left[x_t|a_t, \mathcal{I}_{t-1}\right] - \rho\widehat{\mathbb{E}}\left[x_{t-1}|a_t, \mathcal{I}_{t-1}\right]\right)$$

where $\mathcal{I}_{t-1} = \{a_j, s_j\}_{j=0}^{t-1}$. Then, the measurement equation for the econometrician's state vector $\mathbf{x}_t^E$ is

$$\mathbf{x}_t^E = Q\mathbf{x}_{t-1}^E + RV_t^{'} \tag{11}$$

where $V_t^{'} = (\epsilon_t,\ \eta_t,\ \nu_t)$ and R and Q are defined by

$$R = (1-j)R^0 + jR^1$$
$$Q = (1-j)Q^0 + jQ^1$$

The matrices Q^j and R^j depend on the realized news such that $j = 0$ corresponds to the realization of good news and $j = 1$ indicates the realization of bad news. According to Proposition 4, $j = 0$ if $c_t > c_{t|a_t}$ and $j = 1$ if $c_t < c_{t|a_t}$. Since the econometrician observes productivity and consumption, the observation equation is

$$(a_t,\ c_t)^{'} = T\mathbf{x}_t^E \tag{12}$$

where

$$T = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{1-\rho} & \frac{-\rho}{1-\rho} & 0 \end{bmatrix}$$

The econometricians' filtering problem can, then, be solved with (11), (12), and the cutoff rule in Proposition 4.[27]

4.2 Structural Estimation

The model is estimated through maximum likelihood where a likelihood function depends on the types of news realized. The econometrician can fully recover the regime, i.e., *whether*

[27]For identification Blanchard, L'Huillier, and Lorenzoni (2013) illustrate two special cases, when the signal is perfectly informative or when it is completely uninformative, in which a structural VAR recovers ϵ_t and η_t, and their dynamics effects. In the presence of ambiguity, however, it is not possible, even in these special cases, to simply rely on a structural VAR to recovers shocks in the model. First, consider the case of a fully uninformative signal where $1/\sigma_\nu^2 = 0$ where the range of precisions is given by

$$1/\sigma_\nu^2 = [0, 1/\underline{\sigma}_\nu^2]$$

such that agents would believe either that signal is fully uninformative or that its precision is given by $1/\underline{\sigma}_\nu^2$. In this case, even if the signal is useless, it may affect the agents' belief updating. Similarly, when the signal is fully informative such that $1/\sigma_\nu^2 = \infty$ where the range of precision is defined by

$$1/\sigma_\nu^2 = [1/\bar{\sigma}_\nu^2, \infty],$$

agents might believe that the signal is less than fully informative depending on the types of news they receive. This is an interesting departure from the limit case benchmark. In the limit case, if the signal is fully informative, consumers are able to identify the permanent shock to productivity directly. However, when information quality is given by the range of precisions, even if the signal is fully informative, consumers are not able to recover the permanent shock perfectly.

a noisy signal delivers good or bad news, with the contemporaneously available information as discussed in the previous section. Therefore, the regime can be revealed in each period and the likelihood function can be modified to incorporate that.

Table 5: Parameter Estimates, US 1948:I-2015:II

Parameter	Description	Value	s.e.
ρ	Persistence productivity	0.9904	0.0008
σ_u	Std dev. productivity	0.0080	0.0002
σ_ϵ	Std dev. permanent shock (implied)	0.0001	-
σ_η	Std dev. transitory shock (implied)	0.0080	-
$\underline{\sigma}_\nu$	Std dev. noise shock (high precision)	0.1003	0.0005
$\bar{\sigma}_\nu$	Std dev. noise shock (low precision)	0.0871	0.0005
σ_ν	Std dev. noise shock	0.0875	0.0038
	likelihood	1836.78	

Notes: σ_ϵ and σ_η are obtained with random walk assumption of (8). Hence, no standard errors are given.

Consumption is constructed by taking the first difference of the logarithm of the ratio of NIPA consumption to population whereas productivity is constructed by taking the first difference of the logarithm of the ratio of GDP to employment. Real personal consumption expenditure (PCECC96), real gross domestic product (GDPC1), population (B230RC0Q173SBEA), and employment (LNS12000000Q) series are from 1948:I to 2015:II and are available at the U.S. Bureau of Economic Analysis for the first three series and at the U.S. Bureau of Labor Statistics for the last one. Following Blanchard et al. (2013), I remove secular drift in the consumption-to-productivity ratio from the consumption series. Table 5 shows the estimation results with US consumption and productivity data.[28]

The qualitative implications of the dynamic effects of each shock are as follows. Permanent shocks on productivity slowly and steadily increase productivity and consumption. Transitory shocks on productivity have slowly decreasing effects on productivity and consumption while noisy shocks generate slowly decreasing effects on consumption only. The main takeaway is that the effects are asymmetric such that as depicted in Figure 7 the absolute size of the responses are larger for negative shocks than for positive ones.

The limit case, as defined in Definition 3, refers to the case in which a noisy signal is assumed to be unambiguous. Table 6 reports the parameters obtained when estimating the model by assuming that $\underline{\sigma}_\nu = \bar{\sigma}_\nu$. This limit case indicates that consumers are aware of the exact signal precision and the impulse responses are symmetric to the sign of the shocks. The estimated standard deviation of a noise shock, $\hat{\sigma}_\nu$, in the limit case is shown

[28]For maximum likelihood estimation I initialize the variance covariance matrix of the estimator with a diagonal of 10.

Table 6: Parameter Estimates, US 1948:I-2015:II (the limit case)

Parameter	Description	Value	s.e.
ρ	Persistence productivity	0.9898	0.0009
σ_u	Std dev. productivity	0.0079	0.0002
σ_ϵ	Std dev. permanent shock (implied)	0.0001	-
σ_η	Std dev. transitory shock (implied)	0.0078	-
σ_ν	Std dev. noise shock	0.0980	0.0133
	likelihood	1832.65	

Notes: σ_ϵ and σ_η are obtained with random walk assumption. As they are indirectly recovered, no standard errors are given.

to lie inside the estimated range of precisions in Table 5.

Since this limit model (unambiguous signal) is a special case of the benchmark model (ambiguous signal), a likelihood ratio test can be used to compare the goodness of fit of the two models. Specifically, the "null" model (unambiguous signal) has 3 parameters with a log-likelihood of 1832.65 whereas the "alternative" model (ambiguous signal) has 5 parameters with a log-likelihood of 1836.78 such that the test statistic is $2 \times (1836.78 - 1832.65) = 8.26$ with degrees of freedom equal to 2. Thus, the null model is rejected in favor of the alternative model at a significance level of 0.05.

It has been argued throughout the paper that the type of news that consumers receive play a crucial rule in terms of updating beliefs and choose consumption. Here, I recover the types of news that consumers have received. It shows that, for the whole sample, about half the times (136 out of 269) consumers have received good news. Figure 10 in Appendix B shows that the realization of types of news from 1990:I to 2015:II, as defined in Definition 1, are somewhat persistent. Especially, the mid-1990s to the early-2000s are associated with prolonged periods of consumers receiving good news about the state of the economy whereas the mid to the late-2000s are associated with the bad news regime.[29]

4.3 Recovering States and Shocks via the Kalman Smoother

A useful exercise is to exploit the fact that the econometrician has access to the whole sample (t=1:T) and she is able to estimate states and shocks using the Kalman Smoother. Having more information available, we are able to have better estimates what the states and shocks were. (Figure 11 to Figure 12 in Appendix B report the smoothed estimates of the (unobservable) permanent productivity and shocks in the model.)

Figure 9 plots the smoothed estimate of the permanent component of productivity

[29] L'Huillier and Yoo (2015) study the types of news that consumers received during the U.S. recessions in detail.

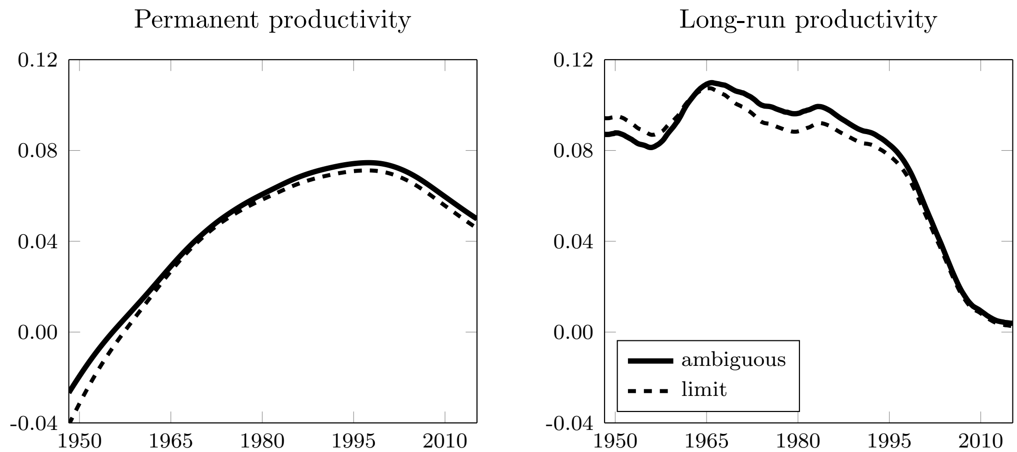

Figure 9: Smoothed Estimates of the Permanent Component of Productivity and of Long-run Productivity for the the Two Specifications

Notes: Left panel depicts the smoothed estimate of x_t and Right panel depicts the smoothed estimate of $x_{t+\infty}$ with ambiguous signals (solid line) and with unambiguous signal(dashed line). I use the parameters estimated in Section 4.2, i.e., for the benchmark ambiguous signal model $\sigma_\nu \in [0.0871, 0.1003]$ and for the limit case unambiguous signal model $\sigma_\nu = 0.980$.

(x_t) and of long-run productivity ($x_{t+\infty}$) for the two models - *the benchmark ambiguous signal model* and *the limit case unambiguous signal model*. The permanent component of productivity (Left panel) is mostly estimated to be higher in the benchmark ambiguous case than in the limit case. Neglecting the presence of ambiguity, thus, implies (slightly) underestimating the state of the economy (measured by its permanent productivity). It also shows that, in terms of the economy's long-run productivity, misspecification of the model (by not taking into account of the presence of ambiguity) results in underestimating the economy's long-run potential from the late-1960s to the late-1990s while overestimating it from the late-1940s to the early 1960s.

5 Concluding remarks

I have provided a simple theory of asymmetric consumption fluctuations due to ambiguous quality of information and agents' aversion toward ambiguity. Methodologically, I have attempted to solve a simple forward looking model of consumption with ambiguous information quality. It allows one to examine the dynamics of consumption when agents face strong uncertainty. In the stylized permanent income model, the closed form solution of consumption dynamics is obtained where consumption is driven by consumers' beliefs about the long run under a worst case belief. Since the consumers' belief updating is consistent with evaluating the boundaries of the range of information precisions, the econometrician's filtering can be expressed as a regime switching model where the regime is fully retrieved by the econometrician. The structural estimation using U.S. data suggests an asymmetric nature of consumption responses to exogenous shocks. However, an obvious caveat is that the result depends on the sample that I use for estimation exercises. For example, estimating the model using the smaller dataset from 1976 to 2015, I do not find that the ambiguity plays a statistically significant role in terms of explaining consumption dynamics.[30]

Throughout the paper, I have made a strict assumption such that there is no learning *about* ambiguity where it remains the same over time and that all agents are alike (no heterogeneity) in terms of their perception toward ambiguity that they face or in the sense that they all receive a common noisy signal. Studies incorporating the interaction between learning and heterogeneity when dealing with strong uncertainty may be the natural extension of this model to explore a future research avenue. For example, a more realistic setup on ambiguity such that agents adhere the α-*maxmin* expected utility with a time varying α may produce some interesting dynamics, e.g. a time-varying skewness of consumption growth.

[30] On the contrary, using the even smaller dataset from 1990 to 2015, I do find that ambiguity plays a significant role in terms of generating asymmetric responses.

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A Proofs

A.1 Proposition 1

Proof. Conditional on $\mathbf{x}_{t|a_t}$, consumers' filtering is given by

$$\begin{aligned}\mathbf{x}_{t|t} &= \mathbf{x}_{t|a_t} + G(s_t - s_{t|a_t}) \qquad (13)\\ &= \mathbf{x}_{t|a_t} + Gs_t - GC_1\mathbf{x}_{t|a_t}\\ &= [I - GC_1]\mathbf{x}_{t|a_t} + Gs_t \qquad (14)\end{aligned}$$

where G is the Kalman gain for the following system of equations:

$$\begin{aligned}\mathbf{x}_t &= A\mathbf{x}_{t-1} + BV_t\\ s_t &= C_1\mathbf{x}_t + D_1W_t\end{aligned}$$

and $\mathbf{x}_t = (x_t,\ x_{t-1},\ z_t)'$, $V_t = (\epsilon_t,\ 0,\ \eta_t)'$, $W_t = \nu_t$, $D_1 = 1$,

$$A = \begin{bmatrix}1+\rho & -\rho & 0\\ 1 & 0 & 0\\ 0 & 0 & \rho\end{bmatrix} \quad B = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix} \quad C_1 = \begin{bmatrix}1 & 0 & 0\end{bmatrix}$$

Similarly, conditional on $\mathbf{x}_{t|t-1}$, $\mathbf{x}_{t|a_t}$ is given by

$$\begin{aligned}\mathbf{x}_{t|a_t} &= \mathbf{x}_{t|t-1} + H(a_t - a_{t|t-1})\\ &= A\mathbf{x}_{t-1|t-1} + Ha_t - HC_2A\mathbf{x}_{t-1|t-1}\\ &= [I - HC_2]A\mathbf{x}_{t-1|t-1} + Ha_t \qquad (15)\end{aligned}$$

where H is the Kalman gain for the following system of equations

$$\begin{aligned}\mathbf{x}_t &= A\mathbf{x}_{t-1} + BV_t\\ a_t &= C_2\mathbf{x}_t + D_2W_t\end{aligned}$$

and $\mathbf{x}_t = (x_t,\ x_{t-1},\ z_t)^{'}$, $V_t = (\epsilon_t,\ 0,\ \eta_t)^{'}$, $W_t = \nu_t$, $D_2 = 0$,

$$A = \begin{bmatrix}1+\rho & -\rho & 0\\ 1 & 0 & 0\\ 0 & 0 & \rho\end{bmatrix} \quad B = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix} \quad C_2 = \begin{bmatrix}1 & 0 & 1\end{bmatrix}$$

Substituting $\mathbf{x}_{t|a,t}$ from (15) into (14)

$$\mathbf{x}_{t|t} = [I - GC_1][I - HC_2]A\mathbf{x}_{t-1|t-1} + [I - GC_1]Ha_t + Gs_t$$

■

A.2 Proposition 2 and 3

Proof. From (13), I have

$$\mathbf{x}_{t|t} = \mathbf{x}_{t|a_t} + G(s_t - s_{t|a_t})$$

where G is a 3×1 column vector such that G_i is the Kalman gain associated with the *i-th* component of $\mathbf{x}_{t|t}$. For example, G_1 is the gain of observing the noisy signal s_t on $x_{t|t}$. Furthermore, $G = \Sigma_X C_1' [C_1 \Sigma_X C_1' R]^{-1}$ where $R = Var(W_t) = \sigma_\nu^2$ and $\Sigma_X = Var_{t-1}(\mathbf{x}_t)$:[31]

$$\Sigma_X = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{21} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{bmatrix}$$

where Σ_{ii} is the $Var_{t-1}(\mathbf{x}_{i,t})$ and Σ_{ij} is the $Cov_{t-1}(\mathbf{x}_{i,t}, \mathbf{x}_{j,t})$. With a little bit of algebra, each component of G can be defined by

$$G_1 = \Sigma_{11}(\Sigma_{11} + \sigma_\nu^2)^{-1} \tag{16}$$
$$G_2 = \Sigma_{21}(\Sigma_{11} + \sigma_\nu^2)^{-1} \tag{17}$$
$$G_3 = \Sigma_{31}(\Sigma_{11} + \sigma_\nu^2)^{-1}$$

Since $\Sigma_{11} > 0$, $0 < \rho < 1$, and $(\Sigma_{11}+\sigma_\nu^2)^{-1} > 0$, $G_1 > 0$ such that if $s_t > s_{t|a_t}$, $x_{t|t} - x_{t|a_t} > 0$ and that if $s_t < s_{t|a_t}$, $x_{t|t} - x_{t|a_t} < 0$. Similarly, as $\Sigma_{21} > 0$ and $(\Sigma_{11} + \sigma_\nu^2)^{-1} > 0$, $G_2 > 0$ such that if $s_t > s_{t|a_t}$, $x_{t-1|t} - x_{t-1|a_t} > 0$ and that if $s_t < s_{t|a_t}$, $x_{t-1|t} - x_{t-1|a_t} < 0$. Finally, given that $\Sigma_{31} = 0$, $G_3 = 0$ such that $z_{t|t} - z_{t|a_t} = 0$, $\forall\ s_t$ and $s_{t|a_t}$.

For consumption, c_t and $c_{t|a_t}$ are given by

$$c_t = \frac{1}{1-\rho}\left(x_{t|t} - \rho x_{t-1|t}\right) \tag{18}$$
$$c_{t|a_t} = \frac{1}{1-\rho}\left(x_{t|a_t} - \rho x_{t-1|a_t}\right)$$

[31] Since $\epsilon_t \perp \eta_{t+j}, \forall j$, $\sigma_{13} = \sigma_{23} = \sigma_{31} = \sigma_{32} = 0$.

Substituting $x_{t|t}$ and $x_{t-1|t}$ from (13) into (18) gives

$$\begin{aligned} c_t &= \frac{1}{1-\rho}\left(x_{t|t} - \rho x_{t-1|t}\right) \\ &= \frac{1}{1-\rho}\left(x_{t|a_t} + G_1(s_t - s_{t|a_t}) - \rho x_{t-1|a_t} - \rho G_2(s_t - s_{t|a_t})\right) \\ &= c_{t|a_t} + \frac{1}{1-\rho}\left((s_t - s_{t|a_t})(G_1 - \rho G_2)\right) \end{aligned} \tag{19}$$

From (16) and (17), it gives

$$\begin{aligned} G_1 - \rho G_2 &= \Sigma_{11}(\Sigma_{11} + \sigma_\nu^2)^{-1} - \rho\Sigma_{21}(\Sigma_{11} + \sigma_\nu^2)^{-1} \\ &= (\Sigma_{11} - \rho\Sigma_{21})(\Sigma_{11} + \sigma_\nu^2)^{-1} \end{aligned}$$

As $\Sigma_{11} > \Sigma_{21}$ and $0 < \rho < 1$, $G_1 - \rho G_2 > 0$. Therefore, when $s_t > s_{t|a_t}$, the second term on the right-hand side of (19) is positive and when $s_t < s_{t|a_t}$, it is negative. Thus, if $s_t > s_{t|a_t}$, $c_t - c_{t|a_t} > 0$. Similarly, if $s_t < s_{t|a_t}$, $c_t - c_{t|a_t} < 0$.

From (19) when $s_t - s_{t|a_t} > 0$, $G_1 - \rho G_2$ should be as small as possible to minimize c_t. Since $G_1 - \rho G_2 = (\Sigma_{11} - \rho\Sigma_{21})(\Sigma_{11} + \sigma_\nu^2)^{-1}$, it is such that $\sigma_\nu = \bar{\sigma}_\nu$. Similarly, when $s_t - s_{t|a_t} < 0$, minimizing c_t requires that $G_1 - \rho G_2$ takes the largest possible values. Thus, when $s_t - s_{t|a_t} < 0$, $\sigma_\nu = \underline{\sigma}_\nu$.

This completes the proof of Proposition 2 and Proposition 3. ■

A.3 Proof to Proposition 4

Proof. Rewrite equation (19):

$$s_t - s_{t|a_t} = \frac{1-\rho}{G_1 - \rho G_2}\left(c_t - c_{t|a_t}\right)$$

Since $\frac{1-\rho}{G_1-\rho G_2} > 0$, $c_t - c_{t|a_t}$ and $s_t - s_{t|a_t}$ should have the same sign: if $c_t > c_{t|a_t}$, $s_t > s_{t|a_t}$ and if $c_t < c_{t|a_t}$, $s_t < s_{t|a_t}$. ■

B Figures

Additional figures discussed in the main body of the paper are given here.

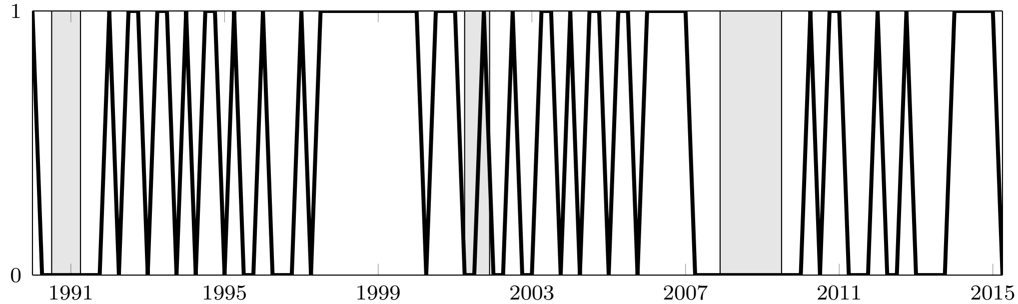

Figure 10: Realized News (1990:I-2015:II)

Notes: The figure depicts the realized news in the economy as defined in Definition 1. 1 denotes the realization of good news and 0 denotes that of bad news. Shaded areas indicate U.S. recessions.

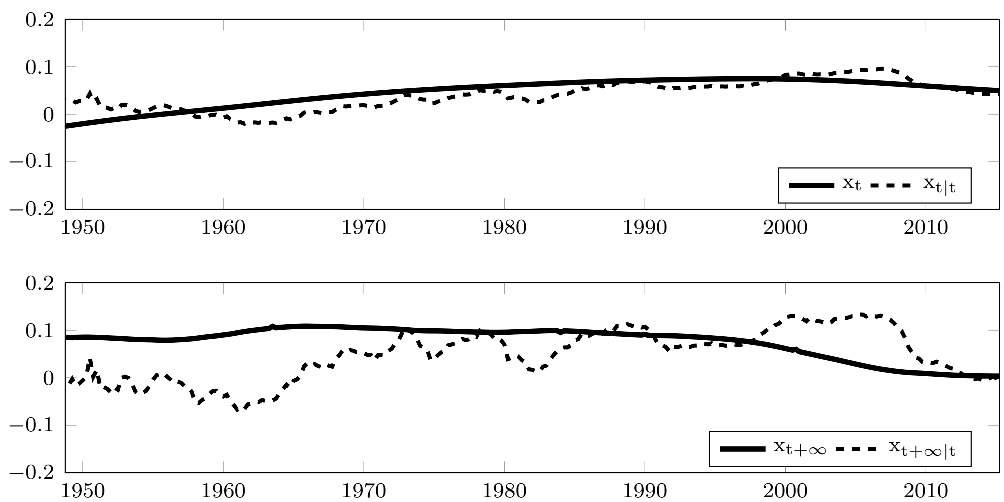

Figure 11: Smoothed Estimates of the Permanent Component of Productivity (x_t) and of Long-run Productivity ($x_{t+\infty}$)

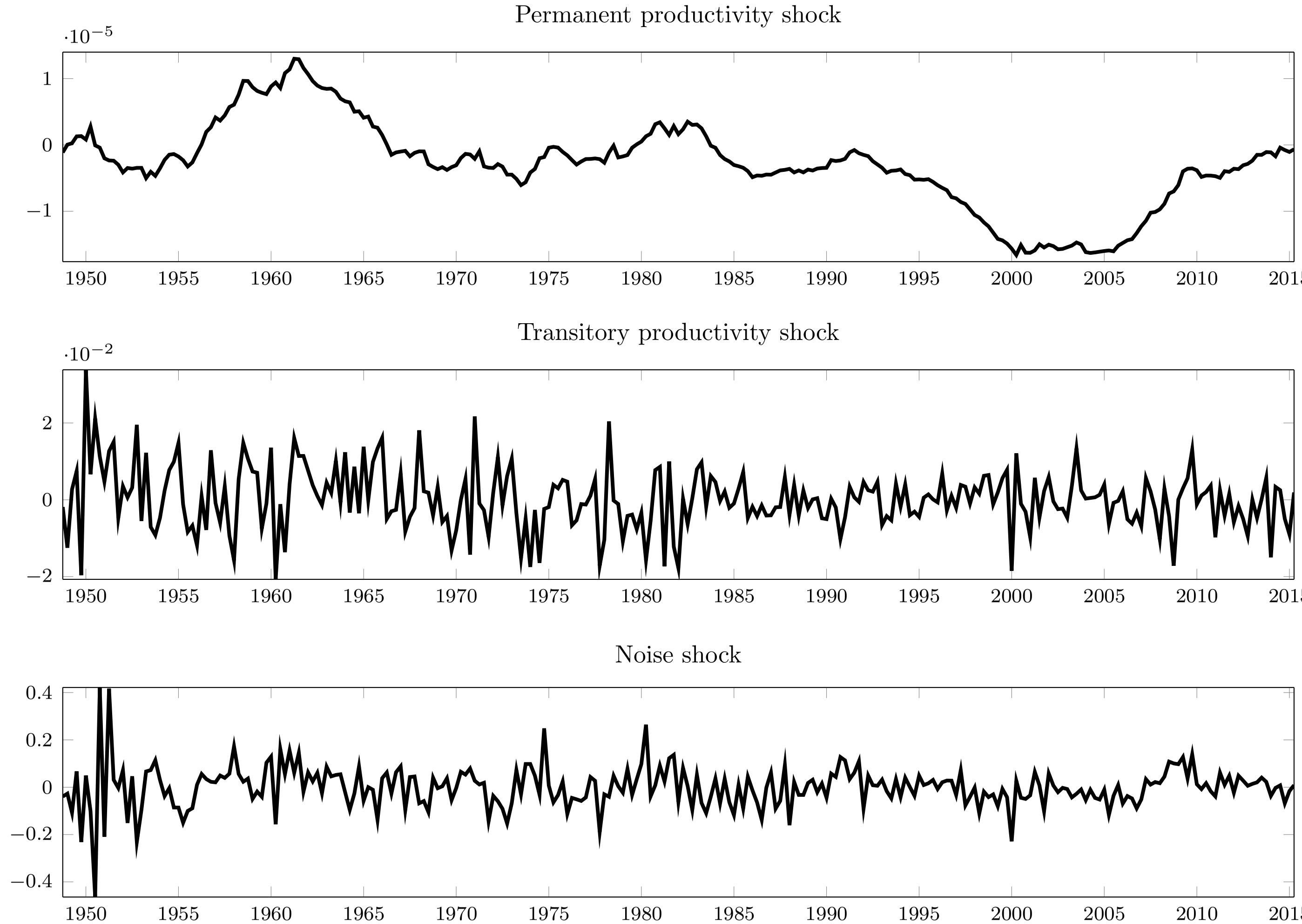

Figure 12: Smoothed Estimates of Three Shocks in the Model

Online Appendix

C Model

This section derives a permanent income consumption model under a worst-case belief from the Small Open Economy *RBC* model similar to Cao and L'Huillier (2015).

C.1 Setup

A representative consumer maximizes multiple priors utility:

$$\widehat{\mathbb{E}}_t \left[\sum_{t=0}^{\infty} \beta^t \log C_t \right]$$

where $\widehat{\mathbb{E}}_t$ is the expectation operator under a worst-case belief that is chosen from a set of conditional probabilities on information quality. The maximization is subject to

$$C_t + B_t = Y_t + Q_t B_{t+1}$$

where B_t is the external debt of the country, Q_t is the price of this debt, and Y_t is the output of the country. Output is produced using only labor

$$Y_t = A_t N$$

where the labor input is assumed constant. The resource constraint is

$$C_t + NX_t = Y_t$$

The price of debt is sensitive to the level of outstanding debt:

$$\frac{1}{Q_t} = R_t = R^* + \psi \left\{ e^{\frac{B_{t+1}}{X_t} - b} - 1 \right\}$$

where b denotes the steady state level of B_{t+1}/X_t.

C.2 Optimality condition

The first order condition from the optimization problem delivers

$$\frac{1}{C_t} = \beta R_t \widehat{\mathbb{E}} \left[\frac{1}{C_{t+1}} \right]$$

For log-linearization, we define endogenous variables c_t, y_t, q_t, b_{t+1}, nx_t as

$$\widehat{c}_t = \log(C_t/X_{t-1}) - \log(\bar{C}/\bar{X})$$

$$y_t = \log(Y_t/X_{t-1}) - \log(\bar{Y}/\bar{X})$$

$$q_t = \log Q_t - \log \bar{Q}$$

$$b_{t+1} = \frac{B_{t+1}}{X_t} - \frac{\bar{B}}{\bar{X}}$$

$$nx_t = \frac{NX_t}{Y_t} - \frac{\bar{NX}}{\bar{Y}}$$

For notational convenience, I also define:

$$C \equiv \frac{\bar{C}}{\bar{X}}, \qquad Y \equiv \frac{\bar{Y}}{\bar{X}}, \qquad B \equiv \frac{\bar{B}}{\bar{X}}, \qquad Q \equiv \bar{Q}, \qquad NX \equiv \frac{\bar{NX}}{\bar{Y}}$$

C.3 Log-linearization

Start from

$$NX_t = B_t - Q_t B_{t+1}$$

Diving both sides by Y_t:

$$NX_t/Y_t = \frac{B_t}{X_{t-1}}\frac{X_{t-1}}{Y_t} - Q_t\frac{B_{t+1}}{X_t}\frac{X_{t-1}}{Y_t}\frac{X_t}{X_{t-1}}$$

leads to

$$nx_t = \frac{1}{Y}b_t - \frac{GQ}{Y}b_{t+1} - \frac{B}{Y}GQ(q_t - y_t + \Delta x_t) - \frac{B}{Y}y_t \tag{20}$$

Dividing both sides of the resource constraints by Y_t:

$$\frac{C_t}{X_{t-1}}\frac{X_{t-1}}{Y_t} + \frac{NX_t}{Y_t} = 1$$

leads to

$$\frac{C}{Y}(\widehat{c}_t - y_t) + nx_t = 0 \tag{21}$$

Substituting nx_t from (20) into (21), I get

$$\frac{C}{Y}(\widehat{c}_t - y_t) + \frac{1}{Y}b_t - \frac{GQ}{Y}b_{t+1} - \frac{B}{Y}GQ(q_t - y_t + \Delta x_t) - \frac{B}{Y}y_t = 0 \tag{22}$$

Dividing the both sides of the production function by X_{t-1}:

$$\frac{Y_t}{X_{t-1}} = \frac{X_t}{X_{t-1}} Z_t N$$

leads to

$$y_t = z_t + \Delta x_t \tag{23}$$

Multiplying both sides of the first order condition by X_{t-1}:

$$\frac{X_{t-1}}{C_t} = \beta R_t \widehat{\mathbb{E}} \left[\frac{X_t}{C_{t+1}} \frac{X_{t-1}}{X_t} \right]$$

leads to

$$\widehat{c}_t = q_t + \widehat{c}_{t+1} + \Delta x_t \tag{24}$$

where $q_t = -r_t$.

Rest of the model is specified similar to Cao and L'Huillier (2015):

$$q_t = -\psi Q b_{t+1} \tag{25}$$

$$c_t = \widehat{c}_t + x_{t-1} \tag{26}$$

Thus, (22) to (26) along with technology processes comprise the log-linearized version of the model.

C.4 Steady states

The following steady state relations hold:

$$Q = \beta$$

$$(1-\beta) \frac{B}{Y} = 1 - C/Y$$

C.5 Closed-form Solution and Limit Result for Consumption

Define a new variable $\hat{b}_t$:

$$\hat{b}_t = b_t + B x_{t-1}$$

Using the definition of the log-deviation of consumption:

$$c_t = \widehat{c}_t + x_{t-1}$$

I make a following conjecture:

$$\begin{aligned} c_t &= D_b \hat{b}_t + D_x \mathbf{x}_t \\ &= D_b \hat{b}_t + D_{x,1} x_t + D_{x,2} x_{t-1} + D_{x,3} z_t \end{aligned} \tag{27}$$

where $\mathbf{x}_t = [x_t,\ x_{t-1},\ z_t]'$. I claim that as $\beta \to 1$ and $\frac{\psi}{(1-\beta)} \to 0$, consumption c_t is only a function of belief about the long-run (BLR) under a worst-case belief:

$$c_t = \frac{1}{1-\rho} \left(\widehat{\mathbb{E}}_t [x_t] - \rho \widehat{\mathbb{E}}_t [x_{t-1}] \right)$$

expressed in the following proposition.

Proposition 5 (Limit Consumption) *As $\beta \to 1$ and $\frac{\psi}{(1-\beta)} \to 0$,*

$$\begin{aligned} \lim_{\beta \to 1} \lim_{\psi \to 0} D_b &= 0 \\ \lim_{\beta \to 1} \lim_{\psi \to 0} D_{x,1} &= \frac{1}{C/Y} \frac{1}{1-\rho} \\ \lim_{\beta \to 1} \lim_{\psi \to 0} D_{x,2} &= \frac{1}{C/Y} \frac{-\rho}{1-\rho} \\ \lim_{\beta \to 1} \lim_{\psi \to 0} D_{x,3} &= 0 \end{aligned}$$

Proof. From (22) and (23), I have

$$0 = Y(z_t + \Delta x_t) + \beta B(\Delta x_t - \psi \beta b_{t+1}) + \beta b_{t+1} - b_t - C\widehat{c}_t$$

and with the definition of c_t and $\widehat{b}_t$, I get

$$\widehat{b}_{t+1} = \frac{1}{(1-\psi\beta B)\beta} \left[\widehat{b}_t + Cc_t - Yz_t - (Y + \beta B \psi \beta B) x_t \right] \tag{28}$$

The Euler equation (24) and the debt equation (25) imply that

$$\widehat{c}_{t+1} - \widehat{c}_t + \Delta x_t - \psi Q b_{t+1} = 0$$

which by using the definition of c_t and $\widehat{b}_t$ becomes

$$c_{t+1} - c_t - \psi\beta\widehat{b}_{t+1} + \psi\beta B x_t = 0 \tag{29}$$

Using the conjecture (27), (29) becomes

$$(D_b - \psi\beta)\widehat{b}_{t+1} + D_x A\mathbf{x}_t + \psi\beta B x_t - c_t = 0$$

and combined with (28)

$$\left[1 - \frac{(D_b - \psi\beta)C}{(1-\psi\beta B)\beta}\right] c_t = \frac{(D_b - \psi\beta)}{(1-\psi\beta B)\beta}\left(\widehat{b}_t - Y z_t\right) + D_x\mathbf{A}\mathbf{x}_t + K x_t \tag{30}$$

where

$$K = -\left[\frac{(D_b - \psi\beta)}{(1-\psi\beta B)\beta}\left(Y + \beta B\psi\beta B\right)\right] + \psi\beta B$$

and

$$\mathbf{A} = \begin{pmatrix} 1+\rho & -\rho & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \rho \end{pmatrix}$$

Rearranging (30) leads to

$$(1-\bar{x})\, c_t = \frac{\bar{x}}{C}\widehat{b}_t - \frac{\bar{x}}{C} Y z_t + D_x\mathbf{A}\mathbf{x}_t - \left[\frac{\bar{x}}{C}\left(Y + \beta B\psi\beta B\right)\right] x_t + \psi\beta B x_t \tag{31}$$

where

$$\bar{x} = \frac{(D_b - \psi Q)\, C}{(1-\psi Q B) G Q}$$

Finding D_b in the limit leads to $D_b = 0$.[32]

[32] I get the following quadratic equation in D_b:

$$D_b^2 + \left[\frac{1}{C} - (1-\psi\beta B)\frac{\beta}{C} - \psi\beta\right] D_b - \frac{\psi\beta}{C} = 0$$

From (31), collecting the terms for x_t:

$$(1-\bar{x})\,D_{x,1} = -\frac{\bar{x}}{C}\,(Y+\beta B\psi\beta B)+\psi\beta B+(1+\rho)D_{x,1}+D_{x,2} \tag{32}$$

Similarly, collecting the terms for x_{t-1}:

$$(1-\bar{x})\,D_{x,2} = -\rho D_{x,1} \tag{33}$$

Finally, for z_t:

$$(1-\bar{x})\,D_{x,3} = \rho D_{x,3}$$

Thus, $D_{x,3}=0$ and from (33)

$$D_{x,2} = \frac{-\rho}{1-\bar{x}}D_{x,1}$$

Substituting $D_{x,2}$ into (32),

$$(1-\bar{x})D_{x,1} = -\frac{\bar{x}}{C}\,(Y+\beta B\psi\beta B)+\psi\beta B+(1+\rho)D_{x,1}-\frac{\rho}{1-\bar{x}}D_{x,1}$$

Then, I can solve for $D_{x,1}$:

$$D_{x,1} = \left(\frac{1-\bar{x}}{1-\rho-\bar{x}}\right)\left(\frac{1}{\bar{x}}\right)\left[\frac{\bar{x}}{C}\,(Y+\beta B\psi\beta B)-\psi\beta B\right]$$

---

where I pick the negative root to ensure the stability of the dynamic system and in the limit $D_b \to 0$:

$$D_b = \frac{-\left(\frac{1}{C}-(1-\psi\beta B)\,\frac{\beta}{C}-\psi\beta\right)-\sqrt{\left(\frac{1}{C}-(1-\psi\beta B)\,\frac{\beta}{C}-\psi\beta\right)^2+\frac{4\psi Q}{C}}}{2}$$

For $C=1$ in the limit:

$$D_b = \frac{-(1-1-0)-\sqrt{(1-1)^2+0}}{2} = 0$$

Similarly, for $C\neq 1$ in the limit:

$$D_b = \frac{-\left(\frac{1}{C}-\frac{\beta}{C}\right)-\sqrt{\left(\frac{1}{C}-\frac{\beta}{C}\right)^2+0}}{2} = 0$$

With the limit conditions,

$$\lim_{\beta\to 1}\lim_{\psi\to 0} D_{x,1} = \left(\frac{1-\bar{x}}{1-\rho-\bar{x}}\right)\left(\frac{1}{C}\right)(Y+\beta B\psi\beta B) - \left(\frac{1-\bar{x}}{1-\rho-\bar{x}}\right)\left(\frac{1}{\bar{x}}\right)\psi\beta B$$
$$= \frac{1}{C/Y}\left(\frac{1}{1-\rho}\right) \tag{34}$$

as $\left(\frac{1-\bar{x}}{1-\rho-\bar{x}}\right)\left(\frac{1}{\bar{x}}\right)\psi\beta B$ goes to zero in the limit.[33] Given $D_{x,1}$ in (34), with the limit conditions, I find $D_{x,2}$:

$$\lim_{\beta\to 1}\lim_{\psi\to 0} D_{x,2} = \lim_{\beta\to 1}\lim_{\psi\to 0}\frac{-\rho}{(1-\bar{x})}D_{x,1} = \frac{1}{C/Y}\left(\frac{-\rho}{1-\rho}\right)$$

Thus, it shows that Proposition 5 holds and when $C/Y=1$,

$$c_t = \frac{1}{1-\rho}\left(\widehat{\mathbb{E}}_t\left[x_t\right] - \rho\widehat{\mathbb{E}}_t\left[x_{t-1}\right]\right)$$

■

D Consumption

Given that $a_t = x_t + z_t$, *beliefs about the long-run* under a worst-case belief is given by

$$\lim_{j\to\infty}\widehat{\mathbb{E}}_t\left[a_{t+j}\right] = \lim_{j\to\infty}\widehat{\mathbb{E}}_t\left[x_{t+j}+z_{t+j}\right]$$

[33] In the limit,

$$\frac{1-\bar{x}}{1-\rho-\bar{x}} = \frac{1}{1-\rho}.$$

Thus, only need to show that $\frac{1}{\bar{x}}\psi\beta B \to 0$ in the limit. Using the definition of $\bar{x}$, I have

$$\frac{1}{\bar{x}}\psi\beta B = \frac{(1-\psi\beta B)\,\beta}{(D_b-\psi\beta)\,C}\left(\frac{1-C}{1-\beta}\right)\psi\beta$$

As $D_b = 0$ in the limit, I have

$$\frac{\psi\beta\,(1-C)\,(1-\psi\beta B)\,\beta}{-\psi\beta C(1-\beta)} = -\frac{1-C}{C}\left(\frac{(1-\psi\beta B)\,\beta\psi\beta}{1-\beta}\right) = 0$$

as long as $C \neq 0$. Since $\frac{(1-\psi\beta B)\beta\psi\beta}{1-\beta} \to 0$ and $\frac{1-C}{C} \leq \infty$.

such that

$$\begin{aligned}
\lim_{j\to\infty}\widehat{\mathbb{E}}_t\left[a_{t+j}\right] &= \lim_{j\to\infty}\widehat{\mathbb{E}}_t\left[\Delta x_{t+j}+\Delta x_{t+j-1}+\cdots+\Delta x_{t+1}+x_t+z_{t+j}\right] \\
&= \lim_{j\to\infty}\widehat{\mathbb{E}}_t\left[\rho^j\Delta x_{t+1}+\rho^j\Delta x_t+\cdots+\Delta x_{t+1}\right]+\lim_{j\to\infty}\widehat{\mathbb{E}}_t\left[x_t\right]++\lim_{j\to\infty}\widehat{\mathbb{E}}_t\left[\rho^{j+1}z_t\right] \\
&= x_{t|t}+\rho\lim_{j\to\infty}\widehat{\mathbb{E}}_t\left[\left(1+\rho+\cdots+\rho^j\right)\Delta x_t\right] \\
&= x_{t|t}+\frac{\rho}{1-\rho}\widehat{\mathbb{E}}_t\left[\Delta x_t\right] \\
&= x_{t|t}+\frac{\rho}{1-\rho}\left(x_{t|t}-x_{t-1|t}\right)=\frac{1}{1-\rho}\left(x_{t|t}-\rho x_{t-1|t}\right)
\end{aligned}$$

where $x_{t|t}=\widehat{\mathbb{E}}_t\left[x_t\right]$ and $x_{t-1|t}=\widehat{\mathbb{E}}_t\left[x_{t-1}\right]$ are the worst case beliefs on current and lagged permanent productivity.

E Econometrician's filtering

Let the state vector $\mathbf{x}_{t|a_t}$ be given by

$$\mathbf{x}_{t|a_t}=\left(x_{t|a_t},\ x_{t-1|a_t},\ z_{t|a_t}\right)'$$

and the dynamics of consumers' beliefs on $\mathbf{x}_{t|a_t}$ be summarized by

$$\begin{bmatrix} x_{t|a_t} \\ x_{t-1|a_t} \\ z_{t|a_t} \end{bmatrix}=\left[I-HC_2\right]A\begin{bmatrix} x_{t-1|t-1} \\ x_{t-2|t-1} \\ z_{t-1|t-1} \end{bmatrix}+H\begin{bmatrix} 1+\rho & -\rho & -\rho \end{bmatrix}\begin{bmatrix} x_{t-1} \\ x_{t-2} \\ z_{t-1} \end{bmatrix}+H\epsilon_t+H\eta_t$$

where A and C_2 are given in Proposition 1 and H represents the gains of observing productivity. Similarly, conditional on expectations $\mathbf{x}_{t|a_t}$, the econometrician's state vector $\mathbf{x}_{t|t}$ becomes

$$\begin{bmatrix} x_{t|t} \\ x_{t-1|t} \\ z_{t|t} \end{bmatrix}=\left[I-GC_1\right]\begin{bmatrix} x_{t|a_t} \\ x_{t-1|a_t} \\ z_{t|a_t} \end{bmatrix}+G\begin{bmatrix} 1+\rho & -\rho & 0 \end{bmatrix}\begin{bmatrix} x_{t-1} \\ x_{t-2} \\ z_{t-1} \end{bmatrix}+G\epsilon_t+G\eta_t+G\nu_t$$

where C_1 is given in Proposition 1 and G represents the gains of observing a noisy signal. Substituting $\mathbf{x}_{t|a_t}$ into $\mathbf{x}_{t|t}$, I have

$$\begin{bmatrix} x_{t|t} \\ x_{t-1|t} \\ z_{t|t} \end{bmatrix} = \mathbf{A} \begin{bmatrix} x_{t-1|t-1} \\ x_{t-2|t-1} \\ z_{t-1|t-1} \end{bmatrix} + \mathbf{B} \begin{bmatrix} x_{t-1} \\ x_{t-2} \\ z_{t-1} \end{bmatrix} + [[I - GC_1]H + G]\,\epsilon_t + [[I - GC_1]H + G]\,\eta_t + G\nu_t$$

where $\mathbf{A} = [I - GC_1]\,[I - HC_2]\,A$ and $\mathbf{B} = \left(H \begin{bmatrix} 1+\rho & -\rho & -\rho \end{bmatrix} + G \begin{bmatrix} 1+\rho & -\rho & 0 \end{bmatrix}\right)$.

Defining the econometrician's state vector as $\mathbf{x}_t^E$:

$$\mathbf{x}_t^E = \left(x_t,\ x_{t-1},\ z_t,\ x_{t|t},\ x_{t-1|t},\ z_{t|t}\right)$$

the transition equation can be summarized by

$$\mathbf{x}_t^E = Q\mathbf{x}_{t-1}^E + R\left(\epsilon_t,\ \eta_t,\ \nu_t\right) \tag{35}$$

where Q and R are given respectively by

$$Q = \begin{bmatrix} \begin{matrix} 1+\rho & -\rho & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \rho \end{matrix} & \mathbf{0} \\ \bar{\mathbf{Q}} & \mathbf{A} \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ & \bar{\mathbf{R}} & \end{bmatrix}$$

with

$$\bar{\mathbf{Q}} = \mathbf{B} \begin{bmatrix} 1+\rho & -\rho & \rho \\ 1+\rho & -\rho & 0 \end{bmatrix}$$

and

$$\bar{\mathbf{R}} = \mathbf{B} \begin{bmatrix} 1+\rho & 0 & 0 \\ 1+\rho & 0 & 0 \end{bmatrix} + \mathbf{B} \begin{bmatrix} 1+\rho & 0 & 0 \\ 1+\rho & 0 & 0 \end{bmatrix} + \mathbf{B} \begin{bmatrix} 1+\rho & 0 & 0 \\ 1+\rho & 0 & 0 \end{bmatrix}$$

As the econometrician observes productivity a_t and consumption c_t, the observation equation is

$$(a_t,\ c_t) = T\mathbf{x}_t^E \tag{36}$$

where

$$T = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/(1-\rho) & \rho/(1-\rho) & 0 \end{bmatrix}$$

Thus, the econoemtrician's filtering problem can be solved by (35) and (36) and the cutoff rule stated in Proposition 4.

F Order effects in belief updating under ambiguity

Updating beliefs sequentially or simultaneously makes no difference on ex-post revised beliefs when the quality of information is certain. However, under ambiguity, such claim may not necessarily be materialized.

The aim of this section is two-fold. First, I show that the simultaneous and sequential belief updating do not necessarily generate identical revised beliefs under ambiguity. Second, I show that the simultaneous belief updating under ambiguity can be described as updating beliefs sequentially when the agents first update beliefs with an unambiguous signal. The discussion is based on the case in which there are two signals and one of which is ambiguous. But it is easy to generalize the discussion with N signals where $N > 2$ and there are $N-1$ unambiguous signals.

Let the process $x \sim N(\theta, \sigma_x^2)$ and agents receive two signals about x:

$$a = x + \eta$$
$$s = x + \nu,$$

where η and ν are i.i.d. Gaussian shocks such that $\eta \sim N(0, \sigma_\eta^2)$, $\nu \sim N(0, \sigma_\nu^2)$. Agents update beliefs about x with the two signals. The key assumption here is that the signal s is ambiguous such that $\sigma_\nu^2 \in [\underline{\sigma}_\nu^2, \bar{\sigma}_\nu^2]$.

Consider the three alternative belief updating schemes. First, agents update beliefs first with the unambiguous signal and then with the ambiguous signal. Second, agents update beliefs simultaneously. Finally, agents first update beliefs with the ambiguous signal and then with the unambiguous signal.

F.1 Sequential updating [seq-1]:

Assume that agents update beliefs sequentially - first with the unambiguous signal and then with the ambiguous signal. Conditional on an ex-ante expectation on x, initial step is to update belief with the observed signal a. For simplicity, assume that the agents' utility is strictly increasing in x.[34] and agents maximize the multiple priors utility:

$$\max_{x \in X} \min_{\omega \in \Omega} \mathbb{E}\left[u(x;\omega)\right]$$

where the set Ω, the priors, is on the range of precisions such that $\Omega = [1/\bar{\sigma}_\nu^2, 1/\underline{\sigma}_\nu^2]$ and $\partial u/\partial x > 0$. While the agents' utility is strictly increasing in x, it also depends on the ambiguity parameter ω as a belief on x is a function of a signal precision. Then, the *maxmin* operation suggests that ω which minimizes the expected utility is chosen to satisfy agent' aversion toward ambiguity.

The procedure to update beliefs is summarized as follows.

updating with a: the updating beliefs with a is given by

$$x|a = \frac{\sigma_\eta^2}{\sigma_x^2 + \sigma_\eta^2}\theta + \frac{\sigma_x^2}{\sigma_x^2 + \sigma_\eta^2}a \tag{37}$$

$$Var(x|a) = \sigma_{x\eta}^2 = \frac{\sigma_x^2\sigma_\eta^2}{\sigma_x^2 + \sigma_\eta^2}$$

In other words, the revised belief with the first signal is just a weighted average of an ex-ante expectation (the unconditional expectations of x) and the observed signal where the weights depends on the precision of fundamental and the noise.

updating with s: conditional on $x|a$, the updating beliefs with the ambiguous signal is given by

$$x|a, s = x|a + \left(\frac{\sigma_{x\eta}^2}{\sigma_{x\eta}^2 + \sigma_\nu^2}\right)(s - x|a) \tag{38}$$

Since $x|a$ and $\sigma_{x\eta}^2$ do not depend on σ_ν^2, the following simple cutoff rule can be applied to update beliefs:

Proposition 6 *With sequential belief updating, when $s > x|a$, agents update beliefs with $\sigma_\nu^2 = \bar{\sigma}_\nu^2$. When $s < x|a$, agents update beliefs with $\sigma_\nu^2 = \underline{\sigma}_\nu^2$.*

[34] The results may not hold if U is a non-monotonic function of x.

Proposition 6 can be proved just by checking the second term in the right-hand side of (38). When the observed signal s is greater than $x|a$, the weight attached to the signal should be as small as possible since the weight is inversely related to σ_ν^2 such that $\sigma_\nu^2 = \bar{\sigma}_\nu^2$. Same logic applies when $s < x|a$.

Intuitively, in terms of comparison between the signal observed (s) and the agents ex-ante expectations ($x|a$), when good news arrives, agents are hesitant to believe that the signal is precise. On the contrary, when bad news are delivered, agents would believe that the signal is very informative.

F.2 Simultaneous updating [sim]:

Agents update beliefs with the signals $S = (a,\ s)'$ where

$$S = Ax + \epsilon$$

with $A = [1,\ 1]'$, $\epsilon = [\eta,\ \nu]'$. $V = Var(\epsilon)$ is a diagonal matrix with σ_η^2 and σ_ν^2 being the diagonal components. Then, the updating of beliefs is given by

$$x|S = \frac{\sigma_\nu^2\sigma_\eta^2}{\sigma_x^2\sigma_\nu^2 + \sigma_x^2\sigma_\eta^2 + \sigma_\nu^2\sigma_\eta^2}\theta + \frac{\sigma_x^2\sigma_\eta^2}{\sigma_x^2\sigma_\nu^2 + \sigma_x^2\sigma_\eta^2 + \sigma_\nu^2\sigma_\eta^2}s + \frac{\sigma_x^2\sigma_\nu^2}{\sigma_x^2\sigma_\nu^2 + \sigma_x^2\sigma_\eta^2 + \sigma_\nu^2\sigma_\eta^2}a \tag{39}$$

where the multiplicative terms for s and a are relative gains of observing the signal s and a, respectively, and σ_ν^2 is chosen to minimize $x|S$. Let $\tilde{s} = s - \theta$ and $\tilde{a} = a - \theta$, then (39) becomes

$$x|S = \theta + \frac{\sigma_x^2\sigma_\eta^2}{\sigma_x^2\sigma_\nu^2 + \sigma_x^2\sigma_\eta^2 + \sigma_\nu^2\sigma_\eta^2}\tilde{s} + \frac{\sigma_x^2\sigma_\nu^2}{\sigma_x^2\sigma_\nu^2 + \sigma_x^2\sigma_\eta^2 + \sigma_\nu^2\sigma_\eta^2}\tilde{a}$$

Lemma 1 *$x|S$ is a monotonic function of σ_ν^2*

Proof. To prove Lemma 1, it is sufficient to show that $\frac{\partial x|S}{\partial \sigma_\nu}$ does not change the sign for $\forall \sigma_\nu^2 \in [\underline{\sigma}_\nu^2, \bar{\sigma}_\nu^2]$ given $\Psi = \{\sigma_x^2, \sigma_\eta^2, \theta, s, a\}$. Taking the derivative of $x|S$ with respect to σ_ν^2,

$$\begin{aligned}\frac{\partial x|S}{\partial \sigma_\nu^2} &= \left(\sigma_x^2\sigma_\nu^2\right)\tilde{s}\left(\sigma_x^2\sigma_\nu^2 + \sigma_x^2\sigma_\eta^2 + \sigma_\eta^2\sigma_\nu^2\right)^{-2}(-1)\left(\sigma_x^2 + \sigma_\eta^2\right) + \\ &+ \left(\sigma_x^2\sigma_\nu^2\right)\tilde{a}\left(\sigma_x^2\sigma_\nu^2 + \sigma_x^2\sigma_\eta^2 + \sigma_\eta^2\sigma_\nu^2\right)^{-2}(-1)\left(\sigma_x^2 + \sigma_\eta^2\right) + \\ &+ \sigma_x^2\tilde{a}\left(\sigma_x^2\sigma_\nu^2 + \sigma_x^2\sigma_\eta^2 + \sigma_\eta^2\sigma_\nu^2\right)^{-2}\left(\sigma_x^2\sigma_\nu^2 + \sigma_x^2\sigma_\eta^2 + \sigma_\eta^2\sigma_\nu^2\right) \\ &= \frac{\sigma_x^4\sigma_\eta^2\left(\tilde{a} - \tilde{s}\right) - \sigma_x^2\sigma_\eta^4\tilde{s}}{\left(\sigma_x^2\sigma_\nu^2 + \sigma_x^2\sigma_\eta^2 + \sigma_\eta^2\sigma_\nu^2\right)^2}\end{aligned} \tag{40}$$

From (40), it is easy to see that the denominator is always positive while the numerator can either be positive or negative depending on the parameters. As σ_ν^2 does not enter into numerator, the sign of $\partial x|S/\partial\sigma_\nu^2$ does not depend on σ_ν^2. $x|S$, therefore, is a monotonic function of σ_ν^2. ■

Proposition 7 *With the simultaneous belief updating, agents update beliefs as if they do it sequentially described in Proposition 6. Specifically, when $s > x|a$, agents update beliefs with $\sigma_\nu^2 = \bar{\sigma}_\nu^2$. When $s < x|a$, the agents update beliefs with $\sigma_\nu^2 = \underline{\sigma}_\nu^2$.*

Proof. From (37), $x|a$ can be written as $x|a = \theta + \left(\frac{\sigma_x^2}{\sigma_x^2+\sigma_\eta^2}\right)\tilde{a}$, where, as before, $\tilde{a} = a - \theta$. It is sufficient to show that when $s > x|a$, $\frac{\partial x|S}{\partial\sigma_\nu^2} < 0$, whereas when $s < x|a$, $\frac{\partial x|S}{\partial\sigma_\nu^2} > 0$. Since the denominator of (40) is always positive, I only need to consider the numerator of (40), which is

$$\sigma_x^4\sigma_\eta^2\left(\tilde{a} - \tilde{s}\right) - \sigma_x^2\sigma_\eta^4\tilde{s} \tag{41}$$

Divide (41) by $\sigma_x^2\sigma_\eta^2$ gives

$$\sigma_x^2\tilde{a} - \left(\sigma_x^2 + \sigma_\eta^2\right)\tilde{s} \tag{42}$$

Dividing (42) by $(\sigma_x^2 + \sigma_\eta^2)$ gives

$$\frac{\sigma_x^2}{\left(\sigma_x^2 + \sigma_\eta^2\right)}\tilde{a} - \tilde{s}$$

Therefore, the sign of $\frac{\partial x|S}{\partial\sigma_\nu^2}$ depends on whether s is greater or less than $\left[\sigma_x^2/\left(\sigma_x^2 + \sigma_\eta^2\right)\right]\tilde{a}$. Since the denominator of (40) is always positive when $\frac{\sigma_x^2}{\left(\sigma_x^2+\sigma_\eta^2\right)}\tilde{a} > \tilde{s}$,

$$\frac{\partial x|S}{\partial\sigma_\nu^2} > 0$$

On the contrary, when $\frac{\sigma_x^2}{\left(\sigma_x^2+\sigma_\eta^2\right)}\tilde{a} < \tilde{s}$,

$$\frac{\partial x|S}{\partial\sigma_\nu^2} < 0$$

The first case coincides with $s < x|a$ and the second case with $s > x|a$. Since $x|S$ is a monotonic function and $\frac{\partial x|S}{\partial\sigma_\nu^2} < 0$, when $s > x|a$, agents update beliefs with $\sigma_\nu^2 = \bar{\sigma}_\nu^2$. Similarly, when $s < x|a$, agents update beliefs with $\sigma_\nu^2 = \underline{\sigma}_\nu^2$.

Thus, simultaneously updating beliefs can be represented as sequentially updating beliefs [seq-1] described in the previous section. ■

F.3 Sequential updating [seq-2]:

Consider the case in which agents update beliefs first with the ambiguous signal and then with the unambiguous signal.

updating with s: the updating of beliefs with s can be given by

$$x|s = \theta + \frac{\sigma_x^2}{\sigma_x^2 + \sigma_\nu^2}(s - \theta)$$
$$Var(x|s) = \hat{\sigma}_{xs}^2 = \frac{\sigma_x^2 \sigma_\nu^2}{\sigma_x^2 + \sigma_{nu}^2}$$

Therefore, the following cutoff criteria would apply to update beliefs:

$$s > \theta \Rightarrow \sigma_\nu^2 = \bar{\sigma}_\nu^2 \tag{43}$$

$$s < \theta \Rightarrow \sigma_\nu^2 = \underline{\sigma}_\nu^2 \tag{44}$$

updating with a: conditional on $x|s$, updating beliefs with the unambiguous signal a can be given by

$$x|s, a = \frac{\sigma_\eta^2}{\hat{\sigma}_{xs}^2 + \sigma_\eta^2} x|s + \frac{\hat{\sigma}_{xs}^2}{\hat{\sigma}_{xs}^2 + \sigma_\eta^2} a$$

where $\hat{\sigma}_{xs}^2$ and $x|s$ are chosen with cutoff rule in (43) and (44).

Proposition 8 *Updating beliefs sequentially defined as above (*[seq-2]*) does not necessarily produce the identical updated beliefs as in the other schemes.*

Belief updating under ambiguity crucially depends on how agents apply cutoff rules. In the previous two cases (*Sim* and *Seq-1*), the cutoff rules apply with the reference level $s = x|a$. However, in *Seq-2*, the reference level to apply cutoff rule is $s = \theta$. Unless $x|a = \theta$, therefore, the updated beliefs *Seq-2* are not the same as the ones obtained from the other cases.