# Testing for Level Shifts in Fractionally Integrated Processes: a State Space Approach*

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**Abstract**

Short memory models contaminated by level shifts have similar long-memory features as those of fractionally integrated processes. This makes it difficult to check if the true data generating process is a pure fractionally integrated process when employing standard estimation methods based on the autocorrelation function or the periodogram. In this paper, we propose a robust testing procedure, based on an encompassing parametric specification, that allows to disentangle the level shift term from the fractionally integrated component. The estimation is carried out on the basis of a state-space methodology and it leads to a robust estimate of the fractional integration parameter also in presence of level shifts. Once the memory parameter is estimated, we use the KPSS test for presence of level shift. The Monte Carlo simulations show how this approach produces unbiased estimates of the memory parameter when shifts in the mean, or other slowly varying trends, are present in the data. Therefore, the subsequent robust version of the KPSS test for the presence of level shifts has proper size and by far the highest power compared to other existing tests. Finally, we illustrate the usefulness of the proposed approach on the daily series of bipower variation and turnover.

**Keywords:** Long Memory, ARFIMA Processes, Level Shifts, State-Space methods, KPSS test.

**JEL Classification:** C10, C11, C22, C80.

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# 1 Introduction

The phenomenon of long memory has been known for years in fields like hydrology and physics. The hydrologist Hurst (1951) was the first to formally study that long periods of dryness of the Nile river were followed by long periods of floods. A formal theory on long memory processes was subsequently formulated by Mandelbrot (1975), who introduced the fractional Brownian motion and studied the so called *self-similarity* property. The introduction of fractional integration in economics and econometrics dates back to Granger (1980) and Granger and Joyeux (1980) who defined the autoregressive fractionally integrated moving average (ARFIMA henceforth) model. Similarly to hydrological and climatological time series, many economic and financial time series show evidence of being neither integrated of order zero (I(0) henceforth) nor integrated of order one (I(1) henceforth). In these circumstances the use of ARFIMA models becomes necessary. Nowadays, a broad range of applications in finance and macroeconomics shows that long-memory models are relevant, see among others Diebold et al. (1991) for exchange rate data, Andersen et al. (2001a) and Andersen et al. (2001b) for financial volatility series, and Baillie et al. (1996) for inflation data. Early papers on the estimation of long-memory models are due to Fox and Taqqu (1986), Dahlhaus (1989), Sowell (1992) and Robinson (1995). In order to carry out inference on the degree of long memory of a given time series, it is standard practice to look at the hyperbolic decay rate of the estimated autocorrelation function or at the explosive behavior of the periodogram close to the origin. However, it is well known that these features can also be generated by non-fractional processes. For example, an I(0) process contaminated by random levels shifts, see Diebold and Inoue (2001) and Granger and Hyung (2004), is characterized by a slow decaying autocorrelation function and a spectral density with a pole in zero. In particular, Mikosch and Starica (2004) stress that when a short memory process is contaminated by level shifts, its autocorrelation function mimics that of an ARFIMA process. Similarly, Dolado et al. (2008) show that the slow hyperbolic decay of the autocorrelation function, typical of the ARFIMA processes, could be confused with that generated by short memory processes whose mean is subject to breaks. In other words, fractional processes represent only a subset of the large family of long-memory processes, although the expression *spurious long-memory* is often used to refer to processes that are long-memory but not fractional.

The literature on testing for fractional integration (*true* long-memory) versus *spurious* long-memory has grown in recent years. Mikosch and Starica (2004) test long-memory versus short-memory plus level shifts (or breaks in the mean) and propose a modified Dickey-Fuller test with shifts. Other tests are based on specific characteristic of fractional processes that are not common to other long-memory processes. For example, Ohanissian et al. (2008) develop a test that exploits the invariance of the fractional parameter to temporal aggregation, that is an indication of the self-similarity property. Shimotsu (2006) proposes two different strategies: the first one is based on sample splitting and subsequent comparison among different estimates of the fractional integration parameter, $d$; the second one is based on a stationary test, such as KPSS and PP tests, performed on the $d^{th}$-differenced data. Perron and Qu (2010) propose a test based on log-periodogram regression with different bandwidths. Alternatively, Qu (2011) compares the spectral domain properties of long-memory and short-memory processes with level shifts at an intermediate range of frequencies. This score-type test is based on the derivatives of the profile local Whittle function and it does not require the specification of the shifting process. Xiaofeng and Xianyang (2010) propose a test to detect a mean shift with unknown dates in the time domain, which can be considered as a parametric version of Qu (2011). More recently, Haldrup and Kruse (2014) have developed a testing strategy based on Hermite polynomial transformations of the series at hand. The test exploits the fact that, under *true* long-memory, the estimates of the fractional parameter decrease at a certain rate as the order of polynomial transformation increases. Finally, Leccadito et al. (2015) perform an extensive Monte Carlo exercise concluding that the test proposed by Qu (2011) has overall the best finite sample performance.

In this paper, we propose a new strategy to test whether an ARFIMA process is contaminated by random level shifts. As opposed to the previous literature, we consider an encompassing specification in which both components, the ARFIMA and the random level shifts, are potentially present. In particular, we rely on a state-space representation of the two-components process, which, coupled with a modified version of the Kalman filter, allows obtaining robust estimates of the ARFIMA parameters as well as of the probability and the size of the random level shifts. Given the estimates of the model parameters, we can test for the absence of the level shift term by adopting a KPSS type of statistic. Specifically, we estimate the parameters of the two-components model with a modified Kalman filter routine, which combines the method proposed by Kim (1994) to deal with level shifts and the state-space approximation of ARFIMA processes introduced by Chan and Palma (1998). Once the parameters of the model are estimated by maximum likelihood, the null hypothesis of absence of the shift is tested by a KPSS statistic applied to the 'filtered' series (i.e. where the fractional component has been removed by $d^{th}$-difference). This procedure doesn't rule out the possibility that a fractionally integrated term and a level shift process are jointly responsible for the observed persistence. A set of Monte Carlo simulations shows that the proposed method has the correct size under the null that the level shift term is not present in the DGP for different specifications of the the ARFIMA component. We find that the Bayesian information criterion, which is adopted to select the optimal lag-length in the short-run component in the ARFIMA term, selects the correct model in more than 90% of the cases, thus controlling for the potential misspecification of the short-run dynamics of the ARFIMA. Interestingly, we find that the KPSS test coupled with the state-space estimation of the two component model has by far the highest power compared to the existing testing strategies, especially for relatively short sample sizes. Since our testing procedure is based on a model that is fully specified both under the null and under the alternative, we also evaluate if the testing procedure is robust to the misspecification of the shifting process. We find that the power remains generally the highest also when the ARFIMA term is contaminated by other slowly moving trend instead of a random level shift process. Finally, the new testing method is carried out on a number of financial time series, such as daily bipower variation and turnover. In most cases, the results suggest that an ARFIMA process with random breaks in the mean generates the observed long-run dependence in the data; different results than those obtained adopting other testing strategies.

The paper is organized as follows. In Section 2, we specify the model as the sum of two unobserved components: an ARFIMA term and a level-shift process. Hence, we discuss the properties of the model, the corresponding state-space representation, the estimation methodology and the KPSS testing statistic. Section 3 reports the results of the Monte Carlo simulations, while Section 4 provides an empirical application and Section 5 concludes the paper.

# 2 An ARFIMA model with level shifts

## 2.1 Model specification

Contrary to existing approaches, the methodology presented in this paper is based on the idea that the observed series may be originated by the joint presence of a fractional and a level shift component. We therefore focus on testing whether the series at hands contains a level shift term rather than looking at the properties that must be fulfilled under the hypothesis of pure fractional integration. Using a fully parametric specification of the model, we can obtain a robust estimate of the ARFIMA parameters both in presence and in absence of the contaminating term by adopting a filtering scheme that is coherent with the assumed encompassing data generating process. This approach presents two advantages. First, the null and the alternative hypothesis are well defined in terms of the model parameters. Second, the presence of level shifts does not automatically exclude the possible presence of a fractional component, but the two terms can co-exist.

We assume that the observed series is given by the sum of two unobserved components

$$y_t = \mu_t + x_t, \quad t = 1, \ldots, T, \tag{1}$$

where $x_t$ follows an ARFIMA$(p, d, q)$ process and $\mu_t$ is the random shift component, modeled as follows

$$\mu_t = \mu_{t-1} + \delta_t, \quad \delta_t = \gamma_t \delta_{1t} + (1 - \gamma_t)\delta_{2t}, \tag{2}$$

where $\delta_t$ is a mixture of Gaussian random variables. In particular, $\delta_{jt} \sim \mathrm{N}(0, \sigma^2_{j\delta})$ for $j = 1, 2$, with $\sigma^2_{1\delta} = \sigma^2_\delta$ and $\sigma^2_{2\delta} = 0$, and the mixture is regulated by a Bernoulli random variable $\gamma_t \sim Bern(\pi)$. The ARFIMA$(p, d, q)$ process $x_t$ is defined as:

$$\Phi(L)(1 - L)^d x_t = \Theta(L)\xi_t, \quad t = 1, \ldots, T, \tag{3}$$

where $\{\xi_t\}_{t=1}^T$ is a sequence of independent Gaussian random variables with zero mean and constant variance $\sigma^2_\xi$, the lag operator $L$ is such that $Ly_t = y_{t-1}$; $\Phi(L) = 1 - \phi_1 L - \ldots - \phi_p L^p$ is the autoregressive polynomial, while $\Theta(L) = 1 + \theta_1 L + \ldots + \theta_q L^q$, is the moving average operator, such that $\Phi(L)$ and $\Theta(L)$ have all their roots outside the unit circle, with no common factors. The long memory property is induced by the term $(1 - L)^d$, which is the fractional difference operator. The parameter $d$ determines the fractional integration degree of the process, also known as memory parameter. If $d > -1/2$, $x_t$ is invertible and has a linear representation. If $d < 1/2$, the process is covariance stationary. Furthermore, for $d > 0$ the autocorrelations of $x_t$ die out at an hyperbolic rate (and indeed are no longer absolutely summable) in contrast to the (much faster) exponential rate for a weakly dependent process. If $d = 0$ the process is an ARMA, also known as short memory process. The full vector of parameters of model (1) contains the ARFIMA parameters plus two extra parameters regulating the random level shift component, namely $\psi = (d, \phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q, \sigma^2_\xi, \sigma^2_\delta, \pi)$.

This formulation nests three different models: (i) when $\pi = 0$, $y_t$ is a pure stationary ARFIMA process with $d < 1/2$; (ii) for $\pi = 1$, $\mu_t$ is a random walk; (iii) if $\pi > 0$ then $y_t$ is an ARFIMA process with random level shifts and the process $y_t$ is non-stationary. Moreover, the non-stationarity degree of the process $\mu_t$ can be characterized in terms of its summability order, see also Berenguer-Rico and Gonzalo (2014).

**Proposition 1.** *Let the process $\mu_t$ be generated by model (2), with $\delta_{jt} \sim N(0, \sigma^2_{j\delta})$ for $j = 1, 2$, with $\sigma^2_{1\delta} = \sigma^2_\delta$ and $\sigma^2_{2\delta} = 0$, and the mixture is regulated by a Bernoulli random variable $\gamma_t \sim Bern(\pi)$. It follows that*

$$\frac{1}{T^{3/2}} \frac{1}{\sigma_\delta \sqrt{\pi}} \sum_{t=1}^{[Tr]} \mu_t \xrightarrow{d} \int_0^r W(r) dr, \tag{4}$$

*so that $\mu_t$ is summable of order 1, i.e. $\mu_t \sim I(1)$ process.*

Proof in Appendix A.1.

Similarly to Examples 3 and 5 in Berenguer-Rico and Gonzalo (2014), the term $\mathcal{L} = \frac{1}{\sigma_\delta \sqrt{\pi}}$ represents a scaling factor of the asymptotic variance which does not depend on $T$. Notably, $\mu_t$ is a random walk with i.i.d. innovations when $\pi = 1$. In this case, $\mathcal{L} = \frac{1}{\sigma_\delta}$ as in Example 3 in Berenguer-Rico and Gonzalo (2014). The main consequence of Proposition 1 is that the process $y_t$ in (1) is I(1) when $\pi \cdot \sigma^2_\delta > 0$, i.e. when the variance of the innovation of the shifting process $\mu_t$ and $\pi$ are both non-zero. This means that, when the random level shift is present, then it asymptotically dominates over the ARFIMA term which is summable of order $d < 1/2$. Indeed, the summability/integration order of $x_t$ is proportional to $\frac{1}{T^{(d+0.5)}}$, which in turn implies that $x_t$ is summable of order $d$, i.e. $x_t \sim$ I$(d)$. Since the ARFIMA model encompasses the class of ARMA processes, the specification in (1) reduces to the case of a short memory ARMA process plus level shifts when $d = 0$.

## 2.2 The testing procedure

We now outline a testing procedure for the absence of level shifts in model (1). Suppose that the true value of the fractional integration parameter, $0 \leq d_0 \leq 1/2$, is available, the observed series (1) can therefore be filtered as follows:

$$\tilde{y}_t := (1 - L)^{d_0} y_t = \tilde{\mu}_t + \tilde{x}_t,$$

where

$$\Delta \tilde{\mu}_t = (1 - L)^{d_0} \delta_t, \quad \Phi(L) \tilde{x}_t = \Theta(L) \xi_t.$$

Under the null hypothesis $\mathbb{H}_0 : \pi \sigma^2_\delta = 0$, i.e. $\mu_t = 0 \; \forall t$, then $y_t \sim$ I$(d_0)$, so that the filtered series, $\tilde{y}_t = \tilde{x}_t$, is a weakly stationary I(0) process. Under the alternative hypothesis, $\mathbb{H}_1 : \pi \sigma^2_\delta > 0$, then

$y_t$ contains a level shift term, (i.e. $\mu_t \neq 0$), and the filtered series $\tilde{y}_t$ is non-stationary given that $\tilde{\mu}_t \sim \mathrm{I}(1-d_0)$. A KPSS test statistic can be then computed as follows:

$$\Psi = \frac{1}{T^2} \frac{\sum_{t=1}^{T} (\sum_{i=1}^{t} \tilde{y}_i)^2}{\tilde{\sigma}_y^2}, \tag{5}$$

where $\tilde{\sigma}_y^2$ is an estimate of the long-run variance of the filtered process $\tilde{y}_t$. Under the null hypothesis, the test has a Cramer-Von-Mises distribution, see Nyblom and Makelainen (1983), Leybourne and McCabe (1994) and Harvey (1997). Unfortunately, the test statistic in (5) is unfeasible since $d_0$ is unknown. Henceforth, the feasible test statistic, is

$$\Psi^* = \frac{1}{T^2} \frac{\sum_{t=1}^{T} (\sum_{i=1}^{t} \tilde{y}_i^*)^2}{\tilde{\sigma}_{y^*}^2}, \tag{6}$$

where $\tilde{y}_t^* = (1-L)^{\hat{d}} y_t$ and $\tilde{\sigma}_{y^*}^2$ is its long run variance and $\hat{d}$ is an estimate of the long-memory parameter. In order to compute a feasible version of the KPSS test statistic, it is therefore necessary to obtain an unbiased estimate of the long memory parameter, $\hat{d}$, under both the null and the alternative hypothesis. The relevant quantiles of the asymptotic distribution of the feasible KPSS test, $\Psi^*$, are tabulated in Shimotsu (2006) and they are very close to those of the standard Cramer-Von-Mises distribution when $0 < d_0 < 0.5$. This means that the uncertainty on the estimate of $d$ plays a minor role asymptotically. Unfortunately, the traditional parametric and semiparametric estimators of the long memory parameter are reliable and efficient under the null hypothesis, i.e. in absence of shifts, but they can be severely biased and inconsistent under the alternative, thus drastically reducing the power of the test, especially in finite samples. Conversely, the state-space methodology outlined in Section 2.3 provides estimates of the ARFIMA parameters that are well centered on the true values both in presence and in absence of the level shift term. Therefore, the estimate of $d$ obtained with the modified Kalman filter routine outlined below can be used in computing the feasible KPSS test statistic, henceforth denoted by $SSF_k$.

### 2.3 Estimation

We propose a robust estimation method to carry out inference on the parameters of model (1). The methodology relies on a modified Kalman filter routine as in Kim (1994). First, model (1) is cast in a state-space form

$$\begin{aligned} y_t &= \mathrm{Z}\alpha_t, & & t = 1, \ldots, T, \\ \alpha_t &= \mathrm{F}\alpha_{t-1} + \mathrm{R}\eta_t, & \eta_t \sim \mathrm{N}(0, \mathrm{Q}^{(j)}), & \quad j = 1, 2, \end{aligned} \tag{7}$$

where the state vector and the system matrices are defined as following

$$\alpha_t = (\mu_t, x_t, \dots, x_{t-m+1})', \qquad \mathrm{Z} = (1, 1, 0, \dots, 0), \qquad \eta_t = (\delta_t, \xi_t)',$$
$$\mathrm{F} = \begin{bmatrix} 1 & 0_{1\times m} \\ 0_{m\times 1} & \mathrm{F}_{22} \end{bmatrix}, \qquad \mathrm{R} = \begin{bmatrix} 1 & 0 \\ 0_{m\times 1} & \mathrm{R}_{22} \end{bmatrix}, \quad \mathrm{Q}^{(j)} = \begin{bmatrix} \sigma^2_{j\delta} & 0 \\ 0 & \sigma^2_{\xi} \end{bmatrix},$$
$$\mathrm{F}_{22} = \begin{bmatrix} \varphi_1 \dots & \dots \varphi_m \\ \mathrm{I}_{m-1} & 0_{(m-1)\times 1} \end{bmatrix}, \quad \mathrm{R}_{22} = (1, 0, \dots, 0)'.$$

Note that the coefficients $\varphi_1, \dots, \varphi_m$ are determined by the infinite autoregressive representation of the ARFIMA process and they are function of the ARFIMA parameters only, while $m > 0$ is the truncation order. Hosking (1981) shows that a stationary ARFIMA$(p, d, q)$ admits infinite AR (and MA) expansions and provides a formula to compute the coefficients of such representations as an infinite convolution of the AR (and MA) filter with the fractional difference operator. Although a long memory process has infinite AR and MA representations, Chan and Palma (1998) propose an approximation based on the truncation up to the $m$-th lag and provide the asymptotic properties of the truncated ML estimator. In particular, Chan and Palma (1998) show that the ML estimator based on the truncated model is consistent, asymptotically Gaussian and efficient. The small sample properties of state space approach has been recently investigated by Grassi and Santucci de Magistris (2014), who have shown the reliability of the methodology for a large number of parameter combinations for the DGP. [1]

Define $\mathrm{S}_{t-1} = i$ as the state at time $t-1$ and $\mathrm{S}_t = j$ as the state at time $t$, with $i, j = 1, 2$. The first element of the pair $\{i, j\}$ denotes the *past regime* and the second one refers to the *present regime*. We denote $\mathrm{Y}_t = \{y_t, \dots, y_1\}$ the information set up to time $t$. Moreover the term $\pi^{(i,j)}_{t|t} := \Pr(\mathrm{S}_{t-1} = i, \mathrm{S}_t = j|\mathrm{Y}_t)$ defines the *real-time* filter probability to switch from $i$ to $j$, so that the *real-time* filter probability to be in $j$ is then equal to $\pi^{(j)}_{t|t} := \Pr(\mathrm{S}_t = j|\mathrm{Y}_t) = \sum_{i=1}^2 \pi^{(i,j)}_{t|t}$. Similarly, the *predictive* filter probability to switch from $i$ to $j$, is $\pi^{(i,j)}_{t|t-1} := \Pr(\mathrm{S}_{t-1} = i, \mathrm{S}_t = j|\mathrm{Y}_{t-1})$, and the *predictive* filter probability to be in state $j$' is $\pi^{(j)}_{t|t-1} := \Pr(\mathrm{S}_t = j|\mathrm{Y}_{t-1}) = \sum_{i=1}^2 \pi^{(i,j)}_{t|t-1}$. Finally, the constant $\lambda_{ij} := \Pr(\mathrm{S}_t = j|\mathrm{S}_{t-1} = i) = \Pr(\mathrm{S}_t = j) = \lambda_j$ denotes the *transition probability*, such that $\lambda_1 = \pi$ and $\lambda_2 = (1 - \pi)$.

The representation (7) differs from the standard state-space representation of an ARFIMA model because the matrix $\mathrm{Q}^{(j)}$ is subject to stochastic changes driven by the shift parameters $\pi$ and $\sigma^2_\delta$. Therefore, the Kalman filter routine required to compute the log-likelihood function associated to model (7) needs to be modified according to Kim (1994) and Kim and Nelson (1999). Differently from the specification adopted in Grassi and Santucci de Magistris (2014), the measurement equation links the levels of the observed variable $y_t$ to the unobserved states. When working with the first difference of $y_t$ as in Grassi and Santucci de Magistris (2014), the innovation to the shift term, $\delta_t$, enters directly in the measurement equation and it is treated as a measurement error. Instead, in the above representation both the ARFIMA term and the level shift are modeled as unobserved state variables and the variances of their innovations enter in the matrix $\mathrm{Q}^{(j)}$. In this way, the measurement equation can be extended with the inclusion of other unobserved compo-

[1] Further details on the state-space representation of ARFIMA processes are presented in Appendix A.2.

nents, if necessary.

The predictive filter for the state vector and its mean squared error (MSE) are

$$\tilde{\alpha}_{t|t-1}^{(i,j)} = \mathrm{F}\tilde{\alpha}_{t-1|t-1}^{(i)}, \qquad \mathrm{P}_{t|t-1}^{(i,j)} = \mathrm{F}\mathrm{P}_{t-1|t-1}^{(i)}\mathrm{F}' + \mathrm{R}\mathrm{Q}^{(j)}\mathrm{R}', \tag{8}$$

where $\tilde{\alpha}_{t|t-1}^{(i,j)} = \mathrm{E}(\alpha_t|\mathrm{Y}_{t-1}, \mathrm{S}_{t-1} = i, \mathrm{S}_t = j)$ and $\mathrm{P}_{t|t-1}^{(i,j)} = \mathrm{Var}(\alpha_t|\mathrm{Y}_{t-1}, \mathrm{S}_{t-1} = i, \mathrm{S}_t = j)$. The corresponding prediction error and its MSE are

$$v_t^{(i,j)} = y_t - \mathrm{Z}\tilde{\alpha}_{t|t-1}^{(i,j)}, \qquad \mathrm{G}_t^{(i,j)} = \mathrm{Z}\mathrm{P}_{t|t-1}^{(i,j)}\mathrm{Z}'. \tag{9}$$

The real-time filter and its MSE for the transition state are

$$\tilde{\alpha}_{t|t}^{(i,j)} = \tilde{\alpha}_{t|t-1}^{(i,j)} + [\mathrm{P}_{t|t-1}^{(i,j)}\mathrm{Z}'/\mathrm{G}_t^{(i,j)}]v_t^{(i,j)}, \qquad \mathrm{P}_{t|t}^{(i,j)} = \mathrm{P}_{t|t-1}^{(i,j)} - \mathrm{P}_{t|t-1}^{(i,j)}\mathrm{Z}'\mathrm{Z}\mathrm{P}_{t|t-1}^{(i,j)}/\mathrm{G}_t^{(i,j)}, \tag{10}$$

where $\tilde{\alpha}_{t|t}^{(i,j)} = \mathrm{E}(\alpha_t|\mathrm{Y}_t, \mathrm{S}_{t-1} = i, \mathrm{S}_t = j)$ and $\mathrm{P}_{t|t}^{(i,j)} = \mathrm{Var}(\alpha_t|\mathrm{Y}_t, \mathrm{S}_{t-1} = i, \mathrm{S}_t = j)$. Given real-time filter probability to be in state "$i$" at time $t-1$, that is $\pi_{t-1|t-1}^{(i)}$, we can compute the predictive filter transition probability

$$\pi_{t|t-1}^{(i,j)} = \lambda_j \pi_{t-1|t-1}^{(i)}, \tag{11}$$

and the predictive filter probability to be in state "$j$", is obtained as follows $\pi_{t|t-1}^{(j)} = \sum_{i=1}^{2} \pi_{t|t-1}^{(i,j)}$. The conditional probability of the observation is obtained as a weighted average of the single conditional Gaussian probabilities

$$\Pr(y_t|\mathrm{Y}_{t-1}) = \sum_{i=1}^{2}\sum_{j=1}^{2} \Pr(y_t^{(i,j)}|\mathrm{Y}_{t-1})\pi_{t|t-1}^{(i,j)}, \tag{12}$$

where the observations are conditionally Gaussian

$$\Pr(y_t^{(i,j)}|\mathrm{Y}_{t-1}) = \left[2\pi\mathrm{G}_t^{(i,j)}\right]^{-1/2} \exp\left[-\frac{v_t^{(i,j)2}}{2\mathrm{G}_t^{(i,j)}}\right]. \tag{13}$$

Given the predictive filter transition probabilities, we can now update the real-time filter transition probabilities

$$\pi_{t|t}^{(i,j)} = \frac{\Pr(y_t^{(i,j)}|\mathrm{Y}_{t-1})\pi_{t|t-1}^{(i,j)}}{\Pr(y_t|\mathrm{Y}_{t-1})}, \tag{14}$$

and we can aggregate them to obtain the real-time filter probability to be in state "$j$" which is $\pi_{t|t}^{(j)} = \sum_{i=1}^{2} \pi_{t|t}^{(i,j)}$. Using (11)-(14) we can then construct the full log-likelihood function

$$\ell(\mathrm{Y}_T|\psi) = \sum_{t=1}^{T} \log\left[\Pr(y_t|\mathrm{Y}_{t-1})\right], \tag{15}$$

where $\psi$ is the set of unknown parameters. Finally, the real-time filter for the state vector to be in state "$j$" and its MSE are

$$\begin{array}{rcl}\tilde{\alpha}_{t|t}^{(j)} & = & \left[\sum_{i=1}^{2}\pi_{t|t}^{(i,j)}\tilde{\alpha}_{t|t}^{(i,j)}\right]/\pi_{t|t}^{(j)}, \\ \mathrm{P}_{t|t}^{(j)} & = & \left[\sum_{i=1}^{2}\pi_{t|t}^{(i,j)}\{\mathrm{P}_{t|t}^{(i,j)}+[\tilde{\alpha}_{t|t}^{(j)}-\tilde{\alpha}_{t|t}^{(i,j)}][\tilde{\alpha}_{t|t}^{(j)}-\tilde{\alpha}_{t|t}^{(i,j)}]'\}\right]/\pi_{t|t}^{(j)},\end{array} \tag{16}$$

and the final filter estimate is

$$\tilde{\alpha}_{t|t}=\sum_{j=1}^{2}\pi_{t|t}^{(j)}\tilde{\alpha}_{t|t}^{(j)}.$$

To summarize, for $t = 1, \ldots, T$, we compute the set of Kalman filter recursions (8)-(10) and (16). In parallel, using (11)-(15) we compute the probabilities and the log-likelihood function which is then maximize with respect to the vector of parameters $\psi$. For more details and derivations see Appendix A.3.

The recursions in (8) are initialized as follows: the first element of the state vector with "diffuse prior" for $i = 1$ (see Harvey (1989), sec. 3.3.4), and with a constant for $i = 2$. The remaining $m$ elements are initialized with the unconditional mean and variance of the stationary long-memory process. Namely,

$$\tilde{\alpha}_{0|0}^{(i)}=\begin{bmatrix}0\\0\end{bmatrix},\quad \mathrm{P}_{0|0}^{(i)}=\begin{bmatrix}\mathrm{P}_{\mu}^{(i)} & 0\\ 0 & \mathrm{P}_{x}^{(i)}\end{bmatrix},\quad i=1,2,$$

with $\mathrm{P}_{\mu}^{(1)} = \kappa$, with $\kappa \to \infty$, and $\mathrm{P}_{\mu}^{(2)} = 0$, while $\mathrm{P}_{x}^{(1)} = \mathrm{P}_{x}^{(2)}$ is initialized as described in Appendix A.2. Similarly, the recursion in (11) is initialized as follows: $\pi_{0|0}^{(1)} = \pi$ and $\pi_{0|0}^{(2)} = 1 - \pi$.

## 3 Monte Carlo analysis

We perform a set of Monte Carlo simulations to evaluate the finite-samples ability of the proposed testing methodology. We study the empirical size and power of the test based on alternative choices of the parameters governing the ARFIMA process $x_t$ and the shifting process $\mu_t$. The proposed method presents the potential pitfall of being fully parametric, meaning that model misspecification may induce severe size distortions and power losses. Hence, the Monte Carlo simulations are carried out to evaluate the robustness of the estimation method to misspecification in $\mu_t$, for which we follow the DGPs used in Qu (2011, p.430). At the same time, we consider the potential problem of the selection of the short-term component of the ARFIMA term. In this case, the AR and MA orders for a low-order ARFIMA(p,d,q) with $p, q \leq 1$ are chosen by minimizing the Bayesian information criterion (BIC), which is shown to be very reliable in the context of fractionally integrated processes, see Beran et al. (1998) and Grassi and Santucci de Magistris (2014).

The performance of the proposed procedure is assessed relatively to several existing tests. In particular, we consider Ohanissian et al. (2008) (ORT), Perron and Qu (2010) (PQ), Qu (2011) (QU), and the three tests of Shimotsu (2006): $\mathrm{SH}_k$ and $\mathrm{SH}_p$ based on KPSS and PP test statistic respectively, and the $\mathrm{SH}_s$ based on the sample splitting.[2] Since the ARMA dynamics worsen the finite sample properties of the Qu (2011) test, this is performed using the *pre-whitening* procedure

[2]Due to space constraints, we don't provide a detailed discussion of these tests. A formal presentation of all these tests can be found in Leccadito et al. (2015).

outlined in Qu (2011) to remove the ARMA terms and keep the empirical size of the test under control. As described in Qu (2011, p.429), a low-order ARFIMA(p,d,q) with $p, q \leq 1$ is fitted on the observed series, $y_t$, thus determining the optimal ARMA lag-order. Subsequently, the series $y_t^*$, i.e. $y_t$ filtered from the ARMA terms, is used to construct the test statistics. Note that the state-space approach instead does not need the *prewhitening* step, since all the parameters of model (7) are jointly estimated maximizing the log-likelihood function in (15). In the next section, we present the results for the empirical size, the empirical power and the case in which both ARFIMA and shifts components are present in the data.

## 3.1 Size

The empirical size of the test is assessed when $y_t$ does not contain the level shift term, i.e. $\pi \cdot \sigma_\delta^2 = 0$, which in turns implies that $\mu_t = 0 \; \forall t \in [0, T]$. In this case, $y_t$ is a stationary ARFIMA process, $(1 - \phi L)(1 - L)^d y_t = (1 - \theta L)\xi_t$, $\xi_t \sim \text{iiN}(0, \sigma_\xi^2)$, with $d = 0.4$ and different combinations of the ARMA terms, as in Qu (2011). In this case, Chan and Palma (1998) prove that the state-space estimator is consistent if $m = T^\nu$ with $\nu > 0$ and asymptotically normal if $\nu \geq 1/2$. The Monte Carlo simulations are based on random samples generated from model (3) with sample sizes equal to $T = 500, 1000, 2000$ and the truncation lag of the infinite AR representation is chosen equal to $m = 30, 45, 60$, respectively as in Grassi and Santucci de Magistris (2014). Table 1 reports the empirical rejection frequencies at 5% nominal level of the various test statistics considered. It emerges that the modified Kalman filter estimates of the two-component model in (7) provides the proper estimates of the ARFIMA parameters also when the level shift process is absent. Indeed, the Monte Carlo average of the estimates of the fractional parameter based on the state-space representation, denoted as $\hat{d}_{SSF}$, is very close to the true value in all cases.[3] Moreover, the Bayesian information criterion selects the correct number of ARMA terms in more than 95% of the cases for most the DGPs considered. At the same time, the estimates of $\pi$ and $\sigma_\delta^2$ are such that their product is very small, meaning that the estimated shift component is negligible if compared to the ARFIMA component. This makes the empirical size of the $\text{SSF}_k$ test very close to the nominal value, with a slight over-rejection rate when $T = 500$. In particular, the rejection rate is higher than 5% when $x_t$ is very persistent, thus making difficult to distinguish between fractional integration and random level shift in samples that are not particularly long. The other reported tests have also good size properties, although the QU, $\text{SH}_k$ and $\text{SH}_p$ tests are slightly conservative. Overall, the empirical size of the proposed testing strategy is in line with that of the other tests and in line with the findings in Qu (2011) and Leccadito et al. (2015).

## 3.2 Power

The major advantage of the proposed procedure arises when looking at the empirical power of the test, i.e. $\mu_t \neq 0$. In line with Qu (2011, p.430), we consider a signal, $x_t$, contaminated by a non-stationary random level shifts trend process, $\mu_t$, $y_t = x_t + \mu_t$, where $\mu_t = \mu_{t-1} + \gamma_t \delta_t$ and

[3] We do not report the Monte Carlo average of the estimates of the other model parameters but they are available upon request to the authors.

| T | $SSF_k$ | $QU_{2\%}$ | $QU_{5\%}$ | ORT | PQ | $SH_k$ | $SH_p$ | $SH_s$ | $\hat{d}_{SSF}$ | $\hat{\pi}\hat{\sigma}^2_\delta$ | BIC |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | ARFIMA(0,0.4,0): | | | | | | |
| 500 | 0.078 | 0.013 | 0.020 | 0.058 | 0.058 | 0.028 | 0.039 | 0.085 | *0.377* | 0.012 | 0.953 |
| 1000 | 0.066 | 0.020 | 0.024 | 0.042 | 0.053 | 0.034 | 0.040 | 0.075 | *0.386* | 0.008 | 0.964 |
| 2000 | 0.060 | 0.019 | 0.024 | 0.061 | 0.045 | 0.033 | 0.036 | 0.064 | *0.388* | 0.006 | 0.970 |
| | | | | | ARFIMA(1,0.4,0), with $\phi = 0.5$: | | | | | | |
| 500 | 0.076 | 0.004 | 0.004 | 0.056 | 0.108 | 0.010 | 0.030 | 0.076 | *0.375* | 0.020 | 0.904 |
| 1000 | 0.056 | 0.010 | 0.011 | 0.078 | 0.061 | 0.015 | 0.036 | 0.068 | *0.385* | 0.016 | 0.950 |
| 2000 | 0.055 | 0.017 | 0.024 | 0.064 | 0.044 | 0.022 | 0.046 | 0.063 | *0.391* | 0.013 | 0.960 |
| | | | | | ARFIMA(1,0.4,0), with $\phi = 0.8$: | | | | | | |
| 500 | 0.078 | 0.006 | 0.015 | 0.061 | 0.087 | 0.071 | 0.000 | 0.112 | *0.450* | 0.028 | 0.968 |
| 1000 | 0.066 | 0.010 | 0.016 | 0.059 | 0.058 | 0.084 | 0.000 | 0.130 | *0.447* | 0.021 | – |
| 2000 | 0.063 | 0.002 | 0.004 | 0.049 | 0.045 | 0.082 | 0.000 | 0.105 | *0.423* | 0.017 | – |
| | | | | | ARFIMA(0,0.4,1), with $\theta = 0.5$: | | | | | | |
| 500 | 0.076 | 0.014 | 0.014 | 0.060 | 0.071 | 0.025 | 0.034 | 0.087 | *0.385* | 0.015 | 0.962 |
| 1000 | 0.064 | 0.020 | 0.027 | 0.060 | 0.047 | 0.035 | 0.033 | 0.066 | *0.387* | 0.011 | 0.969 |
| 2000 | 0.053 | 0.023 | 0.032 | 0.057 | 0.045 | 0.034 | 0.033 | 0.065 | *0.391* | 0.008 | 0.978 |
| | | | | | ARFIMA(0,0.4,1), with $\theta = 0.8$: | | | | | | |
| 500 | 0.067 | 0.017 | 0.024 | 0.062 | 0.060 | 0.031 | 0.023 | 0.075 | *0.370* | 0.016 | 0.963 |
| 1000 | 0.062 | 0.017 | 0.026 | 0.061 | 0.056 | 0.033 | 0.030 | 0.066 | *0.386* | 0.012 | 0.968 |
| 2000 | 0.051 | 0.023 | 0.030 | 0.059 | 0.045 | 0.034 | 0.033 | 0.063 | *0.389* | 0.009 | 0.972 |

***Table 1:*** *Empirical Size. The table reports the empirical rejection rate of several test statistics when $\mu_t = 0$ and $x_t$ is generated according to different ARFIMA specifications with $d = 0.4$. Results are based on 1000 Monte Carlo replications. For each test, the null is true fractional integration (i.e. absence of shifts). $SSF_k$ is the KPSS test based on the state-space representation. QU denotes the Qu (2011) test based on the local Whittle likelihood, with two different trimming choices ($\epsilon = 2\%$ and $\epsilon = 5\%$). ORT is the temporal aggregation test of Ohanissian et al. (2008). PQ is the test of Perron and Qu (2010). $SH_k$ denotes the KPSS test of Shimotsu (2006) based on $d^{th}$-differencing and $SH_p$ is its PhillipsPerron version. $SH_s$ is the Shimotsu (2006) test based on sample splitting with 4 sub-samples. Table also reports the Monte Carlo average of the estimates of the d parameter based on the state-space methodology outlined in Section 2, $\hat{d}_{SSF}$. Finally, the last column reports the proportion of the Monte Carlo replications in which the BIC correctly selects the true ARMA lag-order.*

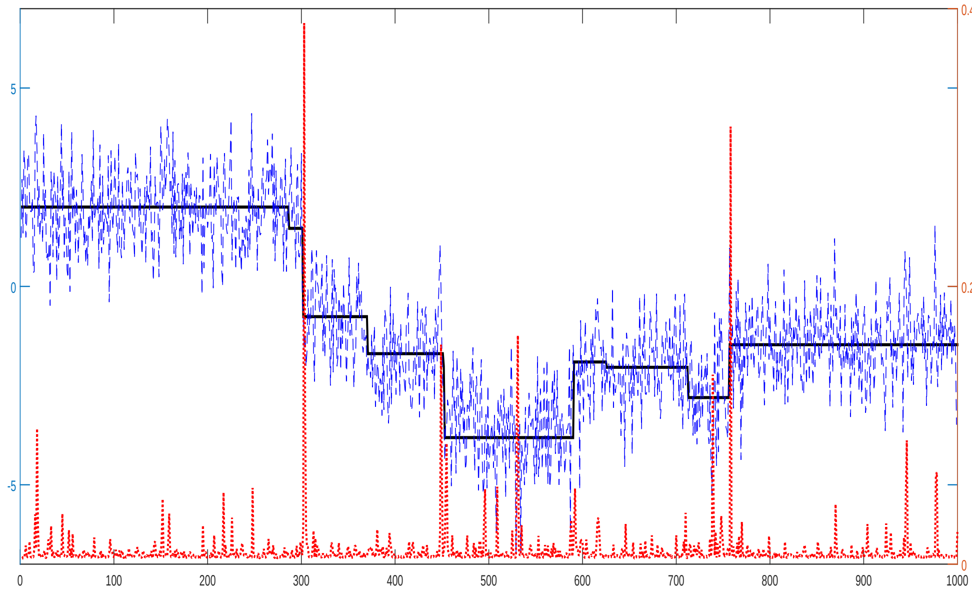
$\delta_t \sim N(0,5)$, $\gamma_t \sim Bern(6.1/T)$. As noted by Qu (2011), when $x_t$ is i.i.d. standard Gaussian, this parameter configuration is such that the implied degree of fractional integration of $y_t$ is close to 0.4. Therefore this setup matches the same degree of long-memory as the one used to analyze the size of the tests in Table 1 when $y_t$ is purely fractional process. In addition to the setup in Qu (2011), where an i.i.d. Gaussian random variable, $x_t$, is added to the shifting process, $\mu_t$, we also consider the possibility that $x_t$ follows an ARFIMA process, since in our setup, the two sources of persistence can co-exist and the state space approach is designed to disentangle them, thus providing reliable parameter estimates in both cases. The main result that emerges from Table 2 is that the proposed testing procedure has very high power for almost all the cases considered and for all sample sizes, as opposed to most existing tests. This result is mainly due to the ability of the modified Kalman filter to provide unbiased estimates of the model parameters when the observed data, $y_t$, is generated by the sum of a non-stationary random level shift process, $\mu_t$, and an ARFIMA process, $x_t$, with different parameter combinations. The highest empirical power is obtained when a white noise process is added to $\mu_t$, which is the case considered in Qu (2011).

| T | $SSF_k$ | $QU_{2\%}$ | $QU_{5\%}$ | ORT | PQ | $SH_k$ | $SH_p$ | $SH_s$ | $\hat{d}_{SSF}$ | VR | BIC |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | i.i.d. Gaussian Noise: | | | | | | |
| 500 | **0.881** | 0.183 | 0.151 | 0.097 | 0.183 | 0.328 | 0.004 | 0.128 | *0.102* | 0.402 | 0.890 |
| 1000 | **0.913** | 0.264 | 0.212 | 0.183 | 0.297 | 0.454 | 0.002 | 0.184 | *0.081* | 0.393 | 0.903 |
| 2000 | **0.955** | 0.571 | 0.374 | 0.223 | 0.422 | 0.669 | 0.001 | 0.284 | *0.079* | 0.408 | 0.914 |
| | | | | | ARFIMA(1,0.2,0), with $\phi = 0.5$: | | | | | | |
| 500 | **0.614** | 0.082 | 0.067 | 0.067 | 0.144 | 0.239 | 0.005 | 0.089 | *0.310* | 0.287 | 0.646 |
| 1000 | **0.781** | 0.193 | 0.151 | 0.122 | 0.169 | 0.350 | 0.003 | 0.110 | *0.231* | 0.274 | 0.847 |
| 2000 | **0.726** | 0.423 | 0.274 | 0.128 | 0.170 | 0.550 | 0.002 | 0.132 | *0.182* | 0.283 | 0.953 |
| | | | | | ARFIMA(1,0.2,0), with $\phi = 0.8$: | | | | | | |
| 500 | **0.262** | 0.011 | 0.017 | 0.042 | 0.071 | 0.074 | 0.015 | 0.047 | *0.381* | 0.141 | 0.750 |
| 1000 | **0.460** | 0.073 | 0.010 | 0.080 | 0.081 | 0.131 | 0.004 | 0.048 | *0.301* | 0.128 | 0.883 |
| 2000 | **0.652** | 0.010 | 0.000 | 0.087 | 0.112 | 0.196 | 0.008 | 0.075 | *0.235* | 0.132 | 0.933 |
| | | | | | ARFIMA(0,0.2,1), with $\theta = 0.5$: | | | | | | |
| 500 | **0.797** | 0.212 | 0.206 | 0.068 | 0.258 | 0.269 | 0.001 | 0.095 | *0.212* | 0.319 | 0.876 |
| 1000 | **0.898** | 0.449 | 0.433 | 0.141 | 0.329 | 0.451 | 0.003 | 0.124 | *0.187* | 0.307 | 0.940 |
| 2000 | **0.945** | 0.692 | 0.667 | 0.118 | 0.495 | 0.624 | 0.002 | 0.135 | *0.191* | 0.319 | 0.967 |
| | | | | | ARFIMA(0,0.2,1), with $\theta = 0.8$: | | | | | | |
| 500 | **0.643** | 0.228 | 0.207 | 0.073 | 0.246 | 0.265 | 0.003 | 0.081 | *0.199* | 0.267 | 0.900 |
| 1000 | **0.783** | 0.431 | 0.423 | 0.110 | 0.349 | 0.438 | 0.002 | 0.102 | *0.197* | 0.255 | 0.960 |
| 2000 | **0.870** | 0.683 | 0.652 | 0.133 | 0.492 | 0.641 | 0.004 | 0.163 | *0.193* | 0.266 | 0.967 |

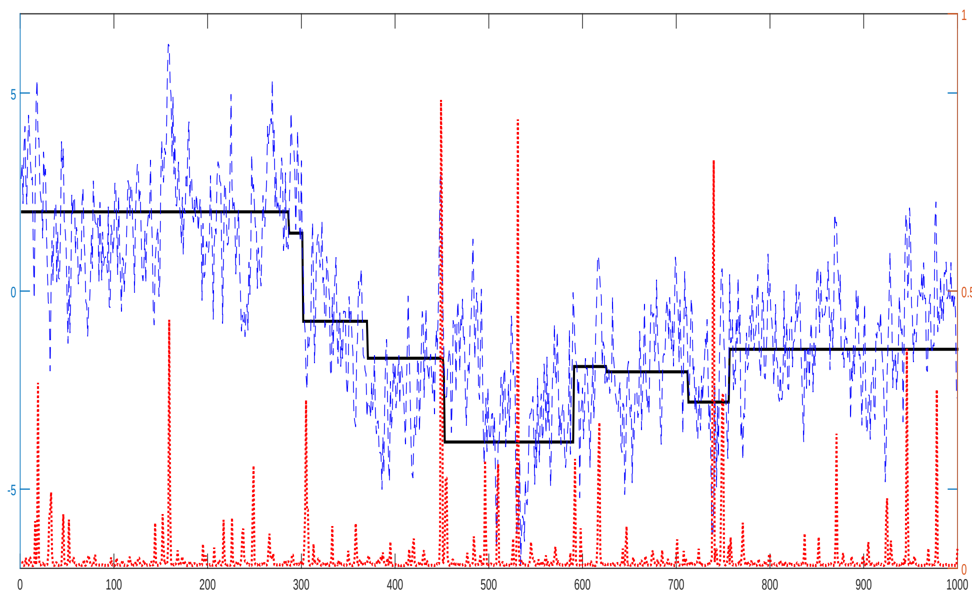
***Table 2:*** *Power. Non-stationary random level shift model. The table reports the empirical rejection rate of several test statistics when $\mu_t$ is a random level shift process and $x_t$ is generated according to different ARFIMA specifications. Results are based on 1000 Monte Carlo replications. For each test, the null is true fractional integration (i.e. absence of shifts). $SSF_k$ is the KPSS test based on the state-space representation. QU denotes the Qu (2011) test based on the local Whittle likelihood, with two different trimming choices ($\epsilon = 2\%$ and $\epsilon = 5\%$). ORT is the temporal aggregation test of Ohanissian et al. (2008). PQ is the test of Perron and Qu (2010). $SH_k$ denotes the KPSS test of Shimotsu (2006) based on $d^{th}$-differencing and $SH_p$ is its PhillipsPerron version. $SH_s$ is the Shimotsu (2006) test based on sample splitting with 4 sub-samples. $\hat{d}_{SSF}$ is the Monte Carlo average of the estimates of the d parameter based on the state-space methodology outlined in Section 2. Table also reports the Monte Carlo average of the variance ratio, $VR$, between the sample variance of $\mu_t$ and to the sample variance of $y_t$. Finally, the last column reports the proportion of the Monte Carlo replications in which the BIC correctly selects the true ARMA lag-order.*

In particular, when $T = 2000$, the empirical rejection rate of the null hypothesis obtained by the $SSF_k$ is close to 90% and the estimates of $d$ are centered around 0, as expected. The highest power obtained by the other tests is that of the KPSS statistic of Shimotsu (2006) which in any case falls well below 100%. The empirical power of the $SSF_k$ remains high also when ARFIMA processes with $d = 0.2$ are considered for $x_t$. In all cases, the estimates of $d$ are centered around the true value, confirming the validity of the state space approach when both the long memory component and the shifts are present. The power is drastically reduced when we consider a highly persistent ARFIMA process with $\phi = 0.8$. Indeed, as indicated by the variance ratio (VR, henceforth) reported in the last column of the table, the variability of the shift process relative to the total variability of $y_t$ is only one third of that of the white-noise case. It follows that it is relatively more difficult to conduct precise inference on the shift process when the ARFIMA series is more persistent, and this impacts on the empirical power of the $\text{SSF}_k$ test. As noted by Grassi and Santucci de Magistris (2014), the estimates of the parameter $d$ become more imprecise as the AR parameter gets closer to 1, but this parameter configuration is rather extreme and not often found in the real data. For what concerns the misspecification of the ARFIMA dynamics, the selection of the correct lag order of the ARFIMA is not a concern as the proportion of models correctly selected by the BIC is generally above 90% when $T \geq 1000$. When $T = 500$, the proportion is around 70% only when $x_t$ follows an ARFIMA(1,d,0) and the estimates of $d$ are slightly upward biased. Interestingly, the power of the $\text{SSF}_k$ test is the highest also in this case, while the power of the other tests slowly increases with $T$. Indeed, the other semi-parametric approaches focus on the properties that the series at hands must fulfill to be generated by a fractionally integrated process while the alternative hypothesis is not necessarily specified in a parametric form. In other words, a rejection of the null hypothesis of fractional integration is not informative on the properties of the data generating process (DGP henceforth) under the alternative. This generally leads to lower empirical powers than those one obtained under a fully specified alternative, and it is particularly true when the sample size is relatively small.

Figure 1 reports two examples of a stochastic process contaminated by level shifts with $\delta_t \sim \text{N}(0, 5)$ and $\gamma_t \sim Bern(6.1/T)$. In Panel a) the process $x_t$ is i.i.d. standard Gaussian, while in Panel b) $x_t$ is more persistent and evolves as an ARFIMA(1,0.2,0) with $\phi = 0.5$. In both cases, the figure plots the real-time filter shift probability, $\pi^{(2)}_{t|t}$, as defined in (14). It clearly emerges that the shift probabilities, as implied by the modified Kalman filter routine outlined in Section 2.3, generally display spikes in correspondence of the break dates, while on average the shift probability is close to zero in the rest of the sample. This means that the Kalman filter methodology could be further exploited to carry out inference on the break dates. However, there are few cases in which $\pi^{(2)}_{t|t}$ assigns a large break probability in absence of shits, for example around $t = 550$ in Panel a) and $t = 150$ in Panel b). This generally happens when the observed value of $y_t$ lies away from the local mean, which may indicate spurious evidence of shifts. Since the process for $x_t$ in Panel b) is more persistent than the one displayed in Panel a), we observe a larger number of spurious spikes in this case. This somehow limits the straightforward applicability of the Kalman filter to estimate the break dates together with the parameter estimates. Alternatively we can use Bayesian methods to estimate $\mu_t$ as in Groen et al. (2013) and Giordani et al. (2007), but this estimation approach is beyond the scope of the present article. Overall, we can conclude that the large powers reported in Table 2 for the $SSP_k$ test are a consequence of the ability of the modified Kalman filter to account for the probability of shifts and to associate large shift probabilities to the break dates in most cases.




(a) $x_t \sim$ i.i.d. N(0,1)




(b) $x_t \sim$ ARFIMA(1,0.2,0) with $\phi = 0.5$

***Figure 1:*** *Observed series and real-time filter shift probability. The blue (dashed) line is the observed series, $y_t = x_t + \mu_t$ , the black (straight) line is the random level shift process, $\mu_t$, and the red (dotted) line is the real-time filter shift probability, $\pi_{t|t}^{(2)}$, as defined in* (14). *Panel a) reports the case in which $x_t$ is i.i.d. standard Gaussian. Panel b) reports the case in which $x_t$ follows an ARFIMA(1,0.2,0) with $\phi = 0.5$.*

## 3.3 Robustness

The estimation/testing methodology proposed in this paper is based on a fully parametric specification of the dynamics of the observed variable as the sum of two components, one long memory one and another characterized by random level shifts. It is therefore important to assess the robustness of the proposed testing methodology to possible misspecifications. We have already seen that the misspecification of the short-run components of the ARFIMA can be successfully controlled by adopting a selection method based on the Bayesian information criterion. What about the misspecification of the dynamics of the random level shift component? In order to assess the robustness of the $\mathrm{SSF}_k$ test to different trend processes, the finite sample properties of the $\mathrm{SSF}_k$ test are also investigated for other types of trends that can characterize the observed series. In other words, the estimation/testing procedure outlined in Section 2.3 is carried out under the following DGPs for $\mu_t$

1. **Stationary random level shifts**: $\mu_t = (1-\gamma_t)\mu_{t-1} + \gamma_t\delta_t$, with $\delta_t \sim \mathrm{N}(0,1)$, $\gamma_t \sim Bern(0.003)$;

2. **Monotonic trend**: $\mu_t = 3t^{-0.1}$;

3. **Non-monotonic trend**: $\mu_t = \sin(4\pi t/T)$;

The good performance of the $\mathrm{SSF}_k$ test is confirmed also when an ARFIMA process is contaminated by a stationary random level shift process, see Table 5 in Appendix B. The power of the $\mathrm{SSF}_k$ test in detecting the presence of the shift process is the highest in almost all cases considered. This is again due to our approachs ability to provide accurate parameter estimates in all cases. Indeed, the estimates of the long memory parameter $d$ are generally centered around the true value also when the $\mu_t$ is misspecified. Also in this case, we note a low empirical power when $x_t$ follows an ARFIMA(1,d,0) with $\phi = 0.8$ as a consequence of the low VR, which is below 10% in all cases and makes very difficult to identify the source of variation generated by the stationary random level shift process. However, the empirical power of the $\mathrm{SSF}_k$ is comparable to that of the semi-parametric alternatives in this case.
Looking at the cases in which $\mu_t$ follows monotonic or non-monotonic trends, we note that the $\mathrm{SSF}_k$ test performs surprisingly well when non-stochastic trends are present in the data, see Tables 6 and 7. Although the model specification is not designed to account for those DGPs, the modified Kalman filter method provides a good tracking of the deterministic trends when they are present in the data. Thus, the empirical power of the $\mathrm{SSF}_k$ test is very high and close to 1 in many cases, while it drops only when a highly persistent ARFIMA process is present in the data. Relatively to the other semi-parametric tests, the power of the $\mathrm{SSF}_k$ test is extremely high for the monotonic trend. For the non-monotonic trend, we observe a good performance of the Qu (2011) test, with the exclusion of the ARFIMA(1,0.2,0) with $\phi = 0.8$. Interestingly, the power of the $SSF_k$ is high even though the VR is relatively low compared to that associated to the random level-shift processes, as shown in the previous tables.

# 4 Empirical application

We now apply the $\mathrm{SSF}_k$ test to a number of financial time series for which evidence of fractional integration has been documented. In particular, we choose daily bipower-variation and stock turnover, which is the trading volume divided by the number of outstanding shares. The sample covers 15

assets traded on NYSE for the period between January 2, 2003 and June 28, 2013, for a total of 2640 observations. As it has been widely shown in the past, the series of realized volatility and bipower-variation are characterized by long-range dependence, or long memory, see Andersen et al. (2001a), Andersen et al. (2001b), Andersen et al. (2003), Andersen et al. (2009) and Martens et al. (2009). Analogously, it has been documented that trading volume also displays the features of a long-range dependent process. For instance, Bollerslev and Jubinski (1999) and Lobato and Velasco (2000) both report strong evidence that volume exhibits long-memory, as measured by significantly positive fractional integration orders. More recently, Rossi and Santucci de Magistris (2013) study the common dynamic dependence between volatility and volume and find evidence of fractional cointegration only for the series belonging to the bank/financial sector, i.e. those that during the financial crises have experienced a large upward level shift. It is therefore of interest to be able to formally test, although in an univariate setup, if volatility and volume are subject to level shifts or if their long-run dependence is more likely generated by a pure fractional process.
The bipower-variation is constructed using log-returns at 1-minute frequencies as

$$BPV_t = \frac{\pi}{2}\left(\frac{M}{M-1}\right)\sum_{j=2}^{M}|r_{t,j-1}|\cdot|r_{t,j}|, \tag{17}$$

where $r_{t,j}$ is the $j$-th log-return on day $t$ and $M = 390$ is the number of intra-daily observations associated to 1-minute frequencies. The $BPV_t$ estimator converges to the daily integrated variance, i.e. the instantaneous variance cumulated over daily horizons, and it is robust to price jumps. For what concerns the daily turnover, they are defined as

$$TRV_t = \frac{V_t}{S_t}, \tag{18}$$

where $V_t$ is the trading volume, i.e. number of shares that have been bought and sold within day $t$ and $S_t$ is the number of shares available for sale by the general trading public at time $t$. The turnover is by construction more robust than trading volume to effects like stock splits and it does not display large upward trends as $V_t$. The empirical analysis is carried out on the log-transformed series, $\log(BPV_t)$ and $\log(TRV_t)$ as the model (1) involves unobserved components that are defined on the entire set of real numbers. Moreover, although the distributions of bipower-variation and turnover are clearly right-skewed, the distribution of their logarithms is approximately Gaussian.

Tables 3 and 4 report the values of the tests for the presence of fractional integration in $\log(BPV_t)$ and $\log(TRV_t)$. For what concerns $\log(BPV_t)$, the $\text{SSF}_k$ test rejects the null hypothesis of absence of shifts for 8 out of 15 stocks at 5% significance level. Interestingly, the highest values of the tests are associated with the companies operating in the financial sector, like Bank of America (BAC), Citygroup (C), JP-Morgan (JPM) and Wells Fargo (WFC). These companies have been subject to a major financial distress during the 2008-2009 financial crisis, and the values of $BPV_t$ have been extremely high for many months in this period. The test of Perron and Qu (2010) also seems to find significant evidence of shifts for three out of four volatility series of the stocks in the bank sector. However, the tests based on semi-parametric specifications are unable to reject the null hypothesis of fractional integration in most cases. This may be a consequence of their rather low power, as it emerged in the Monte Carlo study. Indeed, the local Whittle estimates of $d$ generally lie above the stationary threshold, i.e. $\hat{d} > 0.5$. On the other hand, the estimates obtained with the state-space methodology are still positive but significantly smaller than 0.5 (with the exception of PG), meaning that a large portion of the observed long-run dependence is attributed to the random level shifts (or possibly other trend components).

| | $SSF_k$ | $QU_{2\%}$ | $QU_{5\%}$ | ORT | PQ | $SH_k$ | $SH_p$ | $SH_s$ | $\hat{d}_w$ | $\hat{d}_{SSF}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| BA | 0.638* | 0.868 | 0.558 | 3.035 | -0.298 | -1.606 | 0.118 | 2.881 | 0.652 | 0.420 |
| BAC | 1.735* | 0.653 | 0.569 | 7.415 | 2.493* | -0.802 | 0.226 | 3.169 | 0.711 | 0.412 |
| C | 2.710* | 0.421 | 0.685 | 7.224 | 2.589* | -0.948 | 0.265 | 2.064 | 0.702 | 0.395 |
| CAT | 0.318 | 1.080 | 0.721 | 1.286 | 0.191 | -1.656 | 0.114 | 7.307 | 0.707 | 0.477 |
| FDX | 0.732* | 0.601 | 0.539 | 1.331 | 1.177 | -1.166 | 0.195 | 4.251 | 0.617 | 0.373 |
| HON | 0.355 | 0.771 | 0.460 | 1.099 | -0.360 | -1.195 | 0.097 | 9.070* | 0.645 | 0.404 |
| HPQ | 0.568* | 0.425 | 0.540 | 0.681 | -0.455 | -2.203 | 0.078 | 3.474 | 0.668 | 0.339 |
| IBM | 0.283 | 0.488 | 0.845 | 2.184 | 0.454 | -2.334 | 0.058 | 6.114 | 0.705 | 0.447 |
| JPM | 1.419* | 0.517 | 0.633 | 7.361 | 1.810 | -1.607 | 0.212 | 3.845 | 0.716 | 0.412 |
| PEP | 0.426 | 0.365 | 0.468 | 4.374 | -0.270 | -1.664 | 0.084 | 6.159 | 0.699 | 0.436 |
| PG | 0.166 | 0.565 | 0.670 | 4.548 | 0.749 | -2.098 | 0.058 | 8.335* | 0.673 | 0.499 |
| T | 0.313 | 0.449 | 0.670 | 2.390 | -0.410 | -0.931 | 0.087 | 4.326 | 0.680 | 0.429 |
| TWX | 0.584* | 0.449 | 0.553 | 1.539 | -0.079 | -1.508 | 0.080 | 9.324* | 0.702 | 0.450 |
| TXN | 0.517* | 0.755 | 0.314 | 1.072 | -0.492 | -1.382 | 0.123 | 5.775 | 0.695 | 0.442 |
| WFC | 2.492* | 0.594 | 0.368 | 4.773 | 2.029* | -1.110 | 0.234 | 2.174 | 0.740 | 0.371 |

***Table 3:*** *Empirical application. The table reports the values of several test statistics for the bipower-variations series of 15 assets traded on NYSE. The asterisk denotes rejection of the null at 5% significance level.* $SSF_k$ *is the KPSS test based on the state-space representation. QU denotes the Qu (2011) test based on the local Whittle likelihood, with two different trimming choices (*$\epsilon = 2\%$ *and* $\epsilon = 5\%$*). ORT is the temporal aggregation test of Ohanissian et al. (2008). PQ is the test of Perron and Qu (2010).* $SH_k$ *denotes the KPSS test of Shimotsu (2006) based on* $d^{th}$*-differencing and* $SH_p$ *is its PhillipsPerron version.* $SH_s$ *is the Shimotsu (2006) test based on sample splitting with 4 sub-samples.* $\hat{d}_w$ *and* $\hat{d}_{SSF}$ *are the estimates of the fractional parameter obtained with the local Whittle estimator and the state-space method respectively.*

For what concerns $\log(TRV_t)$, the $SSF_k$ rejects the null hypothesis of absence of shifts for 13 out of 15 stocks at 5% significance level, and the estimated fractional parameter is significantly larger than 0 in all cases. Interestingly, there is an almost unanimous agreement across all tests that the turnover series of BA, HPQ, JPM and PEP present spurious long-memory features, while the assumption of truly long-memory for the $\log(TRV_t)$ of PG is only rejected by the $SH_s$ test. Again, the highest values of the $SSF_k$ test are associated with BAC, C, JPM and WFC. This seems to provide some evidence in favour of the long run relationship between volatility and volume as possibly driven by the joint presence of shifts and not only by a common fractional trend. Indeed, both $\log(BPV_t)$ and $\log(TRV_t)$ may be generated by the combination of a fractional process and a shift (or a potentially non-linear and smooth trend). Thus, the current univariate setup could be extended to the multivariate case and, in particular, to the possibility of jointly modeling common fractional trends and common shifts. This extension, coupled with the definition of an efficient method to track the shifting process given the parameter estimates, is left to future research.

Concluding, Figure 2 plots the observed series of $log(BPV_t)$ of BAC and estimated shift component, $\hat{\mu}_t$, obtained given the $SSF_k$ estimates. The estimated shift process seems to follow the

| | $SSF_k$ | $QU_{2\%}$ | $QU_{5\%}$ | ORT | PQ | $SH_k$ | $SH_p$ | $SH_s$ | $\hat{d}_w$ | $\hat{d}_{SSF}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| BA | 0.767* | 2.382* | 1.926* | 20.43* | 1.859 | -0.356 | 0.794* | 0.753 | 0.394 | 0.275 |
| BAC | 9.357* | 0.918 | 0.749 | 5.359 | 1.673 | -0.066 | 0.563* | 5.351 | 0.831 | 0.262 |
| C | 5.156* | 1.699* | 1.248* | 6.271 | 1.268 | -0.444 | 0.291 | 19.14* | 0.630 | 0.304 |
| CAT | 0.272 | 1.581* | 1.283* | 3.594 | 1.717 | -0.635 | 0.477* | 12.55* | 0.446 | 0.385 |
| FDX | 1.255* | 1.753* | 1.293* | 0.066 | 0.561 | -0.507 | 0.665* | 3.411 | 0.357 | 0.246 |
| HON | 0.523* | 1.025 | 0.641 | 9.005* | 1.879 | -0.898 | 0.362 | 13.30* | 0.410 | 0.346 |
| HPQ | 6.707* | 2.283* | 1.884* | 4.282 | 2.087* | 1.060 | 1.779* | 15.43* | 0.425 | 0.213 |
| IBM | 0.987* | 1.015 | 0.588 | 3.514 | 2.084* | -1.024 | 0.261 | 2.333 | 0.431 | 0.286 |
| JPM | 7.063* | 1.857* | 1.450* | 10.81* | 2.753* | -0.682 | 0.516* | 7.295 | 0.546 | 0.261 |
| PEP | 0.735* | 1.892* | 1.201* | 3.507 | 2.706* | -0.345 | 0.803* | 0.918 | 0.427 | 0.357 |
| PG | 0.393 | 0.871 | 0.703 | 1.598 | 0.714 | -0.736 | 0.372 | 20.84* | 0.444 | 0.375 |
| T | 0.512* | 0.922 | 0.499 | 2.873 | 1.641 | -0.905 | 0.346 | 6.204 | 0.374 | 0.319 |
| TWX | 0.510* | 1.164 | 1.095 | 2.954 | 1.794 | -0.390 | 0.564* | 21.27* | 0.397 | 0.335 |
| TXN | 0.609* | 1.392* | 0.768 | 1.896 | 1.439 | -0.525 | 0.528* | 5.053 | 0.454 | 0.331 |
| WFC | 2.890* | 1.061 | 0.497 | 5.978 | 2.064* | -0.641 | 0.410 | 11.52* | 0.723 | 0.311 |

***Table 4:*** *Empirical application. The table reports the values of several test statistics for the daily turnover series of 15 assets traded on NYSE. The asterisk denotes rejection of the null at 5% significance level. $SSF_k$ is the KPSS test based on the state-space representation. QU denotes the Qu (2011) test based on the local Whittle likelihood, with two different trimming choices ($\epsilon = 2\%$ and $\epsilon = 5\%$). ORT is the temporal aggregation test of Ohanissian et al. (2008). PQ is the test of Perron and Qu (2010). $SH_k$ denotes the KPSS test of Shimotsu (2006) based on $d^{th}$-differencing and $SH_p$ is its PhillipsPerron version. $SH_s$ is the Shimotsu (2006) test based on sample splitting with 4 sub-samples. $\hat{d}_w$ and $\hat{d}_{SSF}$ are the estimates of the fractional parameter obtained with the local Whittle estimator and the state-space method respectively.*

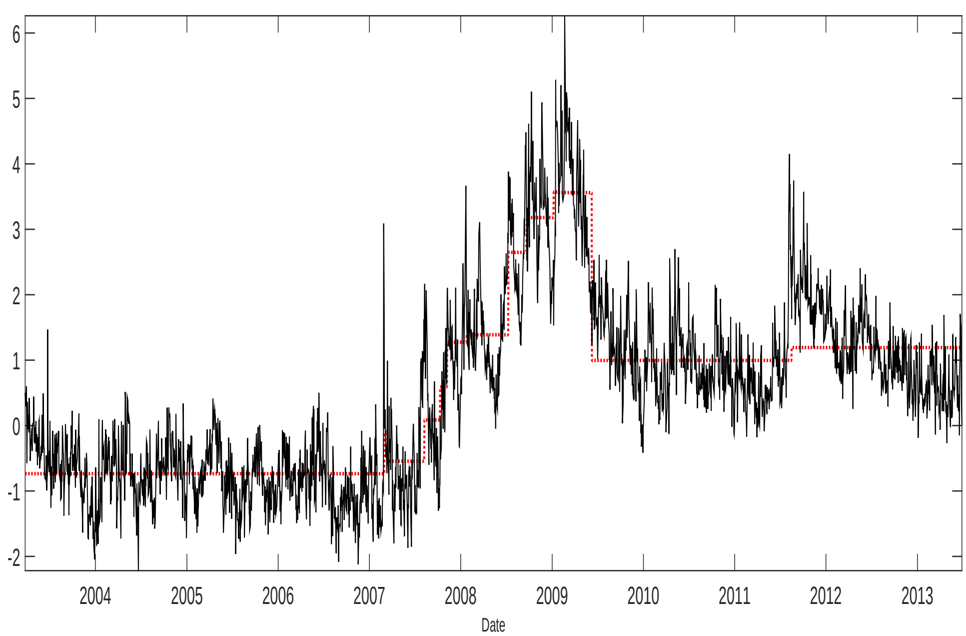



***Figure 2:*** *Observed series of $log(BPV_t)$ of BAC and estimated shift component. The black-solid line is the observed series of $log(BPV_t)$ of BAC, while the red-dotted line is the estimated shift component based on the $SSF_k$ estimates with a realized size that is at least equal to the 95%-th quantile of a Gaussian distribution with variance $\hat{\delta}$.*

largest breaks in the series, which is characterized by a sequence of large increases starting in the summer of 2007, which is the beginning of the 2007-2009 recession period according to NBER. The volatility series reaches the highest levels in late 2008, which is the peak of the subprime financial crisis, while it drops quickly after mid 2009 to a long-run value that is by far larger than the pre-crisis long-run value. Interestingly, the detrended series $\log \tilde{BPV}_t = \log BPV_t - \hat{\mu}_t$, still displays evidence of being a fractional process, with an associated estimate of the fractional parameter $\hat{d}$ equal to 0.44, a value that is extremely close to that reported in Table 3. This evidence not only confirms the ability of the modified Kalman filter to disentangle shifts from the ARFIMA component thus providing unbiased estimates by a straightforward optimization of the log-likelihood function of model 7, but it also provides support to tracking methodology adopted for the shifts process.

# 5 Conclusion

In this paper, we have proposed a robust testing strategy for a fractional process potentially subject to structural breaks. Contrary to the other tests for true fractional integration presented so far in the literature, the focus of our approach is on the level shift process. We propose a flexible state-space parametrization that is able to account for the joint presence of an ARFIMA and a level-shift term. In particular, our parametric approach provides unbiased estimates of the memory parameter for a given time series possibly subject to level shifts or other smoothly varying trends. The testing procedure can be seen as a robust version of the KPSS test for the presence of level shifts. A Monte Carlo study shows that the proposed method performs much better than the other existing tests, especially under the alternative. Interestingly, the modified Kalman filter routine adopted to estimate the model parameters is robust to a variety of different contamination processes and it is reliable also when slowly varying trends characterize the data. From the empirical analysis on a set of US stocks, it emerges that volatility and trading volume are likely to be characterized by the combined presence of both long-memory and level shifts. This result differs from that of other existing tests, which usually over-estimate the long memory parameter and are characterized by low power. The theoretical and empirical results outlined in this paper call for extensions in several directions. For example, the tracking of the shift process, possibly using a smoothing algorithm, would be very informative on the type of trend that characterizes the data and could be exploited for forecasting purposes. Alternatively, a multivariate extension of model (1) would allow to distinguish and test the hypothesis of fractional cointegration in a context characterized by common and idiosyncratic level shifts.

# References

Andersen, T. G., Bollerslev, T., and Diebold, F. X. (2009). Parametric and nonparametric volatility measurement. In Ait-Sahalia, Y. and Hansen, L. P., editors, *Handbook of Financial Econometrics*, volume 1, chapter 2. Elsevier/North-Holland.

Andersen, T. G., Bollerslev, T., Diebold, F. X., and Ebens, H. (2001a). The distribution of stock return volatility. *Journal of Financial Economics*, 61:43–76.

Andersen, T. G., Bollerslev, T., Diebold, F. X., and Labys, P. (2001b). The distribution of exchange rate volatility. *Journal of the American Statistical Association*, 96:42–55.

Andersen, T. G., Bollerslev, T., Diebold, F. X., and Labys, P. (2003). Modeling and forecasting realized volatility. *Econometrica*, 71:579–625.

Baillie, R. T., Chung, C. F., and Tieslau, M. A. (1996). Analysing inflation by the fractionally integrated ARFIMA-GARCH model. *Journal of Applied Econometrics*, 11:23–40.

Beran, J., Bhansali, R., and Ocker, D. (1998). On unified model selection for stationary and nonstationary short- and long-memory autoregressive processes. *Biometrika*, 85:921–934.

Berenguer-Rico, V. and Gonzalo, J. (2014). Summability of stochastic processes: A generalization of integration for non-linear processes. *Journal of Econometrics*, 178:331–341.

Bollerslev, T. and Jubinski, D. (1999). Equity trading volume and volatility: Latent information arrivals and common long-run dependencies. *Journal of Business & Economic Statistics*, 17:9–21.

Chan, N. and Palma, W. (1998). State space modeling of long-memory processes. *Annals of Statistics*, 26:719–740.

Dahlhaus, R. (1989). Efficient parameter estimation for self-similar processes. *Annals of Statistics*, 17:1749–1766.

Diebold, F. X., Husted, S., and Rush, M. (1991). Real exchange rates under the gold standard. *Journal of Political Economy*, 99:1252–1271.

Diebold, F. X. and Inoue, A. (2001). Long memory and regime switching. *Journal of Econometrics*, 105:131–159.

Dolado, J., Gonzalo, J., and Mayoral, L. (2008). Wald tests of i(1) against i(d) alternatives: Some new properties and an extension to processes with trending components. *Studies in Nonlinear Dynamics & Econometrics*, 12:1562–1562.

Fox, R. and Taqqu, M. S. (1986). Large-sample properties of parameter estimates for strongly dependent stationary gaussian series. *Annals of Statistics*, 14:517–532.

Giordani, P., Kohn, R., and van Dijk, D. (2007). A unified approach to nonlinearity, structural change, and outliers. *Journal of Econometrics*, 137:112–133.

Granger, C. W. J. (1980). Long memory relationships and the aggregation of dynamic models. *Journal of Econometrics*, 14:227–238.

Granger, C. W. J. and Hyung, N. (2004). Occasional structural breaks and long memory with application to the S&P 500 absolute stock returns. *Journal of Empirical Finance*, 11:399–421.

Granger, C. W. J. and Joyeux, R. (1980). An introduction to long-memory time series models and fractional differencing. *Journal of Time Series Analysis*, 4:221–238.

Grassi, S. and Santucci de Magistris, P. (2014). When long memory meets the kalman filter: A comparative study. *Computational Statistics and Data Analysis*, 76:301–319.

Groen, J. J. J., Paap, R., and Ravazzolo, F. (2013). Real-time inflation forecasting in a changing world. *Journal of Business & Economic Statistics*, 31:29–44.

Haldrup, N. and Kruse, R. (2014). Discriminating between fractional integration and spurious long memory. CREATES Research Papers 2014-19, School of Economics and Management, University of Aarhus.

Harvey, A. (1989). *Forecasting, Structural Time Series and the Kalman Filter*. Cambridge University Press, Cambridge, UK.

Harvey, A. and Proietti, T. (2005). *Readings in Unobserved Components Models*. OUP Catalogue. Oxford University Press.

Hosking, J. (1981). Fractional differencing. *Biometrika*, 68:165–76.

Hurst, H. (1951). Long term storage capacity of reservoirs. *Transactions of The American Society of Civil Engineers*, 1:519–543.

Kim, C. J. (1994). Dynamic linear models with Markov-switching. *Journal of Econometrics*, 60:1–22.

Kim, C. J. and Nelson, C. R. (1999). *State-Space Models with Regime Switching: Classical and Gibbs-Sampling Approaches with Applications*, volume 1 of *MIT Press Books*. The MIT Press.

Leccadito, A., Rachedi, O., and Urga, G. (2015). True versus spurious long memory: Some theoretical results and a monte carlo comparison. *Econometric Reviews*, 34:452–479.

Leybourne, S. J. and McCabe, B. P. M. (1994). A consistent test for a unit root. *Journal of Business and Economic Statistics*, 12:157–166.

Lobato, I. N. and Velasco, C. (2000). Long memory in stock-market trading volume. *Journal of Business & Economic Statistics*, 18:410–427.

Mandelbrot, B. (1975). Limit theorems of the self-normalized range for weakly and strongly dependent processes. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 31:271–285.

Martens, M., Van Dijk, D., and De Pooter, M. (2009). Forecasting s&p 500 volatility: Long memory, level shifts, leverage effects, day-of-the-week seasonality, and macroeconomic announcements. *International Journal of Forecasting*, 25:282–303.

Mikosch, T. and Starica, C. (2004). Nonstationarities in financial time series, the long-range dependence, and the igarch effects. *The Review of Economics and Statistics*, 86:378–390.

Nyblom, J. and Makelainen, T. (1983). Comparisons of tests for the presence of random walk coefficients in a simple linear model. *Journal of the American Statistical Association*, 78:865–864.

Ohanissian, A., Russell, J. R., and Tsay, R. S. (2008). True or spurious long memory? a new test. *Journal of Business & Economic Statistics*, 26:161–175.

Perron, P. and Qu, Z. (2010). Long-memory and level shifts in the volatility of stock market return indices. *Journal of Business & Economic Statistics*, 28:275–290.

Qu, Z. (2011). A test against spurious long memory. *Journal of Business & Economic Statistics*, 29:423–438.

Robinson, P. M. (1995). Log-periodogram regression of time series with long range dependence. *The Annals of Statistics*, 23:1048–1072.

Rossi, E. and Santucci de Magistris, P. (2013). Long memory and tail dependence in trading volume and volatility. *Journal of Empirical Finance*, 22:94–112.

Shimotsu, K. (2006). Simple (but effective) tests of long memory versus structural breaks. Technical report, Working paper, Queen's University.

Sowell, F. (1992). Maximum likelihood estimation of stationary univariate fractionally integrated time series models. *Journal of Econometrics*, 53:165–188.

Xiaofeng, S. and Xianyang, Z. (2010). Testing for change points in time series. *Journal of the American Statistical Association*, 105:1228–1240.

# A Appendix

## A.1 Summability order of the level shift component $\mu_t$

In order to study the order of integration of the non-linear process $\mu_t$ defined in (2), we use the concept of *summability* as introduced by Berenguer-Rico and Gonzalo (2014).

**Definition 1.** *A stochastic process $\{\mu_t : t \in \mathbb{N}\}$ is said to be summable of order $\beta$, or $S(\beta)$, if there exists a slowly varying function $\mathcal{L}(T)$ and a deterministic sequence $m_t$, such that*

$$S_T = \frac{1}{T^{1/2+\beta}}\mathcal{L}(T)\sum_{t=1}^{T}(\mu_t - m_t) = O_p(1)$$

*where $\beta$ is the minimum real number that makes $S_T$ bounded in probability. Under mild regularity conditions, it holds that if a process $\mu_t$ is $I(\beta)$ with $d \geq 0$, then it is $S(\beta)$.*

The summability condition of the process $\mu_t$ defined in (2) can be obtained by looking at $\mu_t$ as a random walk process (analogously to Examples 3 and 5 in Berenguer-Rico and Gonzalo, 2014), where the innovation is $z_t = \gamma_t \delta_t$, where $\gamma_t \sim Bern(\pi)$ and $\delta_t \sim \mathrm{N}(0, \sigma_\delta^2)$, so that

$$\mathrm{E}(z_t) = E(\gamma_t) \cdot E(\delta_t) = \pi \cdot 0 = 0, \tag{19}$$

$$\begin{aligned} \mathrm{Var}(z_t) &= \mathrm{Var}(\gamma_t) \cdot \mathrm{Var}(\delta_t) + \mathrm{Var}(\delta_t) \cdot \mathrm{E}(\gamma_t)^2 + \mathrm{Var}(\gamma_t) \cdot E(\delta_t)^2 \\ &= p(1-\pi)\sigma_\delta^2 + \pi^2\sigma_\delta^2 + 0 = \pi \cdot \sigma_\delta^2. \end{aligned} \tag{20}$$

Therefore the partial sum

$$\frac{1}{T^{\frac{1}{2}}}\frac{1}{\sigma_\delta\sqrt{\pi}}\sum_{t=1}^{[Tr]} z_t \overset{d}{\to} W(r), \tag{21}$$

so that $z_t \sim S(0)$ and $z_t \sim \mathrm{I}(0)$. Provided that $\mu_t = \sum_{t=1}^{T} z_t$, then

$$\frac{1}{T^{\frac{3}{2}}}\frac{1}{\sigma_\delta\sqrt{\pi}}\sum_{t=1}^{[Tr]} \mu_t \overset{d}{\to} \int_0^r W(r)dr, \tag{22}$$

so that $\mu_t \sim S(1)$ and $\mu_t \sim \mathrm{I}(1)$, but with a slowly varying function equal to $\mathcal{L}(T) = \frac{1}{\sigma_\delta\sqrt{\pi}}$, see Berenguer-Rico and Gonzalo (2014), that does not depend on $T$.

## A.2 Estimation of ARFIMA models by state-space methods

Here we recall how to estimate the ARFIMA model introduced in Section 1 using state-space methods. Following Harvey (1989) and Harvey and Proietti (2005), the *time invariant* state space representation consists of two equations. The first is the *measurement equation*, which relates the univariate time series, $y_t$, to the state vector:

$$y_t = \mathrm{Z}\alpha_t + \varepsilon_t, \qquad t = 1, ..., T, \tag{23}$$

where Z is $1 \times m$ selection vector, $\alpha_t$ is $m \times 1$ state vector with initial values $\alpha_1 \sim \mathrm{N}(\tilde{\alpha}_{1|0}, \mathrm{P}_{1|0})$ and $\varepsilon_t \sim \mathrm{N}(0, \sigma_\varepsilon^2)$ is the measurement error. The second is the *transition equation*, that defines the evolution of the state vector $\alpha_t$ as a first order vector autoregression:

$$\alpha_t = \mathrm{F}\alpha_{t-1} + \mathrm{R}\eta_t, \qquad \eta_t \sim \mathrm{N}(0, \mathrm{Q}), \tag{24}$$

where F is $m \times m$ matrix, R is $m \times g$ selection matrix, and $\eta_t$ is a $g \times 1$ disturbance vector and Q is a $g \times g$ variance-covariance matrix. The two disturbances are assumed to be uncorrelated $\mathrm{E}(\varepsilon_t \eta'_{t-j}) = 0$ for $j = 0, 1, ..., T$.
Let define $\mathrm{Y}_t = \{y_1, \ldots, y_t\}$ as the information set up to time $t$, the state vector $\alpha_t$ and the observations $y_t$, are conditional Gaussian, i.e. $\alpha_t|\mathrm{Y}_{t-1} \sim \mathrm{N}(\tilde{\alpha}_{t|t-1}, \mathrm{P}_{t|t-1})$ and $y_t|\mathrm{Y}_{t-1} \sim \mathrm{N}(\mathrm{Z}\tilde{\alpha}_{t|t-1}, \mathrm{F}_t)$, with mean and variance computed by the Kalman filter (KF) recursions

$$\begin{aligned} v_t &= y_t - \mathrm{Z}\tilde{\alpha}_{t|t-1}, \quad t = 1, \ldots, T, \\ \mathrm{G}_t &= \mathrm{Z}\mathrm{P}_{t|t-1}\mathrm{Z}' + \sigma_\varepsilon^2, \\ \mathrm{K}_t &= \mathrm{F}\mathrm{P}_{t|t-1}\mathrm{Z}'\mathrm{G}_t^{-1}, \\ \tilde{\alpha}_{t+1|t} &= \mathrm{F}\tilde{\alpha}_{t|t-1} + \mathrm{K}_t v_t, \\ \mathrm{P}_{t+1|t} &= \mathrm{F}\mathrm{P}_{t|t-1}\mathrm{F}' - \mathrm{K}_t\mathrm{G}_t\mathrm{K}_t' + \mathrm{R}\mathrm{Q}\mathrm{R}'. \end{aligned} \tag{25}$$

The algorithm is initialized with the unconditional mean $\tilde{\alpha}_{1|0} = 0$ and the unconditional variance $\mathrm{vec}(\mathrm{P}_{1|0}) = (\mathrm{I} - \mathrm{F} \otimes \mathrm{F})^{-1}\mathrm{vec}(\mathrm{RQR}')$. The system matrices are deterministically related with the vector of parameter $\psi$, thus we can construct the log-likelihood function:

$$\ell(\mathrm{Y}_T; \psi) = \sum_{t=1}^{T} \log \Pr(y_t|\mathrm{Y}_{t-1}; \psi) = -\frac{T}{2}\log 2\pi - \frac{1}{2}\sum_{t=1}^{T}\log \mathrm{G}_t - \frac{1}{2}\sum_{t=1}^{T}\frac{v_t^2}{\mathrm{G}_t}. \tag{26}$$

In case we are interested in the "contemporaneous filter" or "real-time estimate" of the state vector, we have that $\alpha_t|\mathrm{Y}_t \sim \mathrm{N}(\tilde{\alpha}_{t|t}, \mathrm{P}_{t|t})$, where

$$\begin{aligned} \tilde{\alpha}_{t|t} &= \tilde{\alpha}_{t|t-1} + \mathrm{P}_{t|t-1}\mathrm{Z}'\mathrm{G}_t^{-1}v_t, \\ \mathrm{P}_{t|t} &= \mathrm{P}_{t|t-1} - \mathrm{P}_{t|t-1}\mathrm{Z}'\mathrm{G}_t^{-1}\mathrm{Z}\mathrm{P}_{t|t-1}. \end{aligned} \tag{27}$$

Looking at equations (25) and (27), we can notice that the filtering (25) can be obtained from (27) as follows

$$\begin{aligned} \tilde{\alpha}_{t+1|t} &= \mathrm{F}\tilde{\alpha}_{t|t}, \\ \mathrm{P}_{t+1|t} &= \mathrm{F}\mathrm{P}_{t|t}\mathrm{F}' + \mathrm{RQR}'. \end{aligned} \tag{28}$$

Equations (27) together with the prediction error $v_t$ and its variance $\mathrm{G}_t$ are known as the "updating step", while the equations (28) are known as the "prediction step". To derive the set of recursions for the model with switching parameters we break down the filtering in those two steps.

The ARFIMA model (3) has the following autoregressive (AR) representation $\varphi(L)x_t = \xi_t$, where

$$\varphi(L) = 1 - \sum_{j=1}^{\infty}\varphi_j L^j = (1-L)^d\frac{\Phi(L)}{\Theta(L)}, \quad (1-L)^d = \sum_{j=0}^{\infty}\frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(-d)}L^j,$$

Its moving average (MA) representation is $x_t = \Psi(L)\xi_t$, where

$$\zeta(L) = 1 + \sum_{j=1}^{\infty} \psi_j L^j = (1-L)^{-d}\frac{\Theta(L)}{\Phi(L)}, \quad (1-L)^{-d} = \sum_{j=0}^{\infty} \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)} L^j.$$

Chan and Palma (1998) show that the exact likelihood function is obtained using the AR (or MA) representation of order $T$. In order to make the state-space methods feasible, they propose to a truncation up to lag $m$. In particular, the truncated AR($m$) representation is

$$\begin{aligned} &\alpha_t = (x_t, \ldots, x_{t-m+1})', \quad \mathrm{Z} = (1, 0, ..., 0), \quad \sigma^2_\varepsilon = 0, \\ &\mathrm{F} = \begin{bmatrix} \varphi_1 \cdots & \varphi_m \\ \mathrm{I}_{m-1} & 0 \end{bmatrix}, \quad \mathrm{R} = (1, 0, ..., 0)', \quad \mathrm{Q} = \sigma^2_\xi. \end{aligned} \tag{29}$$

Similarly, the truncated MA($m$) representation is

$$\begin{aligned} &\alpha_t = (x_t, \tilde{x}_{t|t-1}, \ldots, \tilde{x}_{t+m-1|t-1})', \qquad \mathrm{Z} = (1, 0, ..., 0), \qquad \sigma^2_\varepsilon = 0, \\ &\mathrm{F} = \begin{bmatrix} 0 & \mathrm{I}_m \\ 0 & 0' \end{bmatrix}, \qquad \mathrm{R} = (1, \zeta_1, \ldots, \zeta_m)', \quad \mathrm{Q} = \sigma^2_\xi. \end{aligned} \tag{30}$$

The ML estimator, $\hat{\psi} = \arg\max \ell(\mathrm{Y}_T; \psi)$, based on the truncated representation, as shown to be Consistent, Asymptotically Gaussian and Efficient for $m = T^\beta$ with $\beta \geq 1/2$; see Chan and Palma (1998). Here we adopt the truncated AR($m$) representation with $m = \sqrt{T}$. Note that the initialization of $\mathrm{P}_{1|0}$ requires to invert a $m^2 \times m^2$ matrix, thus to reduce the computational complexity we set $\mathrm{P}_{1|0}$ equal to the Toeplitz matrix of $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{m-1})$, where $\gamma_j$ are the autocovariances of the long memory process.

### A.3 The Kalman filter with level shifts

The standard Kalman filter algorithm can be modified to account for the presence of a level shift process. Following Kim (1994), we show how to obtain the recursive formulas to calculate the transition probabilities and the log-likelihood function for model (7) presented in Section 2.3. The predictive filter transition probability in (11) is obtained as follows

$$\begin{aligned} \pi^{(i,j)}_{t|t-1} &= \Pr(\mathrm{S}_{t-1} = i, \mathrm{S}_t = j | \mathrm{Y}_{t-1}) \\ &= \Pr(\mathrm{S}_t = j | S_{t-1} = i) \Pr(\mathrm{S}_{t-1} = i | \mathrm{Y}_{t-1}) \\ &= \Pr(\mathrm{S}_t = j) \Pr(\mathrm{S}_{t-1} = i | \mathrm{Y}_{t-1}) \\ &= \lambda_j \pi^{(i)}_{t-1|t-1}. \end{aligned}$$

Given our model with two state and one transition probability we can express the four filtering probabilities in compact form

$$\mathrm{vec}(\Pi_{t|t-1}) = \begin{bmatrix} \pi & \pi & 0 & 0 \\ 0 & 0 & \pi & \pi \\ 1-\pi & 1-\pi & 0 & 0 \\ 0 & 0 & 1-\pi & 1-\pi \end{bmatrix} \mathrm{vec}(\Pi_{t-1|t-1}), \tag{31}$$

where $\Pi_t$ is $2 \times 2$ matrix

$$\Pi_t = \begin{bmatrix} \pi_t^{(1,1)} & \pi_t^{(1,2)} \\ \pi_t^{(2,1)} & \pi_t^{(2,2)} \end{bmatrix}. \tag{32}$$

The conditional probability for the observation in expression (12) is obtained as follows

$$\begin{aligned} \Pr(y_t|\mathrm{Y}_{t-1}) &= \textstyle\sum_{i=1}^{2}\sum_{j=1}^{2} \Pr(y_t|\mathrm{S}_{t-1}=i, \mathrm{S}_t=j, \mathrm{Y}_{t-1}) \Pr(\mathrm{S}_{t-1}=i, \mathrm{S}_t=j|\mathrm{Y}_{t-1}) \\ &= \textstyle\sum_{i=1}^{2}\sum_{j=1}^{2} \Pr(y_t^{(i,j)}|\mathrm{Y}_{t-1})\pi_{t|t-1}^{(i,j)} \\ &= \mathrm{vec}(\Omega_t)'\mathrm{vec}(\Pi_{t|t-1}), \end{aligned} \tag{33}$$

where $\Omega_t$ in $2 \times 2$ matrix containing the observations' conditional probabilities

$$\Omega_t = \begin{bmatrix} \omega_t^{(1,1)} & \omega_t^{(1,2)} \\ \omega_t^{(2,1)} & \omega_t^{(2,2)} \end{bmatrix}, \quad \omega_t^{(i,j)} = \Pr(y_t^{(i,j)}|\mathrm{Y}_{t-1}). \tag{34}$$

The expression (14) is obtained as follows:

$$\begin{aligned} \pi_{t|t}^{(i,j)} &= \Pr(\mathrm{S}_{t-1}=i, \mathrm{S}_t=j|\mathrm{Y}_t) \\ &= \Pr(\mathrm{S}_{t-1}=i, \mathrm{S}_t=j|y_t, \mathrm{Y}_{t-1}) \\ &= \frac{\Pr(y_t^{(i,j)}|\mathrm{Y}_{t-1}) \Pr(\mathrm{S}_{t-1}=i,\mathrm{S}_t=j|\mathrm{Y}_{t-1})}{\Pr(y_t|\mathrm{Y}_{t-1})} \\ &= \frac{\Pr(y_t^{(i,j)}|\mathrm{Y}_{t-1})\pi_{t|t-1}^{(i,j)}}{\Pr(y_t|\mathrm{Y}_{t-1})}, \end{aligned}$$

this can be express in compact form

$$\mathrm{vec}(\Pi_{t|t}) = \frac{\mathrm{vec}(\Omega_t) \odot \mathrm{vec}(\Pi_{t|t-1})}{\mathrm{vec}(\Omega_t)'\mathrm{vec}(\Pi_{t|t-1})}, \tag{35}$$

where "$\odot$" is the Hadamard product.

## B Tables

| T | $SSF_k$ | $QU_{2\%}$ | $QU_{5\%}$ | ORT | PQ | $SH_k$ | $SH_p$ | $SH_s$ | $\hat{d}_{SSF}$ | VR | BIC |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | i.i.d. Gaussian Noise: | | | | | | |
| 500 | **0.575** | 0.183 | 0.162 | 0.063 | 0.274 | 0.288 | 0.018 | 0.110 | *0.055* | 0.138 | 0.887 |
| 1000 | **0.767** | 0.392 | 0.353 | 0.156 | 0.532 | 0.525 | 0.007 | 0.168 | *0.035* | 0.225 | 0.913 |
| 2000 | **0.878** | 0.831 | 0.764 | 0.154 | 0.781 | 0.660 | 0.001 | 0.176 | *-0.01* | 0.333 | 0.947 |
| | | | | | ARFIMA(1,0.2,0), with $\phi = 0.5$: | | | | | | |
| 500 | **0.332** | 0.083 | 0.086 | 0.064 | 0.157 | 0.188 | 0.012 | 0.078 | *0.211* | 0.079 | 0.827 |
| 1000 | **0.458** | 0.280 | 0.272 | 0.122 | 0.258 | 0.394 | 0.003 | 0.106 | *0.204* | 0.131 | 0.947 |
| 2000 | 0.649 | **0.675** | 0.653 | 0.144 | 0.465 | 0.639 | 0.002 | 0.167 | *0.209* | 0.200 | 0.963 |
| | | | | | ARFIMA(1,0.2,0), with $\phi = 0.8$: | | | | | | |
| 500 | 0.060 | 0.005 | 0.002 | **0.064** | 0.072 | 0.036 | 0.027 | 0.053 | *0.269* | 0.036 | 0.801 |
| 1000 | 0.067 | 0.008 | 0.012 | **0.090** | 0.083 | 0.082 | 0.010 | 0.055 | *0.220* | 0.059 | 0.945 |
| 2000 | 0.150 | 0.007 | 0.004 | 0.107 | 0.100 | **0.183** | 0.004 | 0.066 | *0.207* | 0.092 | 0.960 |
| | | | | | ARFIMA(0,0.2,1), with $\theta = 0.5$: | | | | | | |
| 500 | **0.212** | 0.086 | 0.088 | 0.059 | 0.154 | 0.177 | 0.016 | 0.073 | *0.190* | 0.100 | 0.963 |
| 1000 | **0.851** | 0.275 | 0.265 | 0.130 | 0.237 | 0.377 | 0.006 | 0.105 | *0.195* | 0.164 | 0.984 |
| 2000 | **0.897** | 0.684 | 0.643 | 0.133 | 0.463 | 0.630 | 0.001 | 0.153 | *0.198* | 0.247 | 0.993 |
| | | | | | ARFIMA(0,0.2,1), with $\theta = 0.8$: | | | | | | |
| 500 | **0.287** | 0.083 | 0.086 | 0.064 | 0.157 | **0.188** | 0.012 | 0.078 | *0.190* | 0.079 | 0.962 |
| 1000 | **0.360** | 0.280 | 0.272 | 0.122 | 0.258 | **0.394** | 0.003 | 0.106 | *0.197* | 0.131 | 0.961 |
| 2000 | **0.581** | **0.675** | 0.653 | 0.144 | 0.465 | 0.639 | 0.002 | 0.167 | *0.199* | 0.200 | 0.963 |

***Table 5:*** *Power - Stationary random level shift model. The table reports the empirical rejection rate of several test statistics when $\mu_t$ is a stationary random level shift process and $x_t$ is generated according to different ARFIMA specifications. Results are based on $M = 1000$ Monte Carlo replications. For each test, the null is true fractional integration (i.e. absence of shifts). $SSF_k$ is the KPSS test based on the state-space representation. QU denotes the Qu (2011) test based on the local Whittle likelihood, with two different trimming choices ($\epsilon = 2\%$ and $\epsilon = 5\%$). ORT is the temporal aggregation test of Ohanissian et al. (2008). PQ is the test of Perron and Qu (2010). $SH_k$ denotes the KPSS test of Shimotsu (2006) based on $d^{th}$-differencing and $SH_p$ is its PhillipsPerron version. $SH_s$ is the Shimotsu (2006) test based on sample splitting with 4 sub-samples. $\hat{d}_{SSF}$ is the Monte Carlo average of the estimates of the d parameter based on the state-space methodology outlined in Section 2. Table also reports the Monte Carlo average of the variance ratio, VR, between the sample variance of $\mu_t$ and to the sample variance of $y_t$. Finally, the last column reports the proportion of the Monte Carlo replications in which the BIC correctly selects the true ARMA lag-order.*

| T | $SSF_k$ | $QU_{2\%}$ | $QU_{5\%}$ | ORT | PQ | $SH_k$ | $SH_p$ | $SH_s$ | $\hat{d}_{SSF}$ | VR | BIC |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | i.i.d. Gaussian Noise: | | | | | | |
| 500 | **0.998** | 0.013 | 0.015 | 0.040 | 0.222 | 0.190 | 0.001 | 0.050 | *0.038* | 0.113 | 0.652 |
| 1000 | **1.000** | 0.010 | 0.013 | 0.058 | 0.498 | 0.210 | 0.001 | 0.032 | *0.027* | 0.116 | 0.681 |
| 2000 | **1.000** | 0.020 | 0.016 | 0.072 | 0.729 | 0.515 | 0.001 | 0.050 | *0.021* | 0.119 | 0.690 |
| | | | | | ARFIMA(1,0.2,0), with $\phi = 0.5$: | | | | | | |
| 500 | **0.531** | 0.006 | 0.012 | 0.069 | 0.101 | 0.089 | 0.006 | 0.067 | *0.321* | 0.060 | 0.536 |
| 1000 | **0.560** | 0.020 | 0.026 | 0.078 | 0.093 | 0.146 | 0.005 | 0.058 | *0.147* | 0.062 | 0.863 |
| 2000 | **0.739** | 0.046 | 0.042 | 0.072 | 0.074 | 0.258 | 0.002 | 0.047 | *.1388* | 0.065 | 0.962 |
| | | | | | ARFIMA(1,0.2,0), with $\phi = 0.8$: | | | | | | |
| 500 | **0.948** | 0.015 | 0.018 | 0.058 | 0.206 | 0.173 | 0.001 | 0.049 | *0.194* | 0.063 | *0.642* |
| 1000 | **0.996** | 0.017 | 0.021 | 0.082 | 0.225 | 0.291 | 0.001 | 0.060 | *0.195* | 0.065 | *0.721* |
| 2000 | **1.000** | 0.045 | 0.036 | 0.065 | 0.271 | 0.449 | 0.001 | 0.055 | *0.199* | 0.068 | *0.978* |
| | | | | | ARFIMA(0,0.2,1), with $\theta = 0.5$: | | | | | | |
| 500 | **0.526** | 0.013 | 0.016 | 0.055 | 0.187 | 0.172 | 0.001 | 0.043 | *0.116* | 0.074 | 0.898 |
| 1000 | **0.561** | 0.013 | 0.015 | 0.090 | 0.197 | 0.257 | 0.001 | 0.049 | *0.148* | 0.075 | 0.901 |
| 2000 | **0.745** | 0.050 | 0.038 | 0.098 | 0.275 | 0.448 | 0.001 | 0.066 | *0.172* | 0.077 | 0.936 |
| | | | | | ARFIMA(0,0.2,1), with $\theta = 0.8$: | | | | | | |
| 500 | **0.372** | 0.015 | 0.018 | 0.058 | 0.206 | 0.173 | 0.001 | 0.049 | *0.160* | 0.063 | 0.920 |
| 1000 | **0.406** | 0.017 | 0.021 | 0.082 | 0.225 | 0.291 | 0.001 | 0.060 | *0.179* | 0.065 | 0.922 |
| 2000 | **0.506** | 0.045 | 0.036 | 0.065 | 0.271 | 0.449 | 0.001 | 0.055 | *0.189* | 0.068 | 0.931 |

***Table 6:*** *Power - Monotonic trend model. The table reports the empirical rejection rate of several test statistics when $\mu_t$ is a monotonic trend and $x_t$ is generated according to different ARFIMA specifications. Results are based on* 1000 *Monte Carlo replications. For each test, the null is true fractional integration (i.e. absence of shifts). $SSF_k$ is the KPSS test based on the state-space representation. QU denotes the Qu (2011) test based on the local Whittle likelihood, with two different trimming choices ($\epsilon = 2\%$ and $\epsilon = 5\%$). ORT is the temporal aggregation test of Ohanissian et al. (2008). PQ is the test of Perron and Qu (2010). $SH_k$ denotes the KPSS test of Shimotsu (2006) based on $d^{th}$-differencing and $SH_p$ is its PhillipsPerron version. $SH_s$ is the Shimotsu (2006) test based on sample splitting with 4 sub-samples. $\hat{d}_{SSF}$ is the Monte Carlo average of the estimates of the d parameter based on the state-space methodology outlined in Section 2. Table also reports the Monte Carlo average of the variance ratio, $VR$, between the sample variance of $\mu_t$ and to the sample variance of $y_t$. Finally, the last column reports the proportion of the Monte Carlo replications in which the BIC correctly selects the true ARMA lag-order.*

| T | $SSF_k$ | $QU_{2\%}$ | $QU_{5\%}$ | ORT | PQ | $SH_k$ | $SH_p$ | $SH_s$ | $\hat{d}_{SSF}$ | VR | BIC |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | i.i.d. Gaussian Noise: | | | | | | |
| 500 | **0.870** | 0.839 | 0.491 | 0.079 | 0.085 | 0.003 | 0.001 | 0.099 | *0.030* | 0.334 | 0.870 |
| 1000 | 0.950 | **0.954** | 0.608 | 0.141 | 0.075 | 0.005 | 0.001 | 0.099 | *0.020* | 0.334 | 0.918 |
| 2000 | 0.973 | **1.000** | 0.722 | 0.162 | 0.095 | 0.015 | 0.001 | 0.124 | *0.032* | 0.335 | 0.941 |
| | | | | | ARFIMA(1,0.2,0), with $\phi = 0.5$: | | | | | | |
| 500 | **0.153** | 0.126 | 0.072 | 0.079 | 0.106 | 0.016 | 0.003 | 0.066 | *0.286* | 0.208 | 0.738 |
| 1000 | **0.543** | 0.286 | 0.162 | 0.109 | 0.096 | 0.030 | 0.001 | 0.081 | *0.231* | 0.203 | 0.887 |
| 2000 | **0.661** | 0.602 | 0.337 | 0.103 | 0.064 | 0.082 | 0.001 | 0.099 | *0.168* | 0.202 | 0.973 |
| | | | | | ARFIMA(1,0.2,0), with $\phi = 0.8$: | | | | | | |
| 500 | **0.024** | 0.008 | 0.008 | 0.062 | 0.110 | 0.022 | 0.016 | 0.066 | *0.343* | 0.083 | 0.772 |
| 1000 | **0.028** | 0.011 | 0.010 | 0.091 | 0.094 | 0.019 | 0.009 | 0.044 | *0.242* | 0.079 | 0.874 |
| 2000 | **0.056** | 0.006 | 0.006 | 0.061 | 0.084 | 0.033 | 0.005 | 0.041 | *0.219* | 0.075 | 0.962 |
| | | | | | ARFIMA(0,0.2,1), with $\theta = 0.5$: | | | | | | |
| 500 | 0.330 | **0.449** | 0.286 | 0.075 | 0.074 | 0.002 | 0.001 | 0.067 | *0.167* | 0.241 | 0.952 |
| 1000 | 0.487 | **0.660** | 0.589 | 0.115 | 0.079 | 0.020 | 0.001 | 0.107 | *0.175* | 0.237 | 0.970 |
| 2000 | 0.833 | **0.962** | 0.921 | 0.131 | 0.040 | 0.186 | 0.001 | 0.133 | *0.176* | 0.237 | 0.971 |
| | | | | | ARFIMA(0,0.2,1), with $\theta = 0.8$: | | | | | | |
| 500 | **0.223** | 0.008 | 0.008 | 0.062 | 0.110 | 0.022 | 0.016 | 0.066 | *0.213* | 0.083 | 0.964 |
| 1000 | **0.369** | 0.011 | 0.010 | 0.091 | 0.094 | 0.019 | 0.009 | 0.044 | *0.191* | 0.079 | 0.972 |
| 2000 | **0.595** | 0.006 | 0.006 | 0.061 | 0.084 | 0.033 | 0.005 | 0.041 | *0.193* | 0.075 | 0.964 |

***Table 7:*** *Power - Nonmonotonic trend model. The table reports the empirical rejection rate of several test statistics when $\mu_t$ is a non-monotonic trend and $x_t$ is generated according to different ARFIMA specifications. Results are based on* 1000 *Monte Carlo replications. For each test, the null is true fractional integration (i.e. absence of shifts). $SSF_k$ is the KPSS test based on the state-space representation. QU denotes the Qu (2011) test based on the local Whittle likelihood, with two different trimming choices ($\epsilon = 2\%$ and $\epsilon = 5\%$). ORT is the temporal aggregation test of Ohanissian et al. (2008). PQ is the test of Perron and Qu (2010). $SH_k$ denotes the KPSS test of Shimotsu (2006) based on $d^{th}$-differencing and $SH_p$ is its PhillipsPerron version. $SH_s$ is the Shimotsu (2006) test based on sample splitting with 4 sub-samples. $\hat{d}_{SSF}$ is the Monte Carlo average of the estimates of the d parameter based on the state-space methodology outlined in Section 2. Table also reports the Monte Carlo average of the variance ratio, $VR$, between the sample variance of $\mu_t$ and to the sample variance of $y_t$. Finally, the last column reports the proportion of the Monte Carlo replications in which the BIC correctly selects the true ARMA lag-order.*