Unstable Diffusion Indexes*

Daniele Massacci

Einaudi Institute for Economics and Finance

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Abstract

This paper presents a comprehensive approach for estimation and inference in diffusion index models subject to structural instability: both the underlying factor model and the resulting factor augmented regression are allowed to experience a structural change with different unknown break dates. As for the factor model, the paper proposes to estimate factors and loadings by principal components, and the unknown break fraction by concentrated least squares: convergence rates for all estimators are obtained and the asymptotic distribution of the estimator for the break date is derived; model selection criteria robust to the unknown break date are also proposed. The properties of the least squares estimator for the diffusion index model are derived and a test to detect structural breaks is introduced. The empirical application shows how the proposed methodology may be used to uncover instabilities in the linkages between bond risk premia and macroeconomic factors.

JEL classification: C22, C55, G12.

Keywords: Diffusion Index Model, Large Factor Model, Structural Break, Least Squares Estimation, Model Selection, Bond Risk Premia.

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1 Introduction

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Factor models are widely used empirical tools to explain the common variations in large scale macroeconomic and financial data. **SENTENCE TO BE IMPROVED + ADD INTUITION BEHIND FACTOR MODELS + ADD EMPIRICAL EXAMPLE FROM INSTABILITY OF STOCK RETURNS**

An extensive literature studies factor models under the maintained assumption of stable factors and loadings over the entire sample period: see Connor and Korajczyk (1986, 1988, 1993), Forni *et al.* (2000, 2004, 2015), Forni and Lippi (2001), Bai and Ng (2002), Stock and Watson (2002), and Bai (2003) for contributions dealing with linear factor models. **THIS PARAGRAPH NEEDS TO BE EXTENDED**

More recently, a number of contributions analyze large dimensional factor models with structural instability in the loadings under the maintained assumption of stable factors over the whole sample period. Breitung and Eickmeier (2011) show that ignoring breaks leads to overestimation of the number of factors and develop statistical tests for the null hypothesis of stability in the factor loadings. Bates *et al.* (2013) study the robustness properties of the least squares estimator as applied to linear factor models under neglected instability. Chen *et al.* (2014), Han and Inoue (2014), and Yamamoto and Tanaka (2015) develop further statistical tools to detect breaks. Chen (2015) considers least squares estimation of the break date. **THIS PARAGRAPH NEEDS TO BE EXTENDED**

Cheng *et al.* (2015). **DISCUSS INTO DETAILS**

DISCUSS WHAT I DO IN THE PAPER AND HOW IT RELATES TO CHENG ET AL. (2015) AND MY WORK WITH THRESHOLD

We start from a factor specification with a break in the factor loadings. Under the maintained assumption that the true number of factors R^0 is known, we propose to estimate the break fraction by concentrated least squares, and factors and loadings by principal components (Bai (1997), and Bai and Ng (2002)). We obtain a number of novel theoretical results. Let N and T denote the cross-sectional and time series dimensions, respectively. We first provide sufficient conditions to ensure that our model is identified from a linear factor representation: formally, for $0.5 < \alpha^0 \leq 1$, we require that at least N^{α^0} cross-sectional units experience a break in the loadings, so that the shift resists to the aggregation induced by the principal components estimator. We then show that the estimator for the break fraction is consistent at a rate equal to $D_{NT}\left(\alpha^0\right) = N^{\left(2\alpha^0-1\right)}T$: this depends on the time series dimension T and on the number of cross-sectional units N^{α^0} subject to structural instability. The convergence rate monotonically increases in α^0 and it is such that $T < D_{NT}\left(\alpha^0\right) \leq NT$: this shows the direct relationship between identification of the model and convergence rate of the estimator for the break fraction. This finding improves over Cheng *et al.* (2015) and Chen (2015): the former do not derive the convergence rate of the estimator for the break fraction; the latter obtains a slower rate. The superconsistency property we show implies that the principal components estimator for factors and loadings has convergence rate equal to $C_{NT} = \min\left\{\sqrt{N}, \sqrt{T}\right\}$: despite the structural instability, the convergence rate C_{NT} is equal to the one derived in Bai and Ng (2002) for linear factor models.

Given the convergence rate $D_{NT}\left(\alpha^0\right)$, we derive the asymptotic distribution of the estimator for the break date. **TO BE COMPLETED**.

Having derived the asymptotic properties of the estimators for factors and loadings when the true number of factors R^0 is known, we next consider the case in which R^0 no longer is known and has to be estimated. Breitung and Eickmeier (2011) show that structural instability in the loadings leads to a factor representation with a higher dimensional factor space. Since the convergence rate C_{NT} of the estimator for factors and loadings is the same as in linear factor models, we make Bai and Ng (2002) information criteria robust to structural instabilities by accounting for the induced higher dimensional factor space representation. **RELATED LITERATURE + COMPARISON WITH CHENG ET AL (2015) TO BE ADDED HERE**.

A widely used tool for empirical work is the diffusion index model of Stock and Watson (1998, 2002), and Bai and Ng (2006). **TO BE COMPLETED**.

As a final methodological contribution, we show how our theoretical results may be extended to the case in which the number of factors (as well as the factor loadings) is subject to structural instability. **TO BE COMPLETED**.

EMPIRICAL APPLICATION TO BE DISCUSSED HERE

The remainder of the paper is organized as follows. Section 2 describes the approximate breakpoint factor model that allows for changes in the factor loadings. Section 3 deals with estimation of factors, loadings and break fraction. Section 4 proposes information criteria to determine the number of factors in the presence of an unknown break fraction. Section 5 shows how our results may be applied to a diffusion index model. Section 6 extends our methodology to an approximate large dimensional factor model with a break in the number of factors. Section 7 performs a comprehensive Monte Carlo analysis. Section 8 applies our methodology to study the linkages between bond risk premia and macroeconomic fundamentals. Finally, Section 9 concludes. Appendix A provides technical proofs.

Concerning notation, $\mathbb{I}(\cdot)$ denotes the indicator function; for a given scalar A, $\mathbf{I}_A$ and $\mathbf{0}_A$ are the $A \times A$ identity matrix and zero matrix, respectively. **TO BE COMPLETED**

2 The Approximate Breakpoint Factor Model

We consider the model

$$\mathbf{x}_t = \mathbb{I}(t/T \leq \pi)\, \mathbf{\Lambda}_1 \mathbf{f}_t + \mathbb{I}(t/T > \pi)\, \mathbf{\Lambda}_2 \mathbf{f}_t + \mathbf{e}_t, \quad t = 1, \ldots, T, \tag{1}$$

where T denotes the time series dimension of the available sample; $\mathbf{x}_t = (x_{1t}, \ldots, x_{Nt})' \in \Re^N$ is the $N \times 1$ vector of observable dependent variables; $\mathbf{f}_t = (f_{1t}, \ldots, f_{Rt})' \in \Re^R$ is the $R \times 1$ vector of latent factors; π is the unknown break fraction; $\mathbf{e}_t = (e_{1t}, \ldots, e_{Nt})' \in \Re^N$ is the $N \times 1$ vector of idiosyncratic errors; $\mathbf{\Lambda}_j = (\boldsymbol{\lambda}_{j1}, \ldots, \boldsymbol{\lambda}_{jN})'$ is the $N \times R$ matrix of factor loadings with $i - th$ row defined as $\boldsymbol{\lambda}_{ji} = (\lambda_{ji1}, \ldots, \lambda_{jiR})'$, for $j = 1, 2$ and $i = 1, \ldots, N$.

The model in (1) extends large dimensional linear factor models to allow for a break in the factor loadings: Cheng *et al.* (2015) refer to this kind of structural instability as to a type-1 change. Given Assumption C3 in Section 3.1 below, we follow Chamberlain and Rothschild (1983) and allow for some degree of correlation in the idiosyncratic components on each side of the breakpoint: (1) then is an *approximate breakpoint factor model*; it is more general than an *exact breakpoint factor model*, which would extend the arbitrage pricing theory of Ross (1976) and would not allow for any correlation in the idiosyncratic components on any side of the breakpoint. The specification in (1) has been studied in Breitung and Eickmeier (2011), Bates *et al.* (2013), Chen *et al.* (2014), Han and Inoue (2014), Yamamoto and Tanaka (2015), Chen (2015) and Cheng *et al.* (2015).

3 Estimation

As in Stock and Watson (2002) and Bates *et al.* (2013), we study estimation of (1) under the assumption that the true number of factors R^0 (i.e., the true dimension of $\mathbf{f}_t$) is known. We extend the theory in Bai and Ng (2002) based on principal components estimation of factor and loadings to allow for concentrated least squares estimation of the break fraction π, as motivated in Bai (1997) and Bai and Perron (1998). The plan is as follows: Section 3.1 states the assumptions; Section 3.2 deals with identification; Section 3.3 describes the estimator; Section 3.4 proves the consistency of the estimator; Section 3.5 derives the convergence rates; and Section 3.6 obtains the asymptotic distribution of the estimator for the break date.

3.1 Assumptions

We group the assumptions into three sets, depending on the role they play to identify and estimate the model, and to derive the convergence rates. Let $\mathbb{I}_{1t}(\pi) = \mathbb{I}(t/T \leq \pi)$ and $\mathbb{I}_{2t}(\pi) = \mathbb{I}(t/T > \pi)$. For $j = 1, 2$, denote $\mathbf{\Lambda}_j^0 = \left(\boldsymbol{\lambda}_{j1}^0, \ldots, \boldsymbol{\lambda}_{jN}^0\right)'$, π^0 and $\mathbf{f}_t^0$ the true values of $\mathbf{\Lambda}_j$, π and $\mathbf{f}_t$, respectively. Define $\mathbf{f}_{jt}^0(\pi) = \mathbb{I}_{jt}(\theta)\,\mathbf{f}_t^0$, for $j, m = 1, 2$ and $t = 1, \ldots, T$, and let $\boldsymbol{\delta}_i^0 = \boldsymbol{\lambda}_{2i}^0 - \boldsymbol{\lambda}_{1i}^0$, for $i = 1, \ldots, N$.

3.1.1 Identification

Assumption I - Breakpoint Factor Model. For $0.5 < \alpha^0 \leq 1$, $\boldsymbol{\delta}_i^0 \neq 0$ for $i = 1, \ldots, N^{\alpha^0}$, and $\sum_{i=N^{\alpha^0}+1}^{N} \boldsymbol{\delta}_i^0 = O(1)$.

Assumption I requires that *at least* N^{α^0} of the N series are subject to instability, for $0.5 < \alpha^0 \leq 1$: this follows up on Bates *et al.* (2013), who show that if at most $N^{0.5}$ series undergo a break then the principal components estimator as applied to the misspecified linear model achieves the Bai and Ng (2002) convergence rate. Assumption I ensures that enough series experience a structural break, so that the instability resists to the aggregation induced by the principal components estimator and (1) is identified from a linear factor model. In the threshold factor model, Massacci (2015) shows that the value of α^0 affects the convergerce rate of the estimator for the threshold parameter in large dimensional factor models: the higher the former, the faster the latter. As shown in Theorem 3.4 below, an analogous result holds for the estimator of the break fraction π: the higher α^0, the faster the convergence rate of the estimator for π. In this paper we do not attempt to estimate α^0: we leave this important and interesting issue to future research.

3.1.2 Consistency

Assumption C1 - Factors. $\mathrm{E}\left\|\mathbf{f}_t^0\right\|^4 < \infty$; for $j = 1,2$, $T^{-1}\sum_{t=1}^{T}\mathbf{f}_{jt}^0(\pi)\,\mathbf{f}_{jt}^0(\pi^0)' \xrightarrow{p} \mathbf{\Sigma}_{j\mathbf{f}}^0(\pi,\pi^0)$ as $T\to\infty$ for all π and some positive definite matrix $\mathbf{\Sigma}_{j\mathbf{f}}^0(\pi,\pi^0)$ such that $\mathbf{\Sigma}_{j\mathbf{f}}^0(\pi^0,\pi^0) - \mathbf{\Sigma}_{j\mathbf{f}}^0(\pi,\pi^0)$ is positive definite for all $\pi \neq \pi^0$.

Assumption C2 - Factor Loadings. For $j = 1,2$ and $i = 1,\ldots,N$, $\left\|\boldsymbol{\lambda}_{ji}^0\right\| \leq \bar{\lambda} < \infty$, and $\left\|\mathbf{\Lambda}_j^{0\prime}\mathbf{\Lambda}_j^0/N - \mathbf{D}_{\mathbf{\Lambda}_j}^0\right\| \to 0$ as $N\to\infty$ for some $R^0\times R^0$ positive definite matrix $\mathbf{D}_{\mathbf{\Lambda}_j}^0$.

Assumption C3 - Time and Cross-Section Dependence and Heteroscedasticity. There exists a positive $M<\infty$ such that for $j=1,2$, for all π and for all (N,T),

(a) $\mathrm{E}(e_{it}) = 0$ and $\mathrm{E}\left|e_{it}\right|^8 \leq M$;

(b) $\mathrm{E}\left[\mathbb{I}_{jt}(\pi)\,\mathbb{I}_{jv}(\pi)\,e_{it}e_{iv}\right] = \tau_{jitv}(\pi)$ with $\left|\tau_{jitv}(\pi)\right| \leq \left|\tau_{jtv}\right|$ for some τ_{jtv} and for all i, and $T^{-1}\sum_{t=1}^{T}\sum_{v=1}^{T}\left|\tau_{jtv}\right| \leq M$;

(c) $\mathrm{E}\left[T^{-1}\sum_{t=1}^{T}\mathbb{I}_{jt}(\pi)\,e_{it}e_{lt}\right] = \sigma_{jil}(\pi)$, $\left|\sigma_{jll}(\pi)\right| \leq M$ for all l, and $N^{-1}\sum_{i=1}^{N}\sum_{l=1}^{N}\left|\sigma_{jil}(\pi)\right| \leq M$;

(d) $\mathrm{E}\left|T^{-1/2}\sum_{t=1}^{T}\mathbb{I}_{jt}(\pi)\,e_{it}e_{lt} - \mathrm{E}\left[\mathbb{I}_{jt}(\pi)\,e_{it}e_{lt}\right]\right|^4 \leq M$ for every (i,l).

Assumption C4 - Weak Dependence between $\mathbf{f}_t^0$ and e_{it}. There exists some positive constant $M<\infty$ such that for all π and for all (N,T),

$$\mathrm{E}\left\{N^{-1}\sum_{i=1}^{N}\left\|T^{-1/2}\left[\sum_{t=1}^{T}\mathbb{I}_{jt}(\pi)\,\mathbf{f}_t^0 e_{it}\right]\right\|^2\right\} \leq M, \quad j=1,2.$$

Assumptions C1 to C4 are the natural extensions of Assumptions A to D imposed on linear factor models in Bai and Ng (2002), and accommodate the presence of the breakpoint. Assumption C1 restricts the sequence $\left\{\mathbf{f}_t^0\right\}_{t=1}^{T}$ so that appropriate second moments exist; it also imposes full rank conditions that exclude multicollinearity in the factors. According to Assumption C2, factor loadings are nonstochastic and each factor has a nonnegligible effect on the variance of $\mathbf{x}_t$ on each side of the breakpoint. Under Assumption C3, limited degrees of time-series and cross-section dependence in the idiosyncratic components as well as heteroscedasticity are allowed. Finally, Assumption C4 provides an upper bound to the degree of dependence between the factors and the idiosyncratic components.

3.1.3 Convergence Rates

Assumption CR - Mixing Condition and Moment Bounds: For $i = 1, \ldots, N$, and $t = 1, \ldots, T$,

(a) $\{\mathbf{f}_t^0, \mathbf{e}_t\}_{t=1}^T$ is $\rho-$mixing, with $\rho-$mixing coefficients satisfying $\sum_{m=1}^{\infty} \rho_m^{1/2} < \infty$;

(b) $\mathrm{E}\left(\left\|\mathbf{f}_t^0 e_{it}\right\|^4\right) \leq C$ and $\mathrm{E}\left(\left\|\mathbf{f}_t^0\right\|^4\right) \leq C$ for some $C < \infty$.

Assumption CR is analogous to Assumption 1 in Hansen (2000): Assumption CR(a) allows for nonstationarity and suitably restricts the memory of the process $\{\mathbf{f}_t^0, \mathbf{e}_t\}_{t=1}^T$; Assumption CR(b) imposes unconditional moment bounds.

3.1.4 Asymptotic Distribution of the Estimator for the Break Date

TO BE DONE

3.2 Identification

Let $\boldsymbol{\Delta}^0 = \left(\boldsymbol{\delta}_1^0, \ldots, \boldsymbol{\delta}_N^0\right)'$ and write the data generating process of $\mathbf{x}_t$ as $\mathbf{x}_t = \boldsymbol{\Lambda}_1^0 \mathbf{f}_t^0 + \mathbb{I}_{2t}\left(\pi^0\right) \boldsymbol{\Delta}^0 \mathbf{f}_t^0 + \mathbf{e}_t$. Define $\mathbf{F}^0 = \left(\mathbf{f}_1^0, \ldots, \mathbf{f}_T^0\right)$ and denote $\tilde{\boldsymbol{\Lambda}}_1 = \left(\tilde{\boldsymbol{\lambda}}_{11}, \ldots, \tilde{\boldsymbol{\lambda}}_{1N}\right)'$ the principal components estimator for $\boldsymbol{\Lambda}_1^0$ from the misspecified linear factor model $\mathbf{x}_t = \boldsymbol{\Lambda}_1 \mathbf{f}_t + \mathbf{e}_t$. Let $\tilde{\mathbf{V}}_1$ be the $R^0 \times R^0$ diagonal matrix of the first R^0 largest eigenvalues of $\hat{\boldsymbol{\Sigma}}_{\mathbf{x}} = (NT)^{-1} \sum_{t=1}^T \mathbf{x}_t \mathbf{x}_t'$ in decreasing order: the underlying optimization problem requires the normalization $N^{-1} \tilde{\boldsymbol{\Lambda}}_1' \tilde{\boldsymbol{\Lambda}}_1 = \mathbf{I}_{R^0}$. The following theorem states the properties of $\tilde{\boldsymbol{\Lambda}}_1$.

Theorem 3.1 *There exists a $R^0 \times R^0$ rotation matrix $\tilde{\mathbf{H}}_1$ with* $\operatorname{rank}\left[\tilde{\mathbf{H}}_1\right] = R^0$ *such that*

$$B_{NT}^2 \left(\frac{1}{N} \sum_{i=1}^N \left\|\tilde{\boldsymbol{\lambda}}_{1i} - \tilde{\mathbf{H}}_1' \boldsymbol{\lambda}_{1i}^0\right\|^2\right) = O_p(1),$$

as $N, T \to \infty$, where

$$B_{NT} = \min\left\{\sqrt{N}, \sqrt{T}, N^{1-\alpha^0}\right\}$$

and

$$\tilde{\mathbf{H}}_1 = \frac{\mathbf{F}^0 \mathbf{F}^{0\prime}}{T} \frac{\boldsymbol{\Lambda}_1^{0\prime} \tilde{\boldsymbol{\Lambda}}_1}{N} \tilde{\mathbf{V}}_1^{-1}.$$

Theorem 3.1 shows that the average squared deviations between the loadings estimated under the null hypothesis of linearity and those that lie in the true loading space vanish as $N, T \rightarrow \infty$ at a rate equal to B_{NT}^2, which drives identification. Under Assumption I, the model in (1) is identified from the standard linear factor model as the rate of convergence B_{NT}^2 of the principal components estimator is slower than it would be under correct model specification: the model in (1) would not be identified if $0 \leq \alpha^0 \leq 0.5$, since in this case $B_{NT}^2 = \min\{N, T\}$, as derived in Bai and Ng (2002) for linear factor models. If $\alpha^0 = 1$ and all cross-sectional units are affected by the breakpoint, $B_{NT}^2 = 1$ and the principal components estimator from the misspecified linear model is asymptotically biased. As proved in Theorem 3.4, the parameter α^0 regulates the convergence rate of the estimator for the unknown break fraction π^0: this resul shows the connection between identification strength and estimation precision.

3.3 Principal Components Estimation

We estimate the common factors and factor loadings by principal components, and the unknown break fraction π^0 by concentrated least squares: see Bai (1997), Bai and Perron (1998), and Bai and Ng (2002). Define the $N \times 2R^0$ matrix of loadings $\mathbf{\Lambda} = (\mathbf{\Lambda}_1, \mathbf{\Lambda}_2)$ and the $R^0 \times T$ matrix of factors $\mathbf{F} = (\mathbf{f}_1, \ldots, \mathbf{f}_T)$. Let $\mathbf{\Lambda}^0 = \left(\mathbf{\Lambda}_1^0, \mathbf{\Lambda}_2^0\right)$ be the true value of $\mathbf{\Lambda}$. The objective function in terms of $\mathbf{\Lambda}$, $\mathbf{F}$ and π is the sum of squared residuals (divided by NT)

$$S(\mathbf{\Lambda}, \mathbf{F}, \pi) = (NT)^{-1} \sum_{t=1}^{T} \left[\mathbf{x}_t - \mathbb{I}_{1t}(\pi)\mathbf{\Lambda}_1\mathbf{f}_t - \mathbb{I}_{2t}(\pi)\mathbf{\Lambda}_2\mathbf{f}_t\right]' \left[\mathbf{x}_t - \mathbb{I}_{1t}(\pi)\mathbf{\Lambda}_1\mathbf{f}_t - \mathbb{I}_{2t}(\pi)\mathbf{\Lambda}_2\mathbf{f}_t\right]: \quad (2)$$

the estimators $\hat{\mathbf{\Lambda}} = \left(\hat{\mathbf{\Lambda}}_1, \hat{\mathbf{\Lambda}}_2\right)$, $\hat{\mathbf{F}} = \left(\hat{\mathbf{f}}_1, \ldots, \hat{\mathbf{f}}_T\right)$ and $\hat{\pi}$ for $\mathbf{\Lambda}^0$, $\mathbf{F}^0$and π^0, respectively, with $\hat{\mathbf{\Lambda}}_j = \left(\hat{\boldsymbol{\lambda}}_{j1}, \ldots, \hat{\boldsymbol{\lambda}}_{jN}\right)'$, for $j = 1, 2$, jointly solve

$$\hat{\mathbf{\Lambda}}, \hat{\mathbf{F}}, \hat{\pi} = \min_{\mathbf{\Lambda}, \mathbf{F}, \pi} S(\mathbf{\Lambda}, \mathbf{F}, \pi).$$

For given $\mathbf{\Lambda}$ and π, and subject to $N^{-1}\left(\mathbf{\Lambda}_j'\mathbf{\Lambda}_j\right) = \mathbf{I}_{R^0}$, for $j = 1, 2$, from (2) we have

$$\hat{\mathbf{f}}_t(\mathbf{\Lambda}, \pi) = N^{-1}\left[\mathbb{I}_{1t}(\pi)\mathbf{\Lambda}_1 + \mathbb{I}_{2t}(\pi)\mathbf{\Lambda}_2\right]'\mathbf{x}_t, \quad t = 1, \ldots, T: \quad (3)$$

replacing $\mathbf{f}_t$ in (2) with $\hat{\mathbf{f}}_t(\mathbf{\Lambda},\pi)$ obtained in (3) leads to the concentrated objective function

$$S_{\mathbf{F}}(\mathbf{\Lambda},\pi)=(NT)^{-1}\sum_{t=1}^{T}\mathbf{x}_t'\left\{\mathbf{I}_N-N^{-1}\left[\mathbb{I}_{1t}(\pi)\mathbf{\Lambda}_1\mathbf{\Lambda}_1'+\mathbb{I}_{2t}(\pi)\mathbf{\Lambda}_2\mathbf{\Lambda}_2'\right]\right\}\mathbf{x}_t, \tag{4}$$

and the estimators for $\mathbf{\Lambda}^0$ and π^0 jointly solve

$$\hat{\mathbf{\Lambda}},\hat{\pi}=\arg\min_{\mathbf{\Lambda},\pi}S_{\mathbf{F}}(\mathbf{\Lambda},\pi).$$

From (4), the estimator for $\mathbf{\Lambda}^0$ for given π is defined as

$$\hat{\mathbf{\Lambda}}(\pi)=\left[\hat{\mathbf{\Lambda}}_1(\pi),\hat{\mathbf{\Lambda}}_2(\pi)\right]=\arg\max_{\mathbf{\Lambda}}V_{\mathbf{F}}(\mathbf{\Lambda},\pi), \tag{5}$$

where

$$\begin{aligned}V_{\mathbf{F}}(\mathbf{\Lambda},\pi)&=(NT)^{-1}\sum_{t=1}^{T}\left\{\mathbf{x}_t'\left[\mathbb{I}_{1t}(\pi)(\mathbf{\Lambda}_1\mathbf{\Lambda}_1')+\mathbb{I}_{2t}(\pi)(\mathbf{\Lambda}_2\mathbf{\Lambda}_2')\right]\mathbf{x}_t\right\}\\&=(NT)^{-1}\left\{\operatorname{tr}\left\{\mathbf{\Lambda}_1'\left[\sum_{t=1}^{T}\mathbb{I}_{1t}(\pi)\mathbf{x}_t\mathbf{x}_t'\right]\mathbf{\Lambda}_1\right\}+\operatorname{tr}\left\{\mathbf{\Lambda}_2'\left[\sum_{t=1}^{T}\mathbb{I}_{2t}(\pi)\mathbf{x}_t\mathbf{x}_t'\right]\mathbf{\Lambda}_2\right\}\right\}.\end{aligned}$$

The problem

$$\max_{\mathbf{\Lambda}}V_{\mathbf{F}}(\mathbf{\Lambda},\pi)$$

is equivalent to

$$\max_{\mathbf{\Lambda}}\left[\mathbf{\Lambda}_1'\hat{\mathbf{\Sigma}}_{1\mathbf{x}}(\pi)\mathbf{\Lambda}_1+\mathbf{\Lambda}_2'\hat{\mathbf{\Sigma}}_{2\mathbf{x}}(\pi)\mathbf{\Lambda}_2\right], \tag{6}$$

where

$$\hat{\mathbf{\Sigma}}_{j\mathbf{x}}(\pi)=\left[(NT)^{-1}\sum_{t=1}^{T}\mathbb{I}_{jt}(\pi)\mathbf{x}_t\mathbf{x}_t'\right],\quad j=1,2: \tag{7}$$

for $j=1,2$, and for given π, the estimator for $\mathbf{\Lambda}_j^0$ solving the problem in (6) is $\hat{\mathbf{\Lambda}}_j(\pi)$, where $\hat{\mathbf{\Lambda}}_j(\pi)$ is equal to $\sqrt{N}$ times the $N\times R^0$ matrix of eigenvectors of $\hat{\mathbf{\Sigma}}_{j\mathbf{x}}(\pi)$ corresponding to its largest R^0 eigenvalues. Replacing $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ in (4) with $\hat{\mathbf{\Lambda}}_1(\pi)$ and $\hat{\mathbf{\Lambda}}_2(\pi)$ leads to the concentrated sum of squared residuals (divided by NT)

$$S_{\mathbf{F\Lambda}}(\pi)=(NT)^{-1}\sum_{t=1}^{T}\mathbf{x}_t'\left\{\mathbf{I}_N-N^{-1}\left[\mathbb{I}_{1t}(\pi)\hat{\mathbf{\Lambda}}_1(\pi)\hat{\mathbf{\Lambda}}_1(\pi)'+\mathbb{I}_{2t}(\pi)\hat{\mathbf{\Lambda}}_2(\pi)\hat{\mathbf{\Lambda}}_2(\pi)'\right]\right\}\mathbf{x}_t: \tag{8}$$

the estimator $\hat{\pi}$ for π^0 then solves

$$\hat{\pi} = \arg\min_{\pi} S_{\mathbf{F\Lambda}}(\pi).$$

Given $\hat{\pi}$, the estimator for $\mathbf{\Lambda}_j^0$ is $\hat{\mathbf{\Lambda}}_j = \hat{\mathbf{\Lambda}}_j(\hat{\pi})$, for $j = 1,2$. Finally, given $\hat{\pi}$ and $\hat{\mathbf{\Lambda}} = \left(\hat{\mathbf{\Lambda}}_1, \hat{\mathbf{\Lambda}}_2\right)$, from (3)

$$\hat{\mathbf{f}}_t = \hat{\mathbf{f}}_t\left(\hat{\mathbf{\Lambda}}, \hat{\pi}\right) = N^{-1}\left[\mathbb{I}_{1t}(\hat{\pi})\hat{\mathbf{\Lambda}}_1 + \mathbb{I}_{2t}(\hat{\pi})\hat{\mathbf{\Lambda}}_2\right]'\mathbf{x}_t, \quad t = 1,\ldots,T.$$

3.4 Consistency

From Theorem 3.1 the two regimes described in (1) are separately identified under Assumption 1. Define the $R^0 \times T$ matrices of regime-specific factors $\mathbf{F}_j^0(\pi) = \left[\mathbb{I}_{j1}(\pi)\mathbf{f}_1^0, \ldots, \mathbb{I}_{jT}(\pi)\mathbf{f}_T^0\right]$, for $j = 1,2$, such that $\mathbf{F}_1^0(\pi) + \mathbf{F}_2^0(\pi) = \left(\mathbf{f}_1^0, \ldots, \mathbf{f}_T^0\right) = \mathbf{F}^0$ and $\mathbf{F}_1^0(\pi^0)\mathbf{F}_2^0(\pi^0)' = \mathbf{0}_{R^0}$. Let $\hat{\mathbf{H}}_{jj}(\pi)$ and $\hat{\mathbf{H}}_{mj}(\pi)$ be the rotation matrices

$$\hat{\mathbf{H}}_{jj}(\pi) = \frac{\mathbf{F}_j^0(\pi^0)\mathbf{F}_j^0(\pi)'}{T}\frac{\mathbf{\Lambda}_j^{0\prime}\hat{\mathbf{\Lambda}}_j(\pi)}{N}\hat{\mathbf{V}}_j(\pi)^{-1}, \quad j = 1,2, \tag{9}$$

$$\hat{\mathbf{H}}_{mj}(\pi) = \frac{\mathbf{F}_m^0(\pi^0)\mathbf{F}_j^0(\pi)'}{T}\frac{\mathbf{\Lambda}_m^{0\prime}\hat{\mathbf{\Lambda}}_j(\pi)}{N}\hat{\mathbf{V}}_j(\pi)^{-1}, \quad j,m = 1,2, \quad j \neq m, \tag{10}$$

where $\hat{\mathbf{V}}_j(\pi)$ is the $R^0 \times R^0$ diagonal matrix of the first R^0 largest eigenvalues of $\hat{\mathbf{\Sigma}}_{j\mathbf{x}}(\pi)$ defined in (7) in decreasing order: for $\pi = \pi^0$ notice that $\hat{\mathbf{H}}_{jj}(\pi)$ and $\hat{\mathbf{H}}_{mj}(\pi)$ reduce to

$$\hat{\mathbf{H}}_{jj}(\pi^0) = \frac{\mathbf{F}_j^0(\pi^0)\mathbf{F}_j^0(\pi^0)'}{T}\frac{\mathbf{\Lambda}_j^{0\prime}\hat{\mathbf{\Lambda}}_j(\pi^0)}{N}\hat{\mathbf{V}}_j(\pi^0)^{-1}, \quad \hat{\mathbf{H}}_{mj}(\pi^0) = \mathbf{0}_{R^0} \quad j,m = 1,2, \quad j \neq m,$$

and $\hat{\mathbf{H}}_{jj}(\pi^0)$ becomes a rotation matrix analogous to the one introduced in Bai and Ng (2002) for linear factor models[1]. The following theorem shows the bias of the principal components estimator induced by the breakpoint when $\pi \neq \pi^0$.

Theorem 3.2 *There exist $R^0 \times R^0$ matrices $\hat{\mathbf{H}}_{jj}(\pi)$ and $\hat{\mathbf{H}}_{mj}(\pi)$ as defined in* (9) *and* (10)*, respectively, with* $\text{rank}\left[\hat{\mathbf{H}}_{jj}(\pi)\right] = R^0$ *for all π and* $\text{rank}\left[\hat{\mathbf{H}}_{mj}(\pi)\right] = R^0$ *for $\pi \neq \pi^0$, and $C_{NT} = \min\left\{\sqrt{N}, \sqrt{T}\right\}$, such that*

$$C_{NT}^2\left[\frac{1}{N}\sum_{i=1}^{N}\left\|\hat{\boldsymbol{\lambda}}_{ji}(\pi) - \hat{\mathbf{H}}_{jj}(\pi)'\boldsymbol{\lambda}_{ji}^0 - \hat{\mathbf{H}}_{mj}(\pi)'\boldsymbol{\lambda}_{mi}^0\right\|^2\right] = O_p(1), \quad \forall\pi, \quad j,m = 1,2, \quad j \neq m.$$

[1] See Bai and Ng (2002), p. 213.

Theorem 3.2 shows that the presence of regimes adds the asymptotic bias $\hat{\mathbf{H}}_{mj}(\pi)' \boldsymbol{\lambda}^0_{mi}$ to the principal components estimator $\hat{\boldsymbol{\lambda}}_{ji}(\pi)$ for the space $\hat{\mathbf{H}}_{jj}(\pi)' \boldsymbol{\lambda}^0_{ji}$ spanned by $\boldsymbol{\lambda}^0_{ji}$. As in linear factor models, the rate of convergence is equal to $C^2_{NT} = \min\{N, T\}$ and therefore depends on the panel structure. Taking into account (10), it follows that for $\pi = \pi^0$,

$$C^2_{NT}\left[\frac{1}{N}\sum_{i=1}^{N}\left\|\hat{\boldsymbol{\lambda}}_{ji}\left(\pi^0\right) - \hat{\mathbf{H}}_{jj}\left(\pi^0\right)' \boldsymbol{\lambda}^0_{ji}\right\|^2\right] = O_p(1), \quad j = 1, 2, \tag{11}$$

which extends the result in Theorem 1 in Bai and Ng (2002) to accommodate the presence of the breakpoint when the break fraction π^0 is known. Theorem 3.2 is central in the proof of the following theorem, which states the consistency of $\hat{\pi}$ as an estimator for π^0.

Theorem 3.3 *Under Assumptions I and C1-C4,* $\hat{\pi} \xrightarrow{p} \pi^0$ *as* $N, T \to \infty$.

Theorems 3.2 and 3.3 imply a number of results analogous to those proved in Theorem 1 in Stock and Watson (2002): these are stated in Corollary 3.1 below.

Corollary 3.1 *For* $j = 1, 2$, *and under Assumptions I and C1-C4, as* $N, T \to \infty$:

(a) $\hat{\boldsymbol{\lambda}}_{ji}(\hat{\pi}) \xrightarrow{p} \hat{\mathbf{H}}_{jj}\left(\pi^0\right)' \boldsymbol{\lambda}^0_{ji}$;

(b) $\hat{\mathbf{f}}_t \xrightarrow{p} \left[\mathbb{I}_{1t}\left(\pi^0\right)\hat{\mathbf{H}}_{11}\left(\pi^0\right) + \mathbb{I}_{2t}\left(\pi^0\right)\hat{\mathbf{H}}_{22}\left(\pi^0\right)\right]^{-1}\mathbf{f}^0_t$;

(c) $\frac{1}{N}\sum_{i=1}^{N}\left\|\hat{\boldsymbol{\lambda}}_{ji}(\hat{\pi}) - \hat{\mathbf{H}}_{jj}\left(\pi^0\right)' \boldsymbol{\lambda}^0_{ji}\right\|^2 \xrightarrow{p} 0$;

(d) $\frac{1}{T}\sum_{t=1}^{T}\left\|\hat{\mathbf{f}}_t - \left[\mathbb{I}_{1t}\left(\pi^0\right)\hat{\mathbf{H}}_{11}\left(\pi^0\right) + \mathbb{I}_{2t}\left(\pi^0\right)\hat{\mathbf{H}}_{22}\left(\pi^0\right)\right]^{-1}\mathbf{f}^0_t\right\|^2 \xrightarrow{p} 0.$

3.5 Convergence Rates

The following theorem states the convergence rates of the concentrated least squares estimator for the break fraction π^0 and of the principal components estimator for the loadings.

Theorem 3.4 *Under Assumptions I, C1-C4 and CR,*

$$D_{NT}\left(\alpha^0\right)\left(\hat{\pi} - \pi^0\right) = O_p(1)$$

with $D_{NT}\left(\alpha^0\right) = N^{\left(2\alpha^0-1\right)}T$*, and*

$$C_{NT}^2\left[\frac{1}{N}\sum_{i=1}^N\left\|\hat{\boldsymbol{\lambda}}_{ji}\left(\hat{\pi}\right)-\hat{\mathbf{H}}_{jj}\left(\pi^0\right)'\boldsymbol{\lambda}_{ji}^0\right\|^2\right]=O_p\left(1\right),\quad j=1,2.$$

Theorem 3.4 states the superconsistency of $\hat{\pi}$ as an estimator for π^0: it extends to an infinite dimensional system the result stated in Proposition 2 in Bai and Perron (1998); it is analogous to Theorem 3.4 in Massacci (2015), which proves a similar result in relation to the concentrated least squares estimator for the threshold parameter in large dimensional threshold factor models. The convergence rate $D_{NT}\left(\alpha^0\right) = N^{\left(2\alpha^0-1\right)}T$ of $\hat{\pi}$ depends on the time series dimension T and the number of cross-sectional units N^{α^0} subject to structural change: the rate $D_{NT}\left(\alpha^0\right)$ monotonically increases in α^0; since $0.5 < \alpha^0 \leq 1$ by Assumption I, then $T < D_{NT}\left(\alpha^0\right) \leq NT$; $D_{NT}\left(\alpha^0\right)$ is unknown since α^0 is unknown; without estimating α^0, the convergence rate of $\hat{\pi}$ is set identified. The higher α^0, the stronger identification of (1) from a linear factor model, and the faster the convergence rate of $\hat{\pi}$ to π^0: this shows the connection between identification and estimation. When $\alpha^0 = 1$, all cross-sectional units are subject to structural instability and the convergence rate is NT. Theorem 3.4 improves over Cheng *et al.* (2015), who do not derive the convergence rate for the estimator of the break fraction obtained through their LASSO-type estimator; it also improves over Chen (2015), who obtains the slower convergence rate $C_{NT} = \min\left\{\sqrt{N},\sqrt{T}\right\}$, which does not depend on the number of cross-sectional units subject to structural instability. Theorem 3.4 implies that the principal components estimator for the loadings has the same convergence rate derived in Bai and Ng (2002) in the case of linear factor models: the estimator for the break point therefore does not affect the estimator of the loadings. Corollary 3.2 below follows from Theorem 3.4.

Corollary 3.2 *Under Assumptions I, C1-C4 and CR,*

$$C_{NT}^2\left[\frac{1}{T}\sum_{t=1}^T\left\|\hat{\mathbf{f}}_t-\left[\mathbb{I}_{1t}\left(\pi^0\right)\hat{\mathbf{H}}_{11}\left(\pi^0\right)+\mathbb{I}_{2t}\left(\pi^0\right)\hat{\mathbf{H}}_{22}\left(\pi^0\right)\right]^{-1}\mathbf{f}_t^0\right\|^2\right]=O_p\left(1\right).$$

Corollary 3.2 shows that the convergence rate C_{NT} also applies to the principal components estimator for the factors; it also shows that the rotation induced by $\hat{\mathbf{f}}_t$ around $\mathbf{f}_t^0$ depends upon the state. Corollary 3.2 justifies the robust Bai and Ng (2002) information criteria proposed in Section 4.

3.6 Asymptotic Distribution of the Estimator for the Break Date

TO BE DONE

4 Model Selection

We now consider the case in which the true number of factors R^0 in (1) (i.e., the true dimension of $\mathbf{f}_t^0$) no longer is known and has to be determined. Breitung and Eickmeier (2011) show that neglecting structural breaks in the factor loadings inflates the estimated number of factors. We rely on Corollary 3.2 and robustify Bai and Ng (2002) selection criteria to account for the unknown breakpoint.

Given (1) and for fixed number of factors R, the loss function in (2) generalizes to

$$S\left(\mathbf{\Lambda}^R, \mathbf{F}^R, \pi\right) = (NT)^{-1} \sum_{t=1}^{T} \left\{ \left[\mathbf{x}_t - \mathbb{I}_{1t}(\pi) \mathbf{\Lambda}_1^R \mathbf{f}_t^R - \mathbb{I}_{2t}(\pi) \mathbf{\Lambda}_2^R \mathbf{f}_t^R\right]' \left[\mathbf{x}_t - \mathbb{I}_{1t}(\pi) \mathbf{\Lambda}_1^R \mathbf{f}_t^R - \mathbb{I}_{2t}(\pi) \mathbf{\Lambda}_2^R \mathbf{f}_t^R\right] \right\}, \tag{12}$$

where $\mathbf{\Lambda}^R = \left(\mathbf{\Lambda}_1^R, \mathbf{\Lambda}_2^R\right)$, $\mathbf{F}^R = \left(\mathbf{f}_1^R, \ldots, \mathbf{f}_T^R\right)$, and where the superscript R denotes the dependence on the number of factors. The loss function in (12) depends on π. From Theorem 3.4, it easily follows that for any *a priori* chosen number of factors $R = \bar{R}$ such that $\bar{R} \geq R^0$, the estimator $\hat{\pi}^{\bar{R}}$ for π^0 is such that $D_{NT}\left(\alpha^0\right)\left(\hat{\pi}^{\bar{R}} - \pi^0\right) = O_p(1)$, with $\hat{\pi}^{R^0} = \hat{\pi}$ (see Lemma **SAY WHICH ONE** in Appendix A.3): in practice, $\bar{R}$ may be chosen as discussed below. Given the convergence rate in Corollary 3.2, this naturally suggests generalizing Bai and Ng (2002) criteria by first setting $\pi = \hat{\pi}^{\bar{R}}$ in (12) to then select $\hat{R}$ factor within each mutually exclusive regime, and therefore $\left(\hat{R} + \hat{R}\right)$ factors in total.

Let $\hat{\mathbf{\Lambda}}^R(\pi)$ and $\hat{\mathbf{F}}^R(\pi)$ be the estimators for $\mathbf{\Lambda}^R$ and $\mathbf{F}^R$, respectively, for any π. Given the loss function in (12), and following Bai and Ng (2002), we want penalty functions $g(N,T)$ to obtain criteria of the form

$$PC(R,R) = S\left[\hat{\mathbf{\Lambda}}^R\left(\hat{\pi}^{\bar{R}}\right), \hat{\mathbf{F}}^R\left(\hat{\pi}^{\bar{R}}\right), \hat{\pi}^{\bar{R}}\right] + (R+R) \cdot g(N,T),$$

which consistently estimate the number of factors R^0 in each regime and therefore $\left(R^0 + R^0\right)$ factors in total: the criterion $PC(R,R)$ accounts for the fact that the structural instability leads to a factor representation with a higher dimensional factor space, namely to a representation with $\left(R^0 + R^0\right)$ factors.

Given a bounded integer $R^{\max} \geq R^0$, the true numbers of factors R^0 is estimated as

$$\hat{R} = \arg \min_{1 \leq R \leq R^{\max}} PC(R, R) :$$

given the convergence rate C_{NT} in Corollary 3.2, this leads to the breakpoint robust Bai and Ng (2002) information criteria

$$IC_{p1}(R, R) = \ln S\left[\hat{\mathbf{\Lambda}}^R\left(\hat{\pi}^{\bar{R}}\right), \hat{\mathbf{F}}^R\left(\hat{\pi}^{\bar{R}}\right), \hat{\pi}^{\bar{R}}\right] + (R + R)\left(\frac{N+T}{NT}\right)\ln\left(\frac{NT}{N+T}\right),$$

$$IC_{p2}(R, R) = \ln S\left[\hat{\mathbf{\Lambda}}^R\left(\hat{\pi}^{\bar{R}}\right), \hat{\mathbf{F}}^R\left(\hat{\pi}^{\bar{R}}\right), \hat{\pi}^{\bar{R}}\right] + (R + R)\left(\frac{N+T}{NT}\right)\ln\left(C_{NT}^2\right), \tag{13}$$

$$IC_{p3}(R, R) = \ln S\left[\hat{\mathbf{\Lambda}}^R\left(\hat{\pi}^{\bar{R}}\right), \hat{\mathbf{F}}^R\left(\hat{\pi}^{\bar{R}}\right), \hat{\pi}^{\bar{R}}\right] + (R + R)\left[\frac{\ln\left(C_{NT}^2\right)}{C_{NT}^2}\right].$$

In practice, to obtain the estimator $\hat{\pi}^{\bar{R}}$ for π^0, we may set $\bar{R} = R^{\max}$. The following theorem states the validity of the proposed information criteria.

Theorem 4.1 *Under Assumptions I, C1-C4 and CR, the criteria* $IC_{p1}(R, R)$*,* $IC_{p2}(R, R)$ *and* $IC_{p3}(R, R)$ *defined in* (13) *consistently estimate the number of factors* R^0*.*

5 Diffusion Index Model

A widely used tool for empirical work is the diffusion index model of Stock and Watson (1998, 2002), and Bai and Ng (2006). Formally, the model is

$$y_{t+h} = \boldsymbol{\beta}'\mathbf{w}_t + \boldsymbol{\gamma}'\mathbf{f}_t + \varepsilon_{t+h}, \quad t = 1, \ldots, T, \quad h \geq 0, \tag{14}$$

where $y_{t+h} \in \Re$ is the scalar dependent variable; $\mathbf{f}_t \in \Re^R$ is the $R \times 1$ vector of latent factors driving the cross-sectional dependence of $\mathbf{x}_t$ in (1); $\mathbf{w}_t \in \Re^K$ is a $K \times 1$ vector of observable variables; $\varepsilon_{t+h} \in \Re$ is the idiosyncratic error; $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ are $R \times 1$ and $K \times 1$ vectors of unknown slope coefficients. We assume that the true number of factors is known, namely $R = R^0$: should that not be the case, the model selection criteria in (13) may be used to consistently estimate the true number of factors R^0.

Our objective is to study the least squares estimator for the $\left(R^0 + K\right) \times 1$ vector of slope coefficients in (14). The properties of the estimator are well known under the linear factor representation: Stock and Watson (2002) prove the consistency; Bai and Ng (2006) establish the convergence rate and the asymptotic distribution. To the very best of our knowledge, we are the first to address the problem of estimating (14) when the factors are extracted from the model with structural instability in (1).

Assumption DI - TO BE COMPLETED Let $\mathbf{z}_t^0 = \left(\mathbf{f}_t^{0\prime}, \mathbf{w}_t^{\prime}\right)^{\prime}$. Then

(a) $T^{-1}\sum_{t=1}^{T} \mathbf{z}_t^0 \mathbf{z}_t^{0\prime} \xrightarrow{p} \boldsymbol{\Sigma}_{\mathbf{z}}^0$, where $\boldsymbol{\Sigma}_{\mathbf{z}}^0$ is a positive definite matrix;

(b) $T^{-1/2}\sum_{t=1}^{T} \mathbf{z}_t^0 \varepsilon_{t+h} \xrightarrow{d} \mathcal{N}\left(\mathbf{0}, \boldsymbol{\Sigma}_{\mathbf{z}\boldsymbol{\varepsilon}}^0\right)$ for any $h > 0$, where $\boldsymbol{\Sigma}_{\mathbf{z}\boldsymbol{\varepsilon}}^0 = \operatorname{p}\lim T^{-1}\sum_{t=1}^{T} \varepsilon_{t+h}^2 \mathbf{z}_t^0 \mathbf{z}_t^{0\prime}$ is a positive definite matrix.

Assumption DI is standard in linear regression models: it suitably extends Assumption Y1 in Stock and Watson (2002) to obtain the asymptotic distribution of the least squares estimator; it is analogous to Assumption E in Bai and Ng (2002). Given the principal components estimator $\hat{\mathbf{f}}_t$ obtained from (1) for the true vector of factors $\mathbf{f}_t^0$, define $\hat{\mathbf{z}}_t = \left(\hat{\mathbf{f}}_t^{\prime}, \mathbf{w}_t\right)^{\prime}$. Let $\boldsymbol{\gamma}^0$ and $\boldsymbol{\beta}^0$ be the true values of $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ in (14), respectively. Define $\widehat{\hat{\boldsymbol{\theta}}} = \left(\widehat{\hat{\boldsymbol{\gamma}}}^{\prime}, \widehat{\hat{\boldsymbol{\beta}}}^{\prime}\right)^{\prime}$, where $\widehat{\hat{\boldsymbol{\gamma}}}$ and $\widehat{\hat{\boldsymbol{\beta}}}$ are the least squares estimators for $\boldsymbol{\gamma}^0$ and $\boldsymbol{\beta}^0$, respectively, obtained by regressing y_{t+h} on $\hat{\mathbf{z}}_t$. Let $\boldsymbol{\theta}_j^0\left(\pi^0\right) = \left\{\left[\hat{\mathbf{H}}_{jj}\left(\pi^0\right)\boldsymbol{\gamma}^0\right]^{\prime}, \boldsymbol{\beta}^{0\prime}\right\}^{\prime}$ with $\hat{\mathbf{H}}_{jj}\left(\pi^0\right)$ defined according to (9), for $j = 1, 2$. Finally, define $\hat{\pi}\left(\pi^0\right) = T^{-1}\sum_{t=1}^{T} \mathbb{I}_{1t}\left(\pi^0\right)$.

Theorem 5.1 *Under Assumptions I, C1-C4, CR and DI, if* $\sqrt{T}/N \to 0$ *then*

$$\sqrt{T}\left\{\widehat{\hat{\boldsymbol{\theta}}} - \left\{\hat{\pi}\left(\pi^0\right)\boldsymbol{\theta}_1^0\left(\pi^0\right) + \left[1 - \hat{\pi}\left(\pi^0\right)\right]\boldsymbol{\theta}_2^0\left(\pi^0\right)\right\}\right\} \xrightarrow{d} \mathcal{N}\left[\mathbf{0}, \boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}}^0\left(\pi^0\right)\right]$$

with

$$\boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}}^0\left(\pi^0\right) = \left[\pi^0 \boldsymbol{\Phi}_{11}^0\left(\pi^0\right) + \left(1-\pi^0\right)\boldsymbol{\Phi}_{22}^0\left(\pi^0\right)\right]^{\prime}\left(\boldsymbol{\Sigma}_{\mathbf{z}}^0\right)^{-1}\boldsymbol{\Sigma}_{\mathbf{z}\boldsymbol{\varepsilon}}^0\left(\boldsymbol{\Sigma}_{\mathbf{z}}^0\right)^{-1}\left[\pi^0 \boldsymbol{\Phi}_{11}^0\left(\pi^0\right) + \left(1-\pi^0\right)\boldsymbol{\Phi}_{22}^0\left(\pi^0\right)\right],$$

where $\boldsymbol{\Phi}_{jj}^0\left(\pi^0\right) = \operatorname{diag}\left[\boldsymbol{\Sigma}_{j\mathbf{f}}^0\left(\pi, \pi^0\right)\mathbf{Q}_{j\boldsymbol{\Lambda}}^0\left(\pi^0\right)\mathbf{V}_j^0\left(\pi^0\right)^{-1}, \mathbf{I}_K\right]$, $\boldsymbol{\Sigma}_{j\mathbf{f}}^0\left(\pi^0, \pi^0\right)$ *is defined in Assumption C1,* $\mathbf{Q}_{j\boldsymbol{\Lambda}}^0\left(\pi^0\right) = \operatorname{p}\lim\left[\boldsymbol{\Lambda}_j^{0\prime}\hat{\boldsymbol{\Lambda}}_j\left(\pi^0\right)\right]/N$ *and* $\mathbf{V}_j^0\left(\pi^0\right) = \operatorname{p}\lim \hat{\mathbf{V}}_j\left(\pi^0\right)$. *A consistent estimator for* $\boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}}^0\left(\pi^0\right)$, *denoted by* $\widehat{\operatorname{Avar}}\left(\widehat{\hat{\boldsymbol{\theta}}}\right)$, *is*

$$\widehat{\operatorname{Avar}}\left(\widehat{\hat{\boldsymbol{\theta}}}\right) = \left(T^{-1}\sum_{t=1}^{T}\hat{\mathbf{z}}_t\hat{\mathbf{z}}_t^{\prime}\right)^{-1}\left(T^{-1}\sum_{t=1}^{T}\hat{\varepsilon}_{t+h}^2\hat{\mathbf{z}}_t\hat{\mathbf{z}}_t^{\prime}\right)\left(T^{-1}\sum_{t=1}^{T}\hat{\mathbf{z}}_t\hat{\mathbf{z}}_t^{\prime}\right)^{-1}. \tag{15}$$

Theorem 5.1 extends Theorem 1 in Bai and Ng (2006) to let the factors be extracted from the model with structural instability in (1). The diffusion index model in (14) is identified up to a rotation because

$$\boldsymbol{\gamma}^{0\prime}\mathbf{f}_t^0 = \mathbb{I}_{1t}\left(\pi^0\right)\boldsymbol{\gamma}^{0\prime}\mathbf{L}_1\mathbf{L}_1^{-1}\mathbf{f}_t^0 + \mathbb{I}_{2t}\left(\pi^0\right)\boldsymbol{\gamma}^{0\prime}\mathbf{L}_2\mathbf{L}_2^{-1}\mathbf{f}_t^0,$$

for some positive definite matrices $\mathbf{L}_1$ and $\mathbf{L}_2$. Theorem 5.1 therefore relates to the difference between $\widehat{\boldsymbol{\gamma}}$ and the space spanned by $\boldsymbol{\gamma}^0$: the rotation induced around $\boldsymbol{\gamma}^0$ depends on the frequency of observations $\hat{\pi}\left(\pi^0\right)$ and $\left[1-\hat{\pi}\left(\pi^0\right)\right]$ in (1) before and after the break, respectively. In particular, Theorem 5.1 establishes the convergence rate and the limiting distribution of the least squares estimator for $\widehat{\boldsymbol{\gamma}}$: these crucially depend upon the results in Theorem 3.4 and Corollary 3.2. The results in Theorem 5.1 could not be established if the factors were estimated by the LASSO-type estimator of Cheng *et al.* (2015), as they do not provide any convergence rate.

The analytical expression for the estimated covariance matrix in (15) is robust to heteroskedasticity. With homoskedastic disturbances the condition $\mathrm{E}\left(\varepsilon_{t+h}^2 \left|\mathbf{z}_t^0\right.\right) = \sigma_\varepsilon^2$ holds for $t=1,\ldots,T$: given the least squares residual $\hat{\varepsilon}_{t+h}$ and $\hat{\sigma}_\varepsilon^2 = T^{-1}\sum_{t=1}^T \hat{\varepsilon}_{t+h}^2$, the estimator in (15) simplifies to

$$\widehat{\mathrm{Avar}}\left(\widehat{\boldsymbol{\theta}}\right) = \left(T^{-1}\sum_{t=1}^T \hat{\mathbf{z}}_t\hat{\mathbf{z}}_t'\right)^{-1}.$$

The formula in (15) is the same as that provided in Theorem 1 in Bai and Ng (2006): structural instability in the factor loadings therefore affects the asymptotic covariance matrix $\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\theta}}}^0\left(\pi^0\right)$ of the least squares estimator $\widehat{\boldsymbol{\theta}}$, but not its estimator $\widehat{\mathrm{Avar}}\left(\widehat{\boldsymbol{\theta}}\right)$.

6 Break in the Number of Factors

6.1 Model

TO BE DONE

We now extend (1) and consider

$$\mathbf{x}_t = \mathbb{I}\left(t/T \le \pi\right)\boldsymbol{\Lambda}_1\mathbf{f}_{1t} + \mathbb{I}\left(t/T > \pi\right)\boldsymbol{\Lambda}_2\mathbf{f}_{2t} + \mathbf{e}_t, \quad t=1,\ldots,T, \tag{16}$$

where $\mathbf{f}_{jt} = (f_{j1t}, \ldots, f_{jRt})' \in \Re^{R_j}$ is the $R_j \times 1$ vector of latent factors, for $j = 1, 2$; $\boldsymbol{\Lambda}_j = (\boldsymbol{\lambda}_{j1}, \ldots, \boldsymbol{\lambda}_{jN})'$ is the $N \times R_j$ matrix of factor loadings with $i - th$ row defined as $\boldsymbol{\lambda}_{ji} = (\lambda_{ji1}, \ldots, \lambda_{jiR})'$, for $j = 1, 2$ and $i = 1, \ldots, N$. The model in (16) generalizes (1) by allowing for breaks in the number of factors, as well as in the factor loadings. By looking at (16), we consider a more general set up than the type-2 instability in Cheng *et al.* (2015), as we do not require any sign restriction on the number of factors: Cheng *et al.* (2015) require sign restrictions on the number of factors, namely $R_1 > R_2$ or $R_1 < R_2$, depending on the prior the econometrician has on the effect of the break on the number of factors; should the prior be wrong, this may lead to model misspecification. Notice that (16) reduces to (1) under the null hypothesis $\mathbf{f}_{1t}^0 = \mathbf{f}_{2t}^0 = \mathbf{f}_t^0$.

6.2 Estimation

TO BE DONE

6.3 Model Selection

TO BE DONE

7 Monte Carlo Analysis

We now assess the empirical validity of the theoretical results obtained in the paper. Starting from the breakpoint factor model with instability in the loadings, Sections 7.1 and 7.2 describe the experiments related to estimation and model selection, respectively. Section 7.3 focuses on the unstable diffusion index model. Section 7.4 considers the factor model with a break in the number of factors. Finally, the results are discussed in Section 7.5.

7.1 Estimation

In line with the results in Section 3, in Experiment 1 we assume a known number of factors. We consider the following Data Generating Process (DGP)

$$x_{it}^s = \mathbb{I}\left(t/T \leq \pi^0\right)\left(\sum_{r=1}^{R^0} \lambda_{1ir}^0 f_{rt}^{0s}\right) + \mathbb{I}\left(t/T > \pi^0\right)\left(\sum_{r=1}^{R^0} \lambda_{2ir}^0 f_{rt}^{0s}\right) + e_{it}^s, \quad i = 1, \ldots, N, \quad t = 1, \ldots, T,$$

where $s = 1, \ldots, S$ refers to the replication, N and T are the cross-sectional and time series dimensions, respectively, and S is the number of replications. We run the experiment in Ox 7.01 (see Doornik, 2012). We set $S = 2000$, $N = 25, 50, 100$ and $T = 100, 200, 400$. We fix the number of factors $R^0 = 2$ In generating the data, we define $\delta_i^0 = \lambda_{2ir}^0 - \lambda_{1ir}^0$, for $r = 1, \ldots, R^0$: we set $\delta_i^0 > 0$ for $i = 1, \ldots, \left[N^{\alpha^0}\right]$ and $\delta_i^0 = 0$ for $i = \left[N^{\alpha^0}\right] + 1, \ldots, N$, where $[\cdot]$ denotes the integer part of the argument. Following the Monte Carlo set up in Breitung and Eickmeier (2011), we fix the factor loadings $\lambda_{1ir}^0 \sim \mathcal{N}(1, 1)$ throughout the replications, for $i = 1, \ldots, N$ and $r = 1, \ldots, R^0$. We control for: (i) the number of cross-sectional units with structural instability $\left[N^{\alpha^0}\right]$, with $0.5 < \alpha^0 \leq 1$; (ii) the magnitude of the breakpoint effect δ_i^0; and (iii) the break fraction π^0.

We generate the factors f_{rt}^{0s} as

$$f_{rt}^{0s} = \rho_f f_{r,t-1}^{0s} + \left(1 - \rho_f^2\right)^{1/2} \epsilon_{frt}^s, \quad f_{r,-50}^{0s} = 0, \quad \epsilon_{frt}^s \sim \text{IID}\mathcal{N}(0, 1), \quad r = 1, \ldots, R^0, \quad t = -49, \ldots, 0, \ldots, T,$$

with ρ_f fixed in repeated samples and drawn as $\rho_f \sim \mathcal{U}(0.05, 0.95)$, so that $\text{E}\left(f_{rt}^{0s}\right) = 0$ and $\text{Var}\left(f_{rt}^{0s}\right) = 1$. As for the idiosyncratic components e_{it}^s, we set

$$e_{it}^s = \rho_e e_{i,t-1}^s + \sigma_{ii}^{1/2} \left(1 - \rho_e^2\right)^{1/2} \epsilon_{e_i t}^s, \quad e_{i,-50}^s = 0, \quad \epsilon_{e_i t}^s \sim \text{IID}\mathcal{N}(0, 1), \quad i = 1, \ldots, N, \quad t = -49, \ldots, 0, \ldots, T,$$

where ρ_e and σ_{ii} are fixed in repeated samples with $\rho_e \sim \mathcal{U}(0.05, 0.95)$ and $\sigma_{ii} \sim \chi(1)$, for $i = 1, \ldots, N$: in this way $\text{Var}(e_{it}^s) = \sigma_{ii}$ for $i = 1, \ldots, N$, and $N^{-1} \sum_{i=1}^N \sigma_{ii} \to 1$ as $N \to \infty$.

We control for the number of cross-sectional units subject to a regime change $\left[N^{\alpha^0}\right]$ by setting $\alpha^0 = 0.60, 1.00$. As for the breakpoint effect, we consider two scenarios: in the homogeneous case, we set $\delta_i^0 = 0.25, 1.00$, for $i = 1, \ldots, \left[N^{\alpha^0}\right]$; we then allow for heterogeneous breaks by setting $\delta_i^0 \sim \mathcal{N}(0, 1)$, for $i = 1, \ldots, \left[N^{\alpha^0}\right]$. Finally, we consider five values for π^0, namely $\pi^0 = 0.15, 0.30, 0.50, 0.70, 0.85$.

To reduce the effect induced by the initial values $f_{r,-50}^{0s} = e_{i,-50}^s = 0$, we discard the first 50 observations in the DGPs for f_{rt}^{0s} and e_{it}^s, for $r = 1, \ldots, R^0$. We estimate the break fraction π^0 by concentrated least squares, and factors and loadings by principal components, as detailed in Section 3.3. Given the convergence rates derived in Theorem 3.4 and Corollary 3.2, the estimator for π^0 is asymptotically independent of that for f_{rt}^{0s}, λ_{1ir}^0 and λ_{2ir}^0. We estimate π^0 by grid search by maximizing the objective function over the set of break fractions $\{5\%, 10\%, 15\%, \ldots, 85\%, 90\%, 95\%\}$; given the estimator $\hat{\pi}^s$ for

π^0, we estimate factors and loadings. We assess the performance of $\hat{\pi}^s$ by computing

$$\text{bias} = S^{-1} \sum_{s=1}^{S} \left(\hat{\pi}^s - \pi^0 \right), \quad \text{RMSE} = \sqrt{S^{-1} \sum_{s=1}^{S} \left(\hat{\pi}^s - \pi^0 \right)^2}.$$

Finally, given the estimator

$$\hat{c}_{it}^s = \mathbb{I}\left(t/T \leq \hat{\pi} \right) \left(\sum_{r=1}^{R^0} \hat{\lambda}_{1ir} \hat{f}_{rt}^s \right) + \mathbb{I}\left(t/T > \hat{\pi} \right) \left(\sum_{r=1}^{R^0} \hat{\lambda}_{2ir} \hat{f}_{rt}^s \right), \quad i = 1, \ldots, N, \quad t = 1, \ldots, T,$$

for the common component

$$c_{it}^{0s} = \mathbb{I}\left(t/T \leq \pi^0 \right) \left(\sum_{r=1}^{R^0} \lambda_{1ir}^0 f_{rt}^{0s} \right) + \mathbb{I}\left(t/T > \pi^0 \right) \left(\sum_{r=1}^{R^0} \lambda_{2ir}^0 f_{rt}^{0s} \right), \quad i = 1, \ldots, N, \quad t = 1, \ldots, T,$$

we report

$$\text{MSE} = S^{-1} \sum_{s=1}^{S} \left[(NT)^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \left(\hat{c}_{it}^s - c_{it}^{0s} \right)^2 \right].$$

7.2 Model Selection

We set $\alpha^0 = 0.60$ and treat the number of factors as unknown: we fix $R^0 = 1, 2, 3$ in Experiments 2, 3 and 4, respectively. The number of factors is estimated by setting $R^{\max} = 8$ through the model selection criteria proposed in Section 4. We assess the criteria by reporting the average number of estimated factors over the replications.

7.3 Diffusion Index Model

In Experiment 5 we consider the linear diffusion index

$$y_t^s = \beta^0 w_t^s + \gamma^0 f_{1t}^{0s} + \varepsilon_t^s, \quad t = 1, \ldots, T,$$

where f_{1t}^{0s} is as in Experiment 1, and the true number of factors $R^0 = 1$ is assumed to be known. We set $\beta_0^0 = \gamma^0 = 1$ fixed in repeated samples and draw ε_t^s as $\varepsilon_t^s \sim \text{IID}\mathcal{N}(0, 1)$. We generate w_t^s as

$$w_t^s = \mu_w (1 - \rho_w) + \rho_w w_{t-1}^s + \left(1 - \rho_w^2 \right)^{1/2} \epsilon_{wt}^s, \quad w_{-50}^s = 0, \quad \epsilon_{wt}^s \sim \text{IID}\mathcal{N}(0, 1), \quad t = -49, \ldots, 0, \ldots, T,$$

with μ_w and ρ_w fixed in repeated samples and generated as $\mu_w \sim \mathcal{N}(0,1)$ and $\rho_w \sim \mathcal{U}(0.05, 0.95)$, respectively: to reduce the effect induced by the initial value $w_{-50}^s = 0$, we discard the first 50 observations in the DGP for w_t^s. We focus on the point identified parameter β^0 and assess the performance of the estimator $\hat{\beta}^s$ from replication s through

$$\text{bias} = S^{-1} \sum_{s=1}^{S} \left(\hat{\beta}^s - \beta^0 \right), \quad \text{RMSE} = \sqrt{S^{-1} \sum_{s=1}^{S} \left(\hat{\beta}^s - \beta^0 \right)^2}.$$

Given **SAY WHAT**, the asymptotic distribution of $\hat{\beta}^s$ is asymptotically normal: we then calculate thesize at 5% level for the null hypothesis $\beta^0 = 1$.

TO BE DONE: EXPERIMENTS 6, 7

$$y_t^s = \mathbb{I}\left(t/T \leq \tau^0\right)\left(\beta_1^0 w_t^s + \gamma_1^0 f_{1t}^{0s}\right) + \mathbb{I}\left(t/T > \tau^0\right)\left(\beta_2^0 w_t^s + \gamma_2^0 f_{1t}^{0s}\right) + \varepsilon_t^s, \quad t = 1, \ldots, T.$$

7.4 Break in the Number of Factors

We now consider the DGP

$$x_{it}^s = \mathbb{I}\left(t/T \leq \pi^0\right)\left(\sum_{r_1=1}^{R_1^0} \lambda_{1ir_1}^0 f_{r_1t}^{0s}\right) + \mathbb{I}\left(t/T > \pi^0\right)\left(\sum_{r_2=1}^{R_2^0} \lambda_{2ir_2}^0 f_{r_2t}^{0s}\right) + e_{it}^s, \quad i = 1, \ldots, N, \quad t = 1, \ldots, T,$$

which allows for a break in the number of factors, as well as in the factor loadings. We fix the number of factors $R_1^0 = 3$ and $R_2^0 = 2$. We set $\lambda_{1ir_1}^0$ fixed in repeated samples and drawn as $\lambda_{1ir_1}^0 \sim \mathcal{N}(1,1)$, for $i = 1, \ldots, N$ and $r_1 = 1, \ldots, R_1^0$. Since $R_2^0 < R_1^0$, we set $\lambda_{2ir_2}^0 = \lambda_{1ir_2}^0 + \delta_i^0$, for $i = 1, \ldots, N$ and $r_2 = 1, \ldots, R_2^0$, with $\delta_i^0 = 0.25, 1.00$ and $\delta_i^0 \sim \mathcal{N}(0,1)$, under the homogeneous and heterogeneous break set up, respectively.

In Experiment 8 we assume the true numbers of factors R_1^0 and R_2^0 are known, and we estimate factors, loadings and break fraction through the procedure detailed in Section 6; we evaluate the estimators using the same approach adopted in Experiment 1 and detailed in Section 7.1. In Experiment 9 we then estimate the number of factors by setting $R_1^{\max} = R_2^{\max} = 8$ through the criteria proposed in Section 6.3, and we assess them by showing the average number of estimated factors over the replications.

7.5 Results

Results from Experiments 1 through 9 are shown in Tables 1 through 9, respectively.

Table 1 about here

Table 2 about here

Table 3 about here

Table 4 about here

Table 5 about here

Table 6 about here

Table 7 about here

Table 8 about here

Table 9 about here

COMMENTS TO BE ADDED HERE.

8 Empirical Application

SECTION TO BE COMPLETED

This section provides an empirical application to illustrate the potential usefulness of our contribution for applied work. We show how our framework may be used to study time variation in bond market risk premia. Fama and Bliss (1987) **TO BE COMPLETED**. Campbell and Shiller (1991) **TO BE COMPLETED**. Cochrane and Piazzesi (2005) **TO BE COMPLETED**. Ludvigson and Ng (2009) **TO BE COMPLETED**. Thornton and Valente (2012) **TO BE COMPLETED**. **MORE TO SAY HERE**. In what follows, Section 8.1 proposes a measure of connectedness, Section 8.2 describes the data and the specification of the models, and Section 8.3 presents the results.

8.1 Data and Model Specification

TO BE DONE

Macroeconomic variables undergo structural instabilities (Breitung and Eickmeier (2012), Chen *et al.* (2014), and Cheng *et al.* (2015)).

8.2 Macro Factors

TO BE DONE

Given the sequence of $N \times 1$ vectors $\{\mathbf{x}_t\}_{t=1}^{T}$, let $\{\omega_r\}_{r=1}^{N}$ be the sequence of eigenvalues of the $N \times N$ covariance matrix $\hat{\mathbf{\Sigma}}_{\mathbf{x}} = (NT)^{-1} \sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t'$. Billio *et al.* (2012) quantify the degree of connectedness amongst the elements of $\mathbf{x}_t$ through the measure[2]

$$C(R) = \frac{\sum_{r=1}^{R} \omega_r}{\sum_{r=1}^{N} \omega_r} :$$

by construction $C(R)$ is increasing in R; for given R, a higher $C(R)$ denotes a higher degree of connectedness amongst the underlying variables. The measure $C(R)$ is a powerful tool to capture the degree of connectedness amongst random variables. However, it suffers from two main drawbacks. First, the number of eigenvalues R is chosen *a priori* and not according to a selection criterion. Second, $C(R)$ refers to the entire time series dimension T and it is unable to detect variations in the degree of connectedness induced by a change-point: the measure $C(R)$ may then not provide an accurate description of the dynamics in the degree of connectedness of the variables of interest[3]. Our theoretical results allow us to build a measure of connectedness that accommodates a breakpoint and that relies on the optimally selected number of eigenvalues.

Let $\{\omega_{jr}(\pi)\}_{r=1}^{N}$ be the sequence of eigenvalue of the $N \times N$ covariance matrix $\hat{\mathbf{\Sigma}}_{j\mathbf{x}}(\pi)$ defined in (7) in decreasing order, for $j = 1, 2$. We generalize $C(R)$ and measure connectedness through

$$C_j\left(\hat{R}\right) = \frac{\sum_{r=1}^{\hat{R}} \omega_{jr}(\hat{\pi})}{\sum_{r=1}^{N} \omega_{jr}(\hat{\pi})}, \quad j = 1, 2.$$

Compared to $C(R)$, the measure $C_j\left(\hat{R}\right)$ has two distinctive features: it allows to quantify connectedness

[2] Billio *et al.* (2012) refer to $C(R)$ as to the Cumulative Risk Fraction.

[3] Billio *et al.* (2012) measure the dynamic degree of connectedness in financial returns by computing $C(R)$ over rolling windows.

on each side of the breakpoint; and the number of eigenvalue $\hat{R}$ is optimally determined according to the selection criteria proposed in Section 4.

8.3 Results

TO BE DONE

9 Conclusions

TO BE DONE

A Proofs of Theorems

A.1 Proofs of Results in Section 3.4

Proof of Theorem 3.1. The proof is analogous to that of Theorem 3.1 in Massacci (2015) and omitted. ■

Proof of Theorem 3.2. The proof is analogous to that of Theorem 3.2 in Massacci (2015) and omitted. ■

Proof of Theorem 3.3. The proof is analogous to that of Theorem 3.3 in Massacci (2015) and omitted. ■

A.2 Proofs of Results in Section 3.5

Let

$$g_{it}^0(\pi_1,\pi_2) = |\mathbb{I}_{2t}(\pi_2) - \mathbb{I}_{2t}(\pi_1)| \left\|\mathbf{f}_t^0 e_{it}\right\|, \quad i=1,\ldots,N, \quad t=1,\ldots,T,$$

$$q_t^0(\pi_1,\pi_2) = |\mathbb{I}_{2t}(\pi_2) - \mathbb{I}_{2t}(\pi_1)| \left\|\mathbf{f}_t^0\right\|, \quad t=1,\ldots,T,$$

$$w_{it}^0(\pi) = \left|\mathbb{I}_{2t}(\pi) - \mathbb{I}_{2t}(\pi^0)\right| \left(\boldsymbol{\delta}_i^{0\prime}\mathbf{f}_t^0\right)^2, \quad i=1,\ldots,N, \quad t=1,\ldots,T,$$

$$w^0(\alpha^0,\pi) = \frac{1}{N^{\alpha^0}}\frac{1}{T}\sum_{i=1}^{N}\sum_{t=1}^{T} w_{it}^0(\pi),$$

$$\mathbf{h}^0(\alpha^0,\pi) = \frac{1}{N^{\alpha^0}}\frac{1}{T}\sum_{i=1}^{N}\sum_{t=1}^{T} \mathbb{I}_{2t}(\pi)\,\boldsymbol{\delta}_i^{0\prime}\mathbf{f}_t^0 e_{it}.$$

Lemma A.1 *There exists a $C_1<\infty$ such that for all $0<\pi_L\leq\pi_1\leq\pi_2\leq\pi_U<1$ and $s\leq 4$,*

$$\mathrm{E}\left\{\left[g_{it}^0(\pi_1,\pi_2)\right]^s\right\} \leq C_1|\pi_2-\pi_1|, \quad i=1,\ldots,N, \quad t=1,\ldots,T, \tag{17}$$

and

$$\mathrm{E}\left\{\left[q_t^0(\pi_1,\pi_2)\right]^s\right\} \leq C_1|\pi_2-\pi_1|, \quad t=1,\ldots,T. \tag{18}$$

Lemma A.2 *There exists a $K<\infty$ such that for all $0<\pi_L\leq\pi_1\leq\pi_2\leq\pi_U<1$,*

$$\mathrm{E}\left\{\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left\{\left[q_t^0(\pi_1,\pi_2)\right]^2 - \mathrm{E}\left\{\left[q_t^0(\pi_1,\pi_2)\right]^2\right\}\right\}\right|^2\right\} \leq K|\pi_2-\pi_1|.$$

Lemma A.3 *There exist constants $0 < B < 1$ and $0 < d < \infty$ such that for all $\eta > 0$ and $\varepsilon > 0$, there exists a $\bar{v} < \infty$ such that for all N and T,*

$$\Pr\left[\inf_{\frac{\bar{v}}{N^{\left(2\alpha^0-1\right)}T}\leq|\pi-\pi^0|\leq B}\frac{w^0\left(\alpha^0,\pi\right)}{|\pi-\pi^0|}<(1-\eta)\,d\right]\leq\varepsilon.$$

Lemma A.4 *For all $\eta > 0$ and $\varepsilon > 0$, there exists some $\bar{v} < \infty$ such that for any $0 < B < 1$,*

$$\Pr\left[\sup_{\frac{\bar{v}}{N^{\left(2\alpha^0-1\right)}T}\leq|\pi-\pi^0|\leq B}\frac{\left\|\mathbf{h}^0\left(\alpha^0,\pi\right)-\mathbf{h}^0\left(\alpha^0,\pi^0\right)\right\|}{|\pi-\pi^0|}>\eta\right]\leq\varepsilon.$$

Proof of Theorem 3.4. Let B and d be defined as in Lemma A.3. Pick $\eta > 0$ small enough so that

$$(1-\eta)\,d-2\eta>0. \tag{19}$$

Let $\mathbb{E}_{NT}$ be the joint event that $\left|\hat{\pi}-\pi^0\right|\leq B$, $\left\|\hat{\boldsymbol{\lambda}}_{ji}'\hat{\mathbf{f}}_t-\boldsymbol{\lambda}_{ji}^{0\prime}\mathbf{f}_t^0\right\|$ is small enough so that (22) below is satisfied, for $j = 1, 2$, $i = 1, \ldots, N$, $t = 1, \ldots, T$, and

$$\inf_{\frac{\bar{v}}{N^{\left(2\alpha^0-1\right)}T}\leq|\pi-\pi^0|\leq B}\frac{w^0\left(\alpha^0,\pi\right)}{|\pi-\pi^0|}\geq(1-\eta)\,d, \tag{20}$$

and

$$\sup_{\frac{\bar{v}}{N^{\left(2\alpha^0-1\right)}T}\leq|\pi-\pi^0|\leq B}\frac{\left\|\mathbf{h}^0\left(\alpha^0,\pi\right)-\mathbf{h}^0\left(\alpha^0,\pi^0\right)\right\|}{|\pi-\pi^0|}\leq\eta. \tag{21}$$

Fix $\varepsilon > 0$ and pick $\bar{v}$, $\bar{N}$ and $\bar{T}$ so that $\Pr\left(\mathbb{E}_{NT}\right)\geq 1-\varepsilon$ for all $N\geq\bar{N}$ and $T\geq\bar{T}$, which is possible under Corollary 3.1, and Lemmas A.3 and A.4. Given $S\left(\boldsymbol{\Lambda},\mathbf{F},\pi\right)$ defined in (2),

$$S\left(\alpha^0,\boldsymbol{\Lambda},\mathbf{F},\pi\right)=\frac{1}{N^{\alpha^0}T}\sum_{t=1}^{T}\left[\mathbf{x}_t-\boldsymbol{\Lambda}_1\mathbf{f}_t-\boldsymbol{\Delta}\mathbf{f}_{2t}\left(\pi\right)\right]'\left[\mathbf{x}_t-\boldsymbol{\Lambda}_1\mathbf{f}_t-\boldsymbol{\Delta}\mathbf{f}_{2t}\left(\pi\right)\right],$$

where $\mathbf{f}_{2t}\left(\pi\right)=\mathbb{I}_{2t}\left(\pi\right)\mathbf{f}_t$ and $\boldsymbol{\Delta}=\boldsymbol{\Lambda}_2-\boldsymbol{\Lambda}_1$. Since $S\left(\alpha^0,\boldsymbol{\Lambda},\mathbf{F},\pi\right)$ is continuous at $(\boldsymbol{\Lambda},\mathbf{F})$, for small enough $\left\|\hat{\boldsymbol{\lambda}}_{ji}'\hat{\mathbf{f}}_t-\boldsymbol{\lambda}_{ji}^{0\prime}\mathbf{f}_t^0\right\|$, for $j = 1, 2$, $i = 1, \ldots, N$, $t = 1, \ldots, T$, it follows that

$$\begin{aligned}
S\left(\alpha^0,\hat{\boldsymbol{\Lambda}},\hat{\mathbf{F}},\pi\right)-S\left(\alpha^0,\hat{\boldsymbol{\Lambda}},\hat{\mathbf{F}},\pi^0\right) &= \frac{1}{N^{\alpha^0}T}\sum_{t=1}^{T}\left[\mathbf{x}_t-\hat{\boldsymbol{\Lambda}}_1\hat{\mathbf{f}}_t-\hat{\boldsymbol{\Delta}}\hat{\mathbf{f}}_{2t}\left(\pi\right)\right]'\left[\mathbf{x}_t-\hat{\boldsymbol{\Lambda}}_1\hat{\mathbf{f}}_t-\hat{\boldsymbol{\Delta}}\hat{\mathbf{f}}_{2t}\left(\pi\right)\right]\\
&\quad-\frac{1}{N^{\alpha^0}T}\sum_{t=1}^{T}\left[\mathbf{x}_t-\hat{\boldsymbol{\Lambda}}_1\hat{\mathbf{f}}_t-\hat{\boldsymbol{\Delta}}\hat{\mathbf{f}}_{2t}\left(\pi^0\right)\right]'\left[\mathbf{x}_t-\hat{\boldsymbol{\Lambda}}_1\hat{\mathbf{f}}_t-\hat{\boldsymbol{\Delta}}\hat{\mathbf{f}}_{2t}\left(\pi^0\right)\right]\\
&= D\left\{\begin{array}{c}\frac{1}{N^{\alpha^0}T}\sum_{t=1}^{T}\left[\mathbf{x}_t-\boldsymbol{\Lambda}_1^0\mathbf{f}_t^0-\boldsymbol{\Delta}^0\mathbf{f}_{2t}^0\left(\pi\right)\right]'\left[\mathbf{x}_t-\boldsymbol{\Lambda}_1^0\mathbf{f}_t^0-\boldsymbol{\Delta}^0\mathbf{f}_{2t}^0\left(\pi\right)\right]\\-\frac{1}{N^{\alpha^0}T}\sum_{t=1}^{T}\left[\mathbf{x}_t-\boldsymbol{\Lambda}_1^0\mathbf{f}_t^0-\boldsymbol{\Delta}^0\mathbf{f}_{2t}^0\left(\pi^0\right)\right]'\left[\mathbf{x}_t-\boldsymbol{\Lambda}_1^0\mathbf{f}_t^0-\boldsymbol{\Delta}^0\mathbf{f}_{2t}^0\left(\pi^0\right)\right]\end{array}\right\}\\
&= D\left[S\left(\alpha^0,\boldsymbol{\Lambda}^0,\mathbf{F}^0,\pi\right)-S\left(\alpha^0,\boldsymbol{\Lambda}^0,\mathbf{F}^0,\pi^0\right)\right],
\end{aligned} \tag{22}$$

for some $D > 0$, where $\hat{\mathbf{f}}_{2t}\left(\pi\right)=\mathbb{I}_{2t}\left(\pi\right)\hat{\mathbf{f}}_t$, $\hat{\boldsymbol{\Delta}}=\hat{\boldsymbol{\Lambda}}_2-\hat{\boldsymbol{\Lambda}}_1$, $\mathbf{f}_{2t}^0\left(\pi\right)=\mathbb{I}_{2t}\left(\pi\right)\mathbf{f}_t^0$ and $\boldsymbol{\Delta}^0=\boldsymbol{\Lambda}_2^0-\boldsymbol{\Lambda}_1^0$: the sign of $S\left(\alpha^0,\hat{\boldsymbol{\Lambda}},\hat{\mathbf{F}},\pi\right)-S\left(\alpha^0,\hat{\boldsymbol{\Lambda}},\hat{\mathbf{F}},\pi^0\right)$ is then equal to the sign of $S\left(\alpha^0,\boldsymbol{\Lambda}^0,\mathbf{F}^0,\pi\right)-S\left(\alpha^0,\boldsymbol{\Lambda}^0,\mathbf{F}^0,\pi^0\right)$. We have

$$\begin{aligned}
S\left(\alpha^0,\boldsymbol{\Lambda}^0,\mathbf{F}^0,\pi\right)-S\left(\alpha^0,\boldsymbol{\Lambda}^0,\mathbf{F}^0,\pi^0\right) &= \frac{1}{N^{\alpha^0}T}\sum_{t=1}^{T}\left[\mathbf{f}_{2t}^0\left(\pi\right)-\mathbf{f}_{2t}^0\left(\pi^0\right)\right]'\boldsymbol{\Delta}^{0\prime}\boldsymbol{\Delta}^0\left[\mathbf{f}_{2t}^0\left(\pi\right)-\mathbf{f}_{2t}^0\left(\pi^0\right)\right]\\
&\quad-2\frac{1}{N^{\alpha^0}T}\sum_{t=1}^{T}\left[\mathbf{f}_{2t}^0\left(\pi\right)-\mathbf{f}_{2t}^0\left(\pi^0\right)\right]'\boldsymbol{\Delta}^{0\prime}\mathbf{e}_t\\
&= S_1\left(\alpha^0,\pi\right)+S_2\left(\alpha^0,\pi\right)
\end{aligned}$$

and

$$\begin{aligned}\frac{S\left(\alpha^0,\mathbf{\Lambda}^0,\mathbf{F}^0,\pi\right)-S\left(\alpha^0,\mathbf{\Lambda}^0,\mathbf{F}^0,\pi^0\right)}{\left|\pi-\pi^0\right|} &= \frac{1}{N^{\alpha^0}T\left|\pi-\pi^0\right|}\sum_{t=1}^{T}\left[\mathbf{f}_{2t}^0\left(\pi\right)-\mathbf{f}_{2t}^0\left(\pi^0\right)\right]'\mathbf{\Delta}^{0\prime}\mathbf{\Delta}^0\left[\mathbf{f}_{2t}^0\left(\pi\right)-\mathbf{f}_{2t}^0\left(\pi^0\right)\right] \\ &\quad -2\frac{1}{N^{\alpha^0}T\left|\pi-\pi^0\right|}\sum_{t=1}^{T}\left[\mathbf{f}_{2t}^0\left(\pi\right)-\mathbf{f}_{2t}^0\left(\pi^0\right)\right]'\mathbf{\Delta}^{0\prime}\mathbf{e}_t \\ &= \frac{S_1\left(\alpha^0,\pi\right)}{\left|\pi-\pi^0\right|}+\frac{S_2\left(\alpha^0,\pi\right)}{\left|\pi-\pi^0\right|}.\end{aligned} \tag{23}$$

Suppose $\pi\in\left[\pi^0+\bar{v}N^{-\left(2\alpha^0-1\right)}T^{-1},\pi^0+B\right]$ and that event $\mathbb{E}_{NT}$ holds. It follows that

$$\begin{aligned}\frac{S_1\left(\alpha^0,\pi\right)}{\pi-\pi^0} &= \frac{1}{N^{\alpha^0}T\left(\pi-\pi^0\right)}\sum_{i=1}^{N}\sum_{t=1}^{T}\left[\mathbf{f}_{2t}^0\left(\pi\right)-\mathbf{f}_{2t}^0\left(\pi^0\right)\right]'\boldsymbol{\delta}_i^0\boldsymbol{\delta}_i^{0\prime}\left[\mathbf{f}_{2t}^0\left(\pi\right)-\mathbf{f}_{2t}^0\left(\pi^0\right)\right] \\ &= \frac{1}{N^{\alpha^0}T\left(\pi-\pi^0\right)}\sum_{i=1}^{N}\sum_{t=1}^{T}\left|\mathbb{I}_{2t}\left(\pi\right)-\mathbb{I}_{2t}\left(\pi^0\right)\right|\left(\boldsymbol{\delta}_i^{0\prime}\mathbf{f}_t^0\right)^2 \\ &= \frac{w^0\left(\alpha^0,\pi\right)}{\pi-\pi^0},\end{aligned} \tag{24}$$

and

$$\begin{aligned}\frac{S_2\left(\alpha^0,\pi\right)}{\pi-\pi^0} &= -2\frac{1}{N^{\alpha^0}T\left(\pi-\pi^0\right)}\sum_{t=1}^{T}\left[\mathbf{f}_{2t}^0\left(\pi\right)-\mathbf{f}_{2t}^0\left(\pi^0\right)\right]'\mathbf{\Delta}^{0\prime}\mathbf{e}_t \\ &= -2\frac{1}{N^{\alpha^0}T\left(\pi-\pi^0\right)}\sum_{i=1}^{N}\sum_{t=1}^{T}\left[\mathbf{f}_{2t}^0\left(\pi\right)-\mathbf{f}_{2t}^0\left(\pi^0\right)\right]'\boldsymbol{\delta}_i^0 e_{it} \\ &\geq -2\frac{1}{\pi-\pi^0}\left\|\frac{1}{N^{\alpha^0}T}\sum_{i=1}^{N}\sum_{t=1}^{T}\left[\mathbf{f}_{2t}^0\left(\pi\right)-\mathbf{f}_{2t}^0\left(\pi^0\right)\right]'\boldsymbol{\delta}_i^0 e_{it}\right\| \\ &= -2\frac{\left\|\mathbf{h}^0\left(\alpha^0,\pi\right)-\mathbf{h}^0\left(\alpha^0,\pi^0\right)\right\|}{\pi-\pi^0}.\end{aligned} \tag{25}$$

By (19) through (25) it follows that for some $D>0$,

$$\frac{S\left(\alpha^0,\hat{\mathbf{\Lambda}},\hat{\mathbf{F}},\pi\right)-S\left(\alpha^0,\hat{\mathbf{\Lambda}},\hat{\mathbf{F}},\pi^0\right)}{\pi-\pi^0}\geq D\left[\frac{w^0\left(\alpha^0,\pi\right)}{\pi-\pi^0}-2\frac{\left\|\mathbf{h}^0\left(\alpha^0,\pi\right)-\mathbf{h}^0\left(\alpha^0,\pi^0\right)\right\|}{\pi-\pi^0}\right]\geq D\left[\left(1-\eta\right)d-2\eta\right]\geq 0.$$

Given the event $\mathbb{E}_{NT}$, if $\pi\in\left[\pi^0+\bar{v}N^{-\left(2\alpha^0-1\right)}T^{-1},\pi^0+B\right]$ then $S\left(\alpha^0,\hat{\mathbf{\Lambda}},\hat{\mathbf{F}},\pi\right)-S\left(\alpha^0,\hat{\mathbf{\Lambda}},\hat{\mathbf{F}},\pi^0\right)>0$. In a similar way, it can be shown that if $\pi\in\left[\pi^0-B,\pi^0-\bar{v}N^{-\left(2\alpha^0-1\right)}T^{-1}\right]$ then $S\left(\alpha^0,\hat{\mathbf{\Lambda}},\hat{\mathbf{F}},\pi\right)-S\left(\alpha^0,\hat{\mathbf{\Lambda}},\hat{\mathbf{F}},\pi^0\right)>0$. As $S\left(\alpha^0,\hat{\mathbf{\Lambda}},\hat{\mathbf{F}},\hat{\pi}\right)-S\left(\alpha^0,\hat{\mathbf{\Lambda}},\hat{\mathbf{F}},\pi^0\right)\leq 0$, if $\mathbb{E}_{NT}$ occurs then $\left|\hat{\pi}-\pi^0\right|\leq\bar{v}N^{-\left(2\alpha^0-1\right)}T^{-1}$: since $\Pr\left(\mathbb{E}_{NT}\right)\geq 1-\varepsilon$ for $N\geq\bar{N}$ and $T\geq\bar{T}$, then $\Pr\left(\left|\hat{\pi}-\pi^0\right|>\bar{v}N^{-\left(2\alpha^0-1\right)}T^{-1}\right)\leq\varepsilon$ for $N\geq\bar{N}$ and $T\geq\bar{T}$: this is sufficient to show that $N^{\left(2\alpha^0-1\right)}T\left(\hat{\pi}-\pi^0\right)=O_p\left(1\right)$. The convergence rate of the estimator for the loadings follows from (11). ■

Proof of Corollary 3.2. Corollary 3.2 easily follows from Theorem 3.4 and the proof is omitted. ■

Proof of Lemma A.1. We show (17): the proof of (18) is analogous. Under Assumption CR(b), for some $C_1<\infty$,

$$\begin{aligned}\mathrm{E}\left\{\left[g_{it}^0\left(\pi_1,\pi_2\right)\right]^s\right\} &= \mathrm{E}\left[\left|\mathbb{I}_{2t}\left(\pi_2\right)-\mathbb{I}_{2t}\left(\pi_1\right)\right|\left\|\mathbf{f}_t^0 e_{it}\right\|^s\right] \\ &= \left|\mathbb{I}_{2t}\left(\pi_2\right)-\mathbb{I}_{2t}\left(\pi_1\right)\right|\mathrm{E}\left(\left\|\mathbf{f}_t^0 e_{it}\right\|^s\right) \\ &\leq C_1\left|\pi_2-\pi_1\right|.\end{aligned}$$

■

Proof of Lemma A.2. Lemma 3.4 in Peligrad (1982) shows that under Assumption CR(a) there exists a $K'<\infty$ such that, taking into account (18) in Lemma A.1,

$$\begin{aligned}\mathrm{E}\left\{\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left\{\left[q_t^0\left(\pi_1,\pi_2\right)\right]^2-\mathrm{E}\left\{\left[q_t^0\left(\pi_1,\pi_2\right)\right]^2\right\}\right\}\right|^2\right\} &\leq K'\frac{1}{T}\sum_{t=1}^{T}\mathrm{E}\left\{\left\{\left[q_t^0\left(\pi_1,\pi_2\right)\right]^2-\mathrm{E}\left\{\left[q_t^0\left(\pi_1,\pi_2\right)\right]^2\right\}\right\}^2\right\} \\ &\leq 2K'\frac{1}{T}\sum_{t=1}^{T}\mathrm{E}\left\{\left[q_t^0\left(\pi_1,\pi_2\right)\right]^4\right\} \\ &\leq 2K'C_1\left|\pi_2-\pi_1\right|:\end{aligned}$$

setting $K = 2K'C_1$ completes the proof of the lemma. ■

Proof of Lemma A.3. Assumption C1 is sufficient to avoid multicollinearity in the factors, so that $\mathrm{E}\left[w^0\left(\alpha^0,\pi\right)\right] = 0$ if and only if $\pi = \pi^0$. It follows that

$$\inf_{\left|\pi-\pi^0\right|\leq B} \mathrm{E}\left[w^0\left(\alpha^0,\pi\right)\right] \geq d\left|\pi-\pi^0\right|, \tag{26}$$

for some $0 < B < 1$ and $0 < d < \infty$. Notice that

$$\begin{aligned}
\mathrm{E}\left\{\left|w^0\left(\alpha^0,\pi\right) - \mathrm{E}\left[w^0\left(\alpha^0,\pi\right)\right]\right|^2\right\} &= \mathrm{E}\left\{\left|\frac{1}{N^{\alpha^0}}\frac{1}{T}\sum_{i=1}^{N}\sum_{t=1}^{T}\left\{w_{it}^0\left(\pi\right) - \mathrm{E}\left[w_{it}^0\left(\pi\right)\right]\right\}\right|^2\right\} \\
&\leq \frac{C_2}{N^{2\alpha^0}}\sum_{i=1}^{N}\mathrm{E}\left\{\left|\frac{1}{T}\sum_{t=1}^{T}\left\{w_{it}^0\left(\pi\right) - \mathrm{E}\left[w_{it}^0\left(\pi\right)\right]\right\}\right|^2\right\},
\end{aligned}$$

for some $C_2 < \infty$, and

$$\begin{aligned}
\mathrm{E}\left\{\left|\frac{1}{T}\sum_{t=1}^{T}\left\{w_{it}^0\left(\pi\right) - \mathrm{E}\left[w_{it}^0\left(\pi\right)\right]\right\}\right|^2\right\} &\leq \left\|\boldsymbol{\delta}_i^0\right\|^4 T^{-1}\mathrm{E}\left\{\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left\{\left[q_t^0\left(\pi,\pi^0\right)\right]^2 - \mathrm{E}\left\{\left[q_t^0\left(\pi,\pi^0\right)\right]^2\right\}\right\}\right|^2\right\}, \quad i = 1,\ldots,N, \\
&\leq \left\|\boldsymbol{\delta}_i^0\right\|^4 T^{-1}K\left|\pi-\pi^0\right|
\end{aligned}$$

by Lemma A.2: since

$$\left\|\boldsymbol{\delta}_i^0\right\| = \left\|\boldsymbol{\lambda}_{2i}^0 - \boldsymbol{\lambda}_{1i}^0\right\| \leq \left\|\boldsymbol{\lambda}_{1i}^0\right\| + \left\|\boldsymbol{\lambda}_{2i}^0\right\| \leq 2\bar{\lambda}, \quad i = 1,\ldots,N, \tag{27}$$

by Assumption C2, it follows that

$$\mathrm{E}\left\{\left|w^0\left(\alpha^0,\pi\right) - \mathrm{E}\left[w^0\left(\alpha^0,\pi\right)\right]\right|^2\right\} \leq \frac{C_2 16\bar{\lambda}^4}{N^{\left(2\alpha^0-1\right)}T}K\left|\pi-\pi^0\right|. \tag{28}$$

For any η and ε, set

$$b = \frac{1-\eta/2}{1-\eta} > 1 \tag{29}$$

and

$$\bar{v} = \frac{8C_2 16\bar{\lambda}^4 K}{\eta^2 d^2\left(1-1/b\right)^2\varepsilon}. \tag{30}$$

Assume N and T large enough so that $\bar{v}\Big/\left[N^{\left(2\alpha^0-1\right)}T\right] \leq B < 1$, otherwise the lemma is trivially satisfied. For $l_N = 1,\ldots,N+1$ and $l_T = 1,\ldots,T+1$, set $\pi_{l_N l_T} = \pi^0 + \bar{v}b^{l_N-1}b^{l_T-1}\Big/\left[N^{\left(2\alpha^0-1\right)}T\right]$, where N and T are integers such that $\pi_{NT} - \pi^0 = \bar{v}b^{N-1}b^{T-1}\Big/\left[N^{\left(2\alpha^0-1\right)}T\right] \leq B < 1$, $B < \pi_{N+1,T} - \pi^0 < 1$ and $B < \pi_{N,T+1} - \pi^0 < 1$ (since $\bar{v}\Big/\left[N^{\left(2\alpha^0-1\right)}T\right] \leq B$ then $NT \geq 1$). By Markov's inequality, (26), (28) and (30),

$$\begin{aligned}
\Pr\left\{\sup_{\substack{1\leq l_N\leq N,\\ 1\leq l_T\leq T}}\left|\frac{w^0\left(\alpha^0,\pi_{l_N l_T}\right)}{\mathrm{E}\left[w^0\left(\alpha^0,\pi_{l_N l_T}\right)\right]} - 1\right| > \frac{\eta}{2}\right\} &\leq \left(\frac{2}{\eta}\right)^2\sum_{l_N=1}^{N}\sum_{l_T=1}^{T}\frac{\mathrm{E}\left\{\left|w^0\left(\alpha^0,\pi_{l_N l_T}\right) - \mathrm{E}\left[w^0\left(\alpha^0,\pi_{l_N l_T}\right)\right]\right|^2\right\}}{\left|\mathrm{E}\left[w^0\left(\alpha^0,\pi_{l_N l_T}\right)\right]\right|^2} \\
&\leq \frac{4}{\eta^2}\sum_{l_N=1}^{N}\sum_{l_T=1}^{T}\frac{C_2 N^{-\left(2\alpha^0-1\right)}T^{-1}16\bar{\lambda}^4 K\left(\pi_{l_N l_T} - \theta^0\right)}{d^2\left(\pi_{l_N l_T} - \theta^0\right)^2} \\
&\leq \frac{4}{\eta^2}\frac{C_2 16\bar{\lambda}^4 K}{d^2\bar{v}}\left(\sum_{l_N=0}^{\infty}\frac{1}{b^{l_N}}\right)\left(\sum_{l_T=0}^{\infty}\frac{1}{b^{l_T}}\right) \\
&= \frac{4}{\eta^2}\frac{C_2 16\bar{\lambda}^4 K}{d^2\bar{v}}\frac{1}{\left(1-1/b\right)^2} \leq \frac{\varepsilon}{2}:
\end{aligned}$$

it follows that for all $1 \leq l_N \leq N$ and $1 \leq l_T \leq T$, and with probability greater than $1-\varepsilon/2$,

$$\left|\frac{w^0\left(\alpha^0,\pi_{l_N l_T}\right)}{\mathrm{E}\left[w^0\left(\alpha^0,\pi_{l_N l_T}\right)\right]} - 1\right| \leq \frac{\eta}{2}. \tag{31}$$

Using (29), for any π such that $\bar{v}\Big/\left[N^{\left(2\alpha^0-1\right)}T\right] \leq \left(\pi-\pi^0\right) \leq B$, there exists some $l_N \leq N$ and $l_T \leq T$ such that $\pi_{l_N l_T} < \pi < \min\left\{\pi_{l_{N+1},l_T}, \pi_{l_N,l_{T+1}}\right\}$ and on the event (31)

$$\frac{w^0\left(\alpha^0,\pi\right)}{\left(\pi-\pi^0\right)} \geq \frac{w^0\left(\alpha^0,\pi_{l_N l_T}\right)}{\mathrm{E}\left[w^0\left(\alpha^0,\pi_{l_N l_T}\right)\right]} \frac{\mathrm{E}\left[w^0\left(\alpha^0,\pi_{l_N l_T}\right)\right]}{\left[\min\left\{\pi_{l_{N+1},l_T}, \pi_{l_N,l_{T+1}}\right\}-\pi^0\right]} \geq \left(1-\frac{\eta}{2}\right) \frac{d\left(\pi_{l_N l_T}-\pi^0\right)}{\left[\min\left\{\pi_{l_{N+1},l_T}, \pi_{l_N,l_{T+1}}\right\}-\pi^0\right]} = (1-\eta)\, d,$$

where we set $\left(\pi_{l_N l_T}-\pi^0\right)\Big/\left[\min\left\{\pi_{l_{N+1},l_T}, \pi_{l_N,l_{T+1}}\right\}-\pi^0\right] = 1/b$: this event has probability greater than $1-\varepsilon/2$ and

$$\Pr\left[\inf_{\frac{\bar{v}}{N^{\left(2\alpha^0-1\right)}T} \leq \left(\pi-\pi^0\right) \leq B} \frac{w^0\left(\alpha^0,\pi\right)}{\left(\pi-\pi^0\right)} < (1-\eta)\, d\right] \leq \frac{\varepsilon}{2},$$

holds. Taking the infimum over $-\bar{v}\Big/\left[N^{\left(2\alpha^0-1\right)}T\right] \geq \left(\pi-\pi^0\right) \geq -B$ allows to prove a similar inequality using the same argument: this completes the proof of the lemma. ■

Proof of Lemma A.4. Given some $C_3 < \infty$ to be determined later, fix $\eta > 0$ and set

$$\bar{v} = \frac{8}{(0.5)^2\,(0.5)^2} \frac{C_1 C_3 \bar{\lambda}^2}{\eta^2 \varepsilon}. \tag{32}$$

For $l_N = 1,\ldots,N$ and $l_T = 1,\ldots,T$, set $\pi_{l_N l_T} - \pi^0 = \bar{v} 2^{l_N-1} 2^{l_T-1}\Big/\left[N^{\left(2\alpha^0-1\right)}T\right] \leq B$. Markov's inequality, (17) in Lemma A.1, (27) and (32) ensure that

$$\begin{aligned}
\Pr\left[\sup_{\substack{1\leq l_N\leq N,\\ 1\leq l_T\leq T}} \frac{\left\|\mathbf{h}^0\left(\alpha^0,\pi_{l_N l_T}\right)-\mathbf{h}^0\left(\alpha^0,\pi^0\right)\right\|}{\left(\pi_{l_N l_T}-\pi^0\right)} > \eta\right] &\leq \frac{1}{\eta^2}\sum_{l_N=1}^{N}\sum_{l_T=1}^{T} \frac{\mathrm{E}\left[\left\|\mathbf{h}^0\left(\alpha^0,\pi_{l_N l_T}\right)-\mathbf{h}^0\left(\alpha^0,\pi^0\right)\right\|^2\right]}{\left(\pi_{l_N l_T}-\pi^0\right)^2} \\
&\leq \frac{1}{\eta^2}\sum_{l_N=1}^{N}\sum_{l_T=1}^{T} \frac{\mathrm{E}\left\{\left\|\frac{1}{N^{\alpha^0}}\frac{1}{T}\sum_{i=1}^{N}\sum_{t=1}^{T}\left[\mathbb{I}_{2t}\left(\pi_{l_N l_T}\right)-\mathbb{I}_{2t}\left(\pi^0\right)\right]\boldsymbol{\delta}_i^{0\prime}\mathbf{f}_t^0 e_{it}\right\|^2\right\}}{\left(\pi_{l_N l_T}-\pi^0\right)^2} \\
&\leq \frac{C_3}{\eta^2}\frac{1}{N^{2\alpha^0}}\frac{1}{T^2}\sum_{l_N=1}^{N}\sum_{l_T=1}^{T}\sum_{i=1}^{N}\sum_{t=1}^{T} \frac{\mathrm{E}\left\{\left\|\left[\mathbb{I}_{2t}\left(\pi_{l_N l_T}\right)-\mathbb{I}_{2t}\left(\pi^0\right)\right]\boldsymbol{\delta}_i^{0\prime}\mathbf{f}_t^0 e_{it}\right\|^2\right\}}{\left(\pi_{l_N l_T}-\pi^0\right)^2} \\
&\leq \frac{C_3}{\eta^2}\frac{1}{N^{2\alpha^0}}\frac{1}{T}\sum_{l_N=1}^{N}\sum_{l_T=1}^{T}\sum_{i=1}^{N} \frac{\left\|\boldsymbol{\delta}_i^0\right\|^2 \mathrm{E}\left\{\left[\mathbb{I}_{2t}\left(\pi_{l_N l_T}\right)-\mathbb{I}_{2t}\left(\pi^0\right)\right]\left\|\mathbf{f}_t^0 e_{it}\right\|^2\right\}}{\left(\pi_{l_N l_T}-\pi^0\right)^2} \\
&\leq \frac{C_3}{\eta^2}\frac{1}{N^{\left(2\alpha^0-1\right)}}\frac{1}{T}\sum_{l_N=1}^{N}\sum_{l_T=1}^{T}\left[4\bar{\lambda}^2\frac{C_1\left(\pi_{l_N l_T}-\pi^0\right)}{\left(\pi_{l_N l_T}-\pi^0\right)^2}\right] \\
&= 4\frac{C_1 C_3\bar{\lambda}^2}{\eta^2\bar{v}}\left[\sum_{l_N=1}^{N}\frac{1}{\left(2^{l_N-1}\right)}\right]\left[\sum_{l_T=1}^{T}\frac{1}{\left(2^{l_T-1}\right)}\right] \\
&\leq \frac{4}{(0.5)^2\,(0.5)^2}\frac{C_1 C_3\bar{\lambda}^2}{\eta^2\bar{v}} \leq \frac{\varepsilon}{2}.
\end{aligned}$$

It follows that for all $1 \leq l_N \leq N$ and $1 \leq l_T \leq T$, and with probability greater than $1-\varepsilon/2$,

$$\frac{\left\|\mathbf{h}^0\left(\alpha^0,\pi_{l_N l_T}\right)-\mathbf{h}^0\left(\alpha^0,\pi^0\right)\right\|}{\left(\pi_{l_N l_T}-\pi^0\right)} \leq \eta,$$

which implies that

$$\Pr\left[\sup_{\frac{\bar{v}}{N^{\left(2\alpha^0-1\right)}T} \leq \left(\pi-\pi^0\right) \leq B} \frac{\left\|\mathbf{h}^0\left(\alpha^0,\pi\right)-\mathbf{h}^0\left(\alpha^0,\pi^0\right)\right\|}{\left(\pi-\pi^0\right)} > \eta\right] \leq \frac{\varepsilon}{2}.$$

Taking the infimum over $-\bar{v}\Big/\left[N^{\left(2\alpha^0-1\right)}T\right] \geq \left(\pi-\pi^0\right) \geq -B$ allows to prove a similar inequality using the same argument, which completes the proof. ■

A.3 Proof of the Result in Section 4

TO BE DONE

A.4 Proof of the Result in Section 5

Lemma A.5 ***TO BE COMPLETED:***

(a) $C_{NT}^2\left\{\frac{1}{T}\sum_{t=1}^{T}\left\{\hat{\mathbf{f}}_t-\left[\mathbb{I}_{1t}\left(\pi^0\right)\hat{\mathbf{H}}_{11}\left(\pi^0\right)+\mathbb{I}_{2t}\left(\pi^0\right)\hat{\mathbf{H}}_{22}\left(\pi^0\right)\right]^{-1}\mathbf{f}_t^0\right\}\varepsilon_{t+h}\right\}=O_p(1);$

(b) $C_{NT}^2\left\{\frac{1}{T}\sum_{t=1}^{T}\left\{\begin{array}{c}\hat{\mathbf{z}}_t\left\{\left[\mathbb{I}_{1t}\left(\pi^0\right)\hat{\mathbf{H}}_{11}\left(\pi^0\right)+\mathbb{I}_{2t}\left(\pi^0\right)\hat{\mathbf{H}}_{22}\left(\pi^0\right)\right]^{-1}\mathbf{f}_t^0-\hat{\mathbf{f}}_t\right\}'\\ \times\left[\mathbb{I}_{1t}\left(\pi^0\right)\hat{\mathbf{H}}_{11}\left(\pi^0\right)+\mathbb{I}_{2t}\left(\pi^0\right)\hat{\mathbf{H}}_{22}\left(\pi^0\right)\right]'\end{array}\right\}\right\}=O_p(1).$

Proof of Theorem 5.1. Adding and subtracting terms, the model may be written as

$$\begin{aligned}
y_{t+h} &= \boldsymbol{\gamma}^{0\prime}\mathbf{f}_t^0+\boldsymbol{\beta}^{0\prime}\mathbf{w}_t+\varepsilon_{t+h}\\
&= \boldsymbol{\gamma}^{0\prime}\left[\mathbb{I}_{1t}\left(\pi^0\right)\hat{\mathbf{H}}_{11}\left(\pi^0\right)+\mathbb{I}_{2t}\left(\pi^0\right)\hat{\mathbf{H}}_{22}\left(\pi^0\right)\right]\hat{\mathbf{f}}_t+\boldsymbol{\beta}^{0\prime}\mathbf{w}_t+\varepsilon_{t+h}\\
&\quad +\boldsymbol{\gamma}^{0\prime}\left[\mathbb{I}_{1t}\left(\pi^0\right)\hat{\mathbf{H}}_{11}\left(\pi^0\right)+\mathbb{I}_{2t}\left(\pi^0\right)\hat{\mathbf{H}}_{22}\left(\pi^0\right)\right]\left\{\left[\mathbb{I}_{1t}\left(\pi^0\right)\hat{\mathbf{H}}_{11}\left(\pi^0\right)+\mathbb{I}_{2t}\left(\pi^0\right)\hat{\mathbf{H}}_{22}\left(\pi^0\right)\right]^{-1}\mathbf{f}_t^0-\hat{\mathbf{f}}_t\right\}\quad,\quad t=1,\ldots,T,\quad h\geq 0.\\
&= \hat{\mathbf{z}}_t'\left[\mathbb{I}_{1t}\left(\pi^0\right)\boldsymbol{\theta}_1^0\left(\pi^0\right)+\mathbb{I}_{2t}\left(\pi^0\right)\boldsymbol{\theta}_2^0\left(\pi^0\right)\right]+\varepsilon_{t+h}\\
&= +\left\{\left[\mathbb{I}_{1t}\left(\pi^0\right)\hat{\mathbf{H}}_{11}\left(\pi^0\right)+\mathbb{I}_{2t}\left(\pi^0\right)\hat{\mathbf{H}}_{22}\left(\pi^0\right)\right]^{-1}\mathbf{f}_t^0-\hat{\mathbf{f}}_t\right\}'\left[\mathbb{I}_{1t}\left(\pi^0\right)\hat{\mathbf{H}}_{11}\left(\pi^0\right)+\mathbb{I}_{2t}\left(\pi^0\right)\hat{\mathbf{H}}_{22}\left(\pi^0\right)\right]'\boldsymbol{\gamma}^0
\end{aligned}$$

The least squares estimator then is

$$\begin{aligned}
\widehat{\boldsymbol{\theta}} &= \begin{pmatrix}\widehat{\boldsymbol{\gamma}}\\ \widehat{\boldsymbol{\beta}}\end{pmatrix}=\left(\sum_{t=1}^{T}\hat{\mathbf{z}}_t\hat{\mathbf{z}}_t'\right)^{-1}\left(\sum_{t=1}^{T}\hat{\mathbf{z}}_t y_{t+h}\right)\\
&= \left(\sum_{t=1}^{T}\hat{\mathbf{z}}_t\hat{\mathbf{z}}_t'\right)^{-1}\left[\sum_{t=1}^{T}\mathbb{I}_{1t}\left(\pi^0\right)\hat{\mathbf{z}}_t\hat{\mathbf{z}}_t'\right]\boldsymbol{\theta}_1^0\left(\pi^0\right)+\left(\sum_{t=1}^{T}\hat{\mathbf{z}}_t\hat{\mathbf{z}}_t'\right)^{-1}\left[\sum_{t=1}^{T}\mathbb{I}_{2t}\left(\pi^0\right)\hat{\mathbf{z}}_t\hat{\mathbf{z}}_t'\right]\boldsymbol{\theta}_2^0\left(\pi^0\right)\\
&\quad +\left(\sum_{t=1}^{T}\hat{\mathbf{z}}_t\hat{\mathbf{z}}_t'\right)^{-1}\left(\sum_{t=1}^{T}\hat{\mathbf{z}}_t\varepsilon_{t+h}\right)\\
&\quad +\left(\sum_{t=1}^{T}\hat{\mathbf{z}}_t\hat{\mathbf{z}}_t'\right)^{-1}\left\{\sum_{t=1}^{T}\left\{\begin{array}{c}\hat{\mathbf{z}}_t\left\{\left[\mathbb{I}_{1t}\left(\pi^0\right)\hat{\mathbf{H}}_{11}\left(\pi^0\right)+\mathbb{I}_{2t}\left(\pi^0\right)\hat{\mathbf{H}}_{22}\left(\pi^0\right)\right]^{-1}\mathbf{f}_t^0-\hat{\mathbf{f}}_t\right\}'\\ \times\left[\mathbb{I}_{1t}\left(\pi^0\right)\hat{\mathbf{H}}_{11}\left(\pi^0\right)+\mathbb{I}_{2t}\left(\pi^0\right)\hat{\mathbf{H}}_{22}\left(\pi^0\right)\right]'\end{array}\right\}\right\}\boldsymbol{\gamma}^0\\
&= \hat{\pi}\left(\pi^0\right)\boldsymbol{\theta}_1^0\left(\pi^0\right)+\left[1-\hat{\pi}\left(\pi^0\right)\right]\boldsymbol{\theta}_2^0\left(\pi^0\right)\\
&\quad +\left(\sum_{t=1}^{T}\hat{\mathbf{z}}_t\hat{\mathbf{z}}_t'\right)^{-1}\left(\sum_{t=1}^{T}\hat{\mathbf{z}}_t\varepsilon_{t+h}\right)\\
&\quad +\left(\sum_{t=1}^{T}\hat{\mathbf{z}}_t\hat{\mathbf{z}}_t'\right)^{-1}\left\{\sum_{t=1}^{T}\left\{\begin{array}{c}\hat{\mathbf{z}}_t\left\{\left[\mathbb{I}_{1t}\left(\pi^0\right)\hat{\mathbf{H}}_{11}\left(\pi^0\right)+\mathbb{I}_{2t}\left(\pi^0\right)\hat{\mathbf{H}}_{22}\left(\pi^0\right)\right]^{-1}\mathbf{f}_t^0-\hat{\mathbf{f}}_t\right\}'\\ \times\left[\mathbb{I}_{1t}\left(\pi^0\right)\hat{\mathbf{H}}_{11}\left(\pi^0\right)+\mathbb{I}_{2t}\left(\pi^0\right)\hat{\mathbf{H}}_{22}\left(\pi^0\right)\right]'\end{array}\right\}\right\}\boldsymbol{\gamma}^0
\end{aligned}$$

so that

$$\begin{aligned}
&\widehat{\boldsymbol{\theta}}-\left\{\hat{\pi}\left(\pi^0\right)\boldsymbol{\theta}_1^0\left(\pi^0\right)+\left[1-\hat{\pi}\left(\pi^0\right)\right]\boldsymbol{\theta}_2^0\left(\pi^0\right)\right\}\\
= &\left(\sum_{t=1}^{T}\hat{\mathbf{z}}_t\hat{\mathbf{z}}_t'\right)^{-1}\left(\sum_{t=1}^{T}\hat{\mathbf{z}}_t\varepsilon_{t+h}\right)\\
&+\left(\sum_{t=1}^{T}\hat{\mathbf{z}}_t\hat{\mathbf{z}}_t'\right)^{-1}\left\{\sum_{t=1}^{T}\left\{\begin{array}{c}\hat{\mathbf{z}}_t\left\{\left[\mathbb{I}_{1t}\left(\pi^0\right)\hat{\mathbf{H}}_{11}\left(\pi^0\right)+\mathbb{I}_{2t}\left(\pi^0\right)\hat{\mathbf{H}}_{22}\left(\pi^0\right)\right]^{-1}\mathbf{f}_t^0-\hat{\mathbf{f}}_t\right\}'\\ \times\left[\mathbb{I}_{1t}\left(\pi^0\right)\hat{\mathbf{H}}_{11}\left(\pi^0\right)+\mathbb{I}_{2t}\left(\pi^0\right)\hat{\mathbf{H}}_{22}\left(\pi^0\right)\right]'\end{array}\right\}\right\}\boldsymbol{\gamma}^0
\end{aligned}$$

or

$$
\begin{array}{ll}
 & T^{1/2}\left\{\widehat{\boldsymbol{\theta}}-\left\{\hat{\pi}\left(\pi^{0}\right)\boldsymbol{\theta}_{1}^{0}\left(\pi^{0}\right)+\left[1-\hat{\pi}\left(\pi^{0}\right)\right]\boldsymbol{\theta}_{2}^{0}\left(\pi^{0}\right)\right\}\right\} \\
= & \left(T^{-1}\sum_{t=1}^{T}\hat{\mathbf{z}}_{t}\hat{\mathbf{z}}_{t}'\right)^{-1}\left(T^{-1/2}\sum_{t=1}^{T}\hat{\mathbf{z}}_{t}\varepsilon_{t+h}\right) \\
 & +\left(T^{-1}\sum_{t=1}^{T}\hat{\mathbf{z}}_{t}\hat{\mathbf{z}}_{t}'\right)^{-1}\left\{T^{-1/2}\sum_{t=1}^{T}\left\{\begin{array}{l}\hat{\mathbf{z}}_{t}\left\{\left[\mathbb{I}_{1t}\left(\pi^{0}\right)\hat{\mathbf{H}}_{11}\left(\pi^{0}\right)+\mathbb{I}_{2t}\left(\pi^{0}\right)\hat{\mathbf{H}}_{22}\left(\pi^{0}\right)\right]^{-1}\mathbf{f}_{t}^{0}-\hat{\mathbf{f}}_{t}\right\}' \\ \times\left[\mathbb{I}_{1t}\left(\pi^{0}\right)\hat{\mathbf{H}}_{11}\left(\pi^{0}\right)+\mathbb{I}_{2t}\left(\pi^{0}\right)\hat{\mathbf{H}}_{22}\left(\pi^{0}\right)\right]'\end{array}\right\}\right\}\boldsymbol{\gamma}^{0}
\end{array}. \tag{33}
$$

Consider

$$
T^{-1/2}\sum_{t=1}^{T}\hat{\mathbf{z}}_{t}\varepsilon_{t+h}=T^{-1/2}\sum_{t=1}^{T}\left(\hat{\mathbf{f}}_{t}',\mathbf{w}_{t}'\right)'\varepsilon_{t+h}
$$

and notice that

$$
\begin{aligned}
T^{-1/2}\sum_{t=1}^{T}\hat{\mathbf{f}}_{t}\varepsilon_{t+h} &= T^{-1/2}\sum_{t=1}^{T}\left[\mathbb{I}_{1t}\left(\pi^{0}\right)\hat{\mathbf{H}}_{11}\left(\pi^{0}\right)+\mathbb{I}_{2t}\left(\pi^{0}\right)\hat{\mathbf{H}}_{22}\left(\pi^{0}\right)\right]^{-1}\mathbf{f}_{t}^{0}\varepsilon_{t+h} \\
&\quad +T^{-1/2}\sum_{t=1}^{T}\left\{\hat{\mathbf{f}}_{t}-\left[\mathbb{I}_{1t}\left(\pi^{0}\right)\hat{\mathbf{H}}_{11}\left(\pi^{0}\right)+\mathbb{I}_{2t}\left(\pi^{0}\right)\hat{\mathbf{H}}_{22}\left(\pi^{0}\right)\right]^{-1}\mathbf{f}_{t}^{0}\right\}\varepsilon_{t+h} \\
&= T^{-1/2}\sum_{t=1}^{T}\left[\mathbb{I}_{1t}\left(\pi^{0}\right)\hat{\mathbf{H}}_{11}\left(\pi^{0}\right)+\mathbb{I}_{2t}\left(\pi^{0}\right)\hat{\mathbf{H}}_{22}\left(\pi^{0}\right)\right]^{-1}\mathbf{f}_{t}^{0}\varepsilon_{t+h}+o_{p}\left(1\right)
\end{aligned}
$$

by Lemma A.5(a) since

$$
\begin{aligned}
 & T^{-1/2}\sum_{t=1}^{T}\left\{\hat{\mathbf{f}}_{t}-\left[\mathbb{I}_{1t}\left(\pi^{0}\right)\hat{\mathbf{H}}_{11}\left(\pi^{0}\right)+\mathbb{I}_{2t}\left(\pi^{0}\right)\hat{\mathbf{H}}_{22}\left(\pi^{0}\right)\right]^{-1}\mathbf{f}_{t}^{0}\right\}\varepsilon_{t+h} \\
= & T^{1/2}\frac{1}{T}\sum_{t=1}^{T}\left\{\hat{\mathbf{f}}_{t}-\left[\mathbb{I}_{1t}\left(\pi^{0}\right)\hat{\mathbf{H}}_{11}\left(\pi^{0}\right)+\mathbb{I}_{2t}\left(\pi^{0}\right)\hat{\mathbf{H}}_{22}\left(\pi^{0}\right)\right]^{-1}\mathbf{f}_{t}^{0}\right\}\varepsilon_{t+h} \\
= & O_{p}\left[\frac{T^{1/2}}{\min\left(N,T\right)}\right]=o_{p}\left(1\right)
\end{aligned}
$$

when $\sqrt{T}/N\rightarrow 0$: it follows that

$$
\begin{aligned}
T^{-1/2}\sum_{t=1}^{T}\hat{\mathbf{z}}_{t}\varepsilon_{t+h} &= T^{-1/2}\sum_{t=1}^{T}\left\{\left\{\left[\mathbb{I}_{1t}\left(\pi^{0}\right)\hat{\mathbf{H}}_{11}\left(\pi^{0}\right)+\mathbb{I}_{2t}\left(\pi^{0}\right)\hat{\mathbf{H}}_{22}\left(\pi^{0}\right)\right]^{-1}\mathbf{f}_{t}^{0}\right\}',\mathbf{w}_{t}'\right\}'\varepsilon_{t+h}+o_{p}\left(1\right) \\
&= T^{-1/2}\sum_{t=1}^{T}\left[\mathbb{I}_{1t}\left(\pi^{0}\right)\hat{\boldsymbol{\Phi}}_{11}\left(\pi^{0}\right)+\mathbb{I}_{2t}\left(\pi^{0}\right)\hat{\boldsymbol{\Phi}}_{22}\left(\pi^{0}\right)\right]^{-1}\mathbf{z}_{t}^{0}\varepsilon_{t+h}+o_{p}\left(1\right),
\end{aligned} \tag{34}
$$

where $\hat{\boldsymbol{\Phi}}_{jj}\left(\pi^{0}\right)=\operatorname{diag}\left[\hat{\mathbf{H}}_{jj}\left(\pi^{0}\right),\mathbf{I}_{K}\right]$ is a $\left(R^{0}+K\right)\times\left(R^{0}+K\right)$ block diagonal matrix, for $j=1,2$. When $\sqrt{T}/N\rightarrow 0$, by Lemma A.5(b) it follows that

$$
T^{-1/2}\sum_{t=1}^{T}\left\{\begin{array}{l}\hat{\mathbf{z}}_{t}\left\{\left[\mathbb{I}_{1t}\left(\pi^{0}\right)\hat{\mathbf{H}}_{11}\left(\pi^{0}\right)+\mathbb{I}_{2t}\left(\pi^{0}\right)\hat{\mathbf{H}}_{22}\left(\pi^{0}\right)\right]^{-1}\mathbf{f}_{t}^{0}-\hat{\mathbf{f}}_{t}\right\}' \\ \times\left[\mathbb{I}_{1t}\left(\pi^{0}\right)\hat{\mathbf{H}}_{11}\left(\pi^{0}\right)+\mathbb{I}_{2t}\left(\pi^{0}\right)\hat{\mathbf{H}}_{22}\left(\pi^{0}\right)\right]'\end{array}\right\}=o_{p}\left(1\right). \tag{35}
$$

From (33), (34) and (35) we thus have

$$
\begin{array}{ll}
 & T^{1/2}\left\{\widehat{\boldsymbol{\theta}}-\left\{\hat{\pi}\left(\pi^{0}\right)\boldsymbol{\theta}_{1}^{0}\left(\pi^{0}\right)+\left[1-\hat{\pi}\left(\pi^{0}\right)\right]\boldsymbol{\theta}_{2}^{0}\left(\pi^{0}\right)\right\}\right\} \\
= & \left(T^{-1}\sum_{t=1}^{T}\hat{\mathbf{z}}_{t}\hat{\mathbf{z}}_{t}'\right)^{-1}\left(T^{-1/2}\sum_{t=1}^{T}\hat{\mathbf{z}}_{t}\varepsilon_{t+h}\right)+o_{p}\left(1\right) \\
= & \left(T^{-1}\sum_{t=1}^{T}\hat{\mathbf{z}}_{t}\hat{\mathbf{z}}_{t}'\right)^{-1}\left\{T^{-1/2}\sum_{t=1}^{T}\left[\mathbb{I}_{1t}\left(\pi^{0}\right)\hat{\boldsymbol{\Phi}}_{11}\left(\pi^{0}\right)+\mathbb{I}_{2t}\left(\pi^{0}\right)\hat{\boldsymbol{\Phi}}_{22}\left(\pi^{0}\right)\right]^{-1}\mathbf{z}_{t}^{0}\varepsilon_{t+h}\right\}+o_{p}\left(1\right).
\end{array}
$$

Since $T^{-1/2}\sum_{t=1}^{T}\mathbf{z}_t^0\varepsilon_{t+h}\xrightarrow{d}\mathcal{N}\left(\mathbf{0},\boldsymbol{\Sigma}_{\mathbf{z}\boldsymbol{\varepsilon}}^0\right)$ by Assumption CR(b), then $T^{1/2}\left\{\widehat{\boldsymbol{\theta}}-\left\{\hat{\pi}\left(\pi^0\right)\boldsymbol{\theta}_1^0\left(\pi^0\right)+\left[1-\hat{\pi}\left(\pi^0\right)\right]\boldsymbol{\theta}_2^0\left(\pi^0\right)\right\}\right\}$ is asymptotically normally distributed. The asymptotic variance matrix is the probability limit of

$$\begin{aligned}&\left(T^{-1}\sum_{t=1}^{T}\hat{\mathbf{z}}_t\hat{\mathbf{z}}_t'\right)^{-1}\\&\times\left\{T^{-1}\sum_{t=1}^{T}\left[\mathbb{I}_{1t}\left(\pi^0\right)\hat{\boldsymbol{\Phi}}_{11}\left(\pi^0\right)+\mathbb{I}_{2t}\left(\pi^0\right)\hat{\boldsymbol{\Phi}}_{22}\left(\pi^0\right)\right]^{-1}\varepsilon_{t+h}^2\mathbf{z}_t^0\mathbf{z}_t^{0\prime}\left\{\left[\mathbb{I}_{1t}\left(\pi^0\right)\hat{\boldsymbol{\Phi}}_{11}\left(\pi^0\right)+\mathbb{I}_{2t}\left(\pi^0\right)\hat{\boldsymbol{\Phi}}_{22}\left(\pi^0\right)\right]^{-1}\right\}'\right\}\\&\times\left(T^{-1}\sum_{t=1}^{T}\hat{\mathbf{z}}_t\hat{\mathbf{z}}_t'\right)^{-1}.\end{aligned}\tag{36}$$

Define $\mathbf{H}_{jj}^0\left(\pi^0\right)=\operatorname{p}\lim\hat{\mathbf{H}}_{jj}\left(\pi^0\right)$ and $\boldsymbol{\Phi}_{jj}^0\left(\pi^0\right)=\operatorname{p}\lim\hat{\boldsymbol{\Phi}}_{jj}\left(\pi^0\right)=\operatorname{diag}\left[\mathbf{H}_{jj}^0\left(\pi^0\right),\mathbf{I}_K\right]$, for $j=1,2$. By Assumption DI(a),

$$\begin{aligned}T^{-1}\sum_{t=1}^{T}\hat{\mathbf{z}}_t\hat{\mathbf{z}}_t' &= T^{-1}\sum_{t=1}^{T}\left[\mathbb{I}_{1t}\left(\pi^0\right)\hat{\boldsymbol{\Phi}}_{11}\left(\pi^0\right)+\mathbb{I}_{2t}\left(\pi^0\right)\hat{\boldsymbol{\Phi}}_{22}\left(\pi^0\right)\right]^{-1}\mathbf{z}_t^0\mathbf{z}_t^{0\prime}\left\{\left[\mathbb{I}_{1t}\left(\pi^0\right)\hat{\boldsymbol{\Phi}}_{11}\left(\pi^0\right)+\mathbb{I}_{2t}\left(\pi^0\right)\hat{\boldsymbol{\Phi}}_{22}\left(\pi^0\right)\right]^{-1}\right\}'+o_p(1)\\&\xrightarrow{p}\left[\pi^0\boldsymbol{\Phi}_{11}^0\left(\pi^0\right)+\left(1-\pi^0\right)\boldsymbol{\Phi}_{22}^0\left(\pi^0\right)\right]^{-1}\boldsymbol{\Sigma}_{\mathbf{z}}^0\left\{\left[\pi^0\boldsymbol{\Phi}_{11}^0\left(\pi^0\right)+\left(1-\pi^0\right)\boldsymbol{\Phi}_{22}^0\left(\pi^0\right)\right]^{-1}\right\}'.\end{aligned}$$

The asymptotic variance matrix is the probability limit of (36) and it is equal to

$$\begin{aligned}\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\theta}}}^0\left(\pi^0\right) &= \left\{\left[\pi^0\boldsymbol{\Phi}_{11}^0\left(\pi^0\right)+\left(1-\pi^0\right)\boldsymbol{\Phi}_{22}^0\left(\pi^0\right)\right]^{-1}\boldsymbol{\Sigma}_{\mathbf{z}}^0\left\{\left[\pi^0\boldsymbol{\Phi}_{11}^0\left(\pi^0\right)+\left(1-\pi^0\right)\boldsymbol{\Phi}_{22}^0\left(\pi^0\right)\right]^{-1}\right\}'\right\}^{-1}\\&\quad\times\left\{\left[\pi^0\boldsymbol{\Phi}_{11}^0\left(\pi^0\right)+\left(1-\pi^0\right)\boldsymbol{\Phi}_{22}^0\left(\pi^0\right)\right]^{-1}\boldsymbol{\Sigma}_{\mathbf{z}\boldsymbol{\varepsilon}}^0\left\{\left[\pi^0\boldsymbol{\Phi}_{11}^0\left(\pi^0\right)+\left(1-\pi^0\right)\boldsymbol{\Phi}_{22}^0\left(\pi^0\right)\right]^{-1}\right\}'\right\}\\&\quad\times\left\{\left[\pi^0\boldsymbol{\Phi}_{11}^0\left(\pi^0\right)+\left(1-\pi^0\right)\boldsymbol{\Phi}_{22}^0\left(\pi^0\right)\right]^{-1}\boldsymbol{\Sigma}_{\mathbf{z}}^0\left\{\left[\pi^0\boldsymbol{\Phi}_{11}^0\left(\pi^0\right)+\left(1-\pi^0\right)\boldsymbol{\Phi}_{22}^0\left(\pi^0\right)\right]^{-1}\right\}'\right\}^{-1}\\&= \left[\pi^0\boldsymbol{\Phi}_{11}^0\left(\pi^0\right)+\left(1-\pi^0\right)\boldsymbol{\Phi}_{22}^0\left(\pi^0\right)\right]'\left(\boldsymbol{\Sigma}_{\mathbf{z}}^0\right)^{-1}\boldsymbol{\Sigma}_{\mathbf{z}\boldsymbol{\varepsilon}}^0\left(\boldsymbol{\Sigma}_{\mathbf{z}}^0\right)^{-1}\left[\pi^0\boldsymbol{\Phi}_{11}^0\left(\pi^0\right)+\left(1-\pi^0\right)\boldsymbol{\Phi}_{22}^0\left(\pi^0\right)\right].\end{aligned}$$

Since $\left[\mathbb{I}_{1t}\left(\pi^0\right)\hat{\mathbf{H}}_{11}\left(\pi^0\right)+\mathbb{I}_{2t}\left(\pi^0\right)\hat{\mathbf{H}}_{22}\left(\pi^0\right)\right]^{-1}=\hat{\mathbf{f}}_t+o_p(1)$ and $\mathbf{z}_t^0=\left(\mathbf{f}_t^{0\prime},\mathbf{w}_t'\right)'$, then

$$\begin{aligned}&T^{-1}\textstyle\sum_{t=1}^{T}\left[\mathbb{I}_{1t}\left(\pi^0\right)\hat{\boldsymbol{\Phi}}_{11}\left(\pi^0\right)+\mathbb{I}_{2t}\left(\pi^0\right)\hat{\boldsymbol{\Phi}}_{22}\left(\pi^0\right)\right]^{-1}\varepsilon_{t+h}^2\mathbf{z}_t^0\mathbf{z}_t^{0\prime}\left\{\left[\mathbb{I}_{1t}\left(\pi^0\right)\hat{\boldsymbol{\Phi}}_{11}\left(\pi^0\right)+\mathbb{I}_{2t}\left(\pi^0\right)\hat{\boldsymbol{\Phi}}_{22}\left(\pi^0\right)\right]^{-1}\right\}'\\=\;&T^{-1}\textstyle\sum_{t=1}^{T}\hat{\varepsilon}_{t+h}^2\hat{\mathbf{z}}_t\hat{\mathbf{z}}_t'+o_p(1).\end{aligned}$$

Therefore, $\widehat{\operatorname{Avar}\left(\widehat{\boldsymbol{\theta}}\right)}=\left(T^{-1}\sum_{t=1}^{T}\hat{\mathbf{z}}_t\hat{\mathbf{z}}_t'\right)^{-1}\left(T^{-1}\sum_{t=1}^{T}\hat{\varepsilon}_{t+h}^2\hat{\mathbf{z}}_t\hat{\mathbf{z}}_t'\right)\left(T^{-1}\sum_{t=1}^{T}\hat{\mathbf{z}}_t\hat{\mathbf{z}}_t'\right)^{-1}$ is a consistent estimator for $\boldsymbol{\Sigma}_{\widehat{\boldsymbol{\theta}}}^0\left(\pi^0\right)$. This completes the proof of Theorem 5.1. ■

Proof of Lemma A.5. TO BE DONE. ■

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Table 1: Experiment 1, Estimation, $R^0 = 2$

TO BE COMPLETED

Panel A: Homogeneous Break, Bias and RMSE, Estimator for π^0, $\alpha^0 = 0.60$

N		25				50				100			
$\delta_i^0 > 0$		0.25		1.00		0.25		1.00		0.25		1.00	
T	π^0	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**
100	0.15	0.3204	0.4280	0.0561	0.1735	0.3267	0.4334	0.0834	0.2147	0.3124	0.4172	0.0477	0.1579
	0.30	0.1697	0.3146	0.0024	0.0536	0.1792	0.3240	0.0032	0.0611	0.1596	0.3038	0.0013	0.0356
	0.50	0.0042	0.2510	-0.0026	0.0332	0.0033	0.2587	-0.0038	0.0361	-0.0031	0.2447	-0.0023	0.0190
	0.70	-0.1605	0.3083	-0.0061	0.0409	-0.1697	0.3170	-0.0079	0.0546	-0.1673	0.3092	-0.0025	0.0272
	0.85	-0.3123	0.4203	-0.0406	0.1356	-0.3208	0.4278	-0.0533	0.1572	-0.3244	0.4278	-0.0288	0.1104
200	0.15	0.2914	0.4189	0.0049	0.0588	0.2894	0.4142	0.0065	0.0612	0.2744	0.4013	0.0010	0.0281
	0.30	0.1376	0.2990	-0.0006	0.0099	0.1387	0.2982	-0.0002	0.0134	0.1234	0.2823	-0.0001	0.0034
	0.50	0.0096	0.2400	-0.0002	0.0054	-0.0059	0.2423	-0.0008	0.0082	-0.0087	0.2268	-0.0001	0.0045
	0.70	-0.1205	0.2873	-0.0005	0.0078	-0.1505	0.3054	-0.0007	0.0087	-0.1349	0.2890	-0.0002	0.0054
	0.85	-0.2735	0.4045	-0.0017	0.0224	-0.2967	0.4202	-0.0034	0.0280	-0.2899	0.4131	-0.0011	0.0169
400	0.15	0.2266	0.3698	-0.0001	0.0025	0.2261	0.3657	-0.0001	0.0030	0.2133	0.3561	0.0000	0.0011
	0.30	0.0840	0.2318	0.0000	0.0016	0.0836	0.2342	0.0000	0.0016	0.0652	0.2053	0.0000	0.0000
	0.50	0.0036	0.1757	0.0000	0.0011	-0.0031	0.1836	0.0000	0.0000	-0.0054	0.1449	0.0000	0.0000
	0.70	-0.0667	0.2134	0.0000	0.0016	-0.0893	0.2367	0.0000	0.0011	-0.0622	0.1939	0.0000	0.0000
	0.85	-0.1943	0.3392	-0.0001	0.0022	-0.2287	0.3678	-0.0001	0.0022	-0.1930	0.3316	0.0000	0.0000

Panel B: Homogeneous Break, MSE, Estimator for c_{it}^{0s}, $\alpha^0 = 0.60$

		Unfeasible Estimator						**Feasible Estimator**					
N		25		50		100		25		50		100	
$\delta_i^0 > 0$		0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00
T	π^0												
100	0.15	0.1175	0.1123	0.1082	0.1083	0.0815	0.0817	0.1395	0.1250	0.1247	0.1201	0.0914	0.0868
	0.30	0.1194	0.1151	0.1086	0.1086	0.0819	0.0820	0.1406	0.1204	0.1252	0.1129	0.0916	0.0836
	0.50	0.1199	0.1169	0.1089	0.1088	0.0821	0.0822	0.1413	0.1209	0.1254	0.1117	0.0917	0.0830
	0.70	0.1202	0.1183	0.1085	0.1083	0.0819	0.0819	0.1416	0.1231	0.1252	0.1125	0.0916	0.0831
	0.85	0.1193	0.1184	0.1087	0.1083	0.0812	0.0811	0.1411	0.1290	0.1248	0.1177	0.0914	0.0851
200	0.15	0.0853	0.0801	0.0744	0.0746	0.0496	0.0499	0.0981	0.0818	0.0835	0.0760	0.0554	0.0502
	0.30	0.0852	0.0810	0.0744	0.0745	0.0499	0.0501	0.0985	0.0815	0.0837	0.0750	0.0553	0.0501
	0.50	0.0855	0.0825	0.0743	0.0744	0.0499	0.0500	0.0988	0.0826	0.0838	0.0746	0.0552	0.0500
	0.70	0.0860	0.0842	0.0743	0.0742	0.0497	0.0497	0.0994	0.0846	0.0837	0.0745	0.0552	0.0498
	0.85	0.0870	0.0861	0.0742	0.0741	0.0494	0.0494	0.0995	0.0871	0.0836	0.0749	0.0551	0.0496
400	0.15	0.0682	0.0630	0.0571	0.0574	0.0336	0.0339	0.0756	0.0631	0.0620	0.0574	0.0365	0.0339
	0.30	0.0683	0.0641	0.0572	0.0574	0.0336	0.0339	0.0752	0.0641	0.0618	0.0574	0.0361	0.0339
	0.50	0.0689	0.0659	0.0572	0.0573	0.0336	0.0338	0.0751	0.0659	0.0615	0.0573	0.0358	0.0338
	0.70	0.0694	0.0676	0.0571	0.0572	0.0337	0.0337	0.0757	0.0676	0.0617	0.0572	0.0360	0.0337
	0.85	0.0698	0.0689	0.0571	0.0570	0.0336	0.0336	0.0770	0.0689	0.0620	0.0570	0.0364	0.0336

Table 1-*Continued*: Experiment 1, Estimation, $R^0 = 2$

TO BE COMPLETED

Panel C: Homogeneous Break, Bias and RMSE, Estimator for π^0, $\alpha^0 = 1.00$													
N		25				50				100			
$\delta_i^0 > 0$		0.25		1.00		0.25		1.00		0.25		1.00	
T	π^0	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**
	0.15	0.3104	0.4199	0.1002	0.2389	0.2924	0.4072	0.0343	0.1424	0.2677	0.3846	0.0206	0.1103
	0.30	0.1548	0.3038	0.0010	0.0859	0.1268	0.2771	-0.0034	0.0221	0.1104	0.2536	-0.0016	0.0157
100	0.50	-0.0038	0.2394	-0.0119	0.0516	-0.0102	0.2199	-0.0033	0.0197	-0.0092	0.1898	-0.0019	0.0116
	0.70	-0.1538	0.2971	-0.0184	0.0635	-0.1379	0.2793	-0.0038	0.0176	-0.1137	0.2450	-0.0022	0.0134
	0.85	-0.3012	0.4113	-0.0415	0.1180	-0.2811	0.3981	-0.0088	0.0408	-0.2596	0.3761	-0.0046	0.0200
	0.15	0.2638	0.3999	0.0176	0.1133	0.2227	0.3605	0.0004	0.0212	0.1904	0.3328	-0.0004	0.0047
	0.30	0.1099	0.2731	-0.0016	0.0245	0.0761	0.2209	-0.0002	0.0047	0.0417	0.1702	-0.0001	0.0019
200	0.50	-0.0052	0.2173	-0.0024	0.0144	-0.0121	0.1615	-0.0004	0.0046	-0.0090	0.1093	-0.0001	0.0027
	0.70	-0.1102	0.2649	-0.0028	0.0160	-0.0795	0.2091	-0.0003	0.0042	-0.0494	0.1589	-0.0001	0.0019
	0.85	-0.2471	0.3824	-0.0048	0.0241	-0.2093	0.3449	-0.0009	0.0083	-0.1703	0.3062	-0.0004	0.0055
	0.15	0.1994	0.3488	0.0000	0.0175	0.1082	0.2465	0.0000	0.0000	0.0686	0.1978	0.0000	0.0011
	0.30	0.0598	0.2041	-0.0002	0.0032	0.0129	0.0975	0.0000	0.0000	0.0021	0.0483	0.0000	0.0000
400	0.50	-0.0101	0.1419	-0.0001	0.0022	-0.0038	0.0588	0.0000	0.0000	-0.0019	0.0293	0.0000	0.0000
	0.70	-0.0471	0.1674	-0.0003	0.0046	-0.0164	0.0868	0.0000	0.0000	-0.0065	0.0471	0.0000	0.0000
	0.85	-0.1558	0.2957	-0.0004	0.0047	-0.0855	0.2145	0.0000	0.0000	-0.0433	0.1450	0.0000	0.0000

Panel D: Homogeneous Break, MSE, Estimator for c_{it}^{0s}, $\alpha^0 = 1.00$													
		Unfeasible Estimator						**Feasible Estimator**					
N		25		50		100		25		50		100	
$\delta_i^0 > 0$		0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00
T	π^0												
	0.15	0.1171	0.1144	0.1081	0.1078	0.0813	0.0811	0.1389	0.1304	0.1257	0.1138	0.0920	0.0838
	0.30	0.1190	0.1167	0.1085	0.1081	0.0817	0.0815	0.1401	0.1239	0.1258	0.1096	0.0918	0.0820
100	0.50	0.1195	0.1177	0.1088	0.1084	0.0820	0.0818	0.1408	0.1234	0.1259	0.1097	0.0912	0.0822
	0.70	0.1197	0.1183	0.1083	0.1078	0.0818	0.0816	0.1414	0.1255	0.1260	0.1092	0.0915	0.0820
	0.85	0.1189	0.1179	0.1085	0.1079	0.0811	0.0808	0.1412	0.1287	0.1259	0.1108	0.0921	0.0818
	0.15	0.0852	0.0827	0.0743	0.0743	0.0495	0.0495	0.0977	0.0859	0.0839	0.0746	0.0551	0.0495
	0.30	0.0851	0.0831	0.0743	0.0742	0.0497	0.0497	0.0977	0.0842	0.0830	0.0743	0.0538	0.0497
200	0.50	0.0854	0.0838	0.0742	0.0741	0.0498	0.0497	0.0981	0.0845	0.0819	0.0742	0.0527	0.0497
	0.70	0.0858	0.0848	0.0742	0.0740	0.0496	0.0495	0.0987	0.0855	0.0824	0.0741	0.0535	0.0495
	0.85	0.0867	0.0861	0.0741	0.0738	0.0494	0.0493	0.0995	0.0874	0.0838	0.0741	0.0548	0.0493
	0.15	0.0682	0.0659	0.0571	0.0572	0.0335	0.0335	0.0753	0.0662	0.0612	0.0572	0.0353	0.0335
	0.30	0.0689	0.0664	0.0572	0.0572	0.0335	0.0335	0.0745	0.0665	0.0593	0.0572	0.0341	0.0335
400	0.50	0.0689	0.0675	0.0571	0.0571	0.0335	0.0335	0.0740	0.0675	0.0586	0.0571	0.0340	0.0335
	0.70	0.0694	0.0685	0.0571	0.0570	0.0336	0.0336	0.0745	0.0686	0.0591	0.0570	0.0341	0.0336
	0.85	0.0697	0.0692	0.0570	0.0569	0.0335	0.0335	0.0764	0.0693	0.0606	0.0569	0.0349	0.0335

Table 1-*Continued*: Experiment 1, Estimation, $R^0 = 2$

TO BE COMPLETED

Panel E: Heterogeneous Break, $\alpha^0 = 0.60$														
Bias and RMSE, Estimator for π^0								MSE, Estimator for c_{it}^{0s}						
								Unfeasible Estimator			Feasible Estimator			
N		25		50		100		25	50	100	25	50	100
T	π^0	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**						
100	0.15	0.0022	0.0286	0.0402	0.1377	0.0292	0.1180	0.1110	0.1063	0.0816	0.1127	0.1143	0.0855
	0.30	-0.0003	0.0052	0.0018	0.0304	0.0009	0.0202	0.1131	0.1067	0.0819	0.1134	0.1088	0.0828
	0.50	0.0000	0.0034	-0.0011	0.0147	-0.0007	0.0092	0.1138	0.1074	0.0821	0.1139	0.1084	0.0825
	0.70	0.0000	0.0030	-0.0023	0.0285	-0.0008	0.0150	0.1140	0.1072	0.0818	0.1142	0.1093	0.0824
	0.85	-0.0018	0.0234	-0.0361	0.1272	-0.0238	0.1053	0.1141	0.1072	0.0810	0.1152	0.1150	0.0845
200	0.15	0.0000	0.0000	0.0009	0.0172	0.0006	0.0187	0.0816	0.0729	0.0498	0.0816	0.0733	0.0499
	0.30	0.0000	0.0000	0.0000	0.0037	0.0000	0.0011	0.0818	0.0731	0.0500	0.0818	0.0732	0.0500
	0.50	0.0000	0.0000	0.0000	0.0022	0.0000	0.0000	0.0824	0.0733	0.0500	0.0824	0.0733	0.0500
	0.70	0.0000	0.0000	-0.0001	0.0025	0.0000	0.0016	0.0830	0.0735	0.0497	0.0830	0.0735	0.0497
	0.85	0.0000	0.0011	-0.0012	0.0210	-0.0003	0.0061	0.0837	0.0735	0.0494	0.0837	0.0738	0.0495
400	0.15	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0658	0.0558	0.0338	0.0658	0.0558	0.0338
	0.30	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0662	0.0561	0.0338	0.0662	0.0561	0.0338
	0.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0670	0.0563	0.0337	0.0670	0.0563	0.0337
	0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0678	0.0565	0.0337	0.0678	0.0565	0.0337
	0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0682	0.0567	0.0336	0.0682	0.0567	0.0336

Panel F: Eterogeneous Break, $\alpha^0 = 1.00$													
Bias and RMSE, Estimator for π^0								MSE, Estimator for c_{it}^{0s}					
								Unfeasible Estimator			Feasible Estimator		
N		25		50		100		25	50	100	25	50	100
T	π^0	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**						
100	0.15	0.0000	0.0016	0.0000	0.0011	0.0000	0.0000	0.1247	0.1019	0.0807	0.1248	0.1019	0.0807
	0.30	-0.0005	0.0016	0.0000	0.0000	0.0000	0.0000	0.1239	0.1029	0.0811	0.1239	0.1029	0.0811
	0.50	-0.0005	0.0016	0.0000	0.0000	0.0000	0.0000	0.1205	0.1041	0.0813	0.1205	0.1041	0.0813
	0.70	0.0000	0.0011	0.0000	0.0000	0.0000	0.0000	0.1163	0.1044	0.0809	0.1163	0.1044	0.0809
	0.85	-0.0005	0.0016	-0.0005	0.0016	0.0000	0.0000	0.1127	0.1045	0.0799	0.1128	0.1045	0.0799
200	0.15	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0975	0.0694	0.0493	0.0975	0.0694	0.0493
	0.30	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0946	0.0701	0.0495	0.0946	0.0701	0.0495
	0.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0909	0.0708	0.0495	0.0909	0.0708	0.0495
	0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0874	0.0716	0.0493	0.0874	0.0716	0.0493
	0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0848	0.0720	0.0490	0.0848	0.0720	0.0490
400	0.15	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0825	0.0527	0.0335	0.0825	0.0527	0.0335
	0.30	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0797	0.0534	0.0335	0.0797	0.0534	0.0335
	0.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0764	0.0543	0.0335	0.0764	0.0543	0.0335
	0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0730	0.0552	0.0335	0.0730	0.0552	0.0335
	0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0704	0.0558	0.0334	0.0704	0.0558	0.0333

Table 2: Experiment 2, Model Selection, $R^0 = 1$, $\alpha^0 = 0.60$

TO BE COMPLETED

Panel A: Homogeneous Break, $IC_{p1}(R, R)$													
		Unfeasible Estimator						Feasible Estimator					
N		25		50		100		25		50		100	
$\delta_i^0 > 0$		0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00
T	π^0												
100	0.15	1.8260	1.8385	1.9145	1.9135	1.0315	1.0315	1.8295	1.8045	2.1040	2.0115	1.0375	1.0360
	0.30	1.8340	1.8420	2.0340	2.0305	1.0360	1.0355	1.8290	1.8175	2.1035	2.0390	1.0370	1.0365
	0.50	1.8380	1.8455	2.0765	2.0675	1.0350	1.0355	1.8280	1.8415	2.1080	2.0850	1.0370	1.0350
	0.70	1.8270	1.8330	2.0275	2.0220	1.0350	1.0360	1.8275	1.8175	2.1060	2.0225	1.0370	1.0370
	0.85	1.8305	1.8345	1.9250	1.9255	1.0295	1.0295	1.8285	1.8025	2.1015	2.0130	1.0360	1.0335
200	0.15	1.9120	1.9175	1.9965	1.9895	1.0145	1.0150	1.9015	1.8975	2.0560	1.9835	1.0170	1.0125
	0.30	1.9065	1.9100	2.0295	2.0250	1.0150	1.0150	1.9015	1.9070	2.0430	2.0130	1.0165	1.0145
	0.50	1.9015	1.9025	2.0510	2.0485	1.0145	1.0150	1.8980	1.9025	2.0530	2.0420	1.0140	1.0155
	0.70	1.8970	1.9020	2.0285	2.0260	1.0180	1.0180	1.8955	1.8980	2.0520	2.0205	1.0165	1.0180
	0.85	1.8990	1.9020	1.9740	1.9690	1.0170	1.0165	1.8965	1.8815	2.0545	1.9610	1.0150	1.0145
400	0.15	1.9715	1.9725	1.9905	1.9780	1.0080	1.0080	1.9715	1.9720	2.0040	1.9735	1.0070	1.0065
	0.30	1.9700	1.9720	2.0140	2.0080	1.0065	1.0065	1.9705	1.9720	2.0045	2.0080	1.0070	1.0065
	0.50	1.9700	1.9710	2.0175	2.0070	1.0095	1.0095	1.9705	1.9710	2.0205	2.0070	1.0065	1.0095
	0.70	1.9725	1.9735	2.0105	2.0060	1.0065	1.0065	1.9720	1.9735	2.0150	2.0060	1.0070	1.0060
	0.85	1.9720	1.9725	1.9815	1.9835	1.0055	1.0060	1.9715	1.9725	2.0015	1.9770	1.0060	1.0065

Panel B: Homogeneous Break, $IC_{p2}(R, R)$													
		Unfeasible Estimator						Feasible Estimator					
N		25		50		100		25		50		100	
$\delta_i^0 > 0$		0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00
T	π^0												
100	0.15	1.6410	1.6455	1.3255	1.3275	1.0005	1.0005	1.6440	1.6170	1.3790	1.3475	1.0015	1.0010
	0.30	1.6515	1.6620	1.3750	1.3815	1.0005	1.0005	1.6430	1.6385	1.3750	1.3590	1.0015	1.0015
	0.50	1.6405	1.6535	1.3980	1.4000	1.0010	1.0010	1.6455	1.6480	1.3730	1.3915	1.0010	1.0010
	0.70	1.6450	1.6480	1.3775	1.3770	1.0010	1.0010	1.6470	1.6260	1.3745	1.3600	1.0010	1.0010
	0.85	1.6395	1.6435	1.3310	1.3295	1.0010	1.0010	1.6465	1.5945	1.3805	1.3485	1.0010	1.0010
200	0.15	1.8295	1.8390	1.5380	1.5425	1.0000	1.0000	1.8265	1.8100	1.5765	1.5310	1.0005	1.0005
	0.30	1.8290	1.8350	1.5630	1.5640	1.0005	1.0005	1.8250	1.8330	1.5715	1.5555	1.0005	1.0005
	0.50	1.8260	1.8290	1.5815	1.5835	1.0005	1.0005	1.8240	1.8270	1.5795	1.5800	1.0005	1.0005
	0.70	1.8260	1.8290	1.5680	1.5700	1.0005	1.0005	1.8235	1.8240	1.5785	1.5605	1.0005	1.0005
	0.85	1.8230	1.8235	1.5465	1.5435	1.0000	1.0000	1.8235	1.7960	1.5810	1.5275	1.0005	1.0005
400	0.15	1.9565	1.9595	1.7285	1.7340	1.0000	1.0000	1.9565	1.9585	1.7400	1.7290	1.0005	1.0005
	0.30	1.9540	1.9590	1.7400	1.7425	1.0000	1.0000	1.9545	1.9590	1.7385	1.7430	1.0005	1.0000
	0.50	1.9575	1.9585	1.7335	1.7385	1.0000	1.0000	1.9570	1.9585	1.7375	1.7380	1.0005	1.0000
	0.70	1.9530	1.9560	1.7305	1.7330	1.0000	1.0000	1.9560	1.9560	1.7275	1.7330	1.0005	1.0000
	0.85	1.9580	1.9595	1.7295	1.7315	1.0000	1.0000	1.9555	1.9570	1.7300	1.7225	1.0005	1.0000

Panel C: Homogeneous Break, $IC_{p3}(R, R)$													
		Unfeasible Estimator						Feasible Estimator					
N		25		50		100		25		50		100	
$\delta_i^0 > 0$		0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00
T	π^0												
100	0.15	2.0785	2.1050	4.7705	4.7575	5.9085	5.9020	2.1230	2.1340	4.9855	4.9980	7.9610	7.9680
	0.30	2.1060	2.1360	4.8855	4.8800	7.7235	7.7215	2.1265	2.1375	4.9975	4.9870	7.9615	7.9555
	0.50	2.1080	2.1210	4.9080	4.9080	7.9360	7.9360	2.1250	2.1275	4.9990	4.9345	7.9630	7.9550
	0.70	2.1020	2.1075	4.8970	4.8925	7.7080	7.7090	2.1160	2.1115	4.9910	4.9990	7.9640	7.9495
	0.85	2.0730	2.0755	4.7745	4.7725	5.9110	5.9070	2.1195	2.0905	4.9970	5.0180	7.9645	7.9655
200	0.15	1.9895	1.9910	4.2440	4.2095	2.2735	2.2700	1.9835	1.9855	4.2375	4.1740	2.5135	2.4975
	0.30	1.9855	1.9900	4.1900	4.1715	2.4185	2.4120	1.9845	1.9895	4.2445	4.1600	2.5080	2.4640
	0.50	1.9850	1.9875	4.1890	4.1715	2.4620	2.4595	1.9855	1.9875	4.2370	4.1760	2.5140	2.4730
	0.70	1.9865	1.9880	4.2080	4.2010	2.4260	2.4245	1.9855	1.9875	4.2420	4.2015	2.5060	2.4625
	0.85	1.9840	1.9855	4.2370	4.2375	2.3035	2.3020	1.9835	1.9790	4.2495	4.1900	2.5055	2.4940
400	0.15	1.9930	1.9940	3.5720	3.5125	1.5045	1.5065	1.9915	1.9940	3.6135	3.4925	1.5175	1.5005
	0.30	1.9920	1.9935	3.6005	3.5380	1.5145	1.5140	1.9915	1.9935	3.6235	3.5370	1.5165	1.5145
	0.50	1.9920	1.9915	3.5760	3.5430	1.5055	1.5060	1.9910	1.9915	3.6150	3.5450	1.5140	1.5060
	0.70	1.9925	1.9925	3.5855	3.5705	1.5090	1.5100	1.9910	1.9925	3.6280	3.5715	1.5170	1.5070
	0.85	1.9910	1.9915	3.5945	3.5875	1.5050	1.5055	1.9905	1.9915	3.6255	3.5715	1.5160	1.5025

Table 2-*Continued*: Experiment 2, Model Selection, $R^0 = 1$, $\alpha^0 = 0.60$

TO BE COMPLETED

Panel D: Eterogeneous Break, $IC_{p1}(R, R)$							
		Unfeasible Estimator			**Feasible Estimator**		
N		25	50	100	25	50	100
T	π^0						
100	0.15						
	0.30						
	0.50						
	0.70						
	0.85						
200	0.15						
	0.30						
	0.50						
	0.70						
	0.85						
400	0.15						
	0.30						
	0.50						
	0.70						
	0.85						

Panel E: Eterogeneous Break, $IC_{p2}(R, R)$							
		Unfeasible Estimator			**Feasible Estimator**		
N		25	50	100	25	50	100
T	π^0						
100	0.15						
	0.30						
	0.50						
	0.70						
	0.85						
200	0.15						
	0.30						
	0.50						
	0.70						
	0.85						
400	0.15						
	0.30						
	0.50						
	0.70						
	0.85						

Panel F: Eterogeneous Break, $IC_{p3}(R, R)$							
		Unfeasible Estimator			**Feasible Estimator**		
N		25	50	100	25	50	100
T	π^0						
100	0.15						
	0.30						
	0.50						
	0.70						
	0.85						
200	0.15						
	0.30						
	0.50						
	0.70						
	0.85						
400	0.15						
	0.30						
	0.50						
	0.70						
	0.85						

Table 3: Experiment 3, Model Selection, $R^0 = 2$, $\alpha^0 = 0.60$

TO BE COMPLETED

Panel A: Homogeneous Break, $IC_{p1}(R, R)$

		Unfeasible Estimator						Feasible Estimator					
N		25		50		100		25		50		100	
$\delta_i^0 > 0$		0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00
T	π^0												
100	0.15	2.7895	2.8140	2.8920	2.8830	2.0245	2.0240	2.8000	2.7930	3.1245	3.0600	2.0410	2.0430
	0.30	2.8035	2.8215	3.0670	3.0610	2.0355	2.0355	2.8035	2.8035	3.1260	3.0775	2.0400	2.0415
	0.50	2.7985	2.8100	3.1030	3.0995	2.0385	2.0390	2.8015	2.8110	3.1240	3.1035	2.0410	2.0405
	0.70	2.7910	2.8020	3.0425	3.0365	2.0325	2.0335	2.8010	2.7865	3.1215	3.0695	2.0420	2.0465
	0.85	2.7855	2.7895	2.9045	2.9035	2.0280	2.0285	2.7960	2.7730	3.1375	3.0930	2.0425	2.0495
200	0.15	2.9145	2.9285	2.9890	2.9750	2.0190	2.0190	2.9105	2.9120	3.0940	3.0350	2.0230	2.0255
	0.30	2.9135	2.9255	3.0615	3.0560	2.0220	2.0220	2.9105	2.9240	3.0940	3.0550	2.0250	2.0220
	0.50	2.9115	2.9200	3.1040	3.0975	2.0215	2.0215	2.9115	2.9200	3.1080	3.0975	2.0260	2.0220
	0.70	2.9140	2.9230	3.0430	3.0420	2.0195	2.0195	2.9105	2.9185	3.0890	3.0490	2.0265	2.0190
	0.85	2.9075	2.9105	2.9915	2.9935	2.0195	2.0200	2.9090	2.8925	3.1100	3.0370	2.0265	2.0285
400	0.15	2.9725	2.9780	3.0300	3.0140	2.0085	2.0085	2.9725	2.9780	3.0420	3.0115	2.0085	2.0110
	0.30	2.9720	2.9780	3.0615	3.0405	2.0090	2.0090	2.9725	2.9780	3.0650	3.0405	2.0075	2.0090
	0.50	2.9725	2.9785	3.0695	3.0590	2.0070	2.0070	2.9710	2.9785	3.0675	3.0590	2.0070	2.0070
	0.70	2.9695	2.9730	3.0710	3.0695	2.0090	2.0090	2.9675	2.9730	3.0635	3.0690	2.0070	2.0090
	0.85	2.9710	2.9735	3.0255	3.0235	2.0075	2.0075	2.9670	2.9735	3.0595	3.0215	2.0080	2.0075

Panel B: Homogeneous Break, $IC_{p2}(R, R)$

		Unfeasible Estimator						Feasible Estimator					
N		25		50		100		25		50		100	
$\delta_i^0 > 0$		0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00
T	π^0												
100	0.15	2.6075	2.6370	2.3035	2.3045	2.0005	2.0005	2.6300	2.6065	2.3950	2.3570	2.0010	2.0015
	0.30	2.6180	2.6405	2.3745	2.3750	2.0000	2.0000	2.6245	2.6155	2.3940	2.3795	2.0010	2.0010
	0.50	2.6190	2.6335	2.4010	2.3975	2.0005	2.0005	2.6280	2.6320	2.4055	2.3880	2.0010	2.0005
	0.70	2.6085	2.6215	2.3685	2.3695	2.0005	2.0005	2.6130	2.6005	2.4010	2.3730	2.0010	2.0010
	0.85	2.6070	2.6115	2.3100	2.3095	2.0010	2.0010	2.6155	2.5650	2.3945	2.3630	2.0010	2.0010
200	0.15	2.8455	2.8660	2.5745	2.5735	2.0005	2.0005	2.8390	2.8395	2.6190	2.5825	2.0005	2.0010
	0.30	2.8410	2.8660	2.6070	2.6090	2.0005	2.0005	2.8360	2.8640	2.6125	2.6060	2.0005	2.0005
	0.50	2.8390	2.8500	2.6225	2.6215	2.0005	2.0005	2.8390	2.8500	2.6220	2.6225	2.0005	2.0005
	0.70	2.8365	2.8460	2.6240	2.6205	2.0005	2.0005	2.8340	2.8425	2.6320	2.6210	2.0005	2.0005
	0.85	2.8285	2.8350	2.5750	2.5730	2.0010	2.0010	2.8280	2.8070	2.6165	2.5920	2.0005	2.0005
400	0.15	2.9485	2.9615	2.7800	2.7830	2.0005	2.0005	2.9485	2.9610	2.7875	2.7800	2.0010	2.0005
	0.30	2.9500	2.9600	2.7950	2.7905	2.0015	2.0015	2.9490	2.9600	2.7965	2.7905	2.0010	2.0015
	0.50	2.9495	2.9575	2.7985	2.7940	2.0005	2.0005	2.9485	2.9575	2.7875	2.7940	2.0005	2.0005
	0.70	2.9475	2.9510	2.8030	2.7995	2.0005	2.0005	2.9460	2.9510	2.7865	2.7995	2.0010	2.0005
	0.85	2.9485	2.9495	2.7785	2.7795	2.0010	2.0010	2.9460	2.9490	2.7915	2.7745	2.0010	2.0010

Panel C: Homogeneous Break, $IC_{p3}(R, R)$

		Unfeasible Estimator						Feasible Estimator					
N		25		50		100		25		50		100	
$\delta_i^0 > 0$		0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00
T	π^0												
100	0.15	3.0850	3.1150	5.7610	5.7570	6.7695	6.7695	3.1425	3.2100	6.0615	6.1385	7.9855	7.9935
	0.30	3.1275	3.1540	5.9040	5.8980	7.8955	7.8990	3.1445	3.1820	6.0595	6.0810	7.9900	7.9805
	0.50	3.1365	3.1570	5.9570	5.9580	7.9825	7.9840	3.1390	3.1560	6.0565	5.9905	7.9900	7.9850
	0.70	3.1165	3.1270	5.8970	5.8980	7.9030	7.9015	3.1355	3.1605	6.0800	6.0920	7.9905	7.9820
	0.85	3.0875	3.0885	5.7575	5.7575	6.7885	6.7875	3.1385	3.1770	6.0705	6.1965	7.9885	7.9920
200	0.15	2.9895	2.9975	5.2045	5.1780	3.3430	3.3320	2.9890	2.9985	5.2345	5.2375	3.5860	3.6590
	0.30	2.9875	2.9940	5.1705	5.1590	3.4765	3.4730	2.9890	2.9945	5.2295	5.1700	3.5775	3.5060
	0.50	2.9885	2.9930	5.1730	5.1600	3.5185	3.5160	2.9880	2.9930	5.2215	5.1615	3.5835	3.5210
	0.70	2.9885	2.9910	5.1950	5.1885	3.4735	3.4760	2.9875	2.9910	5.2220	5.1855	3.5850	3.5125
	0.85	2.9890	2.9900	5.2265	5.2245	3.3505	3.3515	2.9880	2.9865	5.2350	5.2440	3.5890	3.6800
400	0.15	2.9970	2.9980	4.5795	4.5320	2.5390	2.5390	2.9950	2.9980	4.6310	4.5350	2.5510	2.5470
	0.30	2.9950	2.9975	4.5990	4.5640	2.5535	2.5560	2.9950	2.9975	4.6225	4.5650	2.5560	2.5560
	0.50	2.9965	2.9970	4.6125	4.5920	2.5280	2.5280	2.9965	2.9970	4.6380	4.5905	2.5405	2.5280
	0.70	2.9960	2.9965	4.5995	4.5820	2.5400	2.5400	2.9960	2.9965	4.6310	4.5815	2.5410	2.5395
	0.85	2.9950	2.9950	4.6040	4.6045	2.5420	2.5420	2.9955	2.9950	4.6335	4.6035	2.5500	2.5545

Table 3-*Continued*: Experiment 3, Model Selection, $R^0 = 2$, $\alpha^0 = 0.60$

TO BE COMPLETED

Panel D: Eterogeneous Break, $IC_{p1}(R, R)$							
		Unfeasible Estimator			Feasible Estimator		
N		25	50	100	25	50	100
T	π^0						
100	0.15						
	0.30						
	0.50						
	0.70						
	0.85						
200	0.15						
	0.30						
	0.50						
	0.70						
	0.85						
400	0.15						
	0.30						
	0.50						
	0.70						
	0.85						

Panel E: Eterogeneous Break, $IC_{p2}(R, R)$							
		Unfeasible Estimator			Feasible Estimator		
N		25	50	100	25	50	100
T	π^0						
100	0.15						
	0.30						
	0.50						
	0.70						
	0.85						
200	0.15						
	0.30						
	0.50						
	0.70						
	0.85						
400	0.15						
	0.30						
	0.50						
	0.70						
	0.85						

Panel F: Eterogeneous Break, $IC_{p3}(R, R)$							
		Unfeasible Estimator			Feasible Estimator		
N		25	50	100	25	50	100
T	π^0						
100	0.15						
	0.30						
	0.50						
	0.70						
	0.85						
200	0.15						
	0.30						
	0.50						
	0.70						
	0.85						
400	0.15						
	0.30						
	0.50						
	0.70						
	0.85						

Table 4: Experiment 4, Model Selection, $R^0 = 3$, $\alpha^0 = 0.60$

TO BE COMPLETED

Panel A: Homogeneous Break, $IC_{p1}(R, R)$

		Unfeasible Estimator						Feasible Estimator					
N		25		50		100		25		50		100	
$\delta_i^0 > 0$		0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00
T	π^0												
100	0.15	3.4925	3.5005	3.9350	3.9315	3.0235	3.0245	3.5260	3.5300	4.2265	4.2330	3.0470	3.0925
	0.30	3.5285	3.5340	4.1315	4.1270	3.0390	3.0390	3.5290	3.5325	4.2180	4.2055	3.0475	3.0640
	0.50	3.5245	3.5315	4.1950	4.1960	3.0480	3.0480	3.5330	3.5315	4.2335	4.1975	3.0490	3.0485
	0.70	3.5155	3.5200	4.1405	4.1400	3.0370	3.0365	3.5250	3.5170	4.2360	4.1890	3.0475	3.0760
	0.85	3.4960	3.4980	3.9525	3.9540	3.0240	3.0240	3.5155	3.5155	4.2245	4.2700	3.0485	3.1040
200	0.15	3.4960	3.5035	4.0820	4.0780	3.0115	3.0115	3.5090	3.4815	4.2090	4.1455	3.0185	3.0465
	0.30	3.5065	3.5085	4.1770	4.1665	3.0140	3.0140	3.5020	3.5055	4.2060	4.1685	3.0180	3.0170
	0.50	3.5090	3.5130	4.2150	4.2050	3.0145	3.0130	3.5095	3.5130	4.2055	4.2050	3.0150	3.0130
	0.70	3.5040	3.5080	4.1795	4.1740	3.0150	3.0145	3.5135	3.5050	4.2010	4.1785	3.0155	3.0165
	0.85	3.5000	3.5010	4.1020	4.1020	3.0130	3.0130	3.5105	3.4815	4.2065	4.1715	3.0170	3.0545
400	0.15	3.5270	3.5295	4.1670	4.1405	3.0075	3.0075	3.5185	3.5295	4.2065	4.1470	3.0095	3.0080
	0.30	3.5305	3.5390	4.2160	4.1975	3.0095	3.0095	3.5265	3.5390	4.2170	4.1975	3.0080	3.0095
	0.50	3.5285	3.5325	4.2125	4.2070	3.0075	3.0075	3.5280	3.5325	4.2200	4.2070	3.0090	3.0075
	0.70	3.5315	3.5330	4.2000	4.2010	3.0090	3.0090	3.5280	3.5330	4.2165	4.2010	3.0085	3.0090
	0.85	3.5330	3.5395	4.1655	4.1645	3.0075	3.0075	3.5215	3.5375	4.1970	4.1680	3.0090	3.0095

Panel B: Homogeneous Break, $IC_{p2}(R, R)$

		Unfeasible Estimator						Feasible Estimator					
N		25		50		100		25		50		100	
$\delta_i^0 > 0$		0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00
T	π^0												
100	0.15	3.2775	3.2845	3.3380	3.3390	3.0005	3.0005	3.3070	3.2910	3.4640	3.4640	3.0005	3.0000
	0.30	3.3155	3.3190	3.4400	3.4385	3.0000	3.0000	3.3120	3.3070	3.4605	3.4550	3.0005	3.0010
	0.50	3.3075	3.3120	3.4665	3.4635	3.0000	3.0000	3.3150	3.3095	3.4625	3.4625	3.0005	3.0000
	0.70	3.3005	3.3045	3.4300	3.4280	3.0000	3.0000	3.3080	3.2955	3.4605	3.4450	3.0005	3.0005
	0.85	3.2795	3.2790	3.3450	3.3455	3.0000	3.0000	3.3040	3.2790	3.4615	3.4745	3.0005	3.0040
200	0.15	3.3520	3.3555	3.6350	3.6350	3.0010	3.0010	3.3455	3.3355	3.6980	3.6550	3.0010	3.0045
	0.30	3.3515	3.3565	3.6860	3.6810	3.0005	3.0005	3.3525	3.3530	3.7065	3.6815	3.0005	3.0005
	0.50	3.3515	3.3560	3.7095	3.7075	3.0015	3.0015	3.3560	3.3555	3.7180	3.7075	3.0015	3.0020
	0.70	3.3530	3.3530	3.6890	3.6870	3.0010	3.0010	3.3485	3.3500	3.7000	3.6905	3.0010	3.0010
	0.85	3.3520	3.3535	3.6340	3.6300	3.0005	3.0005	3.3485	3.3365	3.6885	3.6650	3.0010	3.0050
400	0.15	3.4215	3.4285	3.8740	3.8765	3.0015	3.0015	3.4145	3.4300	3.8900	3.8770	3.0020	3.0015
	0.30	3.4280	3.4350	3.9075	3.9005	3.0015	3.0015	3.4220	3.4350	3.9075	3.9005	3.0025	3.0015
	0.50	3.4335	3.4345	3.9100	3.9015	3.0015	3.0015	3.4310	3.4345	3.9140	3.9015	3.0015	3.0015
	0.70	3.4300	3.4320	3.9125	3.9075	3.0010	3.0010	3.4310	3.4320	3.9055	3.9075	3.0015	3.0010
	0.85	3.4310	3.4310	3.8940	3.8925	3.0015	3.0015	3.4255	3.4300	3.9020	3.8925	3.0020	3.0015

Panel C: Homogeneous Break, $IC_{p3}(R, R)$

		Unfeasible Estimator						Feasible Estimator					
N		25		50		100		25		50		100	
$\delta_i^0 > 0$		0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00
T	π^0												
100	0.15	3.9860	4.0155	6.8070	6.8125	7.3840	7.3785	4.0705	4.2165	7.1720	7.2975	7.9980	8.0000
	0.30	4.0575	4.0830	6.9860	6.9935	7.9800	7.9805	4.0805	4.1405	7.1610	7.1385	7.9985	7.9930
	0.50	4.0525	4.0725	7.0400	7.0395	7.9960	7.9960	4.0800	4.0765	7.1410	7.0580	7.9990	7.9965
	0.70	4.0395	4.0485	6.9785	6.9800	7.9725	7.9740	4.0685	4.1190	7.1495	7.1555	7.9980	7.9935
	0.85	3.9780	3.9800	6.7970	6.7965	7.3945	7.3950	4.0555	4.1970	7.1710	7.2950	7.9990	7.9990
200	0.15	3.7845	3.7940	6.3435	6.3565	4.2950	4.2850	3.7895	3.8060	6.3705	6.4315	4.5860	4.7180
	0.30	3.7865	3.7885	6.3150	6.3215	4.4665	4.4545	3.7905	3.7910	6.3650	6.3260	4.5790	4.4795
	0.50	3.7930	3.7990	6.2880	6.2905	4.5095	4.5025	3.7890	3.7990	6.3475	6.2910	4.5720	4.5065
	0.70	3.8000	3.8015	6.3225	6.3220	4.4965	4.4935	3.7900	3.7985	6.3645	6.3300	4.5900	4.5075
	0.85	3.7910	3.7955	6.3285	6.3345	4.3140	4.3120	3.7850	3.8110	6.3675	6.4040	4.5980	4.7520
400	0.15	3.7485	3.7520	5.8255	5.8290	3.5340	3.5330	3.7325	3.7495	5.8695	5.8335	3.5630	3.5445
	0.30	3.7420	3.7440	5.8370	5.8315	3.5470	3.5465	3.7335	3.7440	5.8490	5.8315	3.5450	3.5465
	0.50	3.7365	3.7370	5.8230	5.8150	3.5345	3.5370	3.7335	3.7370	5.8410	5.8150	3.5495	3.5370
	0.70	3.7455	3.7480	5.8390	5.8390	3.5345	3.5360	3.7390	3.7480	5.8665	5.8390	3.5565	3.5360
	0.85	3.7430	3.7415	5.8390	5.8370	3.5415	3.5390	3.7305	3.7415	5.8855	5.8370	3.5580	3.5470

Table 4-*Continued*: Experiment 4, Model Selection, $R^0 = 3$, $\alpha^0 = 0.60$

TO BE COMPLETED

Panel D: Eterogeneous Break, $IC_{p1}(R, R)$							
		Unfeasible Estimator			**Feasible Estimator**		
N		25	50	100	25	50	100
T	π^0						
	0.15						
	0.30						
100	0.50						
	0.70						
	0.85						
	0.15						
	0.30						
200	0.50						
	0.70						
	0.85						
	0.15						
	0.30						
400	0.50						
	0.70						
	0.85						

Panel E: Eterogeneous Break, $IC_{p2}(R, R)$							
		Unfeasible Estimator			**Feasible Estimator**		
N		25	50	100	25	50	100
T	π^0						
	0.15						
	0.30						
100	0.50						
	0.70						
	0.85						
	0.15						
	0.30						
200	0.50						
	0.70						
	0.85						
	0.15						
	0.30						
400	0.50						
	0.70						
	0.85						

Panel F: Eterogeneous Break, $IC_{p3}(R, R)$							
		Unfeasible Estimator			**Feasible Estimator**		
N		25	50	100	25	50	100
T	π^0						
	0.15						
	0.30						
100	0.50						
	0.70						
	0.85						
	0.15						
	0.30						
200	0.50						
	0.70						
	0.85						
	0.15						
	0.30						
400	0.50						
	0.70						
	0.85						

Table 5: Experiment 5, Linear Diffusion Index Model, $R^0 = 1$, $\alpha^0 = 0.60$, $\beta^0 = 1$

TO BE COMPLETED

Panel A: Bias

		Homogeneous Break						Heterogeneous Break		
N		25		50		100		25	50	100
$\delta_i^0 > 0$		0.25	1.00	0.25	1.00	0.25	1.00	$\mathcal{N}(0,1)$	$\mathcal{N}(0,1)$	$\mathcal{N}(0,1)$
T	π^0									
	0.15				0.0005	0.0052	0.0054			
	0.30				0.0005	0.0052	0.0054			
100	0.50				0.0006	0.0052	0.0052			
	0.70				0.0006	0.0053	0.0059			
	0.85				0.0007	0.0052	0.0054			
	0.15			0.0012	0.0012	0.0005	0.0005			
	0.30				0.0011	0.0005	0.0005			
200	0.50				0.0009	0.0005	0.0007			
	0.70				0.0011	0.0005	0.0010			
	0.85				0.0011	0.0005	0.0009			
	0.15			0.0004	0.0005	0.0012	0.0012			
	0.30			0.0004	0.0005	0.0012	0.0011			
400	0.50			0.0004	0.0004	0.0012	0.0011			
	0.70			0.0004	0.0004	0.0012	0.0012			
	0.85			0.0004	0.0004	0.0012	0.0013			

Panel B: RMSE

		Homogeneous Break						Heterogeneous Break		
N		25		50		100		25	50	100
$\delta_i^0 > 0$		0.25	1.00	0.25	1.00	0.25	1.00	$\mathcal{N}(0,1)$	$\mathcal{N}(0,1)$	$\mathcal{N}(0,1)$
T	π^0									
	0.15				0.1023	0.0985	0.0991			
	0.30				0.1016	0.0986	0.0996			
100	0.50				0.1021	0.0985	0.1006			
	0.70				0.1020	0.0986	0.1018			
	0.85				0.1019	0.0986	0.1040			
	0.15			0.0704	0.0704	0.0690	0.0690			
	0.30				0.0704	0.0690	0.0692			
200	0.50				0.0707	0.0689	0.0693			
	0.70				0.0706	0.0690	0.0704			
	0.85				0.0706	0.0689	0.0710			
	0.15			0.0502	0.0503	0.0489	0.0490			
	0.30			0.0502	0.0504	0.0489	0.0490			
400	0.50			0.0502	0.0503	0.0489	0.0490			
	0.70			0.0502	0.0503	0.0489	0.0491			
	0.85			0.0502	0.0502	0.0489	0.0494			

Panel C: Size 5% Level

		Homogeneous Break						Heterogeneous Break		
N		25		50		100		25	50	100
$\delta_i^0 > 0$		0.25	1.00	0.25	1.00	0.25	1.00	$\mathcal{N}(0,1)$	$\mathcal{N}(0,1)$	$\mathcal{N}(0,1)$
T	π^0									
	0.15				0.0615	0.0560	0.0575			
	0.30				0.0570	0.0560	0.0560			
100	0.50				0.0570	0.0565	0.0570			
	0.70				0.0585	0.0570	0.0575			
	0.85				0.0600	0.0570	0.0615			
	0.15			0.0500	0.0485	0.0510	0.0510			
	0.30				0.0500	0.0505	0.0490			
200	0.50				0.0495	0.0500	0.0475			
	0.70				0.0475	0.0510	0.0525			
	0.85				0.0500	0.0510	0.0505			
	0.15			0.0620	0.0645	0.0490	0.0490			
	0.30			0.0625	0.0625	0.0480	0.0490			
400	0.50			0.0620	0.0645	0.0485	0.0500			
	0.70			0.0615	0.0625	0.0485	0.0510			
	0.85			0.0620	0.0625	0.0485	0.0520			

Table 6: Experiment 6, Unstable Diffusion Index Model

TO BE COMPLETED

Table 7: Experiment 7, Test for Break

TO BE COMPLETED

Table 8: Experiment 8, Estimation, $R_1^0 = 3$, $R_2^0 = 2$

TO BE COMPLETED

Panel A: Homogeneous Break, Bias and RMSE, Estimator for π^0													
N		25				50				100			
$\delta_i^0 > 0$		0.25		1.00		0.25		1.00		0.25		1.00	
T	π^0	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**
	0.15	0.7639	0.7670	0.7624	0.7681	0.6533	0.6811	0.4130	0.5433	0.5717	0.6141	0.2452	0.4061
	0.30	0.5850	0.5929	0.4604	0.5251	0.4312	0.4785	0.0217	0.0977	0.3590	0.4197	0.0046	0.0391
100	0.50	0.3638	0.3782	0.0846	0.1739	0.2356	0.2845	0.0013	0.0119	0.1832	0.2413	0.0004	0.0047
	0.70	0.1650	0.1840	0.0185	0.0492	0.1069	0.1400	0.0016	0.0102	0.0843	0.1203	0.0010	0.0077
	0.85	0.0510	0.0660	0.0077	0.0230	0.0374	0.0555	0.0018	0.0102	0.0312	0.0493	0.0015	0.0092
	0.15	0.7922	0.7926	0.7918	0.7928	0.7273	0.7373	0.2490	0.4366	0.6444	0.6741	0.0586	0.2089
	0.30	0.6321	0.6333	0.5016	0.5642	0.5087	0.5402	0.0004	0.0137	0.3799	0.4482	0.0001	0.0019
200	0.50	0.4198	0.4239	0.0287	0.1058	0.2576	0.3071	0.0000	0.0000	0.1559	0.2258	0.0000	0.0011
	0.70	0.2102	0.2188	0.0031	0.0179	0.1071	0.1429	0.0002	0.0030	0.0690	0.1076	0.0000	0.0011
	0.85	0.0690	0.0791	0.0028	0.0136	0.0392	0.0567	0.0003	0.0040	0.0283	0.0473	0.0000	0.0000
	0.15	0.7991	0.7992	0.7994	0.7994	0.7730	0.7751	0.0943	0.2729	0.7425	0.7497	0.0035	0.0519
	0.30	0.6466	0.6468	0.5591	0.6015	0.5908	0.6005	0.0000	0.0000	0.4364	0.5049	0.0000	0.0000
400	0.50	0.4429	0.4434	0.0043	0.0420	0.2843	0.3337	0.0000	0.0000	0.1051	0.1882	0.0000	0.0000
	0.70	0.2363	0.2386	0.0004	0.0047	0.0945	0.1360	0.0000	0.0000	0.0400	0.0808	0.0000	0.0000
	0.85	0.0821	0.0878	0.0002	0.0032	0.0344	0.0533	0.0000	0.0000	0.0206	0.0395	0.0000	0.0000

Panel B: Homogeneous Break, MSE, Estimator for c_{it}^{0s}													
		Unfeasible Estimator						**Feasible Estimator**					
N		25		50		100		25		50		100	
$\delta_i^0 > 0$		0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00	0.25	1.00
T	π^0												
	0.15							0.2664	0.2859	0.2044	0.1992	0.1290	0.1210
	0.30							0.2208	0.2357	0.1623	0.1341	0.1161	0.1000
100	0.50							0.2021	0.1811	0.1554	0.1343	0.1126	0.1020
	0.70							0.1937	0.1785	0.1511	0.1385	0.1102	0.1038
	0.85							0.1876	0.1835	0.1470	0.1416	0.1078	0.1046
	0.15							0.2358	0.2597	0.1422	0.1226	0.0804	0.0631
	0.30							0.1855	0.2031	0.1157	0.0886	0.0726	0.0604
200	0.50							0.1678	0.1275	0.1100	0.0926	0.0694	0.0625
	0.70							0.1592	0.1346	0.1064	0.0969	0.0683	0.0641
	0.85							0.1538	0.1452	0.1045	0.0997	0.0675	0.0653
	0.15							0.2193	0.2460	0.1084	0.0766	0.0565	0.0392
	0.30							0.1690	0.1946	0.0928	0.0672	0.0513	0.0405
400	0.50							0.1510	0.1030	0.0869	0.0715	0.0463	0.0424
	0.70							0.1427	0.1159	0.0824	0.0755	0.0462	0.0442
	0.85							0.1371	0.1264	0.0818	0.0786	0.0468	0.0456

Table 8-*Continued*: Experiment 8, Estimation, $R_1^0 = 3$, $R_2^0 = 2$

TO BE COMPLETED

Panel C: Eterogeneous Break													
Bias and RMSE, Estimator for π^0								MSE, Estimator for c_{it}^{0s}					
								Unfeasible Estimator			Feasible Estimator		
N		25		50		100		25	50	100	25	50	100
T	π^0	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**						
100	0.15	0.0443	0.1854	0.0056	0.0655	0.0000	0.0000				0.1311	0.1225	0.0984
	0.30	0.0000	0.0011	0.0000	0.0011	0.0000	0.0000				0.1288	0.1247	0.1002
	0.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000				0.1449	0.1300	0.1022
	0.70	0.0000	0.0000	0.0000	0.0011	0.0000	0.0000				0.1604	0.1346	0.1033
	0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000				0.1715	0.1373	0.1032
200	0.15	0.0036	0.0537	0.0000	0.0000	0.0000	0.0000				0.0859	0.0811	0.0603
	0.30	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000				0.0966	0.0851	0.0618
	0.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000				0.1122	0.0899	0.0634
	0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000				0.1276	0.0948	0.0645
	0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000				0.1391	0.0981	0.0652
400	0.15	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000				0.0684	0.0611	0.0409
	0.30	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000				0.0803	0.0648	0.0420
	0.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000				0.0960	0.0696	0.0435
	0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000				0.1116	0.0742	0.0448
	0.85	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000				0.1234	0.0777	0.0458

Table 9: Experiment 9, Model Selection, $R_1^0 = 3$, $R_2^0 = 2$

TO BE COMPLETED

Panel A: Homogeneous Break, $IC_{p1}(R_1, R_2)$

		Unfeasible Estimator						Feasible Estimator											
N		25		50		100		25				50				100			
$\delta_i^0 > 0$		0.25	1.00	0.25	1.00	0.25	1.00	0.25		1.00		0.25		1.00		0.25		1.00	
								R_1	R_2	R_1	R_2	R_1	R_2	R_1	R_2	R_1	R_2	R_1	R_2
T	π^0																		
100	0.15																		
	0.30																		
	0.50																		
	0.70																		
	0.85																		
200	0.15																		
	0.30																		
	0.50																		
	0.70																		
	0.85																		
400	0.15															3.5850	2.0265	3.7515	2.0360
	0.30															3.1290	2.0120	3.1360	2.0110
	0.50															3.0205	2.0345	3.0205	2.0325
	0.70															3.0080	2.1740	3.0080	2.1665
	0.85															3.0295	2.9470	3.0225	2.8470

Panel B: Homogeneous Break, $IC_{p2}(R_1, R_2)$

		Unfeasible Estimator						Feasible Estimator											
N		25		50		100		25				50				100			
$\delta_i^0 > 0$		0.25	1.00	0.25	1.00	0.25	1.00	0.25		1.00		0.25		1.00		0.25		1.00	
T	π^0							R_1	R_2	R_1	R_2	R_1	R_2	R_1	R_2	R_1	R_2	R_1	R_2
100	0.15																		
	0.30																		
	0.50																		
	0.70																		
	0.85																		
200	0.15																		
	0.30																		
	0.50																		
	0.70																		
	0.85																		
400	0.15															3.3890	2.0040	3.5225	2.0075
	0.30															3.0685	2.0030	3.0720	2.0030
	0.50															3.0045	2.0110	3.0045	2.0105
	0.70															3.0010	2.0790	3.0010	2.0760
	0.85															3.0090	2.7780	3.0035	2.6030

Panel C: Homogeneous Break, $IC_{p3}(R_1, R_2)$

		Unfeasible Estimator						Feasible Estimator											
N		25		50		100		25				50				100			
$\delta_i^0 > 0$		0.25	1.00	0.25	1.00	0.25	1.00	0.25		1.00		0.25		1.00		0.25		1.00	
T	π^0							R_1	R_2	R_1	R_2	R_1	R_2	R_1	R_2	R_1	R_2	R_1	R_2
100	0.15																		
	0.30																		
	0.50																		
	0.70																		
	0.85																		
200	0.15																		
	0.30																		
	0.50																		
	0.70																		
	0.85																		
400	0.15															4.8165	2.7615	5.0855	2.8280
	0.30															3.8005	2.5190	3.8075	2.5120
	0.50															3.4860	2.5605	3.4845	2.5510
	0.70															3.4555	2.8955	3.4540	2.8845
	0.85															3.7280	4.1020	3.7845	4.2970

Table 9-*Continued*: Experiment 9, Model Selection, $R_1^0 = 3$, $R_2^0 = 2$

TO BE COMPLETED

Panel D: Eterogeneous Break, $IC_{p1}(R_1, R_2)$										
		Unfeasible Estimator			Feasible Estimator					
N		25	50	100	25		50		100	
					R_1	R_2	R_1	R_2	R_1	R_2
T	π^0									
	0.15								5.0580	2.1735
	0.30								3.4755	2.0585
100	0.50								3.0640	2.1050
	0.70								3.0430	2.6055
	0.85								3.1295	5.0190
	0.15								4.5000	2.1500
	0.30								3.2575	2.0205
200	0.50								3.0390	2.0590
	0.70								3.0125	2.3320
	0.85								3.1075	3.7400
	0.15								3.7550	2.0400
	0.30								3.1375	2.0120
400	0.50								3.0215	2.0350
	0.70								3.0080	2.1740
	0.85								3.0225	2.8455

Panel E: Eterogeneous Break, $IC_{p2}(R_1, R_2)$										
		Unfeasible Estimator			Feasible Estimator					
N		25	50	100	25		50		100	
T	π^0				R_1	R_2	R_1	R_2	R_1	R_2
	0.15								4.3965	2.0060
	0.30								3.0985	2.0020
100	0.50								3.0030	2.0080
	0.70								3.0005	2.1705
	0.85								3.0045	4.3850
	0.15								3.9240	2.0135
	0.30								3.0895	2.0005
200	0.50								3.0050	2.0070
	0.70								3.0005	2.1185
	0.85								3.0060	3.1235
	0.15								3.5290	2.0080
	0.30								3.0740	2.0030
400	0.50								3.0050	2.0115
	0.70								3.0010	2.0795
	0.85								3.0040	2.6110

Panel F: Eterogeneous Break, $IC_{p3}(R_1, R_2)$										
		Unfeasible Estimator			Feasible Estimator					
N		25	50	100	25		50		100	
T	π^0				R_1	R_2	R_1	R_2	R_1	R_2
	0.15								8.0000	7.8635
	0.30								8.0000	7.9340
100	0.50								7.9910	7.8990
	0.70								7.9985	7.9950
	0.85								7.9495	8.0000
	0.15								7.8470	5.4100
	0.30								5.5640	3.6590
200	0.50								4.4655	3.5515
	0.70								4.5405	4.7555
	0.85								6.4780	7.6400
	0.15								5.1035	2.8410
	0.30								3.8180	2.5320
400	0.50								3.4925	2.5620
	0.70								3.4580	2.8885
	0.85								3.7900	4.2890

Table 10: Empirical Application

TO BE COMPLETED

Table 11: Empirical Application

TO BE COMPLETED