A Dynamic Multi-Level Factor Model with Long-Range Dependence

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Abstract

A dynamic multi-level factor model is proposed that incorporates stationary or nonstationary global and regional factors. In the model, persistence in global and regional common factors as well as innovations allows for the study of fractional cointegrating relationships. Estimation of global and regional common factors is performed in two steps employing canonical correlation analysis and a sequential least-squares algorithm. Selection of number of global and regional factors is discussed. A Monte Carlo simulation is included that provides reliability on the methodology. The method is then applied to the Nord Pool energy market for the analysis of price comovements among different regions of the market.

Keywords: Multi-level factor; long memory; fractional cointegration; electricity prices.
JEL Classification: C12, C22.

1 Introduction

Factor models are extensively used as a dimension reduction tool in the analysis of large economic data sets, and the related inferential theory is derived under various setups, see e.g. Stock and Watson (2002), Bai and Ng (2002) Bai (2003), Bai and Ng (2004), and Bai et al. (2008). While there is a vast literature for estimation of common factors and number of common factors when both cross-section and time-series dimensions are large, available methodologies rely on the existence of only pervasive factors. Only recently has there been interest in decomposing common factor structures into different levels. Wang (2010) considers a stationary $I(0)$ multi-level factor model for which identification is discussed and inferential theory is developed.

In this paper, we consider a dynamic multi-level factor model that allows for both pervasive (or global) and nonpervasive (or regional) common factors in that these common factors are allowed to exhibit fractional long-range dependence without restrictions on them being either stationary $I(0)$ or nonstationary $I(1)$ processes. Model innovations are also allowed to be fractionally integrated. This way, the model can be used for the analysis fractional cointegrating relationships in a wide range of economic applications.

The estimation method is similar in spirit to that of Breitung and Eickmeier (2014): a sequential least-squares algorithm is used for the estimation of global and regional common factors. Based on these plug-in estimates, a conditional-sum-of-squares criterion is later used to estimate the residual fractional integration parameter while those of the factors are estimated using an exact local Whittle approach. We establish the asymptotic behavior of factor structure estimates as well as that of the residual fractional integration parameter and show them to be consistent and asymptotically normally distributed at standard parametric rates. We also discuss selection of the number of global and regional common factors by using the usual information criteria of Bai and Ng (2002) and simple tools from set theory. Monte Carlo simulations show that the methodology works well even in relatively small panels. We then apply the method to the relationship between hourly prices in the bidding areas of Nord Pool power market.

The paper is organized as follows. Next section introduces the model along with the assumption made to study it, contains the estimation strategy and the related inferential theory. Section 3 discusses selection of the number of global and regional factors. Section 4 presents finite-sample studies based on Monte Carlo simulations. Section 5 provides an empirical application to the Nord Pool energy market, and finally Section 6 concludes the paper.

Throughout the paper, $\|A\| = (trace(A'A))^{1/2}$ for a matrix A, and $\to_d$ denotes convergence in distribution. All mathematical proofs are collected in an appendix at the end of the paper.

2 The two-level factor model with long-range dependence

2.1 Model

We introduce a multi-level factor model that allows for fractional long memory both in factors and innovations along with the assumptions to study it. The baseline model considered in this paper is a two-level factor model whose factors can be interpreted as unobserved common shocks and are classified in two types: the first is the global or common factor that is the pervasive top-level factor and affects all economic sectors or regions; the second factor is the regional or sector-specific factor that is the nonpervasive sub-level factor and affects only a particular sector or region. Many macroeconomic applications prefer to label these types of factors as global and regional because there is a close connection with the world (global) and regional economies. In this paper we use the terms global/pervasive common factor and regional/nonpervasive/sector-specific factor indistinctly. The model we consider is given by

$$y_{r,it} = \mu'_{r,i}(L)G_t + \lambda'_{r,i}(L)F_{r,t} + \epsilon_{r,it}, \tag{1}$$

where $r = 1, \ldots, R$ indicates the region or sector, with R fixed, whereas indices $i = 1, \ldots, N_r$ and $t = 1, \ldots, T$ denote as usual the i-th variable of region r and the time period, respectively. $N = N_1 + N_2 + \cdots + N_R$ is the total number of time series. The $\mathbf{r}_G \times 1$ vector $G_t = (g_{1,t}, \ldots, g_{\mathbf{r}_G,t})'$ contains the $\mathbf{r}_G$ unobservable global or common factors and the $\mathbf{r}_F \times 1$ vector $F_{r,t}$ consists of the $\mathbf{r}_F$ unobservable regional or sector-specific factors in the region r. $\mu_{r,i}(L)$ and $\lambda_{r,i}(L)$ are $\mathbf{r}_G$- and $\mathbf{r}_F$-dimensional polynomials in the lag operator L, respectively, for instance, $\mu_{r,i}(L) = \mu_{r,0i} + \mu_{r,1i}L + \cdots + \mu_{r,si}L^s$.

In model 1,

$$\begin{aligned} G_t &= \Delta_t^{-\delta_0} w_t, \\ F_{r,t} &= \Delta_t^{-\vartheta_{r,0}} v_{r,t}, \text{ and} \\ \epsilon_{r,it} &= \Delta_t^{-d_{r,i0}} \rho_{r,i}(L) u_{r,it}, \end{aligned}$$

where $w_t, v_{r,t}$, and, $u_{r,it}$ are zero-mean unobservable white noise sequences and the truncated fractional differencing filter $\Delta_t^{-\zeta}$ is described as follows. With $\Delta = 1 - L$, $\Delta^{-\zeta}$ has the expansion

$$\Delta^{-\zeta} = \sum_{j=0}^{\infty} \pi_j(-\zeta)L^j, \quad \text{where } \pi_j(-\zeta) = \frac{\Gamma(j+\zeta)}{\Gamma(j+1)\Gamma(\zeta)},$$

for $\zeta > 0$ with $\Gamma(\tau) = \infty$ for $\tau = 0, -1, \ldots$, but $\Gamma(0)/\Gamma(0) = 1$. $\Delta_t^{-\zeta}$ truncates latter expansion to $\Delta_t^{-\zeta} = \sum_{j=0}^{t} \pi_j(-\zeta)L^j$. This truncation allows for the study of both the stationary case ($\zeta < 1/2$) and the nonstationary case ($\zeta \geq 1/2$), unlike the untruncated filter that does not converge when $\zeta \geq 1/2$, see Davidson and Hashimzade (2009).

An early problem with multi-level factor models is identification. As also is the case in the literature, our approach imposes a block of zero restrictions on the associated matrix of factor loadings so the system of all R regions is represented as

$$\begin{pmatrix} y_{1,\cdot t} \\ \vdots \\ y_{R,\cdot t} \end{pmatrix} = \begin{pmatrix} \Gamma_1 & \Lambda_1 & 0 & \cdots & 0 \\ \Gamma_2 & 0 & \Lambda_2 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ \Gamma_R & 0 & 0 & \cdots & \Lambda_R \end{pmatrix} \begin{pmatrix} G_t \\ F_{1,t} \\ F_{2,t} \\ \vdots \\ F_{R,t} \end{pmatrix} + \begin{pmatrix} u_{1,\cdot t} \\ \vdots \\ u_{R,\cdot t} \end{pmatrix},$$

or

$$y_t = \Lambda^* \ F_t^* + u_t, \tag{2}$$

where $F_t^* = \left(G_t', F_{1,t}', \ldots, F_{R,t}'\right)'$. Model 2 is then the so-called static representation of the original dynamic factor model.

The intuition behind these types of models is that each process $y_{r,it}$, with $r \in \mathbb{N}$, $i \in \mathbb{N}$, is the sum of a *global common component*, $\chi_{g_{r,t}}$, a *regional common component*, $\chi_{r_{r,it}}$ and an idiosyncratic component, $\xi_{r,it}$. Common components of the region r are driven by the respective $\mathbf{r}_G$ and $\mathbf{r}_F$ vectors of common factors (global and regional), which are loaded with possibly different coefficients and lags as we discussed above. For practical purposes, there may be an interest in measuring certain comovements between countries employing multi-level factors. In that case, the global component would capture common movements in all groups of countries, and the regional component would capture common movements with country's neighbors whereas the specific country component would capture movements that are unique to that specific country. Comovements between countries as captured by these multi-level factors can then be used to measure the connectedness of the countries analyzed. For instance, if the regional component of a specific country weighs in heavier than the global component, the country would seem to be more connected with its neighbors than with all the countries as a whole.

Model 1 can be reorganized as

$$\Delta_t^{d_{r,i0}} y_{r,it} = \mu_{r,i}(L)' \Delta_t^{d_{r,i0}} G_t + \lambda_{r,i}(L)' \Delta_t^{d_{r,i0}} F_{r,t} + u_{r,it},$$

which would then be comparable to the standard two-level factor model proposed by Wang (2010). The main difference is that model 1 covers a wide range of persistence levels in each of its time-varying component. Furthermore, fractional integration parameters quantifying persistence in model 1 are also allowed to be heterogeneous.

Let $\max_i d_{r,i0} = d_{r,max}$ and M denote a generic positive constant. We then use the following assumptions to study the model in (1).

Assumption A. Fractional integration parameters: $\vartheta_{r,0} \in \mathcal{V} = [\underline{\vartheta}_r, \overline{\vartheta}_r] \subseteq [0,1]$ for each $r = 1, \ldots, R$, $\delta_0 \in \mathcal{D} = [\underline{\delta}, \overline{\delta}] \subseteq [0,1]$, $d_{r,max} \leq \max\{\delta_0, \vartheta_{r,0}\}$ and $\max\{\delta_0, \vartheta_{r,0}, d_{r,max}\} - \min\{\underline{\delta}, \underline{\vartheta}_r, \underline{d}_{r,i}\} < 1/2$.

Some of the next assumptions are extensions of Bai (2003) and are based on those provided by Wang (2010).

Assumption B. Factors:

B_1 $G_t = \Delta_t^{-\delta_0} w_t$, where $w_t \sim iid(0, \Sigma_w)$ with Σ_w positive definite and $E||w_t||^4 \leq M$.

B_2 $F_{r,t} = \Delta_t^{-\vartheta_{r,0}} v_{r,t}$, where $v_{r,t} \sim iid(0, \Sigma_{v_r})$ with Σ_{v_r} positive definite and $E||v_{r,t}||^4 \leq M$ for all $r = 1, \ldots, R$.

B_3 Define $H_t = \left[G_t^{0'}, F_{r,t}^{0'}\right]'$. For a fixed $\mathbf{r}$, assume that $T^{-1}\sum_{t=1}^{T} H_t H_t' \xrightarrow{p} \sum_H$for some positive matrix $\sum_H$ with rank $\mathbf{r}_G + \mathbf{r}_1 + \cdots + \mathbf{r}_R$.

Assumption C. Factor loadings:

C_1 $\lambda_{r,i}^0$ is either deterministic such that $||\lambda_{r,i}^0|| \leq M < \infty$, or it is stochastic such that $E||\lambda_{r,i}^0||^4 \leq M < \infty$. In the latter case, $N_r^{-1}\Lambda_r^{0\prime}\Lambda_r^0 \xrightarrow{p} \Sigma_{\Lambda_r} > 0$ for a $\mathbf{r}_F \times \mathbf{r}_F$ non-random matrix Σ_{Λ_r} for all $r = 1, \ldots, R$

C_2 $\mu_{r,i}^0$ is either deterministic such that $||\mu_{r,i}^0|| \leq M$, or it is stochastic such that $E||\mu_{r,i}^0||^4 \leq M < \infty$. In both cases, $N_r^{-1}\mu_r^{0\prime}\mu_r^0 \xrightarrow{p} \Sigma_{\mu_r} > 0$ for a $\mathbf{r}_G \times \mathbf{r}_G$ non-random matrix Σ_{μ_r} for all $r = 1, \ldots, R$

C_3 Rank $([\Gamma_\mathbf{r}\Lambda_\mathbf{r}]) = r_\mathbf{r} + r$.

Assumption D. Idiosyncratic shocks.

D_1 $u_{r,it} \sim iid\left(0, \sigma_{r,i}^2\right)$, $E|u_{r,it}|^8 \leq M$.

D_2 $E\left(u_{r_1,it}u_{r_2,jt}\right) = \tau_{r_1,r_2,ij,t}$, with $|\tau_{r_1,r_2,ij,t}| \leq |\tau_{r_1,r_2,ij}|$ for some $\tau_{r_1,r_2,ij} \geq 0$ and for all t. In addition, $N^{-1}\sum_{r_1=1}^R\sum_{r_2=1}^R\sum_{i=1}^{N_{r_1}}\sum_{j=1}^{N_{r_2}}\tau_{r_1,r_2,ij} \leq M$.

D_3 $E\left(u_{r_1,it}u_{r_2,js}\right) = \tau_{r_1,r_2,ij,ts}$ and $(NT)^{-1}\sum_{r_1=1}^R\sum_{r_2=1}^R\sum_{i=1}^{N_{r_1}}\sum_{j=1}^{N_{r_2}}\sum_{t=1}^T\sum_{s=1}^T|\tau_{r_1,r_2,ij,ts}| \leq M$.

D_4 For every (t,s) , $E\left|N^{1/2}\sum_{r_1=1}^R\sum_{i=1}^{N_{r_1}}\left(u_{r_1,is}u_{r_1,it} - E\left(u_{r_1,is}u_{r_1,it}\right)\right)\right|^4 \leq M$.

Assumption E. Processes $\{u_{r,it}\}$, $\{v_{r,t}\}$, $\{w_t\}$, $\{\lambda_{r,i}\}$, and $\{\mu_{r,i}\}$ are mutually independent groups.

Assumption F. Identification

F_1 $F_r^{\prime}F_r/T = I_{\mathbf{r}_F}$ and $\Lambda_r^{\prime}\Lambda_r$ diagonal (Within region identification).

F_2 $G^{\prime}G/T = I_{\mathbf{r}_G}$ and $\Gamma^{\prime}\Gamma$ diagonal (Between region identification).

F_3 Factors have zero mean, and $\sum_{t=1}^T G_tF_{r,t}^{\prime} = 0$ for $r = 1, \ldots, R$.

Assumption A imposes standard restrictions on the range of allowed values for memory parameters. The model in (1) simultaneously admits a large combination of persistence levels in factors as well as the idiosyncratic terms. Owing to that, Assumption A allows for extensive fractional cointegration analysis and can be useful in understanding the behavior of persistent economic and financial variables involved in a complex system. Our model guarantees a fractional cointegrating relationship in the multi-level factor model when $d_{r,max} <$ max $\{\delta_0, \vartheta_r\}$ for some r but this is not imposed a priori. The condition $\max\{\delta_0 - \underline{\delta}, \vartheta_{r,0} - \underline{\vartheta}_r\} < 1/2$ allows for the use of standard techniques in establishing consistency of the factor structure estimates, cf. Bai and Ng (2004) for example, together with the condition $d_{r,max} \leq$ max $\{\delta_0, \vartheta_r\}$. The requirement $d_{r,i0} - \underline{d} < 1/2$ for all i, is needed for a uniform treatment of the residual memory estimate. Finally the cross requirements between the memory parameters are due to the interplay between the model parameters and ensure that the factor estimation errors employed in the estimation of residual memory parameter are asymptotically negligible.

Assumptions B_1 and B_2 describe the structures of the global and regional factors defined earlier and impose standard moment conditions on their disturbances. These assumptions are more general than those of classical factor analysis in which factors are assumed to be iid, and those of multi-level factor analysis proposed by Wang (2010) who imposes $I(0)$ stationarity in both factors. As usual, even with lagged factors on G_t or $F_{r,t}$, the dynamic model in (1) can be reformulated as a static factor model, see Bai et al. (2008). Many econometric methods are developed under the static framework because it is easily estimated using time domain methods in comparison with the dynamic framework that is generally treated within a frequency domain perspective. In this paper we focus on the static factor model even if derived properties still hold for our dynamic model. Rank condition in Assumption B_3 implies that different factors are not perfectly correlated. Assumption C_1 ensures that global factor G_{mt} has nontrivial contribution to the variance of y_t, $m = 1, \dots, r$ while assumption C_2 ensures that each regional factor $F_{r,jt}$ has a nontrivial contribution to the variance of $y_{r,t}$, $j = 1, \dots, r_F$. Latter means that G_t pervades all variables whereas the regional factor $F_{r,jt}$ pervades only within region r. Rank condition in Assumption C_3 guarantees enough heterogeneity among individual variables within region r when responding to both factors. Such a rank condition is useful in separating identification of global and regional factors. Assumption D allows for limited time-series and cross-section dependence in the idiosyncratic components. Heteroskedasticity in both time and cross-section dimensions is also allowed. Correlation in the idiosyncratic components allow the model to have an approximate factor structure. Finally, Assumption E implies that the global or regional factors can be serially correlated, the factor loadings $(\lambda_{r,i}, \mu_{r,i})$ can be correlated over i in each region r, and the idiosyncratic shocks can have serial and cross-sectional correlations. Although factors, factor loadings and the idiosyncratic shocks are assumed to be independent of each other, regional factors from different regions can still be correlated.

For estimation purposes, we assume the number of factors $\mathbf{r}_G$ and $\mathbf{r}_F$ to be known and fixed at this step. Formal tests or information criteria for such number of factors are, to our knowledge, not yet available even discarding long-range dependence in the factors. Notwithstanding, the number of factors does not affect asymptotic results for the common component, see Bai (2003).

We need the three restrictions given in Assumption F to identify the factors. Assumptions F_1 and F_2 are standard in factor analysis and allow the model to be uniquely identified under such normalizations. Assumption F_3 rules out any possibility of correlation between the global and regional factors which is the same as that the global factors do not contain information about regional factors and vice versa. This assumption enables us to separately identify regional factors and global factors. Wang (2010) discusses restrictions involved in both assumptions.

Following Breitung and Eickmeier (2014) and Proposition 1 in Wang (2010), the factor loadings for the model 2 are identified up to a linear transformation of the loading matrix that preserves the same zero restrictions of the model given by $\Lambda^* Q$ with

$$Q = \begin{pmatrix} Q_{00} & 0 & 0 & \cdots & 0 \\ Q_{10} & Q_{11} & 0 & \cdots & 0 \\ \vdots & & & \ddots & 0 \\ Q_{R0} & 0 & 0 & \cdots & Q_{RR} \end{pmatrix}, \tag{3}$$

where orthonormal global and regional factors within each of the $R+1$ blocks are given by $Q_{00} = \left(T^{-1}\sum_{t=1}^{T} G_t G_t'\right)^{-1/2}$ and $Q_{rr} = \left(T^{-1}\sum_{t=1}^{T} F_{r,t} F_{r,t}'\right)^{-1/2}$ for all r. Matrix 3 imposes that the R blocks of regional factors are uncorrelated with the blocks of global factors.

2.2 Estimation

We adopt the estimation procedure proposed by Breitung and Eickmeier (2014) to estimate the global and regional common factors. A similar procedure related to the sequential PC approach is proposed in Wang (2010) but that of Breitung and Eickmeier (2014) proves to be computationally simpler. We then use an equation-by-equation conditional-sum-of-squares (CSS) estimation based on the regression residuals to estimate the memory parameters $d_{r,i0}$.

Let $\hat{d}_{r,i}$ denote the estimate of the unknown true fractional integration parameter $d_{r,i0}$, that is given by

$$\hat{d}_{r,i} = \underset{d_{r,i} \in D}{\operatorname{argmin}} \, L_{r,i,T}^* \left(d_{r,i}\right),$$

where

$$L_{r,i,T}^* \left(d_{r,i}\right) = \frac{1}{T}\sum_{t=1}^{T} \left(\Delta_t^{d_{r,i}} \hat{\epsilon}_{r,it}\right)^2,$$

and for $r = 1, \dots, R$, $i = 1, \dots, n_r$ and $t = 1, \dots, T$,

$$\hat{\epsilon}_{r,it} \equiv y_{r,it} - \hat{\gamma}_{r,i}' \hat{G}_t - \hat{\lambda}_{r,i}' \hat{F}_{r,t}, \tag{4}$$

where $\hat{\Gamma}^*$ and $\hat{F}_t^*$ in the system representation of (4) are obtained by sequential least squares (SLS) that is proposed by Breitung and Eickmeier (2014), which produces estimates asymptotically equivalent to the maximum likelihood estimates under the assumption that $u_{r,it}$ are iid across i, t and r. We outline the steps of such an algorithm in which the main goal is to minimize the residual sum of squares (RSS) function

$$\begin{aligned} S\left(F^*, \Lambda^*\right) &= \textstyle\sum_{t=1}^{T} \left(y_t - \Lambda^* F_t^*\right)' \left(y_t - \Lambda^* F_t^*\right) \\ &= \textstyle\sum_{r=1}^{R} \sum_{i=1}^{n_r} \sum_{t=1}^{T} \left(y_{r,it} - \gamma_{r,i}' G_t - \lambda_{r,i}' F_{r,t}\right)^2 \end{aligned} \tag{5}$$

by a sequence of two least-squares regressions until RSS achieves a minimum. Algorithm is easily executed as follows:

1. The algorithm is initialized by using initial estimators of the global and regional factors, $\hat{G}^{(0)} = \left(\hat{G}_1^{(0)}, \ldots, \hat{G}_T^{(0)}\right)'$ and $\hat{F}_r^{(0)} = \left(\hat{F}_{r,1}^{(0)}, \ldots, \hat{F}_{r,T}^{(0)}\right)'$. Such estimators can be obtained by using canonical correlation analysis (CCA).

2. Once initial estimators are obtained, the corresponding factor loadings at the initial step are estimated from the time-series regression $y_{r,it} = \mu_{r,i}^{'}\hat{G}_t^{(0)} + \lambda_{r,i}^{'}F_{r,t}^{(0)} + \widetilde{\epsilon}_{r,it}$ that construct the factor loadings matrix, $\hat{\Gamma}^{*(0)}$, in Equation (2).

3. The global and regional factors in the next step, $\hat{G}^{(1)}$ and $\hat{F}_{r,1}^{(1)}$, are updated from the least-squares regression of y_t on $\hat{\Gamma}^{*(0)}$ to obtain $F_t^{*(1)} = \left(\hat{\Gamma}^{*(0)\prime}\hat{\Gamma}^{*(0)}\right)^{-1}\hat{\Gamma}^{*(0)\prime}y_t$.

4. Next, the updated factors $F_t^{*(1)}$ are used to get the associated factor loading matrix, $\hat{\Gamma}^{*(1)}$, as in step 2.

5. Steps 3 and 4 are repeated until RSS converges to a minimum from which $\hat{F}^*$ and $\hat{\Lambda}^*$ are collected.

CCA is a standard tool in multivariate statistics and is a way of measuring the linear relationship between two multidimensional variables. Such an analysis finds two sets of basis vectors, one for each variable, such that the correlations between the projections of the variables onto these bases are mutually maximized. Breitung and Eickmeier (2014) propose CCA to get the initial estimates of the global and regional factors to ensure that the procedure listed above starts in the neighborhood of the global minimum. CCA is carried out in 2 steps. At the first step, in each region, $\mathbf{r} = \mathbf{r}_G + \mathbf{r}_F$ principal components are estimated obtaining R consistent factor spaces of the form $\hat{F}_{r,t}^{+}$ for $r = 1, \ldots, R$ which will eventually share a common component yielding the initial global factor after the second step. Let $\mathscr{H}_t = \left(\hat{F}_{r,t}^{+}, \hat{F}_{s,t}^{+}\right)'$ with $c^0\mathscr{H}_t$ denoting the canonical variables, the CCA (at the second step of the procedure) solves the following maximization problem:

$$\max\left\{c^0\Sigma_{01}c^1 / \left[c^0\Sigma_{00}c^0 \cdot c^1\Sigma_{11}c^1\right]^{1/2}\right\}$$

$$\text{s.t. } c^0\Sigma_{00}c^0 = 1, \text{ and } c^1\Sigma_{11}c^1 = 1,$$

where $\Sigma_{00} = \text{var}\left(\mathscr{H}_t\right),\ \Sigma_{11} = \text{var}\left(\mathscr{H}_{t-1}\right)$ and, $\Sigma_{01} = \text{cov}\left(\mathscr{H}_t, \mathscr{H}_{t-1}\right)$.
The resulting linear combination with the largest canonical correlation will be the estimate of the global factor, $\hat{G}^{(0)}$. Subsequently, we regress original principal components of the region r, $\hat{F}_{r,t}^{+}$, on the estimated global factors in order to find $\hat{F}_r^{(0)}$ for all $r = 1, \ldots, R$.
As pointed out earlier, Λ^{*0} and F_t^{0*} are not separately identifiable. Then in order to

identify the common component $\xi_t^* = \Lambda^* F_t^*$ we choose the nonsingular matrix Q in (3) to preserve identification of the factors. We use the standard normalizations in PC analysis given in Assumption F. Note that even when global and regional factors are orthogonal to each other, the correlation between regional factors from different regions are allowed. We follow the steps proposed by Breitung and Eickmeier (2014) to adapt to the normalization. Such steps consist first of regressing the regional factors $\hat{F}_{r,t}$ on $\hat{G}_t$ in order to get the orthogonalized regional factors, and second, extracting the normalized global and regional factors after running PC analysis of the respective common component.

2.3 Asymptotic inference

We establish asymptotic behavior of the factor structure and residual fractional integration parameter estimates. Let us define the vector that collects the first-differenced global and regional factors as

$$f^* := \Delta F^*.$$

The next theorem presents the consistency of first-differenced factor estimates.

Theorem 2.1 (Consistency of the common components)**.** *Under Assumptions A-F and if* $\Lambda^*\Lambda^{*'}$ *is diagonal with distinct entries,*

$$\min\left\{\sqrt{N_R}, T\right\}\left(\hat{f}^* - f^*\right) = O_p(1).$$

This result basically shows that under the range of allowed memory values in Assumption A, first differencing will produce consistent estimates for both global and regional common factors up to a rotation at the standard parametric rates. A similar result has been established by Bai (2003) that considers only an $I(0)$ setup and by Bai and Ng (2004) in the $I(1)$ case. The additional condition that imposes $\Lambda^*\Lambda^{*'}$ to be diagonal with distinct entries ensures that the factor estimates are asymptotically exactly identified, not just their rotations, see Bai and Ng (2013). Noting that first differencing removes persistence under our setup together with the distinct-entries condition on the diagonal loadings matrix, the result can be directly established from the results provided by Bai (2003) and Bai and Ng (2004), and Bai and Ng (2013) as we show in the appendix.

In the next theorem, we establish asymptotic results for residual memory estimates.

Theorem 2.2 (Asymptotic behavior of residual memory estimates)**.** *Under conditions of Theorem 2.1, as* $(N,T)_j \to \infty$*, for fixed* r*,*

$$T^{1/2}\left(\hat{d}_{r,i} - d_{r,i0}\right) \xrightarrow{d} N\left(0, 6/\pi^2\right).$$

This result is based on the consistent factor estimates whose behavior is established in Theorem 2.1. It states that the residual memory parameters are $\sqrt{T}$-consistent, asymptotically normal and efficient. Though factor structure estimates are used cf. equation (4), under Assumption A and as $N \to \infty$, estimation errors that are due to the use of plug-in factor estimates vanish. This then allows the use of factors as if they were observable in establishing the asymptotics of residual memory estimate, as we show in the appendix.

3 Determining the number of regional and global factors

The model in (1) assumes that the number of factors in each region and the number of global factors is fixed and known. Although a crucial step in the identification of the model is to accurately estimate the numbers of such factors, most of the empirical literature fixes the number of regional and global factors to be one or analyze directly some alternative models considering more factors without using formal information criteria.

Although there are many methodologies to estimate the number of the static factors in one-level factor models, see e.g. Bai and Ng (2002), Alessi et al. (2010), Onatski (2010), Kapetanios (2010) and Ahn and Horenstein (2013), a formal methodology to estimate the number of static factors in a multi-level factor model is not yet available to the best of our knowledge. The only exception is the proposal of Hallin and Liška (2011) who allow for identifying and estimating joint and block-specific common factors in the context of dynamic factor models.

Inspired by the methodology of Hallin and Liška (2011) and using the well-known information criteria of Bai and Ng (2002) (BN2002), we propose a new procedure for identifying the number of regional and global factors under our setup. We retain Assumptions A-F imposed to study the model in (1). Our assumptions are in line with those of BN2002 but we have extra conditions pertaining to the long-range dependence of factors and the idiosyncratic terms as well as to the multi-level factor structure we have.

BN2002 consider an approximate static factor model and suggest a penalty criteria function of the form

$$PC(k) = V\left(k, \hat{F}^k\right) + \; k\, g(N,T), \tag{6}$$

where $V\left(k, \hat{F}^k\right)$ is the sum of squared residuals when k factors are estimated. The essence of the criteria is to find penalty functions, $g(N,T)$, which can consistently estimate the number of static factors r. Assuming that there exists a bounded and positive integer k_{max} number of static factors such that $r \leq k_{max}$, Theorem 2 and Corollary 1 in BN2002 provide necessary conditions to consistently estimate the number of static factors r. BN2002 provide six choices for the penalty function and indicate the corresponding criteria as PC1, PC2, PC3, IC1, IC2, and IC3. Criteria IC1 and IC2 are more often used in empirical applications in the literature.

Because there is no available literature regarding the estimation of the number of factors in presence of long memory, we first focus on a type-II fractionally integrated single-level factor model to discuss how to estimate the number of factors r in a static factor model with long memory by using the same information criteria provided by BN2002. Consider the model 1 with $R = 1$, that is

$$y_{it} = \lambda_i^{'} F_t + \epsilon_{it},$$

where $F_t = \Delta_t^{-\vartheta_0} v_t$, and $\epsilon_{it} = \Delta_t^{-d_{i0}} u_{r,it}$ as before. Let $\varsigma_{i0} \equiv \max(\vartheta_0, d_{i0})$, then fractional differencing each y_i by ς_{i0} we can consistently apply information criteria of BN2002 regardless of the values of ϑ_0 or d_{i0}, otherwise the estimation of r will be dramatically affected when $d_{i0} > 0$. Latter is implied by Assumption C in BN2002. Once fractional differencing is applied, the consistency proof in Theorem 2 by BN2002 is still valid.[1] When $\varsigma_0 \leq 1$, taking first differences would be enough to ensure the consistency of number of factors estimate. Nevertheless, we suggest estimating first ς by using the Extended Local Whittle method (ELW) of Abadir et al. (2007), which covers the stationary and nonstationary regions even beyond the unit root, in order to get $\hat{\varsigma}$ and later, fractional differencing each y_i by $\hat{\varsigma}$. Table 1 reports a small Monte Carlo simulation to illustrate this methodology.

Table 1: Number of static factors under negligence of long memory, first differencing, and fractional differencing (N = 40, T = 300. r=3 and $k_{max} = 10$).

	Neglected memory	First Diff.	Diff. by ς	Neglected memory	First Diff.	Diff. by ς
	$d_i = 0.4\ \forall i$	$\vartheta = 0.8$		$d_i = 0.8\ \forall i$	$\vartheta = 0.4$	
IC1	3	3	3	10	3	3
IC2	3	3	3	10	3	3
IC3	3	3	3	10	3	3
PC1	5	3	3	10	3	3
PC2	5	3	3	10	3	3
PC3	6	3	3	10	3	3
	$d_i = 1\ \forall i$	$\vartheta = 1.5$		$d_i = 1.5\ \forall i$	$\vartheta = 1$	
IC1	10	3	3	10	6	3
IC2	10	3	3	10	5	3
IC3	10	3	3	10	8	3
PC1	10	3	3	10	8	3
PC2	10	3	3	10	8	3
PC3	10	3	3	10	8	3

We now extend the aforementioned methodology to allow for more than one-level in a factor model. Our methodology to estimate the number of static regional

[1] As a matter of fact, it would be only necessary to fractionally difference by d_{i0} to consistently estimate r. However d_{i0} remains unknown until after the model is estimated.

and global factors adopts the method of Hallin and Liška (2011) which identifies and estimates joint and block-specific common factors by using the identification method of Hallin and Liška (2007) in the nature of dynamic factor models. In the case of Hallin and Liška (2011), their joint common factors may be interpreted as a global or pervasive top-level factor in our case whereas the block-specific factor would be the regional or non-pervasive sub-level factor.

For the sake of simplicity, consider only two regions or blocks (B_x, B_y) in model 1 and only one regional factor in each region and one global factor. We can now divide our data into three different factor spaces. Call the marginal factor spaces as those two different spaces spanned by the individual blocks of data B_x and B_y and call the joint factor space as that spanned by the complete block $B_{x\cup y}$. In these spaces, we see that $r_{B_x} = 2$, $r_{B_y} = 2$, and $r_{Bx\cup y} = 3$ given that we have only one regional factor in each region and only one global factor. Latter means that both marginal factor spaces consist of two static factors whereas the joint factor space consists of three static factors. The number of factors in each one of these three factor spaces is consistently estimated by using the information criteria in BN2002 after fractional differencing by $\hat{\varsigma}$ as discussed before. Theorem 2 in BN2002 is still valid after fractional differencing. The simple Venn diagram 1 displays this information. Green sector represents the part of the factor space which is shared by both regions and consists of one static factor (the global factor). The marginal factor space B_x is represented by blue + green sectors having two static factors whereas the marginal factor space B_y is the yellow + green sectors and also has two factors. Naturally, the number of regional static factors is directly obtained after computing the number of global factor by the inclusion-exclusion principle, i.e, $r_{B_x \cup B_y} = r_{B_x} + r_{B_y} - r_{B_x \cap B_y}$, from which we would get $r_{B_x \cap B_y} = 1$ (the global factor).

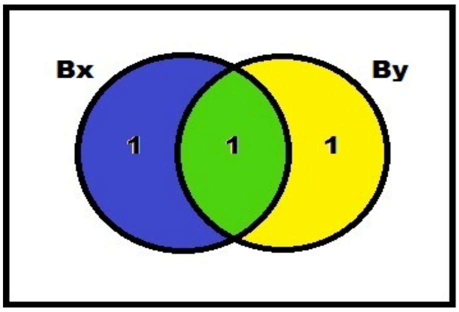

Figure 1: Representation of the three factor spaces spanned by two regions or blocks of data.

Complexity of this methodology increases in the number of regions. Clearly, when we have R regions, the number of blocks to be analyzed will be the power set minus one, $2^R - 1$. Furthermore, we should compute the number of global factors by using each one of the number of factors (cardinalities) estimated in the individual, pairwise, triple-wise, etc. sets by the inclusion-exclusion principle. The number of regional factors in region R would be determined by subtracting the number of factors previously estimated in each one of the intersections where region R interacts with the number of factors previously estimated only in the region R. As an example, consider now three regions, we have the following blocks: B_x, B_y, B_z, $B_{x\cup y}$, $B_{x\cup z}$, $B_{y\cup z}$, and $B_{x\cup y\cup z}$ from which after fractional differencing all the variables in the data set, we compute the number of factors which span each one of the seven blocks. The number of factors in pair-wise blocks will be given by $r_{B_{x\cap y}} = r_{B_x} + r_{B_y} - r_{B_{x\cup y}}$, for instance. The global factor will be given by $r_{B_{x\cap y\cap z}} = r_{B_{x\cup y\cup z}} - r_{B_x} - r_{B_y} - r_{B_z} + r_{B_{x\cap y}} + r_{B_{x\cap z}} + r_{B_{y\cap z}}$ and the number of regional factors of the region x by $r_{B_x} - r_{B_{x\cap y\cap z}} - r_{B_{x\cap y}} - r_{B_{x\cap z}}$, for instance. Naturally, with this methodology it is possible to specify not only the number of factors corresponding to the global and regional levels but also the number of factors in each one of the three pairwise blocks of regions.

4 Finite sample properties

In this section we examine the finite-sample properties of the SLS procedure to investigate the performance of the model in (1) and the methodology proposed to estimate the number of global and regional factors.

4.1 Two-level factor model

We first present four Monte Carlo studies to study the performance of our model. In our simulation studies we are generating a fractional cointegration relationship between $y_{r,it}$ and the global factor (G_t) since we believe such a relationship is likely in several empirical studies.

In the first Monte Carlo study, whose results are presented in Tables 5, 6, and 7, we analyze the performance of our model with $R = 2$, $N_r \in \{20, 80\}$, and different sample sizes with $T = \{150, 1000, 5000\}$, respectively. One global factor and one regional factor in each region are considered for simplicity although more factors are allowed. The global, both regional factors and all the idiosyncratic terms are independently generated by ARFIMA(1,d*,0) processes where d* corresponds to δ, ϑ_r or $d_{i,r}$ as appropriate. Autoregressive parameters are 0.5 for the unobservable factors and 0.1 for the idiosyncratic errors, following Breitung and Eickmeier (2014). $u_{r,it} \overset{iid}{\sim} N(0, 2\phi)$ with ϕ controlling the signal-to-noise-ratio with $\phi = \{5, 2, 0.5\}$, corresponding to low, medium and high signal-to-noise-ratios. $w_t \overset{iid}{\sim} N(0, \sigma_w)$ and $v_{r,t} \overset{iid}{\sim} N(0, \sigma_{v_r})$ controlling the ratio $\frac{\sigma_{v_r}}{\sigma_w}$ to study the relative impact of the factors to each other. Furthermore, all factor loadings are generated as $N(1, 1)$, following Boivin and Ng (2006). All results are based on 1000 replications of the model.

For each experiment, after collecting the estimated regional and global factors, we estimate the memory parameters $\hat{\vartheta}_r$, $\hat{\delta}$ using Extended Local Whittle procedure. We also regress the actual factors (global or regional) on the estimated ones in order to study the reliability of the procedure (R^2_G, $R^2_{R_r}$). Such a correlation coefficient can be considered as a measure of consistency for all t, see Bai (2003). Finally, we present the average of the estimated residual integration orders by the CSS procedure as proposed ($\bar{\hat{d}}_{R_r}$).

On the one hand, memory estimates of the global and regional factors as well as those of the residuals are accurately estimated no matter the sample size or the persistence levels in $d_{r,i0}$. Changes in the level of the signal-to-noise-ratios do not affect the estimated residual integration orders. On the other hand, even when $d_{r,i0} = 0$ the global and regional factors are consistently estimated, when $d_{r,i0} < 0.5$, the accuracy of the global and regional factors is not significantly distorted. Latter suggests that practitioners can estimate the model in (1) without taking fractional differences or first differences in the variables and if $\hat{d}_{r,i} < 0.5$ their global and regional factors will be accurately estimated. Cases for $\hat{d}_{r,i} \geq 0.5$ are discussed in the fourth Monte Carlo simulation. Finally, a low signal-to-noise-ratio makes the regional factors less precisely estimated. Note that such ratios do not affect the accuracy of the estimated global factor. Our findings indicate that it is possible that the use of the canonical correlation procedures is sufficiently robust for specifying the global factor. Using CCA to estimate the number of dynamic factors, Breitung and Pigorsch (2013) point out that CCA is useful for a wide range of stationary or mixing processes and particularly works better than usual PCA methods if the variances of the factors are very different.

In the second Monte Carlo study, for which the results are presented in Table 8, we find that the performance of the model is not affected by increasing the number of regions or varying the persistence of the regional factors. Factors, the idiosyncratic terms and loading factors are generated as before. We now study four regions (R=4) with different persistence levels. We only consider 20 variables ($N_r = 20$) now. All the standard deviations are simpler (0.5,1, and 2). Conclusions are similar to those in the first simulation study.

Conclusions from Tables 5-8 do not change by increasing N_r and reducing T. Latter configuration can be more related with some macroeconomic applications. In this light, we simulate one replication of the model 1 with $R = 2$, $N_r = 300, T = 150$, with the medium signal-to-noise ratio and $\vartheta_{r0} = 0.6$, $\delta_0 = 1$, and $d_{r,i0} = 0.25$. Following Wang (2010), once we get $\hat{G}$,$\hat{F}$,$\hat{\Gamma}$, and $\hat{\Lambda}$ we project true factors on the estimated one to find the rotation matrix,

$$\hat{Q}_G = \left(\hat{G}'\hat{G}\right)^{-1}\hat{G}'\hat{G}.$$

Then we use $\left(\hat{Q}_G\right)^{-1}$ to rotate factor loadings. Figure 2 displays the precision of the projected estimators with the true ones.

The fourth simulation study, Table 9, presents two simple Monte Carlo experiments to show two specific points to be taken into account in our methodology. i) When the memory of the residuals $d_{r,i}$ stay in the nonstationary region, the accuracy of the model in (1) to identify the regional factor decreases considerably when

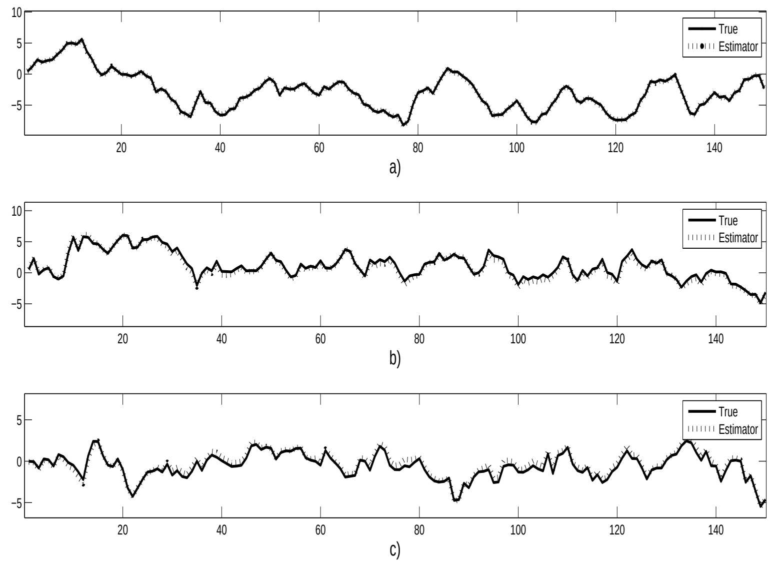

Figure 2: The dashed lines are the estimators for factors projected onto the true ones (solid lines). Global factor in panel a), Regional factors are in panel b) and c).

$d_{r,i}$ increases. However it is still possible to extract well the global factor as well as its memory level and the memory levels of the residuals. Note that the initial values of the global factor is based on CCA but those of regional factors on PCA. In this sense, our findings seem to indicate that CCA may play a key role here in comparison to PCA, although this claim requires further justification and is left for future research. ii) When there is no fractional cointegration, neither the factors nor the memory level of the global and regional factors can be precisely estimated but we can consistently estimate the memory of the residuals. Since we are able to estimate the memory of residuals even in the absence of fractional cointegration, we can analyze still whether $\hat{d}_{r,i} > max\left(\hat{\vartheta}_r, \hat{\delta}\right)$ in order to be sure that global and regional factors are accurately estimated.

4.2 Number of regional and global factors

We now discuss a Monte Carlo experiment to show the reliability of the methodology proposed to estimate the number of regional and global factors in Section 3. We design our simulation study using the same framework as before.

Tables 10-12 show that in general, information criteria IC1, IC2 and IC3 perform better than PC1, PC2 and PC3. Furthermore, the number of factors using IC1, IC2, and IC3 is always consistently estimated when variables are fractionally differenced by $\varsigma_i = max\left(\delta_0, \vartheta_{r,i0}\right)$, in all cases in which $\varsigma = \delta_0$. Only in cases when $d_{r,i0} \leq 1$, the number of factor is accurately estimated taking the first differences in the variables. Original variables can be used only in the specific case when $d_{r,i0} = 0$ although the performance of the number of factors does not diminish considerably in cases when $d_{r,i0} < 0.5$. Since Tables 10 and 11 consider two regions, we have three different blocks of data, $B_{R_1 \cup R_2}$, B_{R_1}, and B_{R_2} as explained before. Table 10 includes the case of only one global factor and one regional factor in each region, consequently the actual number of static factors in each block are $r_{B_{R_1 \cup R_2}} = 3$, $r_{B_{R_1}} = 2$, and $r_{B_{R_2}} = 2$ as represented in Figure 1. Table 11 considers the case of two global factors and two regional factors, then $r_{B_{R_1 \cup R_2}} = 6$, $r_{B_{R_1}} = 4$, and $r_{B_{R_2}} = 4$. When considering three regions in Table 12 with one global factor and one regional factor in each region, we have seven different blocks with the number of static factors as follows $r_{B_{R_1 \cup R_2 \cup R_3}} = 4$, $r_{B_{R_1}} = 2$, $r_{B_{R_2}} = 2$, $r_{B_{R_3}} = 2$, $r_{B_{R_1 \cup R_2}} = 3$, $r_{B_{R_1 \cup R_2}} = 3$, and $r_{B_{R_2 \cup R_3}} = 3$. Finally, the number of global and the regional factors can be obtained by the inclusion-exclusion principle.

5 Application

In this section, we provide an application of our methodology to study comovements in the Nord Pool power market.

Over past few decades a liberalization of power markets has emerged. Power companies produce electricity power from many different sources (hydro, thermal, nuclear, wind, and solar systems) in order to provide competitive prices and ensure production efficiency. From an economic perspective, electricity markets seek to match the supply and demand in order to find a market clearing price. Moreover, spot prices present seasonality at daily and weekly levels by daily activities either on working or non-working days, and at yearly level due to changing weather conditions throughout the year. Such prices also present some irregular cyclical factors which are associated with cyclical movements in macroeconomics or long-term climate trends whereas several spikes are caused by some anticipated special dates (Christmas, National days, etc.) and unanticipated days intrinsically originated in the market. Weron (2007) reports these stylized facts as well as an overview of statistical methods used in the literature.

Another feature that has received considerable attention is the presence of a hyperbolic decay of the autocovariances of electricity prices. In this light, Haldrup and Nielsen (2006) use Phillips-Perron and KPSS tests to suggest that neither an I(0) nor I(1) process is appropriate for electricity prices. They point out that Nord Pool prices are characterized by a high degree of long memory. Along this line, Koopman et al. (2007) consider general seasonal periodic regressions with ARFIMA-GARCH disturbances to analyze daily spot prices.

Although daily average prices are widely studied in the literature due to the role played in the so-called day-ahead market, it would be also of interest to disaggregate electricity prices in order to strengthen the respective prediction as Ramanathan et al. (1997) stress. In this regard, Raviv et al. (2015) also point out that the daily average of the disaggregate hourly forecasts contain useful information to study the daily average price in the Nord Pool market. In addition, it is habitually overlooked when modeling the hourly prices that the vector of 24 hourly prices is determined simultaneously in the day-ahead market. Latter means that a proper form of the data set would be a panel of prices with a natural ordering in the cross section dimension instead of a single time series since consecutive prices are determined simultaneously.

Examining in detail the hourly electricity prices implies the study of a complex dependence structure in the market which has not been extensively considered in the literature. A natural way to take into account such dependence is with a Vector Autoregressive (VAR) approach that would lead to the so-called 'curse of dimensionality'. Hence it is also of interest to reduce dimensionality. Panel data and factor models are standard tools to analyze high dimensional data and have been recently used in electricity markets (see e.g., Alonso et al. (2011), Dordonnat et al. (2012) and Raviv et al. (2015)).

The 'Panel Analysis of Nonstationarity in Idiosyncratic and Common components' (PANIC) is an alternative way to study the complexity of electricity prices. In this regard, Ergemen et al. (2015) very recently delve into the complex dynamics of Nord Pool electricity prices by considering the models proposed by Ergemen and Velasco (2015) and Ergemen (2015) which allow for fractionally integrated panels with fixed effects and cross-section dependence. We may say in a sense that the application of this paper is in line with one of the findings in Ergemen et al. (2015) that suggests a fractional cointegrating relationship in the panel of electricity prices and their main unobservable common factor although they do not consider an energy market divided by some regions but instead work with a reference price for the whole energy system – the system price.

A possible limitation of aforementioned studies of Elspot market is the use of these system prices which are the unconstrained equilibrium price for the entire Nordic region disregarding the available transmission capacity between the bidding areas. However, Elspot market is divided into several bidding areas due to system prices not clearing all regions within the Nordic market.

Another possible drawback when analyzing the entire Nordic market is that empirical studies that include factor models in their analysis assume that the common factors affect all regions of the system without taking into account some regional specific characteristics. In principle it is natural to extract common factors of each specific bidding area of the Nord Pool market and analyze them separately. However it is not clear whether there exist global common factors affecting all areas in such a case which provoke severe loss of efficiency when trying to identify common factors. Thence our interest in studying the hourly prices dynamics by considering each bidding area of Nord Pool market.

In the present paper, the data set under consideration are $R = 12$ balanced panels consisting of $N_r = 24$ hourly prices for each day for the period January 1, 2012, to December 31, 2014, and thus yielding a total of $T = 1096$ daily observations in each panel. We consider 12 panels since we analyze 12 bidding areas: Five

Norwegian bidding areas (NO1-NO5), Western Denmark (DK1), Eastern Denmark (DK1), four Swedish bidding areas (SE1-SE4), and Finland (FI). All bidding areas are connected. The series are downloaded from the Nord Pool ftp server. The prices are denominated in Euros per Mwh of load. Following Ergemen et al. (2015), the series are prefiltered by

$$y_{it} = \alpha_{i0} + \alpha_{i1}\, t + \alpha_{i2}\, D_t + \mathbb{B}_t' A_i + \alpha_{i3}\, \cos(\frac{2\pi t}{365}) + \alpha_{i4}\, \cos(\frac{2\pi t}{7}) + \alpha_{i5}\, \cos(\frac{2\pi t}{3.5}) + y_{it}^*, \quad (7)$$

where $\mathbb{B}_t$ is a vector of shift dummies which captures level changes caused by structural breaks. D_t is a dummy variable for holidays that takes the value of 1 if any of the countries participating in the Nord Pool system suspends or reduces normal business activities by custom or law, and 0 otherwise. The data for non-working days in each of the countries of the Nord Pool System is extracted from Bloomberg, which is then incorporated into the analysis due to the strong effect of holidays in the electricity market, see Ergemen et al. (2015) for more details.

As explained in Section 3, the number of different blocks from the regional data considerably increases with the number of regions. In our case, since the number of bidding areas to be analyzed are 12, we would have $2^{12} - 1 = 4,095$ different blocks. To avoid such complexity, we take advantage of the correlation showed by the daily regional prices to establish only four regions as follows: Region 1 = (DK1, DK2), Region 2 = (NO1, NO2, NO5), Region 3 = (NO3, NO4, SE1, SE2, SE3, SE4), and Region 4 = FI. Table 2 shows the correlation matrix of the daily prices whereas Figure 3 displays the map of Nord Pool market with these four regions. Note that each region consists of neighboring bidding areas. The bidding area corresponding to Finland presents much more spikes than any other area and may decrease such correlations.

Table 2: Correlation matrix of the daily prices in each bidding area of Nord Pool power market.

	DK1	DK2	NO1	NO2	NO3	NO4	NO5	SE1	SE2	SE3	SE4	FI
DK1	1.00											
DK2	0.93	1.00										
NO1	0.57	0.61	1.00									
NO2	0.56	0.57	0.98	1.00								
NO3	0.68	0.73	0.87	0.84	1.00							
NO4	0.66	0.72	0.88	0.85	0.99	1.00						
NO5	0.54	0.57	0.99	0.99	0.85	0.85	1.00					
SE1	0.72	0.77	0.84	0.81	0.98	0.97	0.82	1.00				
SE2	0.72	0.77	0.84	0.81	0.98	0.97	0.82	1.00	1.00			
SE3	0.74	0.79	0.83	0.80	0.96	0.96	0.80	0.99	0.99	1.00		
SE4	0.79	0.86	0.79	0.75	0.91	0.90	0.76	0.93	0.93	0.95	1.00	
FI	0.66	0.70	0.66	0.63	0.81	0.80	0.63	0.83	0.83	0.85	0.78	1.00

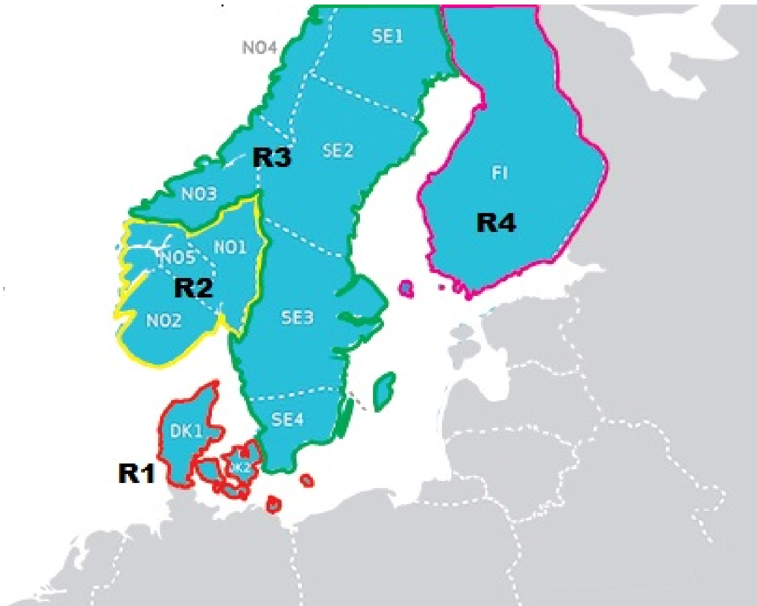

Figure 3: Nord Pool power market divided by the new regions.

For each of the four regions we compute the hourly regional prices by taking the simple average of hourly prices corresponding to bidding areas that define the region. Naturally, we still have 24 hourly prices. It is possible to consider a weighted average of the prices by considering the available transmission capacity but we work with the simplest average to focus on the main ideas.
We estimate the memory, $\varepsilon_{r,i}$, of each one of the hourly regional prices with the Extended Local Whittle procedure. Each hourly regional price is fractionally differenced by its respective estimated memory, $\hat{\varepsilon}_{r,i}$, so that the number of global and regional static factors can be estimated as described in Section 3.

To estimate the number of regional and global factors, we use the procedure proposed by Alessi et al. (2010). This procedure improves the penalization in the criteria IC1 and IC2 of BN2002 introducing a tuning multiplicative constant in the penalty function under the same set of assumptions that lead to heteroskedasticity-robust inference. Theorem 2 in BN2002 is still valid and consequently our methodology can also be applied with the information criteria of Alessi et al. (2010).

Table 3 presents the number of factors estimated in each one of the 15 blocks. Furthermore, using the inclusion-exclusion principle, we get the number of global and regional factors. Note that for computing the number of regional static factors, we need to compute the number of static factors in each one of the blocks of regions. Edward's diagram in Figure 4 displays how the static factors are accommodated in each one of the blocks. Particularly, such a diagram shows that we find one global factor and two regional factors in each one of the regions. Note that the sum of all the static factors identified in the Edward's diagram corresponds to the number of static factors estimated in the quadruple-wise block $B_{R_1 \cup R_2 \cup R_3 \cup R_4}$.

Table 3: Number of static factors in the 15 blocks formed.

Individual blocks		Pairwise blocks		Triple-wise blocks		Quadruple-wise block	
r_{R_1}	7	$r_{B_{R_1} \cup B_{R_2}}$	11	$r_{B_{R_1} \cup B_{R_2} \cup B_{R_3}}$	14	$r_{B_{R_1} \cup B_{R_2} \cup B_{R_3} \cup B_{R_4}}$	16
r_{R_2}	7	$r_{B_{R_1} \cup B_{R_3}}$	12	$r_{B_{R_1} \cup B_{R_3} \cup B_{R_4}}$	14		
r_{R_3}	7	$r_{B_{R_1} \cup B_{R_4}}$	11	$r_{B_{R_1} \cup B_{R_2} \cup B_{R_4}}$	14		
r_{R_4}	7	$r_{B_{R_2} \cup B_{R_3}}$	11	$r_{B_{R_2} \cup B_{R_3} \cup B_{R_4}}$	14		
		$r_{B_{R_2} \cup B_{R_4}}$	11				
		$r_{B_{R_3} \cup B_{R_4}}$	11				

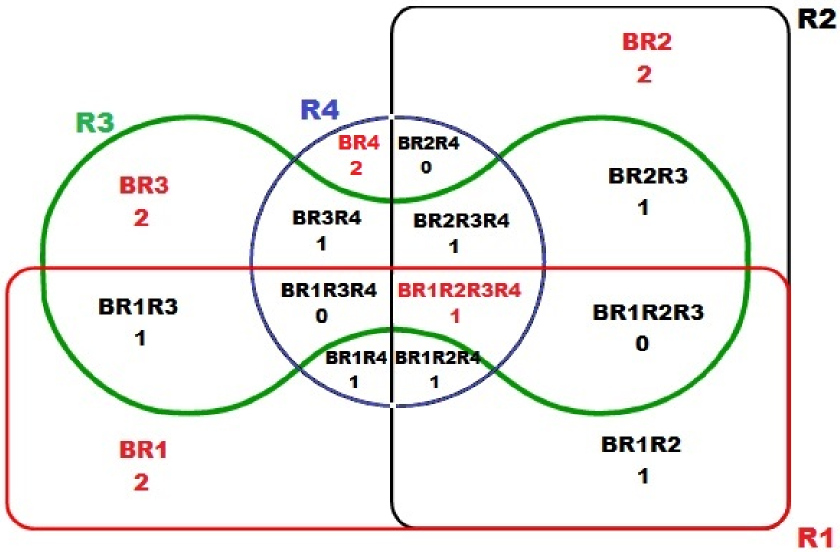

Figure 4: Number of factors in the 15 blocks.

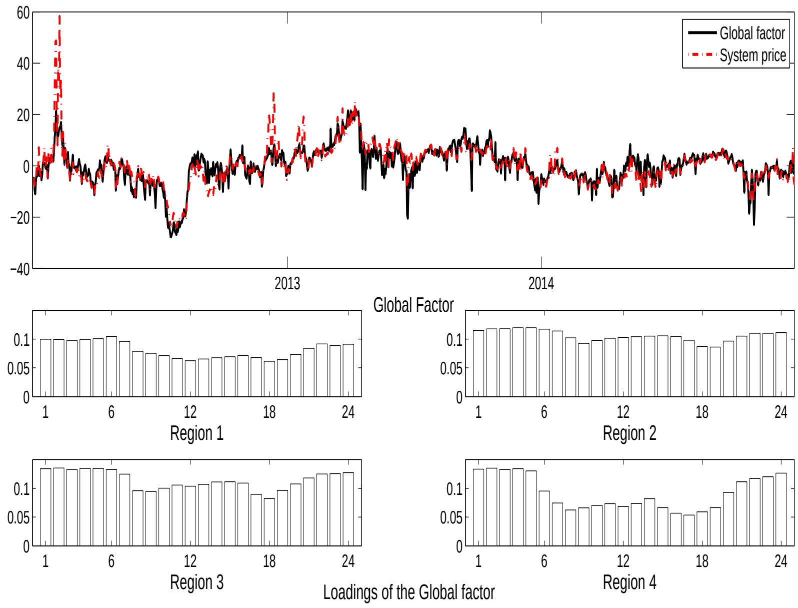

Figure 5: Global common component of Nord Pool bidding areas

Now we estimate the model 1 by the methodology proposed in Section 2.2. We fix the number of global factor to be one and the regional factors to be two in all four regions. Figure 5 shows the global factor and its loadings for each region. The first panel in Figure 5 also displays the filtered system daily prices by the same filtering model 7 as before.

As seen from Figure 5, the global factor tends to be highly persistent. The global factor loadings show a regular behavior among bidding areas. Loadings are positive and larger overnight indicating that the global factor plays a key role from 12 a.m.-7 a.m. and from 10 p.m.-12 p.m. Levels of the loadings are similar across regions. Figure 5 shows that the global factor fits well to the filtered system prices. Furthermore, the correlation between the global factor and the filtered system price is around 0.85. The estimated memory of the filtered system prices is 0.77 whereas the estimated memory of the global factor is 0.82. Our findings indicate that the global factor may be interpreted as the system price even when we have reduced from 12 to only 4 bidding regions and the model is able to capture well the persistence of the system prices. The so-called system price of the Nord Pool market is based on the equlibrium between the aggregated supply and demand curves ignoring the available transmission capacity among the bidding areas. It is used as the Nordic Reference price for trading and clearing of most financial contracts.

Figures 6 and 7 present the regional loadings and both regional factors of each region.

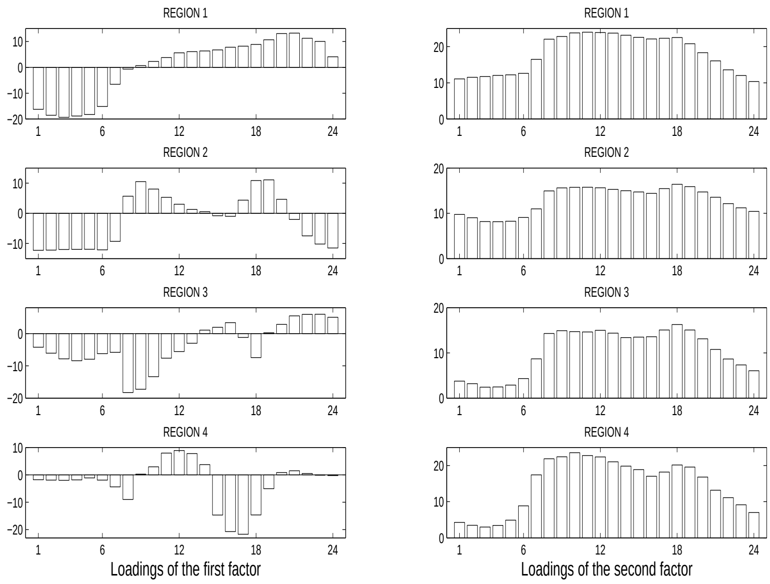

Figure 6: Loadings of the regional factors

Figure 6 indicates that the regional factors play a key role during working hours explaining much more of the variability, mainly the second regional factors.

Consequently, for each hourly price, the commonality of the regional factor implies a small subtraction over the commonality of this hour considering only the global factor. On the other hand, during working hours, the commonality of regional actors explains more of the variability than the global factor which only represents a small correction. Furthermore, it is apparent that regional factors also seem to be highly persistent. Table 4 shows the correlation among first regional factor as well as second regional factors. All correlations are low indicating that regional factors differ region by region.

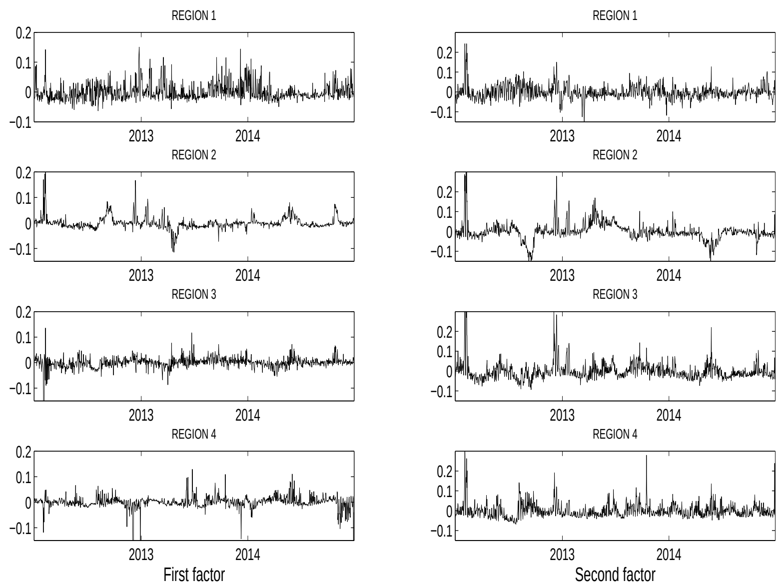

Figure 7: Regional factors

We study whether a fractional cointegration relationship exists in our analysis that ensures that the memory of the residuals stay in the stationary region to verify that our model is consistently estimated. We collect the global and regional factors $(\hat{G}_t, \hat{F_{r,t}})$ and estimate the fractional memory parameters with the Extended Local Whittle procedure. Figure 8 displays that the global factor is more persistent than regional factors. Regional factors of Region 2 are more persistent than the other regions while the Danish region, Region 1, shows less persistence in both regional factors.

Fractional cointegration relationship is confirmed given that for each region and each hour of the day $\hat{d}_{r,i} \leq \hat{\delta}$. Figure 9 shows that persistence levels of the residuals of model in (1) have decreased once we have taken into account the strong dependence of the hourly electricity prices analyzed with the global and regional factors estimated.

To conclude, the number of factors in each one of the blocks represented in the Figure 4 can be used to extend the model in (1). For instance, consider the case

Table 4: Correlation among regional factors.

	First regional factors			
	Region 1	Region 2	Region 3	Region 4
Region 1	1			
Region 2	0.13	1		
Region 3	0.35	0.01	1	
Region 4	0.01	0.06	0.02	1
	Second regional factors			
	Region 1	Region 2	Region 3	Region 4
Region 1	1			
Region 2	0.33	1		
Region 3	0.63	0.60	1	
Region 4	0.50	0.32	0.66	1

of the three first bidding regions of the Nord Pool power market, R1, R2 and R3. We may extend our model as

$$\begin{pmatrix} y_{R_1,\cdot t} \\ y_{R_2,\cdot t} \\ y_{R_3,\cdot t} \end{pmatrix} = \begin{pmatrix} \Gamma_1 & \Lambda_1 & 0 & 0 & \kappa_{1,12} & 0 & \kappa_{1,13} \\ \Gamma_2 & 0 & \Lambda_2 & 0 & \kappa_{2,12} & \kappa_{2,23} & 0 \\ \Gamma_3 & 0 & 0 & \Lambda_3 & 0 & \kappa_{3,23} & \kappa_{3,13} \end{pmatrix} \begin{pmatrix} G_{123,t} \\ F_{1,t} \\ F_{2,t} \\ F_{3,t} \\ F_{12,t} \\ F_{23,t} \\ F_{13,t} \end{pmatrix} + \begin{pmatrix} u_{1,\cdot t} \\ u_{2,\cdot t} \\ u_{3,\cdot t} \end{pmatrix},$$

where we would add to model 1, the common factors, $F_{12,t}$, $F_{23,t}$, and $F_{13,t}$, corresponding to blocks $B_{R_1 \cap R_2}$, $B_{R_2 \cap R_3}$, and $B_{R_1 \cap R_3}$, respectively. The third block in the loading matrix, κ's, would be the respective blocks loading factors. In future research, we plan to focus on estimating this kind of extended models in order to analyze in depth the interaction among blocks of regions in a multi-level factor model. In principle, the methodology proposed in this paper can be used to estimate the new model after incorporating more steps in the procedure, however it would be also necessary to add more restrictions and assumptions in order to identify the model.

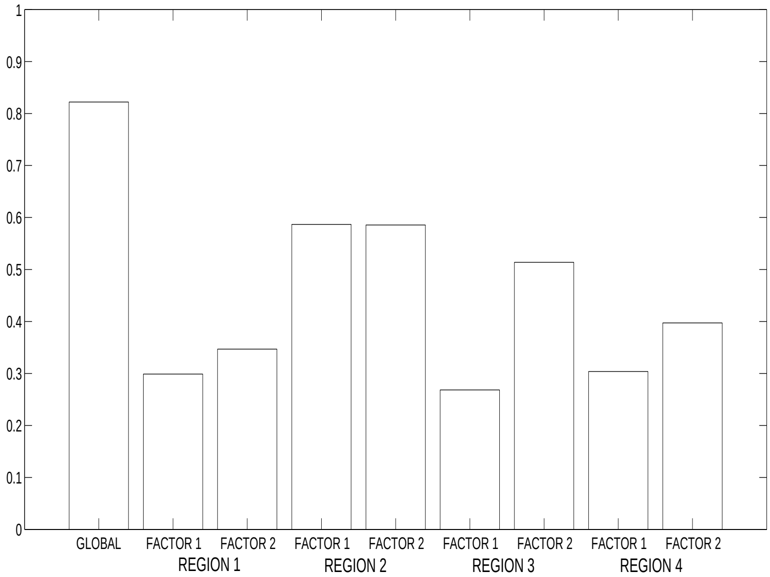

Figure 8: Memory estimates of the global and regional factors. The number of Fourier frequencies used is m = $T^0.7$ with T = 1096 corresponding to m = 134. The standard error of the univariate estimates is 0.043.

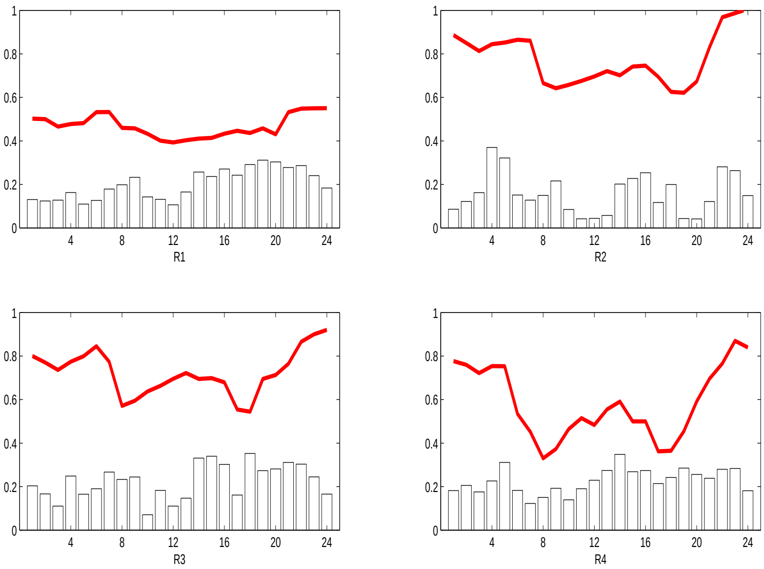

Figure 9: Comparative of the residual integration order estimates (bars) with the memory estimates of the filtered regional prices for each region (red line).

6 Concluding remarks

In this paper we have considered a dynamic multi-level factor model which allow for both pervasive and nonpervasive common factors. In these factors we allow to exhibit fractional long-range dependence without restrictions on them being either stationary I(0) or nonstationary I(1) processes. Model innovations are also allowed to be fractionally integrated. Our model can be used for an extensive fractional cointegrating analysis in a wide range of economic or financial applications.

We have suggested a methodology that consists in two steps: i) use sequential least squares to minimize the total of sum of squared residuals in order to get the global and regional common components, and ii) minimize the conditional sum-of-squares in order to estimate the residual memory estimates of the model. We have established asymptotic results for both steps.

Based on the baseline model, we have also suggested a simple methodology to estimate the number of regional and global factors by using the popular information criteria of BN2002. This methodology consists of designing all combinations of data blocks to estimate the number of factors which span each of the respective factor spaces. Then, the inclusion-exclusion principle is used to determine the number of regional and global factors.

To illustrate our methodology, we study the complex dynamics of Nord Pool power market in a large panel of hourly observations. We have defined and studied four bidding regions of Nord Pool power market to apply our methodology. We have found only one global factor which drives the commonality among regions and two regional factors in each one of the regions. Our findings suggest a fractional cointegration relationship between hourly regional prices of Nord Pool and the global factor. Furthermore, we have discussed that the global factor can be interpreted as the so-called system price of the Nord Pool market which is based on the sale and purchase contracts ignoring the available transmission capacity among the bidding areas.

The present paper can be directly extended to consider interactions among blocks. Particularly, when analyzing the Nord Pool power market, the blocks of the bidding regions can provide important information among regions that is not possible to capture directly from the regional factors neither the global factor.

Appendix

A Proof of Theorem 1

Note that first differencing leads the series y_{it} to become asymptotically stationary under Assumption A because $\Delta x_{it} \sim I(\max\{\delta_0, \vartheta_{r,0}, d_{r,i0}\} - 1)$, with $d_{max} \leq$

$\max\{\delta_0, \vartheta_{r,0}\}$ and $\max\{\delta_0 - \underline{\delta}, \vartheta_{r,0} - \underline{\vartheta}, d_{r,i0} - \underline{d}\} < 1/2$ so that

$$\frac{1}{N}\frac{1}{T-1}\sum_{i=1}^{N}\sum_{t=2}^{T}(\Delta y_{it})^2 \to_p \sigma_y^2 > 0,$$

which implies that standard techniques that consider $I(0)$ setups can be borrowed from the literature in establishing the following results.

For notational simplicity, let us drop the dependence on asterisks and denote f_t^* by f_t. The result that

$$\min\left\{\sqrt{N_R}, T\right\}(\hat{f}_t - f_t H_f) = O_p(1), \textit{ for a fixed } t,$$

with

$$H_f' = \hat{V}^{-1}(\hat{f}' f/(T-1))(\Lambda'\Lambda/N_R), \quad \hat{V} = \hat{\Lambda}'\hat{\Lambda}/N_R,$$

can be shown proceeding the same way as in the proof of Lemma 2 by Bai and Ng (2004) which follows the steps in Bai (2003) whose Assumptions A-G are satisfied since the number of regions, R, is assumed fixed, also see Wang (2010) for an extended discussion.

Next, we want to establish, adopting Bai and Ng (2013), that

$$H_f = I_r + O_p(\varsigma_{NT}^{-2}) \tag{8}$$

with $\varsigma_{NT} = \min\left\{\sqrt{N_R}, \sqrt{T}\right\}$ if $\Lambda'\Lambda$ is diagonal with distinct entries. We first check that

$$\begin{aligned}\frac{\hat{f}' f}{T-1} &= \frac{(\hat{f} - fH_f)' f}{T-1} + \frac{H_f' f' f}{T-1} \\ &= \frac{H_f' f' f}{T-1} + O_p(\varsigma_{NT}^{-2}) \end{aligned}\tag{9}$$

since $(\hat{f} - fH_f)' f/(T-1) = O_p(\varsigma_{NT}^{-2})$ by Lemma B.2 of Bai (2003). Right-multiplying both sides of (9) by H_f gives

$$\frac{\hat{f}' f H_f}{T-1} = \frac{H_f' f' f H_f}{T-1} + O_p(\varsigma_{NT}^{-2}). \tag{10}$$

Then,

$$\frac{\hat{f}' f H_f}{T-1} = \frac{\hat{f}'(fH_f - \hat{f} + \hat{f})}{T-1} = O_p(\varsigma_{NT}^{-2}) + I_r \tag{11}$$

because $\hat{f}'(fH_f - \hat{f})/(T-1) = O_p(\varsigma_{NT}^{-2})$ as above and $\hat{f}'\hat{f}/(T-1) = I_r$ under the identifying restriction.

Equating 10 and 11,

$$
\begin{aligned}
I_r &= H_f' \overbrace{\frac{f'f}{T-1}}^{=I_r} H_f + O_p(\varsigma_{NT}^{-2}) \\
&= H_f' H_f + O_p(\varsigma_{NT}^{-2}),
\end{aligned}
$$

so asymptotically H_f is an orthogonal matrix with eigenvalues equal to 1 or -1.

Next, we show that H_f is diagonal. From (9) and using $f'f/(T-1) = I_r$, we have that

$$
\begin{aligned}
H_f' &= \hat{V}^{-1}(\hat{f}' f/(T-1))(\Lambda'\Lambda/N_R), \\
&= \hat{V}^{-1} H_f' (\Lambda'\Lambda/N_R) + O_p(\varsigma_{NT}^{-2}). \qquad (12)
\end{aligned}
$$

Multiply (12) on both sides by $\hat{V}$ and transpose to get

$$
(\Lambda'\Lambda/N_R) H_f = H_f \hat{V} + O_p(\varsigma_{NT}^{-2}), \qquad (13)
$$

which shows that asymptotically H_f is a matrix containing the eigenvectors of $(\Lambda'\Lambda/N)$ that is diagonal with distinct eigenvalues by assumption. So then, each eigenvalue is associated with a unique unitary eigenvector, and this establishes that H_f is asymptotically diagonal. Without loss of generality, we can assume the eigenvalues of H_f are 1's and in that case, (8) is shown. Furthermore, from (13),

$$
(\Lambda'\Lambda/N_R) = \hat{V} + O_p(\varsigma_{NT}^{-2}).
$$

□

B Proof of Theorem 2

The CSS criterion based on (4) can be written, suppressing the dependence on asterisks as before, as

$$
L_{N,T}(\delta) = \frac{1}{T}\left(\Delta_t^{d-1}\left(\hat{f}_t - f_t H_f\right) + \Delta_t^{d-1}(f_t H_f)\right)'\left(\Delta_t^{d-1}\left(\hat{f}_t - f_t H_f\right) + \Delta_t^{d-1}(f_t H_f)\right). \qquad (14)
$$

We argue that the squared estimation-error term in (14),

$$
\frac{1}{T}\left(\Delta_t^{d-1}\left(\hat{f}_t - f_t H_f\right)\right)'\left(\Delta_t^{d-1}\left(\hat{f}_t - f_t H_f\right)\right)
$$

is negligible as $(N,T)_j \to \infty$ because under Assumption E, Assumption D is necessary and sufficient for the conditions imposed in Lemma A.1 of Bai and Ng (2004). So the results in Lemma A.2 of Bai and Ng (2004) hold under Assumption A, see similar arguments used in the proofs of Theorems 4 and 5 of

Ergemen and Velasco (2015). Therefore, denoting $\xi = \max\{\delta_0, \vartheta_{r,0}, d_{r,max}\} - \min\{\underline{\delta}, \underline{\vartheta}_r, \underline{d}_{r,i}\}$,

$$\max_{1\leq k\leq T} \frac{1}{T} \left\| \sum_{t=1}^{k} \left(\hat{f}_t - f_t H_f \right) \right\| = O_p\left((NT)^{-1/2} + T^{\xi-1/2} \right).$$

Then the $\sqrt{T}-$consistency of the memory estimate can be established from

$$\frac{1}{T} \left(\Delta_t^{d-1} \hat{f}_t \right)' \left(\Delta_t^{d-1} \hat{f}_t \right)$$

following exactly the same steps followed by Hualde and Robinson (2011) and Ergemen and Velasco (2015).

Finally the cross-term in (14) is bounded and of smaller size by Cauchy-Schwarz inequality, and the proof is then complete. □

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Table 5: Monte Carlo simulation results with $T = 150$ and $R = 2$

			$N_r = 20$								$N_r = 80$							
ϕ	σ_{v_r}	σ_w	$\hat{\vartheta}_1$	$\hat{\vartheta}_2$	$\hat{\delta}$	R^2_G	R^2_{R1}	R^2_{R2}	$\bar{\hat{d}}_{R1}$	$\bar{\hat{d}}_{R2}$	$\hat{\vartheta}_1$	$\hat{\vartheta}_2$	$\hat{\delta}$	R^2_G	R^2_{R1}	R^2_{R2}	$\bar{\hat{d}}_{R1}$	$\bar{\hat{d}}_{R2}$
$\vartheta_{r0} = 0.4$ $\delta_0 = 0.6$ $d_{r,i0} = 0$																		
0.5	2	1	0.429	0.426	0.642	0.994	0.967	0.965	0.044	0.045	0.434	0.439	0.642	0.999	0.966	0.969	0.044	0.044
0.5	1	1	0.428	0.432	0.633	0.994	0.951	0.954	0.044	0.045	0.432	0.432	0.646	0.999	0.966	0.968	0.044	0.044
0.5	1	2	0.422	0.427	0.644	0.999	0.951	0.953	0.045	0.044	0.431	0.437	0.644	1.000	0.968	0.966	0.044	0.045
2	2	1	0.427	0.438	0.628	0.975	0.952	0.955	0.044	0.044	0.426	0.435	0.641	0.994	0.966	0.966	0.045	0.044
2	1	1	0.410	0.414	0.624	0.975	0.904	0.904	0.045	0.044	0.430	0.431	0.634	0.994	0.955	0.954	0.044	0.045
2	1	2	0.414	0.412	0.635	0.994	0.905	0.905	0.045	0.044	0.427	0.431	0.641	0.999	0.954	0.952	0.044	0.044
5	2	1	0.387	0.384	0.633	0.984	0.815	0.816	0.046	0.045	0.422	0.423	0.641	0.996	0.931	0.931	0.045	0.044
5	1	1	0.379	0.385	0.633	0.984	0.818	0.817	0.045	0.044	0.429	0.421	0.639	0.996	0.930	0.929	0.045	0.044
5	1	2	0.388	0.383	0.636	0.984	0.818	0.818	0.046	0.045	0.418	0.423	0.646	0.996	0.930	0.930	0.045	0.045
$\vartheta_r 0 = 0.6$ $\delta_0 = 1$ $d_{r,i0} = 0.25$																		
0.5	2	1	0.637	0.642	1.023	0.999	0.914	0.913	0.235	0.234	0.640	0.637	1.027	1.000	0.915	0.921	0.236	0.236
0.5	1	1	0.636	0.634	1.026	0.999	0.908	0.906	0.235	0.235	0.643	0.640	1.028	1.000	0.912	0.913	0.236	0.235
0.5	1	2	0.636	0.639	1.029	0.999	0.907	0.906	0.235	0.235	0.640	0.640	1.024	1.000	0.911	0.915	0.236	0.236
2	2	1	0.632	0.638	1.012	0.995	0.909	0.908	0.235	0.236	0.632	0.638	1.028	0.999	0.915	0.915	0.236	0.236
2	1	1	0.618	0.626	1.014	0.995	0.869	0.873	0.235	0.235	0.634	0.639	1.025	0.999	0.907	0.904	0.236	0.236
2	1	2	0.618	0.623	1.021	0.999	0.872	0.870	0.236	0.235	0.635	0.640	1.032	1.000	0.902	0.906	0.235	0.235
5	2	1	0.597	0.593	1.021	0.997	0.816	0.818	0.235	0.235	0.635	0.633	1.023	0.999	0.888	0.886	0.235	0.236
5	1	1	0.599	0.597	1.018	0.997	0.815	0.813	0.235	0.235	0.633	0.631	1.026	0.999	0.889	0.890	0.235	0.236
5	1	2	0.593	0.599	1.018	0.997	0.820	0.814	0.235	0.235	0.637	0.641	1.028	0.999	0.891	0.891	0.235	0.235
$\vartheta_r 0 = 0.8$ $\delta_0 = 1$ $d_{r,i0} = 0.45$																		
0.5	2	1	0.839	0.838	1.031	0.998	0.826	0.827	0.441	0.440	0.835	0.837	1.031	1.000	0.822	0.820	0.440	0.441
0.5	1	1	0.835	0.838	1.029	0.998	0.821	0.808	0.440	0.440	0.842	0.844	1.032	1.000	0.815	0.821	0.440	0.441
0.5	1	2	0.835	0.833	1.028	0.998	0.820	0.821	0.441	0.441	0.840	0.842	1.029	1.000	0.825	0.825	0.440	0.440
2	2	1	0.833	0.838	1.017	0.993	0.822	0.813	0.440	0.440	0.848	0.841	1.033	0.998	0.822	0.833	0.441	0.440
2	1	1	0.829	0.822	1.018	0.993	0.799	0.808	0.439	0.441	0.836	0.838	1.031	0.998	0.820	0.812	0.441	0.441
2	1	2	0.822	0.830	1.029	0.998	0.804	0.809	0.440	0.440	0.838	0.837	1.022	1.000	0.819	0.814	0.440	0.441
5	2	1	0.804	0.804	1.016	0.995	0.748	0.743	0.440	0.440	0.845	0.849	1.027	0.999	0.806	0.801	0.441	0.441
5	1	1	0.810	0.806	1.023	0.996	0.760	0.758	0.441	0.441	0.847	0.843	1.025	0.999	0.804	0.813	0.441	0.441
5	1	2	0.804	0.805	1.018	0.996	0.744	0.760	0.441	0.439	0.845	0.846	1.037	0.999	0.787	0.813	0.440	0.441

Table 6: Monte Carlo simulation results with $T = 1000$ and $R = 2$

			$N_r = 20$								$N_r = 80$							
ϕ	σ_{v_r}	σ_w	$\hat{\vartheta}_1$	$\hat{\vartheta}_2$	$\hat{\delta}$	R^2_G	R^2_{R1}	R^2_{R2}	$\bar{\hat{d}}_{R1}$	$\bar{\hat{d}}_{R2}$	$\hat{\vartheta}_1$	$\hat{\vartheta}_2$	$\hat{\delta}$	R^2_G	R^2_{R1}	R^2_{R2}	$\bar{\hat{d}}_{R1}$	$\bar{\hat{d}}_{R1}$
									$\vartheta_{r0} = 0.3$		$\delta_0 = 0.5$		$d_{r,i0} = 0$					
0.5	2	1	0.304	0.306	0.505	0.995	0.989	0.988	0.062	0.062	0.304	0.302	0.506	0.999	0.992	0.992	0.062	0.062
0.5	1	1	0.298	0.300	0.507	0.995	0.975	0.976	0.062	0.062	0.305	0.303	0.508	0.999	0.989	0.989	0.062	0.062
0.5	1	2	0.301	0.300	0.508	0.999	0.976	0.977	0.062	0.061	0.303	0.303	0.508	1.000	0.988	0.989	0.062	0.062
2	2	1	0.300	0.303	0.500	0.982	0.977	0.977	0.062	0.062	0.302	0.301	0.505	0.996	0.989	0.989	0.062	0.062
2	1	1	0.294	0.295	0.506	0.982	0.930	0.930	0.062	0.061	0.301	0.301	0.509	0.996	0.978	0.977	0.062	0.062
2	1	2	0.294	0.294	0.508	0.995	0.930	0.930	0.061	0.062	0.301	0.304	0.505	0.999	0.977	0.978	0.062	0.062
5	2	1	0.283	0.284	0.503	0.989	0.851	0.851	0.062	0.062	0.299	0.298	0.505	0.997	0.956	0.956	0.062	0.062
5	1	1	0.287	0.281	0.507	0.989	0.854	0.852	0.062	0.061	0.296	0.300	0.504	0.997	0.956	0.956	0.062	0.062
5	1	2	0.284	0.283	0.503	0.989	0.853	0.850	0.062	0.061	0.299	0.297	0.507	0.997	0.956	0.956	0.062	0.062
									$\vartheta_r 0 = 0.5$		$\delta_0 = 0.9$		$d_{r,i0} = 0.25$					
0.5	2	1	0.508	0.506	0.901	1.000	0.953	0.949	0.261	0.261	0.508	0.510	0.904	1.000	0.954	0.953	0.261	0.261
0.5	1	1	0.507	0.508	0.903	1.000	0.945	0.945	0.261	0.261	0.510	0.509	0.903	1.000	0.949	0.949	0.261	0.261
0.5	1	2	0.509	0.508	0.903	1.000	0.945	0.943	0.261	0.261	0.507	0.508	0.901	1.000	0.953	0.954	0.261	0.261
2	2	1	0.506	0.506	0.900	0.999	0.944	0.950	0.260	0.261	0.510	0.509	0.902	1.000	0.952	0.952	0.261	0.261
2	1	1	0.502	0.499	0.898	0.999	0.918	0.923	0.261	0.261	0.508	0.510	0.903	1.000	0.948	0.946	0.260	0.261
2	1	2	0.503	0.502	0.900	1.000	0.922	0.923	0.261	0.261	0.506	0.505	0.903	1.000	0.946	0.945	0.261	0.261
5	2	1	0.491	0.490	0.899	0.999	0.874	0.874	0.261	0.261	0.501	0.506	0.901	1.000	0.935	0.935	0.261	0.261
5	1	1	0.491	0.491	0.900	0.999	0.875	0.874	0.261	0.260	0.503	0.505	0.901	1.000	0.936	0.930	0.261	0.261
5	1	2	0.491	0.492	0.900	0.999	0.873	0.874	0.261	0.261	0.504	0.507	0.904	1.000	0.938	0.934	0.261	0.261
									$\vartheta_r 0 = 0.7$		$\delta_0 = 0.9$		$d_{r,i0} = 0.45$					
0.5	2	1	0.713	0.713	0.902	0.999	0.863	0.868	0.465	0.464	0.715	0.717	0.905	1.000	0.869	0.861	0.464	0.465
0.5	1	1	0.710	0.710	0.901	1.000	0.862	0.855	0.464	0.464	0.715	0.715	0.902	1.000	0.862	0.868	0.464	0.464
0.5	1	2	0.712	0.712	0.904	1.000	0.862	0.862	0.465	0.465	0.714	0.714	0.903	1.000	0.868	0.858	0.465	0.464
2	2	1	0.713	0.710	0.897	0.998	0.862	0.852	0.464	0.464	0.715	0.717	0.903	1.000	0.860	0.866	0.465	0.464
2	1	1	0.707	0.706	0.897	0.998	0.846	0.846	0.465	0.465	0.714	0.713	0.902	1.000	0.856	0.866	0.465	0.465
2	1	2	0.708	0.708	0.902	0.999	0.849	0.850	0.465	0.464	0.714	0.712	0.903	1.000	0.858	0.860	0.464	0.465
5	2	1	0.697	0.700	0.896	0.999	0.825	0.818	0.465	0.464	0.713	0.713	0.904	1.000	0.857	0.852	0.465	0.465
5	1	1	0.698	0.698	0.900	0.999	0.827	0.827	0.465	0.465	0.709	0.711	0.903	1.000	0.849	0.849	0.465	0.465
5	1	2	0.697	0.696	0.898	0.999	0.824	0.821	0.464	0.464	0.713	0.712	0.900	1.000	0.862	0.848	0.465	0.464

Table 7: Monte Carlo simulation results with $T = 5000$ and $R = 2$

			$N_r = 20$								$N_r = 80$							
ϕ	σ_{v_r}	σ_w	$\hat{\vartheta}_1$	$\hat{\vartheta}_2$	$\hat{\delta}$	R^2_G	R^2_{R1}	R^2_{R2}	$\bar{\hat{d}}_{R1}$	$\bar{\hat{d}}_{R2}$	$\hat{\vartheta}_1$	$\hat{\vartheta}_2$	$\hat{\delta}$	R^2_G	R^2_{R1}	R^2_{R2}	$\bar{\hat{d}}_{R1}$	$\bar{\hat{d}}_{R1}$
								$\vartheta_{r0} = 0.24$		$\delta_0 = 0.44$		$d_{r,i0} = 0$						
0.5	2	1	0.246	0.246	0.447	0.996	0.994	0.994	0.064	0.064	0.246	0.243	0.449	0.999	0.997	0.997	0.064	0.064
0.5	1	1	0.246	0.244	0.446	0.996	0.982	0.982	0.064	0.064	0.246	0.246	0.448	0.999	0.994	0.994	0.064	0.064
0.5	1	2	0.244	0.245	0.448	0.999	0.982	0.982	0.064	0.064	0.246	0.247	0.449	1.000	0.994	0.994	0.064	0.064
2	2	1	0.244	0.244	0.446	0.984	0.982	0.982	0.064	0.064	0.245	0.246	0.448	0.996	0.994	0.994	0.064	0.064
2	1	1	0.241	0.242	0.446	0.984	0.938	0.939	0.064	0.064	0.246	0.244	0.447	0.996	0.983	0.983	0.064	0.064
2	1	2	0.241	0.240	0.447	0.996	0.937	0.937	0.064	0.064	0.244	0.244	0.449	0.999	0.983	0.983	0.064	0.064
5	2	1	0.234	0.234	0.447	0.990	0.861	0.859	0.064	0.064	0.243	0.244	0.448	0.998	0.962	0.962	0.064	0.064
5	1	1	0.236	0.236	0.448	0.990	0.859	0.859	0.064	0.064	0.244	0.243	0.448	0.998	0.962	0.962	0.064	0.064
5	1	2	0.235	0.235	0.446	0.990	0.860	0.860	0.064	0.064	0.243	0.242	0.446	0.998	0.962	0.962	0.064	0.064
								$\vartheta_r0 = 0.44$		$\delta_0 = 0.84$		$d_{r,i0} = 0.25$						
0.5	2	1	0.449	0.450	0.845	1.000	0.967	0.969	0.264	0.264	0.449	0.448	0.846	1.000	0.971	0.970	0.264	0.264
0.5	1	1	0.448	0.449	0.843	1.000	0.963	0.963	0.264	0.264	0.450	0.449	0.847	1.000	0.967	0.971	0.264	0.264
0.5	1	2	0.448	0.447	0.845	1.000	0.960	0.960	0.264	0.264	0.448	0.449	0.845	1.000	0.969	0.968	0.264	0.264
2	2	1	0.448	0.448	0.846	1.000	0.961	0.963	0.264	0.264	0.448	0.448	0.845	1.000	0.970	0.970	0.264	0.264
2	1	1	0.444	0.443	0.843	1.000	0.940	0.939	0.264	0.264	0.448	0.449	0.845	1.000	0.964	0.964	0.264	0.264
2	1	2	0.444	0.444	0.845	1.000	0.941	0.941	0.264	0.264	0.448	0.448	0.845	1.000	0.963	0.965	0.264	0.264
5	2	1	0.438	0.438	0.844	1.000	0.899	0.895	0.264	0.264	0.446	0.446	0.847	1.000	0.953	0.952	0.264	0.264
5	1	1	0.438	0.438	0.846	1.000	0.900	0.897	0.264	0.264	0.446	0.447	0.846	1.000	0.952	0.954	0.264	0.264
5	1	2	0.438	0.440	0.845	1.000	0.899	0.898	0.264	0.264	0.447	0.448	0.845	1.000	0.952	0.954	0.264	0.264
								$\vartheta_r0 = 0.65$		$\delta_0 = 0.84$		$d_{r,i0} = 0.45$						
0.5	2	1	0.656	0.653	0.845	1.000	0.888	0.885	0.466	0.466	0.654	0.655	0.844	1.000	0.876	0.885	0.466	0.466
0.5	1	1	0.654	0.654	0.846	1.000	0.882	0.877	0.466	0.466	0.654	0.654	0.845	1.000	0.889	0.876	0.466	0.466
0.5	1	2	0.653	0.655	0.846	1.000	0.876	0.867	0.466	0.466	0.655	0.654	0.845	1.000	0.877	0.870	0.466	0.466
2	2	1	0.652	0.654	0.844	0.999	0.869	0.881	0.466	0.466	0.656	0.654	0.845	1.000	0.873	0.881	0.466	0.466
2	1	1	0.650	0.650	0.844	0.999	0.873	0.866	0.466	0.466	0.653	0.654	0.844	1.000	0.870	0.870	0.466	0.466
2	1	2	0.649	0.651	0.844	1.000	0.874	0.856	0.466	0.466	0.654	0.656	0.846	1.000	0.875	0.874	0.466	0.466
5	2	1	0.645	0.645	0.845	0.999	0.859	0.854	0.466	0.466	0.652	0.653	0.845	1.000	0.878	0.870	0.466	0.466
5	1	1	0.645	0.646	0.845	0.999	0.855	0.851	0.466	0.466	0.653	0.653	0.845	1.000	0.872	0.872	0.466	0.466
5	1	2	0.645	0.644	0.845	0.999	0.854	0.855	0.466	0.466	0.653	0.652	0.845	1.000	0.871	0.877	0.466	0.466

Table 8: Monte Carlo simulation results with $N_r = 20$ and $R = 4$

σ	ϑ_{10}	ϑ_{20}	ϑ_{30}	ϑ_{40}	δ_0	$d_{r,i0}$	$\hat{\vartheta}_1$	$\hat{\vartheta}_2$	$\hat{\vartheta}_3$	$\hat{\vartheta}_4$	$\hat{\delta}$	R_g^2	R_{R1}^2	R_{R2}^2	R_{R3}^2	R_{R4}^2	$\bar{\hat{d}}_{R1}$	$\bar{\hat{d}}_{R2}$	$\bar{\hat{d}}_{R3}$	$\bar{\hat{d}}_{R4}$
T=250																				
0.5	0.3	0.4	0.5	0.6	0.6	0	0.28	0.37	0.47	0.57	0.59	0.99	0.92	0.92	0.92	0.92	0.06	0.06	0.06	0.06
1	0.3	0.4	0.5	0.6	0.6	0	0.29	0.39	0.48	0.58	0.60	1.00	0.97	0.96	0.96	0.94	0.06	0.06	0.06	0.06
2	0.3	0.4	0.5	0.6	0.6	0	0.29	0.39	0.48	0.58	0.60	1.00	0.98	0.98	0.96	0.95	0.06	0.06	0.06	0.06
0.5	0.5	0.6	0.7	0.8	1	0.25	0.48	0.58	0.68	0.79	0.98	1.00	0.91	0.90	0.86	0.83	0.25	0.25	0.25	0.25
1	0.5	0.6	0.7	0.8	1	0.25	0.50	0.59	0.69	0.80	0.98	1.00	0.94	0.92	0.89	0.84	0.25	0.25	0.25	0.25
2	0.5	0.6	0.7	0.8	1	0.25	0.50	0.60	0.70	0.79	0.98	1.00	0.96	0.93	0.88	0.84	0.25	0.25	0.25	0.25
T=1000																				
0.5	0.2	0.3	0.4	0.5	0.5	0	0.20	0.30	0.40	0.50	0.50	0.99	0.92	0.94	0.95	0.94	0.06	0.06	0.06	0.06
1	0.2	0.3	0.4	0.5	0.5	0	0.20	0.30	0.40	0.50	0.51	1.00	0.98	0.98	0.98	0.96	0.06	0.06	0.06	0.06
2	0.2	0.3	0.4	0.5	0.5	0	0.20	0.30	0.41	0.50	0.51	1.00	0.99	0.99	0.98	0.97	0.06	0.06	0.06	0.06
0.5	0.4	0.5	0.6	0.7	0.9	0.25	0.40	0.50	0.60	0.71	0.91	1.00	0.93	0.92	0.90	0.85	0.26	0.26	0.26	0.26
1	0.4	0.5	0.6	0.7	0.9	0.25	0.40	0.51	0.61	0.71	0.91	1.00	0.96	0.95	0.92	0.87	0.26	0.26	0.26	0.26
2	0.4	0.5	0.6	0.7	0.9	0.25	0.40	0.51	0.61	0.71	0.91	1.00	0.98	0.95	0.92	0.86	0.26	0.26	0.26	0.26
T=5000																				
0.5	0.15	0.25	0.35	0.45	0.45	0	0.14	0.24	0.34	0.44	0.45	0.99	0.93	0.94	0.95	0.96	0.06	0.06	0.06	0.06
1	0.15	0.25	0.35	0.45	0.45	0	0.15	0.24	0.34	0.44	0.45	1.00	0.98	0.98	0.98	0.98	0.06	0.06	0.06	0.06
2	0.15	0.25	0.35	0.45	0.45	0	0.15	0.25	0.35	0.44	0.45	1.00	0.99	0.99	0.99	0.98	0.06	0.06	0.06	0.06
0.5	0.35	0.45	0.55	0.65	0.85	0.25	0.34	0.44	0.55	0.65	0.86	1.00	0.94	0.94	0.92	0.88	0.26	0.26	0.26	0.26
1	0.35	0.45	0.55	0.65	0.85	0.25	0.35	0.45	0.55	0.66	0.86	1.00	0.98	0.96	0.93	0.88	0.26	0.26	0.26	0.26
2	0.35	0.45	0.55	0.65	0.85	0.25	0.35	0.45	0.55	0.65	0.86	1.00	0.99	0.97	0.93	0.88	0.26	0.26	0.26	0.26

Table 9: Monte Carlo simulation results with $N_r = 20, T = 5000, R = 2$

ϑ_r	δ	d	$\hat{\vartheta}_1$	$\hat{\vartheta}_2$	$\hat{\delta}$	R^2_G	R^2_{R1}	R^2_{R2}	$\bar{\hat{d}}_{R1}$	$\bar{\hat{d}}_{R2}$
Experiment: Nonstationary d_{ri}										
0.7	0.8	0.6	0.71	0.71	0.81	0.99	0.80	0.81	0.6	0.6
1	1.2	0.8	1.01	1.01	1.20	0.99	0.66	0.66	0.82	0.82
1.3	1.5	1	1.31	1.31	1.50	1.00	0.48	0.50	1.03	1.03
1.5	1.5	1	1.5	1.5	1.5	1.00	0.02	0.17	1.06	1.06
Experiment: Non-fractional cointegration										
0.3	0.5	0.6	0.39	0.39	0.50	0.82	0.60	0.61	0.66	0.66
0.5	0.7	1	0.85	0.85	0.85	0.18	0.07	0.07	0.98	0.98

Table 10: Number of Common Factors. 2 regions. 3 blocks of data. (N = 40, T = 500 and k_{max} = 10). 1 Global factor and 1 regional factor in each region.

	Neglected memory			First Difference			Fractional Differencing using δ_0		
	$r_{B_{R_1 \cup R_2}}$	$r_{B_{R_1}}$	$r_{B_{R_2}}$	$r_{B_{R_1 \cup R_2}}$	$r_{B_{R_1}}$	$r_{B_{R_2}}$	$r_{B_{R_1 \cup R_2}}$	$r_{B_{R_1}}$	$r_{B_{R_2}}$
$d_{r,i0} = 1.5$, $\delta_0 = 2$, and $\vartheta_{r0} = 1.8$.									
IC1	10	10	10	6.7	5.5	5.5	3.0	2.0	2.0
IC2	10	10	10	6.6	5.2	5.3	3.0	2.0	2.0
IC3	10	10	10	7.4	6.2	6.2	3.0	2.0	2.0
PC1	10	10	10	8.4	9.7	9.8	3.0	6.4	6.4
PC2	10	10	10	8.3	9.7	9.7	3.0	6.1	6.2
PC3	10	10	10	8.8	9.9	9.9	3.0	6.9	6.9
$d_{r,i0} = 0.4$, $\delta_0 = 1$, and $\vartheta_{r0} = 0.7$.									
IC1	3.31	2.25	2.23	3.00	2.00	2.00	3.00	2.00	2.00
IC2	3.27	2.23	2.21	3.00	2.00	2.00	3.00	2.00	2.00
IC3	3.43	2.29	2.26	3.00	2.00	2.00	3.00	2.00	2.00
PC1	4.65	7.47	7.45	3.00	4.40	4.40	3.00	5.48	5.49
PC2	4.53	7.27	7.28	3.00	4.11	4.10	3.00	5.21	5.20
PC3	4.97	7.84	7.84	3.00	4.95	4.96	3.00	6.06	6.03
$d_{r,i0} = 0.6$, $\delta_0 = 1$, and $\vartheta_{r0} = 0.8$.									
IC1	6.75	5.51	5.57	3.00	2.00	2.00	3.00	2.00	2.00
IC2	6.56	5.22	5.27	3.00	2.00	2.00	3.00	2.00	2.00
IC3	7.37	6.20	6.15	3.00	2.00	2.00	3.00	2.00	2.00
PC1	8.42	9.77	9.73	3.00	3.96	3.94	3.00	5.35	5.35
PC2	8.28	9.69	9.64	3.00	3.70	3.67	3.00	5.07	5.10
PC3	8.85	9.89	9.87	3.00	4.54	4.53	3.00	5.91	5.90

Table 11: Number of Common Factors. 2 regions. 3 blocks of data. (N = 40, T = 500 and k_{max} = 10). 2 Global factors and 2 regional factors in each region.

	Neglected memory			First Difference			Fractional Differencing using δ_0		
	$r_{B_{R_1 \cup R_2}}$	$r_{B_{R_1}}$	$r_{B_{R_2}}$	$r_{B_{R_1 \cup R_2}}$	$r_{B_{R_1}}$	$r_{B_{R_2}}$	$r_{B_{R_1 \cup R_2}}$	$r_{B_{R_1}}$	$r_{B_{R_2}}$
$d_{r,i0} = 1.5$, $\delta_0 = 2$, and $\vartheta_{r0} = 1.8$.									
IC1	10.00	10.00	10.00	6.78	4.83	4.80	6.00	4.00	4.00
IC2	10.00	10.00	10.00	6.68	4.77	4.75	6.00	4.00	4.00
IC3	10.00	10.00	10.00	7.06	4.99	4.98	6.00	4.00	4.00
PC1	10.00	10.00	10.00	7.84	9.12	9.11	6.00	6.95	6.99
PC2	10.00	10.00	10.00	7.73	8.99	8.98	6.00	6.74	6.75
PC3	10.00	10.00	10.00	8.15	9.40	9.39	6.00	7.40	7.45
$d_{r,i0} = 0.4$, $\delta_0 = 1$, and $\vartheta_{r0} = 0.7$.									
IC1	6.05	4.08	4.08	6.00	4.00	4.00	6.00	4.00	4.00
IC2	6.04	4.07	4.07	6.00	4.00	4.00	6.00	4.00	4.00
IC3	6.08	4.10	4.10	6.00	4.00	4.00	6.00	4.00	4.00
PC1	6.47	7.90	7.92	6.00	5.21	5.22	6.00	6.13	6.12
PC2	6.40	7.73	7.73	6.00	4.98	5.00	6.00	5.89	5.87
PC3	6.66	8.22	8.21	6.00	5.68	5.68	6.00	6.60	6.61
$d_{r,i0} = 0.6$, $\delta_0 = 1$, and $\vartheta_{r0} = 0.8$.									
IC1	8.77	7.74	7.73	6.00	4.00	4.00	6.00	4.00	4.00
IC2	8.59	7.39	7.37	6.00	4.00	4.00	6.00	4.00	4.00
IC3	9.24	8.41	8.53	6.00	4.00	4.00	6.00	4.00	4.00
PC1	9.38	9.89	9.89	6.00	4.88	4.90	6.00	5.98	6.01
PC2	9.27	9.84	9.85	6.00	4.67	4.69	6.00	5.74	5.79
PC3	9.68	9.95	9.96	6.00	5.33	5.35	6.00	6.45	6.49

Table 12: Number of Common Factors. 3 regions. 7 blocks of data. (N = 40, T = 500 and k_{max} = 10). 1 Global factor and 1 regional factor in each region. Estimation is performed in first differences. $d_{r,i0} = 0.6$, $\delta_0 = 1$, and $\vartheta_{r0} = 0.8$.

	$r_{B_{R_1 \cup R_2 \cup R_3}}$	$r_{B_{R_1}}$	$r_{B_{R_2}}$	$r_{B_{R_3}}$	$r_{B_{R_1 \cup R_2}}$	$r_{B_{R_1 \cup R_2}}$	$r_{B_{R_2 \cup R_3}}$
IC1	4.00	2.00	2.00	2.00	3.00	3.00	3.00
IC2	4.00	2.00	2.00	2.00	3.00	3.00	3.00
IC3	4.00	2.00	2.00	2.00	3.00	3.00	3.00
PC1	4.00	5.38	5.35	5.32	3.00	3.00	3.00
PC2	4.00	5.09	5.09	5.07	3.00	3.00	3.00
PC3	4.00	5.92	5.88	5.90	3.00	3.00	3.00